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CONSTRAINT ALGEBRA FOR INTERACTING QUANTUM SYSTEMS

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ABSTRACT

We consider relativistic constrained systems interacting with external fields. We provide physical arguments to support the idea that the quantum constraint algebra should be the same as in the free quantum case. For systems with ordering ambiguities this principle is essential to obtain a unique quantization. This is shown explicitly in the case of a relativistic spinning particle, where our assumption about the constraint algebra plus invariance under general coordinate transformations leads to a unique S-matrix.

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1. - Recently, gauge invariant systems based on singular relativistic Lagrangians have attracted a lot of interest, and the Hamiltonian BRS approach (invented by Batalin, Fradkin and Vilkovisky [1,2]) has become the basic tool for most modern investigations^{F1)}.

Classically, one assumes that all first-class constraints satisfy a Lie algebra

$$\{F_\alpha, F_\beta\} = C_{\alpha\beta}^\gamma F_\gamma \quad (1 \leq \alpha, \beta, \gamma \leq m) \quad (1)$$

which is realized in terms of graded Poisson brackets^{F1)}, where $C_{\alpha\beta}^\gamma$ are constants. Next, the procedure consists in introducing three extra sets of canonical variables, one of which consists in m pairs of canonically-conjugated ghosts (η^α, p_α) . The fundamental object is the BRS operator^{F2)}

$$\Omega^{\min} = \eta^\alpha F_\alpha - \frac{1}{2} (-)^{\epsilon_\beta} \eta^\beta \eta^\gamma C_{\gamma\beta}^\alpha p_\alpha \quad (2)$$

which fulfils the PB relation

$$\{\Omega^{\min}, \Omega^{\min}\} = 0. \quad (3)$$

Now quantization can be introduced. A possibility is to work in the framework of the path-integral method (as in the original paper of Fradkin and Vilkovisky [1]). Alternatively, one can follow the standard operator procedure. In the latter method, one expects naively that Eq. (1) becomes

$$[F_\alpha, F_\beta] = C_{\alpha\beta}^\gamma F_\gamma \quad (4)$$

where the square brackets denote graded commutators^{F1)}, and Eq. (3) turns into the nilpotence condition

$$\Omega^{\min 2} = 0. \quad (5)$$

We like to stress that Eq. (5) is essential to make the theory physically consistent, since it is responsible for the decoupling of the unphysical degrees of freedom from the S-matrix.

F1) For a comprehensive review, we refer to Ref. [3].

F2) The relation of $\Omega^{\min}$ to the complete form of the BRS operator (involving further variables) is well known. See Ref. [3].

We consider throughout the case of a vanishing classical Hamiltonian $H_c = 0$, where the BRS operator is supposed to fix the S-matrix in a unique way (Fradkin and Vilkovisky theorem [1-3]).

Actually, in the transition from classical to quantum theory, attention should be paid to the ordering of operators^{F3)}. Presently, Eq. (4) might be modified by extra terms which are precisely due to operator-ordering effects. Nevertheless, a compelling requirement is that the definition (2) of the BRS operator and its fundamental property (5) should be preserved at the quantum level. Let us now investigate which modifications of Eq. (4) are compatible with these conditions.

It is standard to define

$$\bar{F}_\alpha = [\Omega^{\text{min}}, P_\alpha] = F_\alpha + F_\alpha^{\text{ghost}} \quad (6)$$

where F_α^{ghost} is a bilinear expression in the ghost variables. As a consequence of Eqs. (2) and (5), the $\bar{F}_\alpha$ satisfy the Lie algebra

$$[\bar{F}_\alpha, \bar{F}_\beta] = C_{\alpha\beta}^\gamma \bar{F}_\gamma \quad (7)$$

where no extra term is allowed. Now, since

$$[F_\alpha, F_\beta^{\text{ghost}}] = 0 \quad (8)$$

Eq. (7) can be rewritten as

$$[F_\alpha, F_\beta] - C_{\alpha\beta}^\gamma F_\gamma = C_{\alpha\beta}^\gamma F_\gamma^{\text{ghost}} - [F_\alpha^{\text{ghost}}, F_\beta^{\text{ghost}}] \quad (9)$$

Owing to the fact that the left-hand side of Eq. (9) contains only physical variables, while only unphysical degrees of freedom appear in its right-hand side, it follows that Eq. (9) is actually an equality between C-number terms. As a result, Eq. (4) can be modified by a C-number term only.

For a free string, we deal with the celebrated Virasoro algebra. The equivalent of Eq. (4) gets in this case an extra C-number term. In the critical dimension, there is a complete cancellation [as shown in Eq. (9)] with the ghost contribution, so that Eqs. (7) are unchanged and Ω is still nilpotent.

^{F3)} Ordering problems affect also in a more subtle but physically equivalent way the path-integral approach [4].

In this paper we wish to consider constrained quantum systems interacting with external fields, e.g., electromagnetic field $A_\mu(x)$ and gravitational field $g_{\mu\nu}(x)$. In this case, the ordering problem takes a completely new dimension. Due to the presence in the constraints of interaction terms, like $p_\mu A^\mu(x)$ and $p_\mu p_\nu g^{\mu\nu}(x)$, one has a fundamental quantum ambiguity which involves now q-number terms.

Since $A_\mu(x)$ and $g_{\mu\nu}(x)$ appear in the starting Lagrangian, they modify the form of the constraints. We introduce the key dynamical assumption that the form of the BRS operator, as given by Eq. (2), remains unchanged; only the explicit form of the constraints is affected by the presence of the external fields. The nilpotence condition (5) together with Eqs. (6)-(9) tells that the quantum constraint algebra in the presence of external fields remains the same as in the free quantum case.

We shall see that this principle, familiar to people acquainted with current algebra, is a powerful tool to handle the ordering ambiguities.

2. - A particularly interesting example of constraint algebra is provided by spinning systems, like the spinning particle and superstrings.

It exhibits two remarkable features:

i) all bosonic constraints ϕ_α are given by graded commutators of certain fermionic constraints χ_α , i.e.

$$\phi_\alpha = f_\alpha^{\beta\gamma} [\chi_\beta, \chi_\gamma] \quad (10)$$

ii) the bosonic constraints ϕ_α depend quadratically on the canonical momentum p_μ , whereas the fermionic ones χ_α have at most a linear dependence on p_μ .

As a consequence, all we need to know is the explicit form of χ_α , for the ϕ_α are dictated by Eq. (10).

Now, for operators which are at most linear in p_μ the ordering ambiguities come in a much milder form, and can be completely fixed by using arguments like general co-ordinate invariance and gauge invariance. This has already been noticed and developed in two preceding papers dealing with non-relativistic supersymmetric quantum mechanics [5,6]. There is however a fundamental difference. In the latter non-relativistic case, the basic equations corresponding to Eq. (10) are those of the SUSY algebra which is present only in a subclass of Lagrangians. On the other hand, in the relativistic case under discussion, Eqs. (10) are dictated by the

internal consistency of the theory. More explicitly, Eqs. (10) do not come from a particular choice of parameters imposed by the "outside" SUSY requirement, but follow directly from the nilpotence Eq. (5).

3. - We wish now to illustrate our point of view on the simplest spinning system: the relativistic quantum spinning particle (for simplicity, we work in Euclidean space-time).

Let us consider first the free quantum case [7]. The constraints are

$$\chi = p_\mu \xi^\mu + m \xi_5 \quad (11a)$$

$$\varphi = p_\mu p^\mu + m^2 \quad (11b)$$

where p_μ is the bosonic momentum and ξ^μ , ξ_5 are real Grassmann variables. They obey the graded canonical algebra

$$[\chi^\mu, p_\nu] = i \delta^\mu_\nu \quad (12a)$$

$$\{\xi^\mu, \xi_\nu\} = \delta^\mu_\nu \quad (\mu, \nu = 1, \dots, 4) \quad (12b)$$

$$\{\xi_5, \xi_5\} = 1 \quad (12c)$$

(In this section [,] brackets mean ordinary commutators, while { , } brackets are anticommutators). The constraints (12) obey the quantum algebra^{F4)}

$$\{\chi, \chi\} = \varphi \quad (13)$$

Finally, the BRS operator is given by

$$\Omega^{min} = \eta^1 \varphi + \eta^2 \chi + \frac{i}{2} p_1 (\eta^2)^2 \quad (14)$$

We now introduce the external fields $A_\mu(x)$ and $g_{\mu\nu}(x)$, and follow closely the mathematical scheme of Ref. [6]. The canonical formalism requires the use of the fermionic variables

^{F4)} It is clear that in the present case there is no difference between the classical and quantum constraint algebra.

$$\xi^a = e^a_\mu(x) \xi^\mu \quad (a = 1, \dots, 4) \quad (15)$$

where $e^a_\mu(x)$ are the vierbein. The non-vanishing fundamental brackets are

$$[\chi^\mu, p_\nu] = i \delta^\mu_\nu \quad (16a)$$

$$\{\xi^a, \xi^b\} = \delta^{ab} \quad (16b)$$

$$\{\xi_5, \xi_5\} = 1. \quad (16c)$$

Classically, the fermionic constraint reads now

$$\chi = \pi_\mu \xi^\mu + m \xi_5 \quad (17)$$

with

$$\pi_\mu = p_\mu + \frac{i}{2} \omega_{\mu,ab}(x) \xi^a \xi^b + A_\mu(x) \quad (18)$$

where $\omega_{\mu,ab}(x)$ is the spin-connection.

Let us now consider the quantum mechanical modification of Eqs. (17) and (18). The definition (18) of π_μ is unambiguous, owing to the antisymmetry of $\omega_{\mu,ab}$. Invariance under both general co-ordinate transformations and local Lorentz rotations implies

$$\chi = \frac{1}{2} \{\pi_\mu, \xi^\mu\} + m \xi_5. \quad (19)$$

At this point, we make use of our fundamental assumption that the constraint algebra is still given by Eq. (13). This allows us to compute ϕ in its unambiguously-ordered form

$$\begin{aligned} \phi = & \frac{1}{2} g^{-1/4} \pi_\mu g^{1/2} g^{\mu\nu} \pi_\nu g^{-1/4} - \frac{1}{8} R + \\ & + m^2 - \frac{1}{2} F_{\mu\nu} e^\mu_a e^\nu_b \xi^a \xi^b \end{aligned} \quad (20)$$

where R is the curvature scalar. We stress that, as a consequence of our approach, the form of ϕ is uniquely defined. It is pleasant to notice that the correct Dirac gyromagnetic ratio is completely determined without any minimality assumption.

One can next proceed to compute Ω which, as already stated, has the same form as in the free-case. Applying then the now standard procedure of Fradkin et al. [1-3], one is finally led to a unique S-matrix (the arbitrariness in the effective Hamiltonian affects only off-shell amplitudes).

It is worthwhile translating our result in the language of path-integrals, where the ordering ambiguity appears in its very definition [4]. One can pass from the operator formalism to the path-integral by recalling the well-known equivalence between Weyl-ordering and middle-point prescription [4]. In this respect, it is useful to write our result for ϕ in the form

$$\varphi = \frac{1}{2} \left(\pi_\mu g^{\mu\nu} \pi_\nu - F_{\mu\nu} e^\mu_a e^\nu_b \xi^a \xi^b + m^2 \right)_{W0} + \Delta V \quad (21)$$

where the Weyl-ordering is both in bosonic and fermionic variables^{F5)} and the effective potential ΔV is

$$\Delta V = \frac{1}{8} g^{\mu\nu} \left(\Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} + \frac{1}{2} \omega_{\mu,ab} \omega_{\nu,ab} \right). \quad (22)$$

4. - In this paper we have considered relativistic constrained systems interacting with external fields. We have introduced and motivated the principle that the quantum constraint algebra should be the same as in the free quantum case. We have then studied explicitly the example of a relativistic spinning particle interacting with external fields, focusing our attention on the ordering ambiguities. We have seen that our dynamical principle combined with invariance arguments fixes completely these ambiguities in the S-matrix.

It looks interesting to apply our approach to extended systems, like the relativistic string. In this respect, it is encouraging to notice that at the classical level the presence of an external massless background does not change the constraint algebra for the bosonic string [8]. At the quantum level, the question is still open and we hope that by our method, ordering ambiguities for spinning systems like superstrings can be dealt with. Work is in progress in this direction.

^{F5)} The Weyl-ordering with respect to bosonic and fermionic variables is defined in Ref. [5], where the main properties are discussed.

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