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A LATTICE CALCULATION OF THE PION'S FORM FACTOR AND STRUCTURE FUNCTION

G. Martinelli
CERN - Geneva

and

C.T. Sachrajda
Physics Department, The University
Southampton SO9 5NH, United Kingdom

ABSTRACT

In this paper we present the results of a lattice computation of the pion's form factor at several values of momentum transfer and the lowest two moments of the pion's structure function. The results are very satisfying, having small statistical errors and being consistent with experimental results. In addition, we present all the details of the Monte Carlo simulation and an outline of the analysis of the results.

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1. - INTRODUCTION

In Ref. [1] we have presented the results for the first two moments of the pion's structure function, computed using lattice QCD. These were the first results of our project to study many features of hadron structure on a $20 \times 10^2 \times 40$ lattice using Wilson fermions in the quenched approximation at $\beta = 6/g_0^2 = 6.0$. In this paper, we present results for the pion's electromagnetic form factor at several values of momentum transfer (see Fig. 5 below), improved results for the first two moments of the structure function [Eqs. (6.14) and (6.15)] and some results (albeit with large errors) for $\langle \xi^2 \rangle$, the second moment of the quark distribution amplitude of the pion (Tables 4 and 5). In addition we discuss the details of the calculations and the relevant properties of the 15 configurations which we use.

We use the Wilson formulation of lattice QCD [2] for which the action is given by

$$\begin{aligned}
 S &= \sum_f \bar{\psi}^f \Delta(U, K) \psi^f + S_{Gluon}(U) \\
 &= \sum_f \sum_x \left(\sum_{\mu} [K \bar{\psi}^f(x) (1 + \gamma^{\mu}) U_{\mu}(x) \psi^f(x + \hat{\mu}) + K \bar{\psi}^f(x + \hat{\mu}) (1 - \gamma^{\mu}) U_{\mu}^{\dagger}(x) \psi^f(x)] \right. \\
 &\quad \left. - \bar{\psi}^f(x) \psi^f(x) \right) + \frac{1}{6} \sum_{\text{plaquettes}} \text{Tr}[U_p + U_p^{\dagger}]
 \end{aligned}
 \tag{1.1}$$

where all colour and spin indices have been omitted for simplicity. For our calculations, the flavour label, f , distinguishes between the up or down quark, and we take the same value of the Wilson parameter K in both cases. $\beta = 6/g_0^2$ where g_0 is the bare coupling constant and U_p is the product of gauge fields belonging to the elementary plaquette P . The Euclidean γ -matrices are Hermitian and are defined by

$$\gamma^4 = \gamma_{BD}^0 \quad \underline{\gamma} = i\gamma_{BD}
 \tag{1.2}$$

where the subscript BD indicates the basis defined in Ref. [3].

All the calculations discussed in this paper require the evaluation of matrix elements of local operators, of the form $\langle \pi | O | \pi \rangle$ in the case of the form factor and moments of structure functions, and of the form $\langle o | O | \pi \rangle$ in the case of $\langle \xi^2 \rangle$, where $|\pi\rangle$ and $|o\rangle$ represent a single pion and vacuum state respectively. For form factors, O is the electromagnetic current, whereas for moments of structure

functions or $\langle \xi^2 \rangle$, O is the corresponding lowest twist operator. The choice of suitable lattice operators, from which the continuum matrix elements can be determined, is governed by their renormalization properties, which in turn depend on the transformation properties of the operators under the hypercubic group. This question will be discussed in detail in Section 2. One important consequence of this discussion is that for all matrix elements of the form $\langle \pi | O | \pi \rangle$, at least one of the pions must be chosen to have a non-zero component of momentum in one spatial direction. It is for this reason that we have chosen to work on a lattice which is larger in one spatial direction, enabling us to decrease the minimum momentum which the pion can have, and hence to decrease the sensitivity of our calculations to the granularity of the lattice.

In Section 3, we explain the procedure which we use to compute the operator matrix elements. Section 4 contains a determination of the pion and rho masses, obtained using 15 gauge configurations at three values of the Wilson parameter K . From these we determine K_c (the value of K at the chiral limit), and the value of the lattice spacing in physical units.

The main results of this paper are contained in Sections 5, 6 and 7. Section 5 deals with the electromagnetic form factor of the pion $F(q^2)$, defined by

$$\langle \pi(p) | V^\mu(0) | \pi(p') \rangle = (p + p')^\mu F(q^2) \quad (1.3)$$

where V^μ is the electromagnetic current $2/3\bar{u}\gamma^\mu u - 1/3\bar{d}\gamma^\mu d$ (working in the quenched approximation we have no contribution from heavier quarks), and $q^2 \equiv (p' - p)^2$. On the lattice only a discrete set of values is possible for the momenta, and hence we are only able to evaluate $F(q^2)$ for the corresponding momentum transfers.

In Section 6 we present our results for $\langle x \rangle$ and $\langle x^2 \rangle$, the first two moments of the momentum distribution function of the valence quarks in the pion. These results are a little more precise than those presented in Ref. [1]. This improvement is due to the fact that by dividing the correlation functions (see Section 3) involving the appropriate operator of lowest twist by that involving the lattice conserved vector current at zero momentum transfer [for which we know $F(0) = 1$], we are able to obtain $\langle x \rangle$ and $\langle x^2 \rangle$ without any need to fit the time dependence of either correlation function to one or more falling exponentials.

Section 7 contains our attempts to determine $\langle \xi^2 \rangle$, the second moment of the pion's distribution amplitude $\phi(\xi, q^2)$. The distribution amplitude contains the non-perturbative information required to predict hard exclusive processes in QCD, such as hadronic form factors or elastic scattering at large momentum transfers [4-8]. q^2 is the square of the momentum transfer and $\xi \equiv x - x_{q^-}$, where in a physical

gauge x_q and $x_{\bar{q}}$ are interpreted as the fractions of momenta carried by the valence quark and antiquark. In Ref. [9] we evaluated $\langle \xi^2 \rangle$ on a $10^3 \times 20$ lattice and obtained a result $\langle \xi^2 \rangle_{\overline{MS}} = 0.34 \pm 0.17$ at $\sqrt{-q^2} \approx 2$ GeV. The large error was primarily due to the fact that the computation involved the numerical subtraction of terms which diverge as inverse powers of the lattice spacing. We have repeated our calculations on the $20 \times 10^2 \times 40$ lattice and have obtained results which are consistent with those found earlier, but with even larger fluctuations. This is clearly a very difficult quantity to determine on the lattice. Section 8 contains our conclusions.

2. - LATTICE RENORMALIZATION OF COMPOSITE OPERATORS

In this section, we discuss the renormalization properties of the operators whose matrix elements we wish to compute.

Because Euclidean invariance (and also chiral symmetry in the case of Wilson fermions) is not a symmetry of the lattice, the mixing of operators under renormalization is generally more complicated on the lattice than in continuum renormalization schemes. In general, composite operators can mix with other operators of equal or lower dimension. In the latter case, the mixing coefficients diverge as powers of the lattice spacing a , and cannot be computed in perturbation theory. In all the cases to be discussed below, it is possible to avoid these power divergences by choosing the "Lorentz" indices of the operator judiciously, or by taking suitable linear combinations of operators, so that their transformation properties under the hypercubic group prevent any mixing with operators of lower dimension. In addition, we need to know how to relate the operator matrix elements calculated using the lattice renormalization scheme to those renormalized using some standard continuum scheme (such as the $\overline{MS}$ scheme). We now consider in turn each of the operators in which we are interested, starting with the vector current, for which the problem of power divergences does not arise.

2.1 The Vector Current, $\bar{q}(x)\gamma^\mu q(x)$

To determine the electromagnetic form factor of the pion, we have computed the matrix elements of the local vector current,

$$V_\mu^{LOC}(x) = \bar{q}(x)\gamma_\mu q(x) \quad (2.1)$$

In gauge-invariant renormalization schemes (such as the $\overline{MS}$ scheme), this operator is conserved. This is not the case on the lattice; nevertheless, one can show that V_μ^{LOC} is multiplicatively renormalized (in the limit $a \rightarrow 0$), since the only con-

tinuum operator with equal or lower dimension with which it could mix is V_μ^{LOC} itself [10,11]. On the lattice, using the Wilson action of Eq. (1.1), the conserved current is^{*}

$$\hat{V}_\mu(x) = \frac{1}{2} [\bar{q}(x)(\gamma_\mu - 1)U_\mu(x)q(x + \hat{\mu}) + \bar{q}(x + \hat{\mu})(\gamma_\mu + 1)U_\mu^+(x)q(x)] \quad (2.2)$$

As a check of our calculational procedures, we will compute the renormalization constant of this operator numerically and verify that, within our errors, it is equal to 1 (see Section 5). In addition, the knowledge that this operator has a renormalization constant equal to 1 can be used to reduce the systematic errors in the determination of the moments of the structure function.

For later use, we also define the "extended" vector current,

$$V_\mu^{\text{EXT}}(x) = \frac{1}{2} [\bar{q}(x)\gamma_\mu U_\mu(x)q(x + \hat{\mu}) + \bar{q}(x + \hat{\mu})\gamma_\mu U_\mu^+(x)q(x)] \quad (2.3)$$

The matrix elements of $\hat{V}_\mu$, V_μ^{LOC} and V_μ^{EXT} between on-shell states are related through the equation

$$\langle \alpha | \hat{V}_\mu | \beta \rangle = Z_V^{\text{LOC}}(g_0) \langle \alpha | V_\mu^{\text{LOC}} | \beta \rangle = Z_V^{\text{EXT}}(g_0) \langle \alpha | V_\mu^{\text{EXT}} | \beta \rangle \quad (2.4)$$

The renormalization constants Z_V^{LOC} and Z_V^{EXT} are finite functions of the bare lattice coupling g_0 and can be computed in perturbation theory. On the lattice we can measure these constants by comparing the matrix elements of V_μ^{LOC} or V_μ^{EXT} with those of $\hat{V}_\mu$. In Section 5 we will present the results of such a non-perturbative determination of the renormalization constants as well as a comparison with the perturbative results.

^{*}) In some cases it is convenient or necessary to use $\frac{1}{2} [\hat{V}_\mu(x) + \hat{V}_\mu(x-\mu)]$ which transforms irreducibly. For the applications in this paper, the two definitions are equivalent, since we sum over positions.

2.2 The Operator $O_2^{\mu\nu} \equiv i^2 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \psi$ - traces

In order to calculate $\langle x \rangle$, the average momentum carried by the valence quarks in the pion, we need to compute the matrix element

$$\langle \pi(p) | O_2^{\mu\nu}(0) | \pi(p) \rangle = p^\mu p^\nu A_2(p) \quad (2.5)$$

where $|\pi(p)\rangle$ is a pion state with momentum p and

$$O_2^{\mu\nu}(0) = i^2 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \psi - \text{traces} \quad (2.6)$$

(with suitable flavour indices and symmetrized over the Lorentz indices). We now discuss the two cases $\mu = \nu$ and $\mu \neq \nu$ in turn.

The three linearly-independent operators with $\mu = \nu$ transform like one of the three-dimensional irreducible representations of the hypercubic group (in the notation of Ref. [12], where the character table for this group can be found). The subtraction of the trace removes the singlet component [which is proportional to $1/a(\bar{\psi}\psi)$] and hence eliminates the power divergence. Such subtractions have, in previous cases, led to significant numerical cancellations and hence to large statistical errors [9].

The subtraction of power divergences can be avoided by choosing μ to be different from ν . The six operators $O_2^{\mu\nu}$ with $\mu \neq \nu$ transform like the six-dimensional irreducible representation of the hypercubic group, and for this reason there is no mixing with lower-dimensional operators. In addition, there is no numerical subtraction of divergent terms in the calculation itself, and hence the choice $\mu \neq \nu$ is likely to give results with a smaller statistical error. The price to be paid for choosing to have $\mu \neq \nu$ is that the pion must be given a non-zero component of momentum in at least one spatial direction, otherwise the matrix element (2.5) vanishes. Finally, as was also the case for the vector current, a finite multiplicative renormalization is needed to relate the lattice operator to the corresponding continuum operator defined in a given renormalization scheme. For the $\overline{MS}$ scheme, this renormalization is given by

$$\langle \pi(p) | O_2^{\mu\nu}(0) | \pi(p) \rangle_{\overline{MS}} = \langle \pi(p) | O_2^{\mu\nu} | \pi(p) \rangle_{LATT} \left(1 + \frac{\alpha_s}{4\pi} C_F \delta \right) \quad (2.7)$$

where $\delta = -2.5$ for $\mu = \nu$ [13,14] and $\delta = -1.2$ for $\mu \neq \nu$ [13]^{*)}. At $\beta = 6.0$ ($g_0 = 1$)

*) We disagree with the result in Ref. [15] which gave $\delta = +17.1$.

the perturbative corrections are negligible and the matrix elements in the two schemes can be taken to be equal.

2.3 The Operator $G_2^{\mu\nu} = F^{\mu\rho}(0)F^{\nu\rho}(0)$ -traces

The average fraction of momentum carried by gluons in a pion can be evaluated from the matrix element

$$\langle \pi(p) | F^{\mu\rho}(0)F^{\nu\rho}(0) - \text{traces} | \pi(p) \rangle \quad (2.8)$$

where $F_{\mu\nu}$ is the field strength tensor. To avoid subtractions of power divergences, following the discussion of the previous subsection, we will take $\mu \neq \nu$.

We define the lattice field strength tensor to be (see also Ref. [12])

$$F^{\mu\nu}(x) = \frac{1}{4a^2} \sum_{4 \text{ plaquettes}} \frac{1}{2i} (U - U^\dagger) \quad (2.9)$$

where the sum extends over the four plaquettes in the $\mu\nu$ plane which have the point x at one corner. The matrices U are the products of the four link variables for each plaquette taken in an anticlockwise sense. With our definition of $F^{\mu\nu}$, no power divergences are encountered when evaluating the matrix element (2.8). This is no longer true if other definitions of $F^{\mu\nu}$ are used. The essential point is that $F^{\mu\nu}$ defined in Eq. (2.9) transforms like the direct sum of two three-dimensional irreducible representations of the hypercubic group ($\underline{E+B}$ and $\underline{E-B}$, each of which transform irreducibly). If one does not symmetrize over the four plaquettes, then this is not the case. Suppose, for example, that we define $F^{\mu\nu}$ using just one of the plaquettes (this is an equally good definition at the tree level). At the quantum level, the matrix element in Eq. (2.8) will diverge as inverse powers of the lattice spacing. In fact, $F^{\mu\nu}$ defined by means of a single plaquette transforms like a 24-dimensional reducible representation of the hypercubic group and $\{F^{\mu\rho}F^{\nu\rho}, \mu \neq \nu\}$ like a 192-dimensional reducible representation. This 192-dimensional representation contains the singlet, so that $F^{\mu\rho}F^{\nu\rho}$ can mix with the identity, the mixing coefficients containing power divergences. By using the definition (2.9) we automatically avoid this problem^{*}.

The perturbative corrections which relate the matrix element (2.9) as measured on the lattice with that in the $\overline{MS}$ scheme (or some other continuum scheme) are currently being calculated [17].

^{*}) Note that a different definition was used in computing spin-spin forces for heavy quark systems [16]. We believe that these calculations should be repeated using Eq. (2.9) as the definition of the field strength tensor.

2.4 The Operator $O_3^{\mu\nu\rho} \equiv i^3 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \overleftrightarrow{D}^\rho \psi$ -traces

The evaluation of the second moment of the momentum distribution of the valence partons requires the calculation of the matrix element

$$\langle \pi(p) | O_3^{\mu\nu\rho}(0) | \pi(p) \rangle = p^\mu p^\nu p^\rho A_3(p) \quad (2.10)$$

where

$$O_3^{\mu\nu\rho}(0) = i^3 \bar{\psi} \gamma^\mu \overleftrightarrow{D}^\nu \overleftrightarrow{D}^\rho \psi - \text{traces} \quad (2.11)$$

(with suitable flavour indices and symmetrized over the Lorentz indices). Also the matrix element of

$$\hat{O}_3^{\mu\nu\rho}(0) = i^3 \bar{\psi} \gamma^\mu \gamma^5 \overleftrightarrow{D}^\nu \overleftrightarrow{D}^\rho \psi - \text{traces} \quad (2.12)$$

taken between the pion state and the vacuum, is proportional to $\langle \xi^2 \rangle$ (see Section 7), the second moment of the momentum distribution relevant for hard exclusive processes.

The transformation properties of $\hat{O}_3^{\mu\nu\rho}$ (and hence also of $O_3^{\mu\nu\rho}$) under the hypercubic group have been discussed in detail in Refs. [9] and [15]. Taking $\mu = \nu = \rho$, we find that $O_3^{\mu\mu\mu}$ transforms like the four-dimensional vector representation $((\frac{1}{2}, \frac{1}{2})$ in the notation of Ref. [12]), and hence can mix with lower dimensional operators such as $\bar{\psi} \gamma^\mu \psi$. We will therefore not study the matrix elements of $O_3^{\mu\mu\mu}$, but rather of $O^{411}-O^{433}$ (which transforms like the eight-dimensional irreducible representation). To have a non-vanishing matrix element (2.10), we shall give the pion a component of momentum in the 1 direction. The subtraction in $O^{411}-O^{433}$ involves a subtraction of power divergences, and hence we expect our results for $\langle x^2 \rangle$ to have larger statistical errors than for $\langle x \rangle$.

The operator O^{412} , which transforms like the $(\frac{1}{2}, \frac{1}{2})$ four-dimensional irreducible representation (in the notation of Ref. [12]) is also free of power divergences. In this case, however, we must give the pion non-zero components of momentum in the 1 and 2 directions, increasing the systematic errors which grow as $a|p|$ increases [the granularity of the lattice becomes apparent once $a|p|$ is $O(1)$]. We point out also that the symmetrization over the indices, implicit in the definition of O^{412} , involves a cancellation of power divergences. Nevertheless, we evaluate $\langle \xi^2 \rangle$ by using both the operators $\hat{O}^{411}-\hat{O}^{433}$ and $\hat{O}^{412}$. We can do this because $\langle \xi^2 \rangle$ is derived from the matrix element $\langle 0 | \hat{O}_3^{\mu\nu\rho}(0) | \pi \rangle$ by calculating a two-point correlation function (see next section) and in this case it is straightforward to vary the pion's momentum.

The perturbative corrections required to obtain the results in the $\overline{\text{MS}}$ scheme, say, from those measured on the lattice, have not been evaluated for the operator $O_3^{411}-O_3^{433}$. Nevertheless, since tadpole diagrams tend to give the largest contribution [9,15], we estimate that we should multiply the lattice results for the matrix elements of all the $O_3^{\mu\nu\rho}$ and $\hat{O}_3^{\mu\nu\rho}$ operators by a factor of 1.13 (with an estimated uncertainty of about 0.05).

For the quantities which are obtained from matrix elements of the form $\langle \pi|O|\pi \rangle$ (e.g., $\langle x \rangle$, $\langle x^2 \rangle$ and the electromagnetic form factor), the optimal use of computing resources requires us to have one of the two pion states at the same momentum in all cases (this will be explained in detail in the next section). In view of the above discussion, for one of the two pion states we have chosen $p_1 \neq 0$ (and as small as is consistent with our lattice in order to minimize systematic errors) and $p_2 = p_3 = 0$. This means that for the calculation of $\langle x^2 \rangle$ we have to use the operator $O_3^{411}-O_3^{433}$, while all the other matrix elements do not require any subtractions.

3. - OPERATOR MATRIX ELEMENTS, CORRELATION FUNCTIONS AND QUARK PROPAGATORS

In this section, we describe the procedure we used to obtain matrix elements of the form $\langle o|O(o)|\pi \rangle$ and $\langle \pi|O(o)|\pi \rangle$. In the former case the procedure is now standard and we illustrate it briefly by studying the matrix element of the pseudo-scalar density $J_\pi \equiv \bar{\psi}\gamma^5\psi$. In Section 7, when we evaluate $\langle \xi^2 \rangle$, this procedure will be applied to other operators. We start with the two-point correlation function

$$K(t_x) = \sum_{\underline{x}} \langle 0|T[J_\pi^+(0)J_\pi(\underline{x}, t_x)]|0 \rangle e^{i\underline{p} \cdot \underline{x}} \quad (3.1)$$

For sufficiently large t_x , so that the contribution from states heavier than the pion is suppressed, we have

$$K(t_x) \simeq \frac{e^{-ET/2}}{2E} \cosh((t - T/2)E) \quad | \langle \pi|J_\pi|0 \rangle |^2 \quad (3.2)$$

where $E = \sqrt{m_\pi^2 + \underline{p}^2}$ and T is the length of the lattice in the time direction. For most of our calculations, the contribution from the backward-moving pion can be neglected (i.e., $T \gg t$), so that

$$K(t_x) \simeq \frac{e^{-Et_x}}{2E} \quad | \langle \pi|J_\pi|0 \rangle |^2 \quad (3.3)$$

By fitting the computed values of $K(t_x)$ to the expression in (3.3), the values of E

and $|\langle \pi | J_\pi(0) | 0 \rangle|^2$ can be obtained.

In the quenched approximation, $K(t_x)$ is given by

$$K(t_x) = \int [dU] e^{-S(U)} \sum_{\underline{x}} e^{i p \underline{x}} \text{Tr} [S(0, x) \gamma^5 S(x, 0) \gamma^5] \quad (3.4)$$

where $S(x, 0)$ is the quark propagator from x to the origin in the presence of the background field U . The right-hand side of (3.4) is represented by the diagram in Fig. 1a. Equation (3.4), together with the relation

$$S(0, x) = \gamma^5 S^+(x, 0) \gamma^5 \quad (3.5)$$

implies that it is sufficient to know the set of propagators $\{S(x, 0)\}$ in order to compute $K(t_x)$. These are obtained by solving the equation

$$\sum_x \Delta(y, x) S(x, 0) = \delta_{y, 0} \quad (3.6)$$

where $\Delta(y, x)$ is the kernel of the fermionic action

$$\Delta(y, x) = -K \sum_{\mu} [(1 + \gamma^{\mu}) U_{\mu}(y) \delta_{x, y + \hat{\mu}} + (1 - \gamma^{\mu}) U_{\mu}^+(y - \hat{\mu}) \delta_{x, y - \hat{\mu}}] + \delta_{x, y} \quad (3.7)$$

We use the Gauss-Siedel algorithm to solve (3.6).

For the computation of the operator matrix elements of the form $\langle \pi | 0(0) | \pi \rangle$ we start with the three-point correlation function^{*)}

$$C(t_x, t_y) = \sum_{\underline{x}, \underline{y}} e^{i p \underline{x}} \langle 0 | T [J_{\pi}^+(0) O(\underline{y}, t_y) J_{\pi}(\underline{x}, t_x)] | 0 \rangle \quad (3.8)$$

Inserting complete sets of states between the operators in (3.8), taking $t_x > t_y > 0$ and using translational invariance, we have, for sufficiently large t_y and $t_x - t_y$,

*) For the purposes of illustration we consider here the calculation of forward matrix elements. In Section 5 this discussion will be generalized to the case where there is a non-zero momentum transfer at y .

$$C(t_x, t_y) \simeq \frac{e^{-Et_x}}{4E^2} \quad | \langle \pi | J_\pi(0) | 0 \rangle |^2 \langle \pi | O(0) | \pi \rangle \quad (3.9)$$

Thus the computation of $C(t_x, t_y)$ and $K(t_x)$ enables us to determine $\langle \pi | O(0) | \pi \rangle$.

In the quenched approximation the computation of $C(t_x, t_y)$ requires the evaluation of the diagram of Fig. 1b. If, for example, we take $O(y)$ to be the local vector current $\bar{\psi}(y)\gamma^\mu\psi(y)$, then $C(t_x, t_y)$ is given by

$$C(t_x, t_y) = \int [dU] e^{-S(U)} \sum_{\underline{x}, \underline{y}} e^{i\underline{p}\underline{x}} \text{Tr}[S(y, x)\gamma^5 S(x, 0)\gamma^5 S(0, y)\gamma^\mu] \quad (3.10)$$

where again we have taken the pseudoscalar density as the interpolating operator for the pion. It is now clear that the set of propagators $\{S(x, 0)\}$ is not sufficient to evaluate $C(t_x, t_y)$. For this reason, we also generate a set of "generalized quark propagators" [18,19], $\{S_1(y, 0; t_x, \underline{p})\}$, with the insertion of a pion source with momentum $\underline{p}$ at time co-ordinate t_x . The S_1 's are defined by the relations

$$S_1(y, 0; t_x, \underline{p}) = \sum_{\underline{x}} e^{i\underline{p}\underline{x}} S(y, x)\gamma^5 S(x, 0) \quad (3.11)$$

and (3.10) can now be rewritten in the form (see Fig. 1b)

$$C(t_x, t_y) = \int [dU] e^{-S(U)} \sum_{\underline{y}} \text{Tr}[S_1(y, 0; t_x, \underline{p})\gamma^5 S(0, y)\gamma^\mu] \quad (3.12)$$

The S_1 's, together with the quark propagators $S(x, 0)$, are sufficient to evaluate $C(t_x, t_y)$ and hence $\langle \pi | O | \pi \rangle$. The S_1 's are generated by solving a similar equation to (3.6),

$$\sum_{\underline{y}} \Delta(z, y) S_1(y, 0; t_x, \underline{p}) = \gamma^5 S(z, 0) \delta_{t_z, t_x} e^{i\underline{p}\underline{z}} \quad (3.13)$$

using the same numerical techniques as for the propagator $S(x, 0)$.

We have generated the sets of propagators $\{S(x,o)\}$ and $\{S_1(y,o; t_x, \underline{p})\}$ for 15 $SU(3)_C$ configurations on a $20 \times 10^2 \times 40$ lattice, at a value of the coupling constant given by $\beta = 6/g_0^2 = 6.0$. We have chosen to have one longer spatial direction in order to be able to work with a pion with a small, but non-zero, momentum $\underline{ap} = (\pi/10, 0, 0)$. The value of t_x in $S_1(y,o; t_x, \underline{p})$ was chosen to be 15, so that t_y and $t_x - t_y$ can both be sufficiently large ($\gtrsim 5$) to isolate the pion's contribution in the correlation function (3.8). In addition, with this choice of t_x we also have room to vary t_y to check that $C(t_x, t_y)$ is indeed independent of t_y [see Eq. (3.9)]. The quark propagators were generated for three different values of the quark mass (corresponding to the following values of the Wilson parameter: $K = 0.1515, 0.1530$ and 0.1545). The results for the matrix elements are then extrapolated to the chiral limit. Finally we record that the 15 configurations were generated on a $10^3 \times 20$ lattice using the Metropolis algorithm (8 hits) and the Wilson action for the gauge fields. An initial thermalisation of 2000 sweeps and a separation of 250 sweeps between configurations were allowed. The lattice was then duplicated in the time and one space direction.

4. - π AND ρ MASSES AND THE VALUE OF THE LATTICE SPACING

To translate the results obtained on the lattice in dimensionless units at rather large values of the quark masses into physical quantities, we have to go through the following two steps:

- i) The determination of the value of the lattice spacing at the given value of the coupling constant $\beta = 6/g_0^2$. This is needed not only to set the scale for dimensionful quantities, but also to determine the scale at which the operators are renormalized.
- ii) The extrapolation of our results to the (almost) chiral limit. Since computations with very light pions are not practicable, we follow the standard procedure of performing our calculations at several (in our case three) non-zero values of the quark mass. We then extrapolate to the physical limit. In so doing, we need to know K_c , the value of the Wilson parameter at which $m_\pi^2 = 0$.

Since m_ρ is expected to be linearly related to m_π^2 , perhaps the most direct way to determine the lattice spacing (a) is to plot m_ρ versus m_π^2 and extrapolate to the physical limit. In Table 1 we present the values of m_π and m_ρ obtained from the two-point correlation functions (3.1) at zero momentum, with the pseudoscalar density $\bar{\psi}\gamma^5\psi$ as the interpolating operator for the pion and the local vector current $\bar{\psi}\gamma^\mu\psi$ for the rho. The fits were obtained using Eq. (3.3) for the range

$t_x \in (9-14)$ and Eq. (3.2) for the symmetric combination of $t_x \in (9-14)$ plus (26-31). The masses obtained by using different time intervals lie within the quoted errors. The values we obtain are compatible with those obtained in previous simulations at $\beta = 6.0$ [20-22]. A comparison for the pion case is shown in Fig. 2. Our central values of the pion and rho masses at $K = 0.1545$ are a little higher (although consistent) with those previously found.

In Fig. 3 we plot m_ρ against m_π^2 using the values from the last two columns of Table 1. The linear behaviour of m_ρ with m_π^2 is well satisfied and we find^{*)}

$$a^{-1} = (2.2 \pm 0.4) \text{GeV} \quad (4.1)$$

This is at present our best estimate of a^{-1} ; however, it should be remembered that in addition to the statistical error in (4.1), there is also a systematic uncertainty. For example, since m_p/m_ρ (where m_p is the proton's mass) in the quenched approximation is always found to be too large [23], had we used m_p rather than m_ρ to determine a^{-1} , we would have obtained a smaller result (probably by about 20% or so). We will extend our calculations to include nucleons in a future publication. Below we shall present one other estimate of a^{-1} .

At this point we should explain how we estimate the errors for all the quantities which we measure. We cluster the 15 configurations into three sets of five configurations. Each measured quantity (X , say) is determined from the full 15 configurations (giving the result $\bar{X}$) and from each of the three clusters (giving the results X_1 , X_2 and X_3), the results from the three clusters being treated as independent. The quoted result is then

$$X = \bar{X} \pm \Delta X \quad (4.2)$$

with

$$(\Delta X)^2 = \frac{1}{2} \left[\frac{1}{3} \sum_{i=1}^3 (X_i)^2 - \left(\frac{1}{3} \sum_{i=1}^3 X_i \right)^2 \right] \quad (4.3)$$

The stability of the results under different clusterings is monitored.

From the measured values of the pion mass, using the fact that m_π^2 is linearly related to $1/K$, one can obtain K_c , the value of the Wilson parameter in the chiral limit ($m_\pi^2 = 0$). Previous determinations, using this method at $\beta = 6.0$, have found

^{*)} The corresponding result obtained using the first two columns of Table 1 is $a^{-1} = 2.6 \pm 0.5 \text{ GeV}$.

$K_c = 0.1564$ with the small error of 2 on the last digit. From our 15 configurations we obtain a result with a much larger error

$$K_c = 0.1582 \pm 0.0015 \quad (4.4)$$

We stress that the result for K_c is very sensitive to small fluctuations in the measured values of the pion mass.

A second way of obtaining K_c , proposed in Ref. [22], is to use the ratio of correlation functions

$$2\rho = \frac{\sum_{\underline{x}} \langle \nabla_t A_4^+(\underline{x}, t) J_\pi(0) \rangle}{\sum_{\underline{x}} \langle J_\pi^+(\underline{x}, t) J_\pi(0) \rangle} \quad (4.5)$$

where A_μ is the axial-vector current $\bar{\psi}\gamma_\mu\gamma_5\psi$. 2ρ is also proportional to m_ρ and hence to $(1/K - 1/K_c)$. The measured values of the numerator and denominator in (4.5) are highly correlated in such a way that the fluctuations in the ratio are very small. Averaging over the results obtained from the time intervals $t = 9-14$ plus $26-31$, we find $2\rho = 0.232(4)$ for $K = 0.1515$, $2\rho = 0.158(4)$ for $K = 0.1530$ and $2\rho = 0.090(6)$ for $K = 0.1545$, where the number in parentheses is the error on the last digit. The result for K_c which we obtain in this way is

$$K_c = 0.1563 \pm 0.0002 \quad (4.6)$$

in perfect agreement with those found in Refs. [20-22]. This calculation illustrates the common feature that the fluctuations in ratios of correlation functions such as (4.5), in which the exponentials cancel, are very much smaller than those in the correlations functions themselves. It is this property which enables us to measure many operator matrix elements with good precision.

Since the value of K_c given in (4.6) is different from the central value in (4.4), we should check what value of a^{-1} is obtained by plotting m_ρ against $1/K$ (the behaviour should be linear), and imposing $m_\rho(1/K_c) = 0.77$ GeV. The result is $a^{-1} = 1.9 \pm 0.3$ GeV, in agreement with (4.1), and we retain (4.1) as our estimate of a^{-1} .

We end this section by presenting in Figs. 4a,b the pion propagator at rest and with $p_1 = \pi/10$ (the smallest value consistent with our lattice), $2\pi/10$ and $3\pi/10$ (in each case with $p_2 = p_3 = 0$ for $K = 0.1530$). For $K = 0.1515$ (0.1545), the results are more (less) accurate, but the following discussion holds for all three values of K . The pion propagators will be relevant when we study the electro-

magnetic form factor at non-zero momentum transfer. The straight line for the propagator with the pion at rest corresponds to the best fit in the range 9-14. The straight lines in the other cases are determined using Eq. (3.3) with $E = \sqrt{\underline{p}^2 + m^2}$, and with the values of m and $|\langle \pi | J_\pi(0) | 0 \rangle|^2$ obtained from the fit with the pion at rest. The almost negligible scatter of points away from the straight lines in the cases $p_1 = 0, \pi/10$ demonstrates the highly-correlated nature of the results for different values of t_x . It is reassuring to see that the propagator with $p_1 = \pi/10$ has the correct exponential behaviour with such good precision, since, as explained in Section 3, we have chosen to generate the S_1 "generalized quark propagators" with an insertion of a pion with this momentum. As we increase the momentum, the sensitivity of the measured quantities to the granularity of the lattice increases and we expect the systematic errors to grow. This is borne out by the propagators with $p_1 = 2\pi/10$ and $3\pi/10$. Clearly it will be very difficult to measure accurately any quantity which involves the propagation of a pion with momentum $3\pi/10$.

5. - THE ELECTROMAGNETIC FORM FACTOR OF THE PION

For the computation of the electromagnetic form factor of the pion we start with the three-point correlation function $C(t_x, t_y)$ of Eq. (3.8) generalized to non-zero momentum transfers

$$C(t_x, t_y) = \sum_{\underline{x}, \underline{y}} e^{i\underline{p}\underline{x}} e^{i\underline{q}\underline{y}} \langle 0 | T[J_\pi^+(0) V_\mu(y) J_\pi(x)] | 0 \rangle \quad (5.1)$$

where V_μ is the electromagnetic current (e.g., the conserved, local or extended current defined in Section 2). For sufficiently large values of t_y and $t_x - t_y$, (5.1) reduces to

$$C(t_x, t_y) = \frac{e^{-E(t_x - t_y)} e^{-E' t_y}}{4EE'} |\langle \pi | J_\pi(0) | 0 \rangle|^2 \cdot Z_V^{-1} \cdot (p + p')_\mu F(q^2) \quad (5.2)$$

where $E = \sqrt{m_\pi^2 + \underline{p}^2}$, $E' = \sqrt{m_\pi^2 + (\underline{p} + \underline{q})^2}$ and p' is the four-vector $(E', \underline{p} + \underline{q})$. $F(q^2)$ is the electromagnetic form factor and Z_V is the renormalization constant for the electromagnetic current being used.

We start by presenting our results for Z_V^{CONST} , the renormalization constant for the conserved current. We find

$$\begin{aligned} Z_V^{CONS} &= 0.99 \pm 0.05 \quad \text{for } K = 0.1515 \\ Z_V^{CONS} &= 1.00 \pm 0.05 \quad \text{for } K = 0.1530 \end{aligned}$$

and

(5.3)

$$Z_V^{CONS} = 1.11 \pm 0.06 \quad \text{for } K = 0.1545$$

in agreement with the expected result, $Z_V^{CONS} = 1$. The results in (5.3) were obtained from our computation of the correlation function (5.1) at zero momentum transfer ($\underline{q} = 0$), using Eq. (5.2) with the values of $E (= E')$ and $|\langle \pi | J_\pi(0) | 0 \rangle|^2$ obtained from the two-point correlation function with $J_\pi = \bar{u}\gamma^5 d$. The results in (5.3) provide a very reassuring check of our numerical procedures. It must be remembered, however, that many of the systematic errors generally present in our computations are absent in the calculation of the matrix elements of $\hat{V}_\mu$. A useful indication of some of these systematic errors is provided by computations of Z_V^{LOC} [using the local vector current (2.1)] and Z_V^{EXT} [using the extended vector current (2.3)] and by comparing the results to those expected from perturbation theory. The best determination of Z_V^{LOC} and Z_V^{EXT} is obtained by assuming that $Z_V^{CONS} = 1$ and using Eq. (2.4). For the local current we find

$$\begin{aligned} Z_V^{LOC} &= 0.72 \pm 0.01 \quad \text{for } K = 0.1515 \\ Z_V^{LOC} &= 0.71 \pm 0.01 \quad \text{for } K = 0.1530 \end{aligned}$$

and

(5.4)

$$Z_V^{LOC} = 0.70 \pm 0.01 \quad \text{for } K = 0.1545$$

to be compared with the (one-loop) perturbative result [24-26]

$$Z_V^{LOC} = 0.83 \quad (5.5)$$

Using the extended current we find

$$\begin{aligned} Z_V^{EXT} &= 0.87 \pm 0.01 \quad \text{for } K = 0.1515 \\ Z_V^{EXT} &= 0.85 \pm 0.01 \quad \text{for } K = 0.1530 \end{aligned}$$

and

(5.6)

$$Z_V^{EXT} = 0.84 \pm 0.01 \quad \text{for } K = 0.1545$$

whereas the perturbative result is [13,14,26]

$$Z_V^{EXT} = 0.92 \tag{5.7}$$

In Ref. [22], using two-point correlation functions, Z_V^{LOC} was found to be 0.57(2) and Z_V^{EXT} was found to be 0.69(2). The results in (5.4) [(5.6)] were obtained by dividing the three-point correlation function (5.1) corresponding to the local (extended) current at zero momentum transfer by that corresponding to the conserved current^{*}). The differences between (5.4) and (5.5) and between (5.6) and (5.7) provide an indication of the systematic errors present in our calculations.

For the remainder of this section we present and discuss our results for $F(q^2)$ with $q^2 \neq 0$, obtained from (5.1) using the local vector current $\bar{\psi}\gamma^\mu\psi$ with $\mu = 1$ or 4 [27,28]. Since we have generated the generalized propagators S_1 for a pion with momentum $\underline{p} = (\pi/10, 0, 0)$, the momentum of the final state pion is fixed. We have evaluated the matrix element $\langle \pi(p') | V_\mu^{LOC}(0) | \pi(p) \rangle$, and hence the corresponding form factor, for the five cases $\underline{p}' = 2\underline{p}, 3\underline{p}, 0, -\underline{p}$ and $(\pi/10, \pi/5, 0)$. The corresponding values of q^2 vary very slowly with K . The results are presented in Tables 2a-2c. For $K = 0.1530$ and 0.1545 we only present the results for those momentum transfers for which the errors are not overwhelming. The most precise results are obtained for the case $K = 0.1515$, and these are plotted in Fig. 5. For reference we also plot the expression (suggested by the vector dominance model) $1/(1+(-q^2)/m_\rho^2)$. The results are encouraging, the form factor falls with q^2 and is "measured" reasonably accurately for at least four of the five momentum transfers. For the fifth case (that with $-q^2 a^2 = 0.17$), the incoming pion has a momentum $\underline{p} = (3\pi/10, 0, 0)$, so that it is not surprising that the corresponding errors are large.

*) The Lorentz index (μ) was chosen to be 1 for all the currents. The statistical errors are tiny; the numerator and denominator are clearly highly correlated.

Of course we are really interested in the form factor of a real pion, and not the one corresponding to $K = 0.1515$, which has a mass of about 1 GeV. We must therefore extrapolate to the chiral limit. In Table 3 we present the results using a linear extrapolation in $1/K$ to $K_c = 0.1563$, the value found in Section 4 (linear extrapolations in K or m_π^2 yield consistent results). The only values of momentum transfer for which we have useful information about the form factor at $K = 0.1545$ correspond to $\underline{p}' = -\underline{p}$ and $\underline{0}$, and indeed it is only in these two cases that we end up with results with reasonably small statistical errors.

In Fig. 6 we plot the results of Table 3 vs. $\sqrt{-q^2}$ using $a^{-1} = 2.2$ GeV [Eq. (4.1)]. Recalling that there is an error of 0.4 on this value of a^{-1} , the points can be moved horizontally accordingly. In the same figure we also plot the results from the NA7 [29] and Cornell [30] experiments^{*}). The results (in particular for the two most reliable points $\underline{p}' = \underline{0}$ and $\underline{p}' = -\underline{p}$) are very satisfying.

We end this section with a brief discussion of the validity of the formula

$$F(q^2) = \frac{1}{1 + (-q^2)/m_\rho^2} \quad (5.8)$$

for $\sqrt{-q^2} \approx 0.65$ GeV and $\sqrt{-q^2} \approx 1.37$ GeV where we have our most reliable points. Here we are changing K , and hence m_π , whilst keeping q^2 almost constant. At $\sqrt{-q^2} \approx 1.37$ GeV (corresponding to $\underline{p}' = -\underline{p}$), the change in $F(q^2)$ as we vary K is much slower than required by Eq. (5.8); indeed, we observe almost no variation of $F(q^2)$ with K at this value of momentum transfer. At $\sqrt{-q^2} \approx 0.65$ GeV (corresponding to $\underline{p}' = \underline{0}$), on the other hand, the decrease of $F(q^2)$ as K increases (that is, as m_ρ decreases) is perfectly consistent with Eq. (5.8). If (5.8) were exact, we would obtain $F(q^2) \approx 0.77$ at $K = 0.1515$, $F(q^2) \approx 0.72$ at $K = 0.1530$ and $F(q^2) \approx 0.68$ at $K = 0.1545$. Our results (see Tables 2a-2c) agree well with these numbers when we use the local vector current with Lorentz index $\mu = 4$, but are slightly lower when we use the current with $\mu = 1$. So it would seem that we have some evidence, albeit rather weak, that the form (5.8) may be valid for the form factor at $\sqrt{-q^2} \approx 0.65$ GeV, and rather stronger evidence that it is not valid at momentum transfer as large as $\sqrt{-q^2} \approx 1.4$ GeV. We have also fitted our points at $K = 0.1515$, using the expression in Eq. (5.8), leaving m_ρ as a free parameter. We find $m_\rho = 0.46$ to be compared to 0.55 (see Table 1).

^{*}) The Cornell experiments determine the pion's form factor indirectly, by studying the electroproduction processes $ep \rightarrow e\pi^+n$ and $en \rightarrow e\pi^-p$.

6. - THE DEEP INELASTIC STRUCTURE FUNCTION OF THE PION

The calculation of the first two moments of the pion's structure function has been described in Ref. [1]. In this paper, in Sections 2-4, we have presented a more detailed discussion of many of the ingredients required for our calculations, as well as a description of the properties of our 15 configurations. We define $\langle x \rangle_{LATT}$ and $\langle x^2 \rangle_{LATT}$ in terms of the three-point correlation functions (3.8) as follows:

$$C^{(2)}(t_x, t_y) = | \langle \pi | J_\pi(0) | 0 \rangle |^2 \frac{p^2 \langle x \rangle_{LATT}}{2E} e^{-15E} \quad (6.1)$$

when $O(y) = O_2^{41}(y)$ and

$$C^{(3)}(t_x, t_y) = | \langle \pi | J_\pi(0) | 0 \rangle |^2 \frac{p^2 \langle x^2 \rangle_{LATT}}{E} e^{-15E} \quad (6.2)$$

when $O(y) = O_3^{411} - \frac{1}{2}(O_3^{422} + O_3^{433})$, where $p = \pi/10$ and $E = \sqrt{m_\pi^2 + p^2}$.

In Ref. [1] we presented results based on (6.1) and (6.2), where E and $| \langle \pi | J_\pi(0) | 0 \rangle |^2$ were determined from the two-point correlation function (3.1) following the procedure described in Section 3. Within our errors, the results were independent of the Wilson parameter K , and our best estimate of the results for $\langle x \rangle_{LATT}$ and $\langle x^2 \rangle_{LATT}$ in the chiral limit (allowing for some uncertainty for the extrapolation to $m_\pi^2 \approx 0$) were

$$\langle x \rangle_{LATT} = 0.64 \pm 0.10 \quad (6.3)$$

and

$$\langle x^2 \rangle_{LATT} = 0.26 \pm 0.10 \quad (6.4)$$

at a renormalization scale of about 2.2 GeV. Using perturbation theory we expressed (6.3) and (6.4) in standard continuum renormalization schemes (such as $\overline{MS}$) for which the coefficient function ≈ 1 , at a scale of about 7 GeV where the experimental data are available. The results quoted in Ref. [1] were

$$\langle x \rangle_{CONT} = \int_0^1 x V_\pi(x) dx = 0.49 \pm 0.08 \quad (6.5)$$

and

$$\langle x^2 \rangle_{CONT} = \int_0^1 x^2 V_\pi(x) dx = 0.20 \pm 0.08 \quad (6.6)$$

where $V_\pi(x)$ is the distribution function of the valence quarks in the pion $V_\pi = u_v + d_v$. Equations (6.5) and (6.6) are in good agreement with the experimental results obtained from the Drell-Yan process with lepton pair masses in the continuum region between the J/ψ and the γ :

$$\langle x \rangle_{EXP} = 0.4 \pm 0.1 \quad \text{Ref. [31]}$$

$$\langle x \rangle_{EXP} = 0.35 \pm 0.05 \quad \text{Ref. [32]} \quad (6.7)$$

$$\langle x \rangle_{EXP} = 0.4 - 0.5 \quad \text{Ref. [33]}$$

$$\langle x \rangle_{EXP} \sim 0.34 \quad \text{Ref. [34]}^{*)}$$

and $\langle x^2 \rangle_{CONT}$ agrees well with the experimental observation that the distribution function $V_\pi(x) \sim (1-x)^a$ as $x \rightarrow 1$, with the exponent a being close to 1.

Here we present an alternative way to determine $\langle x \rangle_{LATT}$ and $\langle x^2 \rangle_{LATT}$ (and other operator matrix elements), one in which some of the systematic errors associated with the fits used to determine E and $|\langle \pi | J_\pi(0) | 0 \rangle|^2$ may cancel. Let $C^{(2)}(t_x, t_y)$, $C^{(3)}(t_x, t_y)$ and $C^{(\hat{V})}(t_x, t_y)$ be the three-point correlation functions (3.8) corresponding to the operators O_2^{41} , $O_3^{411-\frac{1}{2}}(O_3^{422} + O_3^{433})$ and the conserved vector current $\hat{V}^1$ (with Lorentz index $\mu = 1$)^{**} respectively. Then

$$\frac{C^{(2)}(t_x, t_y)}{C^{(\hat{V})}(t_x, t_y)} = 2E \langle x \rangle_{LATT} \quad (6.8)$$

$t_x - t_y \quad \text{large}$
 $t_y \quad \text{large}$

and

*) Here we have used the distribution function $xV_\pi(x) = Ax^\alpha(1-x)^\beta$ with parameters $\alpha = 0.44$ and $\beta = 1.14$ and fixed A by imposing $\int xV_\pi(x)dx = 2$.

**) Alternatively the conserved vector current with $\mu = 4$ can be used with obvious modifications of (6.8) and (6.9).

$$\frac{C^{(3)}(t_x, t_y)}{C^{(\hat{V})}(t_x, t_y)} = 4pE \langle x^2 \rangle_{LATT}$$

$$t_x - t_y \text{ large}$$

$$t_y \text{ large} \quad (6.9)$$

with $p = \pi/10$ and $E = \sqrt{m_\pi^2 + p^2}$. Using (6.8) and (6.9), we find (averaging over results obtained for $t_y = 6$ to 9),

$$\begin{aligned} \langle x \rangle_{LATT} &= 0.65 \pm 0.01 \quad \text{at} \quad K = 0.1515 \\ \langle x \rangle_{LATT} &= 0.63 \pm 0.02 \quad \text{at} \quad K = 0.1530 \\ \langle x \rangle_{LATT} &= 0.63 \pm 0.09 \quad \text{at} \quad K = 0.1545 \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} \langle x^2 \rangle_{LATT} &= 0.25 \pm 0.05 \quad \text{at} \quad K = 0.1515 \\ \langle x^2 \rangle_{LATT} &= 0.26 \pm 0.06 \quad \text{at} \quad K = 0.1530 \\ \langle x^2 \rangle_{LATT} &= 0.23 \pm 0.05 \quad \text{at} \quad K = 0.1545 \end{aligned} \quad (6.11)$$

These results are in agreement with those presented in Ref. [1], but have slightly lower errors. Again it can be seen that any dependence on K must be very mild. Extrapolating to the chiral limit [$K_c \approx 0.1563$, Eq. (4.6)], assuming a linear behaviour in K , we find that at a renormalization scale of about 2.2 GeV,

$$\langle x \rangle_{LATT} = 0.60 \pm 0.08 \quad (6.12)$$

and

$$\langle x^2 \rangle_{LATT} = 0.23 \pm 0.06 \quad (6.13)$$

to be compared with (6.3) and (6.4). Most of the error in Eqs. (6.3) and (6.4) is due to the linear extrapolation to $K_c = 0.1563$ [Eq. (4.6)]. It is clear that, had we assumed a constant behaviour, as we did in Ref. [1], we would have found a much smaller error. Using perturbation theory to rewrite these results in continuum schemes for which the coefficient function is close to 1 (such as the $\overline{MS}$ scheme), and at a renormalization scale $\mu \sim 7$ GeV, we find

$$\langle x \rangle_{CONT} = 0.46 \pm 0.07 \quad (6.14)$$

and

$$\langle x^2 \rangle_{CONT} = 0.18 \pm 0.05 \quad (6.15)$$

to be compared with (6.5) and (6.6). Equations (6.14) and (6.15) are our best estimates of the first two moments of the pion's structure function. Although the statistical errors on the final results are only slightly smaller than in (6.5) and (6.6), the new method allows us to make a linear extrapolation to the chiral limit with reasonably small errors. We stress both the simplicity of the procedure of dividing by the three-point correlation function corresponding to the conserved current [Eqs. (6.8) and (6.9)], and the fact that by taking this ratio the errors are reduced.

7. - THE SECOND MOMENT OF THE PION'S DISTRIBUTION AMPLITUDE

In Ref. [9] we have presented our results of a calculation of the second moment of the pion's distribution amplitude performed on a $10^3 \times 20$ lattice at $\beta = 6.0$,

$$\langle \xi^2 \rangle_{LATT} = 0.26 \pm 0.13 \quad (7.1)$$

where the subscript LATT signifies that we are in the lattice renormalization scheme at a renormalization scale of about 2 GeV. In the $\overline{MS}$ scheme at the same renormalization point, (7.1) becomes

$$\langle \xi^2 \rangle_{MS} = 0.34 \pm 0.17 \quad (7.2)$$

(7.2) is in good agreement with the results found using QCD sum rules [35] ($\langle \xi^2 \rangle \approx 0.4 \pm 15\text{--}20\%$ at $\mu^2 \approx 1.5 \text{ GeV}^2$) and with Ref. [36], but is in disagreement with the previous lattice calculation [37], which found $\langle \xi^2 \rangle = 1.58 \pm 0.23$ at $Q \approx 1 \text{ GeV}$ (as discussed in Ref. [9], the likely explanation of this discrepancy is the wrong treatment of power divergences in Ref. [37]). The reason for the large error in (7.1) and (7.2) is the large numerical subtraction present due to the cancellation of power divergences.

We have repeated the evaluation of $\langle \xi^2 \rangle$ using our 15 configurations on the $20 \times 10^2 \times 40$ lattice. The results can be summarized briefly as being consistent with those of Ref. [9] within rather large errors, so that we have little to add to our previous analysis.

In order to determine $\langle \xi^2 \rangle$, we evaluate the following ratios of correlation functions

$$R_1(t_x) = \frac{\sum_{\underline{x}} e^{i\underline{p}\underline{x}} \langle 0 | [\hat{O}^{411}(x) - \hat{O}^{433}(x)] \bar{\psi}(0) \gamma^5 \psi(0) | 0 \rangle}{\sum_{\underline{x}} e^{i\underline{p}\underline{x}} \langle 0 | \bar{\psi}(x) \gamma^4 \gamma^5 \psi(x) \bar{\psi}(0) \gamma^5 \psi(0) | 0 \rangle} \quad (7.3)$$

and

$$R_2(t_x) = \frac{\sum_{\underline{x}} e^{i\underline{p}\underline{x}} \langle 0 | \hat{O}^{412}(x) \bar{\psi}(0) \gamma^5 \psi(0) | 0 \rangle}{\sum_{\underline{x}} e^{i\underline{p}\underline{x}} \langle 0 | \bar{\psi}(x) \gamma^4 \gamma^5 \psi(x) \bar{\psi}(0) \gamma^5 \psi(0) | 0 \rangle} \quad (7.4)$$

where the operators $\hat{O}^{\mu\nu\rho}$ are defined in Eq. (2.12) and $\underline{p} = (\pi/10, 0, 0)$ with $\underline{p}' = (\pi/10, \pi/5, 0)$. $\underline{p}$ and $\underline{p}'$ are the smallest momenta for which the matrix elements $\langle 0 | \hat{O}^{411}(0) - \hat{O}^{433}(0) | \pi(\underline{p}) \rangle$ and $\langle 0 | \hat{O}^{412}(0) | \pi(\underline{p}') \rangle$ do not vanish. For sufficiently large t_x , the correlation functions are dominated by the single pion intermediate state and we have

$$R_1(t_x) = \text{constant} = (p^1)^2 \langle \xi^2 \rangle = \frac{\pi^2}{100} \langle \xi^2 \rangle \quad (7.5)$$

and

$$R_2(t_x) = \text{constant} = p^1 p^2 \langle \xi^2 \rangle = \frac{\pi^2}{50} \langle \xi^2 \rangle \quad (7.6)$$

We have also evaluated ratios of the correlation functions (7.3) and (7.4) with the pseudoscalar density $\bar{\psi}(0) \gamma^5 \psi(0)$ replaced by the axial current $\bar{\psi}(0) \gamma^4 \gamma^5 \psi(0)$ in both the numerator and denominator [the results are still given by (7.5) and (7.6)]. The errors in this latter case are larger (which was also true in Ref. [9]), so that we restrict ourselves to presenting the results using (7.3) and (7.4) only.

The results for $\langle \xi^2 \rangle$ obtained by measuring $R_1(t_x)$ and $R_2(t_x)$ are presented in Tables 4 and 5 respectively. We present the results for $t_x = 5, 6$ and 7 as well as those obtained by averaging over the range $5 \leq t_x \leq 10$. The last one is obtained by linear extrapolation of the entries for $K = 0.1515, 0.1530$ and 0.1545 to the chiral limit $K \approx 0.1563$. It is clear from the tables that the fluctuations are large (due to the cancellation of power divergences; see Section 2), and that the extrapolation to the chiral limit also has significant uncertainties, which cannot be fully accounted for in the quoted errors. Although on the average (see the last columns in Tables 4 and 5) the measured results agree well with those found in Ref. [9], the large fluctuations from one value of t_x to another lead us to conclude that we cannot improve on our earlier result for $\langle \xi^2 \rangle$ [Eq. (7.1)]. $\langle \xi^2 \rangle$ is clearly a very difficult quantity to measure on the lattice.

8. - CONCLUSIONS AND OUTLOOK

In this paper we have presented the results of our computations of the pion's form factor at several values of momentum transfer (see Fig. 5) as well as improved results for the first two moments of the pion's structure function [Eqs. (6.14) and (6.15)]. We stress that these computations were performed from first principles. The results are very satisfying; they have reasonably small (in many cases very small) statistical errors and are consistent with experimental errors. We are thus encouraged to continue our study of hadronic structure on the lattice, and are at present using the quark propagators for the 15 configurations discussed in this paper to study further properties of hadrons, such as the proton's decay amplitude in Grand Unified Theories, the proton's distribution amplitude and matrix elements of higher twist operators between pion states. In addition we are generating the generalized propagators necessary to extend the calculations of this paper to nucleons.

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WILSON PARAMETER K	$t_x = 9-14$		$t_x = 9-14$ PLUS 26-31	
	$m_\pi a$	$m_\rho a$	$m_\pi a$	$m_\rho a$
0.1515	0.50 ± 0.03	0.55 ± 0.05	0.52 ± 0.02	0.58 ± 0.04
0.1530	0.43 ± 0.03	0.48 ± 0.06	0.45 ± 0.03	0.52 ± 0.04
0.1545	0.36 ± 0.04	0.42 ± 0.09	0.38 ± 0.04	0.48 ± 0.06

Table 1: The masses of the pion and the mesons obtained from the two-point correlation function.

$p'a$	$-q^2 a^2$	$F(q^2)$ using $\mu = 1$	$F(q^2)$ using $\mu = 4$
$2\bar{p}a$	0.054	0.73 ± 0.08	0.40 ± 0.50
$3\bar{p}a$	0.17	0.21 ± 0.27	0.26 ± 0.53
$\underline{0}$	0.09	0.66 ± 0.05	0.76 ± 0.03
$-\bar{p}a$	0.39	-	0.34 ± 0.03
$(\pi/10, \pi/5, 0)$	0.32	0.31 ± 0.09	0.44 ± 0.10

Table 2a: Values of the electromagnetic form factor for $K = 0.1515$

$\underline{p}'a$	$-q^2a^2$	$F(q^2)$ using $\mu = 1$	$F(q^2)$ using $\mu = 4$
$2\underline{p}a$	0.047	0.65 ± 0.15	-
$\underline{0}$	0.09	0.63 ± 0.04	0.72 ± 0.03
$-\underline{p}a$	0.39	-	0.30 ± 0.02
$(\pi/10, \pi/5, 0)$	0.31	0.23 ± 0.06	0.40 ± 0.19

Table 2b: Values of the eletromagnetic form factor for $K = 0.1530$

$\underline{p}'a$	$-q^2a^2$	$F(q^2)$ using $\mu = 1$	$F(q^2)$ using $\mu = 4$
$\underline{0}$	0.085	0.59 ± 0.04	0.72 ± 0.10
$-\underline{p}a$	0.39	-	0.33 ± 0.03
$(\pi/10, \pi/5, 0)$	0.30	0.12 ± 0.24	-

Table 2c: Values of the electromagnetic form factor for $K = 0.1545$

$\underline{p}'a$	$-q^2a^2$	$F(q^2)$ using $\mu = 1$	$F(q^2)$ using $\mu = 4$
$2\underline{p}a$	~ 0.05	0.47 ± 0.31	-
$\underline{0}$	~ 0.09	0.55 ± 0.08	0.65 ± 0.09
$-\underline{p}a$	0.39	-	0.30 ± 0.04
$(\pi/10, \pi/5, 0)$	~ 0.31	0.09 ± 0.10	0.31 ± 0.38

Table 3: Values of the electromagnetic form factor obtained by extrapolating the results at $K = 0.1515$, 0.1530 and 0.1545 linearly in $1/K$ to $K_c = 0.1564$. No useful information is obtained in the case $\underline{p}'a = -3\underline{p}a$.

K	$t_x = 5$	$t_x = 6$	$t_x = 7$	Average of $5 \leq t_x \leq 10$
0.1515	0.28 ± 0.15	0.00 ± 0.05	0.15 ± 0.03	0.12 ± 0.05
0.1530	0.37 ± 0.16	0.05 ± 0.05	0.17 ± 0.03	0.16 ± 0.06
0.1545	0.57 ± 0.19	0.12 ± 0.14	0.17 ± 0.03	0.23 ± 0.09
K_c	0.72 ± 0.22	0.17 ± 0.16	0.19 ± 0.04	0.28 ± 0.05

Table 4: $\langle \xi^2 \rangle$ determined from measured values of $R_1(t_x)$ at $K = 0.1515$, 0.1530 and 0.1545 , and the values obtained by linear extrapolation to K_c .

K	$t_x = 5$	$t_x = 6$	$t_x = 7$	Average of $5 \leq t_x \leq 10$
0.1515	0.13 ± 0.04	0.24 ± 0.09	0.10 ± 0.11	0.17 ± 0.06
0.1530	0.10 ± 0.04	0.29 ± 0.10	0.11 ± 0.12	0.14 ± 0.09
0.1545	0.06 ± 0.05	0.41 ± 0.11	0.10 ± 0.20	0.03 ± 0.17
K_c	0.02 ± 0.05	0.50 ± 0.13	0.11 ± 0.22	0.00 ± 0.09

Table 5: $\langle \xi^2 \rangle$ obtained from measured values of $R_2(t_x)$ at $K = 0.1515$, 0.1530 and 0.1545 , and the values obtained^x by linear extrapolation to K_c .

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FIGURE CAPTIONS

- Fig. 1 : Diagrams which have to be evaluated in the calculation of the correlation functions. The crosses $\times$ signify the insertion of the pion's interpolating operator J_π , and the closed circle $\bullet$ the insertion of the operator whose matrix element we wish to calculate.
- Fig. 2 : A comparison of our results for $m_\pi a$ with the results from Refs. [20]-[22] as a function of $1/K_{\text{eff}} = \frac{1}{2}(1/K_1 + 1/K_2)$ is given (K_1 and K_2 being the Wilson parameters for the two quarks in the pion).
- Fig. 3 : The pion's propagator at $K = 0.1530$ for a pion at rest and at three other momenta. For the pion at rest, the straight line represents the best fit in the range $9 \leq t_x \leq 14$. The straight lines in the other three cases are determined from that for the pion at rest using $E = \sqrt{m_\pi^2 + \underline{p}^2}$. For some of the points in the figure, the statistical errors are also reported.
- Fig. 4 : m_ρ versus m_π^2 . The straight line represents the best linear fit, from which we derive $a^{-1} = 2.2 \pm 0.4$ GeV.

Fig. 5 : The pion's form factor at $K = 0.1515$. The curve is the result suggested by the vector dominance model $1/(1+(-q^2)/m_\rho^2)$.

Fig. 6 : Our predictions for the pion's form factor, together with the experimental results. It should be remembered that, in addition to the vertical error bars, there is an uncertainty in our evaluation of a^{-1} ($a^{-1} = 2.2 \pm 0.4$ GeV).

$$K(t_x) = \sum_{\underline{x}} \quad \begin{array}{c} \text{Diagram: A circle with two vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'x'. Two directed edges connect them: the top edge is labeled 'S' with an arrow pointing right, and the bottom edge is labeled 'S' with an arrow pointing left.} \end{array} \quad e^{ip \cdot \underline{x}}$$

$$\begin{aligned} C(t_x, t_y) &= \sum_{\underline{x}, \underline{y}} \quad \begin{array}{c} \text{Diagram: A circle with three vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'x'. A third vertex at the top is labeled 'y'. Three directed edges connect them: the top edge from '0' to 'y' is labeled 'S' with an arrow pointing right; the right edge from 'y' to 'x' is labeled 'S' with an arrow pointing left; the bottom edge from 'x' to '0' is labeled 'S' with an arrow pointing left.} \end{array} \quad e^{ip \cdot \underline{x}} \\ &= \sum_{\underline{y}} \quad \begin{array}{c} \text{Diagram: A circle with two vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'y'. Two directed edges connect them: the top edge is labeled 'S' with an arrow pointing right, and the bottom edge is labeled 'S_1' with an arrow pointing left.} \end{array} \end{aligned}$$

Fig. 1

Fig. 2

Fig. 3

Fig. 4a

Fig. 4b

Fig. 5

Fig. 6