SUPERSINGLETONS

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ABSTRACT

In the light of their application to the supermembrane, we construct the three-dimensional action and transformation rules of the singleton fields of $OSp(4/N)$. We stress, in particular, the role of fermionic gauge invariance and superconformal invariance. The latter guarantees renormalizability.

Singletons

It is now known that there is a connection [1,2] between the eleven-dimensional supermembrane [3] and "singleton" representations of $OSp(4/N)$, the superextension of the anti-de Sitter group $SO(3,2)$. These remarkable boson and fermion representations of $SO(3,2)$, which have no Poincaré analogue, were pointed out by Dirac in 1963 [4], a year after he put forward the idea of a relativistic membrane [5]. In fact, they constitute the first ever example of a supermultiplet! It is gratifying to find, a quarter of a century later, that the eleven-dimensional supermembrane allows a merger of all these ideas. The purpose of the present paper, with membrane applications in mind, is to construct the three-dimensional action and transformation rules for these singletons, emphasizing in particular the role both of superconformal invariance and of the fermionic gauge invariance. (The latter is closely related to the fermionic Siegel invariance on the three-dimensional world-volume of the supermembrane.)

Singletons, and their supersymmetry, have been discussed in some depth by Fronsdal and others [6,7,8,9] and we shall draw upon their results. To the best of our knowledge, however, they have not written down the supersymmetric action and transformation rules in the explicitly three-dimensional language appropriate to the super-membrane, preferring instead the five-dimensional cone formalism. Nor have they considered extended supersymmetry. A three-dimensional analysis has been carried out by Nicolai and Sezgin [10] for both simple and extended supersymmetry but our results disagree with theirs both for the action and for the supersymmetry transformation rules.

Let us begin by recalling some properties of the anti-de Sitter group $SO(3,2)$. Unitary irreducible representations are denoted $D(E_0, s)$ where E_0 is the lowest energy eigenvalue (in units of the AdS (curvature) $^{1/4}$) and s is its spin, where

$$E_0 \geq s + 1 \quad \text{if } s \geq 1; \quad E_0 \geq s + 1/2 \quad \text{if } s = 0, 1/2. \quad (1)$$

In the limit of zero curvature all these representations go over into the familiar massless representations of the Poincaré group with the exception of $D(1/2, 0)$ and $D(1, 1/2)$. These are the singleton representations, termed "Rac" and "Di" by Fronsdal, which together, in fact, form a supermultiplet of the simple supergroup $Osp(4/1)$. N of them can form an ultrashort spin 0-spin $1/2$ supermultiplet of the extended supergroup $Osp(4/N)$ transforming as the spinor representations of $SO(N)$. (Supermembrane consid-

erations restrict us to $N \leq 8$.)

We shall first quote our results for $N=1$ and then explain them before generalizing to higher N . The free action is

$$S_0 = -\frac{1}{2} \int d^3 \xi / h [h^{ij} \phi_i \phi_j + \frac{g^2}{4} \phi^2 + i \bar{\chi} (1 - \gamma) \not{D} \chi - \Lambda^2] \quad (2)$$

where h_{ij} is the metric of a three-dimensional spacetime with signature $(-, +, +)$ and topology $S^1 \times S^2$. In coordinates $\xi^i = (\xi^0, \xi^1, \xi^2) = (t, \vartheta, \varphi)$

$$ds^2 = h_{ij} d\xi^i d\xi^j = -dt^2 + a^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (3)$$

where

$$0 \leq \text{at} \leq 2\pi, \quad 0 \leq \vartheta \leq \pi, \quad \text{and} \quad 0 \leq \varphi \leq 2\pi.$$

'a' is a constant parameter with the dimensions of mass. The Rac field is denoted by the scalar $\phi(\xi)$, the DI field by the four-component Majorana $\chi(\xi)$ and $\Lambda(\xi)$ is an auxiliary field. The 4×4 matrices γ^a ($a=0,1,2$) and γ^3 obey the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \quad m, n = (0, 1, 2, 3) \quad (4)$$

where η^{mn} is the four-dimensional Minkowski metric with signature $(-, +, +, +)$. The adjoint is given by $\bar{\chi} = \chi^\dagger \gamma^0$. We also define

$$\gamma = -\gamma_0 \gamma_1 \gamma_2 \quad (5)$$

so that $(1 \pm \gamma)/2$ act as projection operators. $\not{D} = \gamma_a^i \gamma^a D_i$ where γ_a^i is the dreibein and the symbol D_i denotes the covariant derivative

$$D_i = \partial_i - \frac{1}{4} \omega_i^{ab} \gamma_{ab} \quad (6)$$

appropriate to the $S^1 \times S^2$ background.

The metric (3) on $S^1 \times S^2$ admits ten conformal Killing vectors ζ_i^a .

$$\nabla_i \zeta_j^a + \nabla_j \zeta_i^a = \frac{2}{3} \nabla_k \zeta^k h_{ij} \quad (7)$$

which generate the algebra of $SO(3,2)$. The action (2) is correspondingly invariant under the conformal transformations

$$\begin{aligned} \delta \phi &= \left(-\zeta^i \partial_i - \frac{1}{6} \nabla_i \zeta^i \right) \phi \\ \delta \chi &= \left(-\zeta^i D_i - \frac{1}{3} \nabla_i \zeta^i - \frac{1}{4} \nabla_i \zeta_j^j \gamma^i \right) \chi \\ \delta \Lambda &= \left(-\zeta^i D_i - \frac{1}{2} \nabla_i \zeta^i \right) \Lambda \end{aligned} \quad (8)$$

The $S^1 \times S^2$ spacetime also admits conformal Killing spinors ϵ satisfying

$$(1 - \gamma)(D_i + \frac{a}{2} \gamma_i \gamma_3) \epsilon = 0 \quad (9)$$

$$(1 + \gamma)(\gamma^i D_i + \frac{a}{2} \gamma_3) \epsilon = 0 \quad (10)$$

and

where ϵ is a four-component Majorana spinor. The action is correspondingly invariant under the supersymmetry transformations

$$\begin{aligned} \delta \phi &= i \bar{\epsilon} (1 - \gamma) \chi \\ \delta \chi &= [(\not{D} - \frac{a}{2} \gamma_3) \phi + \Lambda] \epsilon \\ \delta \Lambda &= i \bar{\epsilon} (1 - \gamma) \not{D} \chi \end{aligned} \quad (11)$$

One may readily verify, using (9) and (10), that the commutator of two supersymmetry transformations (11) correctly yields a conformal transformation (8) off shell. In fact, the algebra closes on $OSp(4/1)$. Since supersymmetry requires equal number of Bose and Fermi degrees of freedom, however, and since we are employing four-component spinors, there ought to be another symmetry which reduces by half the number of fermionic degrees of freedom. This is indeed the case. The action (2) is also invariant under the fermionic gauge transformation

$$\delta \chi = (1 + \gamma) \lambda \quad (12)$$

where $\lambda(\xi)$ is an arbitrary four-component Majorana spinor.

The field equations obtained by varying (2) admit the solutions

$$\begin{aligned}\varphi(t, \theta, \varphi) &= \sum_{l, m} [A_{lm}^+ e^{i(l+1/2)t} + A_{lm}^- e^{-i(l+1/2)t}] Y_{lm}(\theta, \varphi) \\ \chi(t, \theta, \varphi) &= \sum_{l, m} [B_{lm}^+ e^{i(l+1/2)t} + B_{lm}^- e^{-i(l+1/2)t}] Y_{lm}(\theta, \varphi) \\ A(t, \theta, \varphi) &= 0\end{aligned}\quad (13)$$

with $Y_{lm}(\theta, \varphi)$ the standard spherical harmonics and $Y_{jm}(\theta, \varphi)$ the spherical spinors of Ref[8]. Note that φ is antiperiodic in $\theta \rightarrow \theta + 2\pi n$ ($n = \text{integer}$) but χ is periodic. The supersymmetry transformation rules thus require ε to be antiperiodic also, an explicit solution of (9) and (10) being given in the orthonormal frame $v_i^a = \text{diag}(1, a^{-1}, a^{-1}, \sin\theta)$ by

$$\varepsilon(t, \theta, \varphi) = \left(\cos \frac{t}{2} + \gamma_0 \gamma_3 \sin \frac{t}{2} \right) \left(\cos \frac{\theta}{2} - \gamma_1 \gamma_3 \sin \frac{\theta}{2} \right) \left(\cos \frac{\varphi}{2} + \gamma_1 \gamma_2 \sin \frac{\varphi}{2} \right) \varepsilon_0 \quad (14)$$

where ε_0 is a constant spinor. For this reason, the authors of Refs [7,10] prefer to define the theory on the double-cover of S^1 , with $0 \leq t < 4\pi$, though the necessity for this is not clear to us.

The $OSp(4/1)$ symmetry also permits self-interactions of the singletons but of a very restricted form, namely

$$S_1 = \int d^3\xi \sqrt{-h} \left[-g \Lambda \varphi^3 + \frac{1}{2} g \bar{\chi} (1 - \gamma) \chi \varphi^2 \right] \quad (15)$$

where g is a dimensionless coupling constant. On-shell, this becomes

$$S_1 = \int d^3\xi \sqrt{-h} \left[\frac{1}{2} g^2 \varphi^4 + \frac{1}{2} g \bar{\chi} (1 - \gamma) \chi \varphi^2 \right]. \quad (16)$$

The total action $S_0 + S_1$ is thus renormalizable.

Conformal Invariance

Most particle physicists are familiar with the conformal group in flat Minkowski space. It is the group of coordinate transformations which leave invariant the Minkowski light-cone. In the case of three-dimensional Minkowski space, M_3 , it is $SO(3,2)$. In the present context, however, the space-time is curved with topology $S^1 \times S^2$, but still admits $SO(3,2)$ as its conformal group i.e. as the group which leaves invariant the three-dimensional light-cone $h_{ij} dx^i dx^j = 0$. Hence it admits the ten conformal Killing vectors of (7) $K^{[mn]} = \zeta^{[mn]} b_1^{(m, n=0,1,2,3,s)}$ which, in the (t, θ, φ) coordinates of (3) are given explicitly by

$$\begin{aligned}K^{0s} &= \partial_t \\ K^{12} &= \partial_\varphi \\ K^{1s} &= -\cos\varphi \partial_\theta + \sin\varphi \cot\theta \partial_\varphi \\ K^{23} &= -\sin\varphi \partial_\theta - \cos\varphi \cot\theta \partial_\varphi \\ K^{01} &= \sin\theta \cos\varphi \cos\partial_t + \sin\theta \cos\theta \cos\varphi \partial_\theta - \sin\theta \sin\varphi \csc\theta \partial_\varphi \\ K^{02} &= \sin\theta \sin\varphi \cos\partial_t + \sin\theta \cos\theta \sin\varphi \partial_\theta + \sin\theta \cos\varphi \csc\theta \partial_\varphi \\ K^{03} &= \cos\theta \cos\partial_t - \sin\theta \sin\theta \partial_\theta \\ K^{31} &= -\sin\theta \sin\varphi \cos\partial_t + \cos\theta \cos\theta \cos\varphi \partial_\theta - \cos\theta \sin\varphi \csc\theta \partial_\varphi \\ K^{32} &= -\sin\theta \sin\theta \sin\varphi \partial_t + \cos\theta \cos\theta \sin\varphi \partial_\theta + \cos\theta \cos\varphi \csc\theta \partial_\varphi \\ K^{33} &= -\cos\theta \sin\theta \partial_t - \sin\theta \cos\theta \partial_\theta\end{aligned}\quad (17)$$

where we have set $a = 1$. The metric itself is left invariant only by the $U(1) \times SO(3)$ subgroup of $SO(3,2)$ for which K^{0s} and (K^{12}, K^{23}, K^{31}) are the Killing vectors.

The failure to discriminate between these different kinds of conformal invariance is, we believe, a source of confusion in the singleton literature. In particular, the φ^2 "mass" term appearing in the action (2) would be incompatible with conformal invariance if the action were on M_3 but is essential for conformal invariance on $S^1 \times S^2$. Moreover, the coefficient $a^2/4$ is uniquely fixed. (This is the first of our disagreements with Sezgin and Nicolai [10]. These authors have no mass term but nevertheless claim conformal invariance.) Another way to see this is to appeal to Weyl invariance. A well-known theorem states that any theory which is both generally covariant and Weyl invariant is automatically conformally invariant when the metric is restricted to flat Minkowski space. A less well-known theorem generalizes this result to any fixed background geometry which admits conformal Killing vectors. Indeed, the conformal transformations (8) may be obtained from those combined Weyl, Lorentz and general coordinate transformations on φ, χ and Λ chosen to leave the dreibein v_i^a invariant. Now a Weyl-invariant action is given, in d dimensions, by

$$S = -\frac{1}{2} \int d^d x \left[h^{\mu\nu} \partial_\mu \partial_\nu \phi + \xi R \phi^2 + i \bar{\chi} (\gamma_5 - \gamma_4) \not{\partial} \chi - A^2 \right] \quad (18)$$

where $\mu, \nu = 1, \dots, d$ and $\xi = (d-2)/4(d-1)$. It is invariant under

$$v^\mu \rightarrow \Omega v^\mu, \quad h_{\mu\nu} \rightarrow \Omega^2 h_{\mu\nu}, \quad \phi \rightarrow \Omega^{\frac{2-d}{2}} \phi, \quad \chi \rightarrow \Omega^{\frac{1-d}{2}} \chi, \quad A \rightarrow \Omega^{\frac{d}{2}} A. \quad (19)$$

If we specialize to $S^1 \times S^2$ then $d=3$, $\xi = \frac{1}{8}$ and $R=2a^2$. Hence the coefficient $a^2/4$ in (2).

The same theorems apply, *mutatis mutandis*, when we include fermions and superconformal invariance. Thus our singleton action could have been derived from a Weyl invariant coupling of the ϕ and χ fields to an external three-dimensional conformal supergravity, and then specializing to the bosonic $S^1 \times S^2$ background. Note that superconformal invariance fixes uniquely the conditions (9) and (10) on the supersymmetry parameter. Note, once again, the terms involving 'a' which are absent on M_3 but are crucial on $S^1 \times S^2$. This also accounts for our decision to quote the singleton action and transformation rules in terms of four-component spinors, rather than two component spinors. The conditions (9) and (10) would tangle them up in a rather complicated way. In either case, our rules differ from those of Sezgin and Nicolai [10].

The metric (3) on $S^1 \times S^2$ is conformally flat. In spaces of more than three dimensions, the criterion for conformal flatness is the vanishing of the Weyl tensor. In three dimensions, however, the Weyl tensor vanishes identically and the criterion for conformal flatness is the vanishing of the Cotton tensor C^{ij} defined by [12,13,14]

$$\nabla^i h C^{ij} = \epsilon^{ijk} \nabla_k \left(R^j_l - \frac{1}{2} \delta^j_l R \right) \quad (20)$$

C^{ij} is traceless, covariantly conserved and symmetric by virtue of the Bianchi identities². One may verify that it vanishes on $S^1 \times S^2$. This means that, locally (i.e. on each coordinate patch), one may choose a conformal factor Ω and from (19) a "conformal gauge"

$$h'_{ij} = \Omega^2 h_{ij}, \quad \phi' = \Omega^{-1/2} \phi, \quad \chi' = \Omega^{-1} \chi, \quad A' = \Omega^{3/2} A \quad (21)$$

for which h'_{ij} is a flat metric and for which the action in terms of ϕ' , χ' and A' is just the flat space action without the ϕ'^2 "mass" term. This flat space still has $S^1 \times S^2$ topology, however, and describes compactified Minkowski space rather than the usual Minkowski space with $\mathbb{R}^3$ topology. From now on we will always work with the $S^1 \times S^2$ metric (3), and the action (2).

The classical conformal (or Weyl) invariance of (2) (or (18)) raises the question of conformal [15] (or Weyl [16]) anomalies in the quantum theory. However, since we are in an odd dimension, these will be zero in dimensional regularization at odd-loop order [17], in particular at one loop. There is, however, a discrete symmetry anomaly arising from fermions in odd dimensions [18]. In the present case of three-dimensional gravity, it induces a one-loop Chern-Simons correction to the action proportional to $\epsilon^{ijk} \text{Tr} [R_{ij} \omega_k + \frac{2}{3} \omega_i \omega_j \omega_k]$ where the trace is over Lorentz indices. In the supersymmetric case, the induced action is precisely the action of three-dimensional conformal supergravity [19]. Interestingly enough, the action yields the Cotton tensor C^{ij} of (20), or its supersymmetric generalization, when varied with respect to h_{ij} treating the spin connections in second order formalism. It is thus a moot point whether one should include these corrections to the singleton action.

In order to construct a theory in one time and p space dimensions which is invariant under the conformal group $SO(p+1,2)$, one may employ an old trick due to Mack and Salam [20] and generalized to the superconformal group by Ferrara [21]. One first writes down manifestly $SO(p+1,2)$ invariant field equations defined on the $p+2$ dimensional hypersurfaces of the space $\mathbb{R}^{p+3}$ with metric $(-, +, +, \dots, +, -)$ given by

$$-(y^0)^2 + (y^1)^2 + \dots + (y^{p+1})^2 - (y^{p+2})^2 = -L^2. \quad (22)$$

To recover the lower dimensional theory one then considers the field equations on the cone hypersurface $L^2 = 0$, and restricts the fields to be homogeneous with degree of homogeneity fixed by the conformal weight. This procedure respects the $SO(p+1,2)$ invariance while reducing the dependence from $p+3$ coordinates to $p+1$. In the next section we shall derive our three-dimensional results starting from the five-dimensional cone formalism due to Fronsdal et al [7,8].

Five-dimensional cone derivation

Let us introduce natural coordinates y^m ($m=0,1,2,3$) on $\mathbb{R}^4$ and a metric η_{mn} with signature $(-, +, +, -)$. The free field equations are

$$\begin{aligned} \eta^{mn} \partial_m \partial_n \hat{\phi} &= 0 \\ (x - \frac{3}{2}) \hat{\chi} &= 0 \\ \hat{A} &= 0 \end{aligned} \quad (23)$$

where $x = \frac{1}{4} [\Gamma^m, \Gamma^n] (y_{mn}^0 - y_{mn}^1)$

(24)

and where the Γ^m 's satisfy the Clifford algebra

$$\{\Gamma^m, \Gamma^n\} = 2\eta^{mn} \quad (25)$$

We shall use the representation

$$\begin{aligned} \Gamma^m &= \gamma^m, \quad m=0, \dots, 3 \\ \Gamma^s &= \gamma^s, \quad \gamma^s = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad \text{with} \quad \gamma^0 \gamma^m \gamma^0 = -\gamma^m \end{aligned}$$

where the γ^m 's satisfy the Clifford algebra of $d=4$ Minkowski space. In order that the Dirac kinetic term be $SO(3,2)$ invariant the adjoint must be defined as

$$\bar{\chi} = \chi^\dagger \Gamma^0 \Gamma^s \quad (26)$$

$\hat{\chi}$ is a Majorana spinor i.e. $\bar{\chi} = \hat{\chi}^T C$.

$\hat{\phi}$, $\hat{\chi}$ and $\hat{A}$ are homogeneous functions of degrees $-\frac{1}{2}$, $-\frac{3}{2}$ and $-\frac{3}{2}$ respectively:

$$\begin{aligned} \gamma^m \partial_m \hat{\phi} &= -\frac{1}{2} \hat{\phi} \\ \gamma^m \partial_m \hat{\chi} &= -\frac{3}{2} \hat{\chi} \\ \gamma^m \partial_m \hat{A} &= -\frac{3}{2} \hat{A} \end{aligned} \quad (27)$$

We now reparametrize $\mathbb{R}^6$ as follows,

$$\begin{aligned} y^0 &= Y \sin \alpha t \\ y^s &= Y \cos \alpha t \\ y^1 &= r \sin \vartheta \cos \varphi \\ y^2 &= r \sin \vartheta \sin \varphi \\ y^3 &= r \cos \vartheta \end{aligned} \quad (28)$$

with orthonormal frames defined by the fünfbein

$$E_t^0 = 1, \quad E_\phi^1 = r^{-1}, \quad E_\varphi^2 = r^{-1} \sin \vartheta, \quad E_r^3 = 1 \quad \text{and} \quad E_Y^s = 1. \quad (29)$$

Restricting ourselves to the cone $Y=r$, (27) becomes

$$\begin{aligned} r \partial_r \hat{\phi} &= -\frac{1}{2} \hat{\phi} \\ r \partial_r \hat{\chi} &= -\frac{3}{2} \hat{\chi} \\ r \partial_r \hat{A} &= -\frac{3}{2} \hat{A} \end{aligned} \quad (30)$$

(30) allows us to define new fields ϕ, χ and A which are functions only of coordinates ϑ, φ and t ,

$$\begin{aligned} \hat{\phi} &= (r)^{-1/2} \phi \\ \hat{\chi} &= (r)^{-3/2} \chi \\ \hat{A} &= (r)^{-3/2} A \end{aligned} \quad (31)$$

Expressing (23) in terms of the new coordinates, orthonormal frames and fields we find

$$\begin{aligned} -\hbar \partial_t (\hbar^2 \partial_t \phi) - \frac{\hbar^2}{4} \phi &= 0 \\ (1 - Y) \not{\partial} \chi &= 0 \\ A &= 0 \end{aligned} \quad (32)$$

But these are just the field equations obtained from the free action (2). Thus we obtain the $SO(3,2)$ conformal invariant singleton field equations on $S^1 \times S^2$.

We next consider the supersymmetry transformations. In terms of the natural coordinates on $\mathbb{R}^6$, we have

$$\begin{aligned} \delta \hat{\phi} &= i \bar{\epsilon} \not{\partial} \hat{\chi} \\ \delta \hat{\chi} &= (\Gamma^m \partial_m \hat{\phi} + \hat{A}) \epsilon \\ \delta \hat{A} &= i \bar{\epsilon} \not{\partial} \hat{\chi} \end{aligned} \quad (33)$$

where $\bar{\epsilon} = \epsilon^\dagger \gamma^0 \gamma^3$, $\gamma = \gamma^m \gamma^n \eta_{mn}$ and ϵ is a constant Majorana spinor, i.e.

$$\partial_m \epsilon = 0 \quad (34)$$

Expressing (33) and (34) in terms of coordinates r, θ, ϕ, t on the cone and orthonormal frames (29) we recover the supersymmetry transformation rules (11) and

$$\begin{aligned} (D_t - \frac{\theta}{2} \gamma_t \gamma_3) \epsilon &= 0 \\ (D_\theta + \frac{\theta}{2} \gamma_\theta \gamma_3) \epsilon &= 0 \\ (D_\phi + \frac{\theta}{2} \gamma_\phi \gamma_3) \epsilon &= 0 \end{aligned} \quad (35)$$

where D_t , D_θ and D_ϕ are given by (6). One may verify that (35) implies (9) and (10).

The above five-dimensional trick involves the introduction of spinors with twice the required number of components and a fermionic gauge invariance is required. On the cone the D1 field equation is invariant under the fermionic gauge transformation

$$\delta \hat{\chi} = \gamma \hat{\chi} \quad (36)$$

where $\hat{\chi}$ is an arbitrary homogeneous function of degree $-\frac{3}{2}$. Changing coordinates and orthonormal frames gives (12).

The interaction terms of (15) were also derived using the five-dimensional cone formalism of Fronsdal (8).

Extended Supersymmetry

Singleton representations exist for all N [10], with the supersymmetry parameter belonging to the vector representation of $SO(N)$ and with the Rac and D1 states belonging to the two inequivalent spinor representations for N even and to the same spinor representation for N odd. In the special case of $N=8$, triality allows us to assign ϵ , φ and χ to any permutation of 8_v , 8_s and 8_c . Let us consider this example in more detail. The on-shell action and transformation rules are

$$S = -\frac{1}{2} \int d^2 \xi \sqrt{-h} [h^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^A + \frac{g^2}{4} \phi^A \phi^A + i \bar{\chi}^A (1 - \gamma) \partial_\mu \chi^A] \quad (37)$$

$$\delta \phi^A = i \bar{\epsilon}^A (1 - \gamma) \chi^B \Gamma_{BA}^A \quad (38)$$

$$\delta \chi^A = [(\delta - \frac{g}{2} \gamma^3) \phi^B] \epsilon^A \Gamma_{BA}^A$$

where the $SO(8)$ Γ matrices satisfy

$$\begin{aligned} (\Gamma_{BA}^A, \Gamma_{AC}^A) &= \delta_{BC}^{AB} \\ (\Gamma_{AB}^A, \Gamma_{BC}^A) &= \delta_{AC}^{AB} \end{aligned} \quad (39)$$

The indices i , A and $\hat{A}$ range over 1 to 8 and denote the 8_v , 8_s and 8_c representations, respectively. Since the spinor representations of $SO(8)$ are real, ϕ^A is real and χ^A is a Majorana spinor.

If we were to try to construct an off-shell formulation for $N=8$, we might expect transformation rules of the form

$$\begin{aligned} \delta \phi^A &= i \bar{\epsilon}^A (1 - \gamma) \chi^B \Gamma_{BA}^A \\ \delta \chi^A &= [(\delta - \frac{g}{2} \gamma^3) \phi^B + \Lambda^B] \epsilon^A \Gamma_{BA}^A \end{aligned} \quad (40)$$

$$\delta \Lambda^B = i \bar{\epsilon}^B (1 - \gamma) \partial_\mu \chi^A \Gamma_{AB}^A$$

In analogy with those of $N=1$ in (11). However, as the following commutator illustrates

$$\{\delta_1, \delta_2\} \hat{\phi}_A = \dots + i \bar{\epsilon}_2^A (1 - \gamma) \epsilon_1^B \Gamma_{AB}^A \Lambda^B, \quad (41)$$

the algebra does not close. Similar conclusions are reached for all $N > 1$. In the absence of an off-shell formulation, we have also been unable to construct any self-interactions akin to (15) for $N > 1$. This seems to suggest that an off-shell formulation and self-interactions for singletons exist only in the case of simple $N=1$ supersymmetry. In this respect we agree with Sezgin and Nicolai [10].

Supermembranes

In Ref[2], a stable solution of the eleven-dimensional supermembrane equations was found for which spacetime is the product of four-dimensional AdS and a compact seven-dimensional Einstein space M_7 and for which the three-dimensional membrane occupies the $S^1 \times S^2$ boundary of AdS at spatial infinity: the Membrane at the End of the Universe. In spherical coordinates (t, θ, φ, r) on AdS, this means identifying t, θ, φ with the world - volume coordinates and taking the limit $r \rightarrow \infty$. The symmetry of this vacuum is $OSp(4/N)$ where $0 \leq N \leq 8$ is the number of Killing spinors on M_7 , e.g. $N=8$ for $M_7 = S^7$ [22]. General arguments were given to show that, in a physical gauge, the eight bosonic fluctuations δx^I and the eight four-component Majorana spinor fluctuations $\delta \psi^I$ of the 11-dimensional superspace coordinates x^I and ψ^I fall into the singleton representations of $OSp(4/N)$. Indeed the supersymmetry parameter was precisely that of equation (35). Moreover, as a consequence of the Siegel symmetry on the world volume of the membrane, there remained a fermionic gauge invariance which was precisely that of equation (12)². Owing to the uncertainty in the singleton action and transformation rules (which the present paper has now remedied), their complete derivation starting from those of the supermembrane was not given. Indeed, one may wonder about the appearance of the mass term in (2) starting from the membrane. To understand this, it is easier to work backwards and note that a change of variables like (u) with conformal factor $\Omega = ar$ leaves the action (2) invariant but replaces 'a' by $1/r$. The mass term thus disappears in the $r \rightarrow \infty$ limit. However, this means that in the reverse process, discussed in ref [2], one must acquire the mass term. We shall return to this elsewhere. For the time-being we content ourselves to note that, contrary to what one might have expected, (super)conformal invariance also has a part to play in supermembranes as well as superstrings.

As remarked upon in Refs[1] and [2], singleton representations of $OSp(4/8)$ have already occurred in the literature of 11-dimensional supergravity. They appear in the harmonic expansion of the 11-dimensional fields on $AdS \times S^7$ [22,23], but can apparently be gauged away everywhere (except on the boundary of AdS?). It was conjectured, therefore, that this may signal a form of "bootstrap" in which the membrane yields the field theory which in turn yields the membrane [1,2]. This idea has recently resurfaced in the work of Townsend [24] who gives a "topological defect" or "soliton" interpretation of super p-branes in the $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ sequences and conjectures (but does not find) a similar solution for the $\mathbb{O}$ sequence for which the theory must be its own soliton! Might the Membrane at the End of the Universe be just this missing solution? If so, it is interesting to note that, in contrast to the $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ sequences, the $\mathbb{O}$ sequence requires spacetime compactification!

If, generalizing the idea of Refs[1] and [2] and the present paper, we regard all super p-branes as boundaries of AdS spacetime we are restricted to those dimensions which admit AdS supergroups. These are strictly limited [25,26] as shown in the table below

SUPERGROUP	DIMENSION	P
$OSp(p/2) \times OSp(q/2)$	3	1
$OSp(N/4)$	4	2
$SU(2,2/(N/2))$	5	3
$F(4)$	6	4
$O^*Sp(8/N)$	7	5

This would lead to the restriction $1 \leq p \leq 5$. But this is precisely the same restriction achieved by totally different methods [27]. Are they, by any chance, related? Note, however, that in order to identify the corresponding singleton representations with fluctuations in the superspace coordinates (x, θ) we require them to be scalars and spinors. This occurs only in the cases $d=3$ and $d=4$ [28] i.e. for strings and membranes.

Returning to the case of the eleven-dimensional supermembrane,

we finish with two remarks. First, motivated by the observation that two-singleton states correspond to massless one-particle states of arbitrarily high spin, Flato and Fronsdal have conjectured that singletons may play the role of "preons" [7]. It would be interesting to reexamine this idea in the light of the supermembrane. Secondly, the major objection to quantum membranes, as opposed to strings, is that the three-dimensional action is non-renormalisable [29]. Yet we have seen that about the very special $S^1 \times S^2$ membrane vacuum, corresponding to the $AdS \times M^7$ spacetime, the self-interactions of the quantum fluctuations are severely limited by the superconformal $OSp(N/4)$ symmetry. For $N \leq 1$, the theory must at worst be renormalizable and for $N > 1$ seems even to be a free field theory!

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Footnotes

- f1) One sometimes finds the statement in the physics literature that the only compact manifolds admitting conformal Killing vectors are those isomorphic to spheres. By a theorem of Yano and Nagano [11], this is true for Einstein spaces, but $S^1 \times S^2$ is not Einstein.
- f2) We are grateful to S. Deser for bringing Ref. [12] to our attention.
- f3) It is amusing to note that the fermionic gauge invariance so important for superstrings and supermembranes can thus be traced back to 1969 [20] !