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STRINGS ON ORBIFOLDS: AN INTRODUCTION

Hans-Peter Nilles

CERN - Geneva

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1. Introduction

These lectures intend to give an elementary introduction to four-dimensional string models as obtained in orbifold compactifications [1], including the basic calculational techniques. To follow the lectures the reader should have some acquaintance with the heterotic string [2]. In Section 2 we start with reviewing basic facts about lattices [3] and θ -functions [4], followed by a discussion of one-loop modular invariance and its implications for heterotic string theory. Section 4 discusses the concept of an orbifold as a geometrical space, while in Section 5 some general properties of strings on orbifolds are discussed. Finally, in Section 6 we explain one example, the Z -orbifold, in detail including the partition function and the constraints from modular invariance.

2. Lattices and theta-functions

Let us discuss briefly the main facts about lattices that will be useful for us [3]. Consider an N -dimensional vector space (V) with basis e_i , $1 \leq i \leq N$. A lattice $\Lambda \in V$ is defined as a set of points of the form

$$\Lambda = \left\{ x = \sum_{i=1}^N n_i e_i \mid n_i \in \mathbb{Z} \right\} \quad (2.1)$$

The lattice is called integral if $x \cdot y$ is integral for every $x, y \in \Lambda$ ($x \cdot y$ denotes the inner product in V). The lattice is called even if for every $x \in \Lambda$ $x \cdot x$ is even.

For any lattice $\Lambda \subset V$ we can define the dual lattice - denoted Λ^* . It is the set of points $y \in V$ for which $x \cdot y \in \mathbb{Z}$ for all $x \in \Lambda$.

$$\Lambda^* = \{y \in V / x \cdot y \in \mathbb{Z} \quad \forall x \in \Lambda\} \quad (2.2)$$

A basis e^{*i} , $i = 1, \dots, N$ satisfies $e_i \cdot e^{*j} = \delta_i^j$. Moreover, if we define

$$g_{ij} = e_i \cdot e_j \quad (2.3)$$

the inverse is given by

$$(g_{ij})^{-1} = g^{ij} = e^{*i} \cdot e^{*j} \quad (2.4)$$

If the lattice is integral, it is contained in its dual. A lattice is called self-dual if $\Lambda = \Lambda^*$. The volume of the lattice is defined as the volume of its fundamental cell $\sqrt{|\det g_{ij}|}$. A lattice is called unimodular, if $|\det g_{ij}| = 1$. Since Λ and Λ^* have reciprocally related numbers of points per unit volume, an integral and unimodular lattice can be shown to be self-dual.

As a simple example let us discuss $SU(3)$ -group lattices. The root lattice is spanned by the two simple roots $e_1 = (\sqrt{2}, 0)$ and $e_2 = (-1/\sqrt{2}, \sqrt{3}/2)$.

Figure 1

Observe that $g_{ij} = e_i \cdot e_j$ coincides with the Cartan matrix of $SU(3)$. The

lattice Γ spanned by e_1 and e_2 is integral and even. Its dual Γ^* is spanned by $e_1^* = (1/\sqrt{2}, 1/\sqrt{6})$ and $e_2^* = (0, \sqrt{2/3})$

Figure 2

Since Γ is integral, it is contained in Γ^* . Γ^* is the weight lattice of $SU(3)$. The volume of Γ is $\sqrt{3}$ and the volume of Γ^* is $1/\sqrt{3}$.

An even simpler example can be found within the rectangular $O(2N)$ lattices. The root lattice (Λ_R) consists of the points $(n_1, \dots, n_N)$ with $n_i \in \mathbb{Z}$ and $\sum n_i \in 2\mathbb{Z}$, again an even integral lattice. Its dual, the weight lattice of $O(N)$, consists of the points $(n_1, \dots, n_N)$ where n_i are half integers. It can be decomposed into the following conjugacy classes:

scalar	:	$(n_1, \dots, n_N)$	$\sum n_i$ even
vector	:	$(n_1, \dots, n_N)$	$\sum n_i$ odd
spinor	:	$(n_1 + \frac{1}{2}, \dots, n_N + \frac{1}{2})$	$\sum n_i$ even
antispinor	:	$(n_1 + \frac{1}{2}, \dots, n_N + \frac{1}{2})$	$\sum n_i$ odd.

Take as an example $O(10)$. The states in the scalar conjugacy class with length² two correspond to 40 roots. The states in the vector class with $p^2 = 1$ to the fundamental (10) of $O(10)$ and the shortest vectors in the spinor (antispinor) class with $p^2 = N/4$ to the $16(\overline{16})$ of $O(10)$.

Let us next discuss partition functions of the lattices. Define $q = e^{2\pi i \tau}$ and

$$Z = \sum_{p \in \Lambda} q^{1/2 p^2} \quad (2.5)$$

and we want to work this out for the $O(2N)$ lattices.

$$Z = \sum_{\{n_i\}} q^{1/2 \sum_{i=1}^N n_i^2} \frac{1}{2} (1 + (-1)^{\sum n_i}) \quad (2.6)$$

corresponds to the root lattice. This can be written as

$$Z = \frac{1}{2} \left[\prod_{i=1}^N \sum_{n_i \in \mathbb{Z}} q^{n_i^2/2} + \prod_{i=1}^N \sum_{n_i} q^{n_i^2/2} e^{i\pi n_i} \right] \quad (2.7)$$

Jacobi theta-functions are defined as

$$\vartheta \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] (0|1) = \vartheta \left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix} \right] = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}(n+\theta)^2} e^{2\pi i (n+\theta)\phi} \quad (2.8)$$

Specializing to $\theta = 0$ we deduce for the expression of (2.7)

$$Z = \frac{1}{2} \left[\vartheta^N \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] + \vartheta^N \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] \right] \quad (2.9)$$

while the partition function of the $O(2N)$ weight lattice is given by

$$Z = \vartheta^N \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] + \vartheta^N \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] \quad (2.10)$$

Observe that $\theta = \frac{1}{2}$ corresponds to the spinor conjugacy classes.

Very useful is the product representation of θ -functions

$$\frac{\vartheta\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right]}{\eta} = e^{2\pi i \theta \phi} q^{\theta/2 - 1/24} \times \prod_{n=1}^{\infty} (1 + q^{n+\theta-1/2} e^{2\pi i \phi}) (1 + q^{n-\theta-1/2} e^{-2\pi i \phi}) \quad (2.11)$$

where

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (2.12)$$

is Dedekind's η -function. The transition from product to sum-representation is a statement about bosonization in two-dimensional field theory. Parametrizing the two-dimensional world-sheet by $\sigma = \sigma_1 + \tau \sigma_2$ with $\tau = \tau_1 + i\tau_2$ (2.11) can be seen as the partition function of a complex fermion with boundary conditions

$$\begin{aligned} \psi(\sigma_1 + 2\pi, \sigma_2) &= -e^{-2\pi i \theta} \psi(\sigma_1, \sigma_2) \\ \psi(\sigma_1, \sigma_2 + 2\pi) &= -e^{-2\pi i \phi} \psi(\sigma_1, \sigma_2) \end{aligned} \quad (2.13)$$

For details see Ref. [5].

It will later be useful to know the behaviour of θ -functions under the change of the argument τ . We have

$$\begin{aligned} \vartheta\left[\begin{smallmatrix}\theta\\ \phi-\theta-1/2\end{smallmatrix}\right](\tau+1) &= e^{-i\pi(\theta^2-\theta)} \vartheta\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](\tau) \\ \vartheta\left[\begin{smallmatrix}-\phi\\ \theta\end{smallmatrix}\right](-1/\tau) &= (-i\tau)^{1/2} e^{-2\pi i \theta \phi} \vartheta\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](\tau) \end{aligned} \quad (2.14)$$

The corresponding transformations for η are

$$\begin{aligned}\eta(\tau+1) &= e^{i\pi/12} \eta(\tau) \\ \eta(-1/\tau) &= (-i\tau)^{1/2} \eta(\tau)\end{aligned}\tag{2.15}$$

The two transformations $\tau \rightarrow \tau+1$ and $\tau \rightarrow -1/\tau$ generate the modular transformations

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d} \quad ad-bc=1 \tag{2.16}$$

and a, b, c, d integers.

Coming back to our lattice discussion, one might observe that the transformation $\tau \rightarrow -1/\tau$ corresponds to a transition from a certain lattice to its dual. It is just a Fourier transformation and we have the Poisson resummation formula

$$\sum_{p \in \Gamma} e^{i\pi\tau p^2} = \frac{1}{\text{Vol}(\Gamma')} \sum_{k \in \Gamma^*} e^{-i\pi k^2/\tau} \tag{2.17}$$

The reader can check this explicitly for (2.9) and (2.10) by using (2.14) and computing the volume of $O(2N)$ lattices. The reciprocity of the number of points per unit volume for a lattice and its dual is therefore just expressed through the $\tau \rightarrow -1/\tau$ transformation. A self-dual lattice should therefore have a partition function that is invariant under this transformation (apart from the fact that the volume of such a lattice is unity). For a further check reconsider again $SU(3)$

$$Z_{\text{root}} = \sum_{n,m} e^{i\pi\tau \frac{1}{2}(n^2+3m^2)} \frac{1}{2} (1 + (-1)^{n+m}) \tag{2.18}$$

and

$$Z_{\text{weight}} = \sum_{n,m} e^{i\pi z (\frac{1}{2}n^2 + \frac{1}{6}m^2)} \frac{1}{2} (1 + (-1)^{n+m}) \quad (2.19)$$

and show the validity of (2.17).

For group lattices, there is always this duality of root and weight lattices. What about self-dual lattices? The rectangular lattices consisting of scalar and vector conjugacy classes of $O(2N)$ are self-dual

$$Z = \mathcal{D}^N \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.20)$$

Even rarer are the even self-dual lattices. They only exist in $8n$ dimensions. In $d = 8$ there is only one such lattice: the E_8 root lattice. That such a thing can happen with E_8 can already be guessed from the fact that fundamental and adjoint representations coincide and that therefore root and weight lattices have to be the same. But let us be more explicit. Consider first the $O(16)$ root lattice

$$Z = \frac{1}{2} \left[\mathcal{D}^8 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathcal{D}^8 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right] \quad (2.21)$$

Then observe that the shortest vectors in the spinor conjugacy class $(n_1 + \frac{1}{2}, \dots, n_8 + \frac{1}{2})$ have $(\text{length})^2$ two. The lattice

$$Z = \frac{1}{2} \left[\mathcal{D}^8 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathcal{D}^8 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \mathcal{D}^8 \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \mathcal{D}^8 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right] \quad (2.22)$$

is even integral and self-dual. Vectors with $p^2 = 2$ come with multiplicity 240; the number of E_8 roots. Equation (2.22) defines the root-lattice of E_8 . The self-duality of the lattice can best be seen by showing invariance of (2.22) under the transformation $\tau \rightarrow -1/\tau$. In $d = 16$ there are two even integral self-dual lattices: the $E_8 \times E_8$ root lattice and a $\text{spin}(32)/Z_2$ lattice given by

$$\frac{1}{2} \left[\mathcal{D}^{16} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathcal{D}^{16} \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \mathcal{D}^{16} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \mathcal{D}^{16} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right] \quad (2.23)$$

States with $p^2 = 2$ correspond to the roots of $O(32)$. In higher dimensions

these lattices appear more frequently. Already in $d = 24$ one has 24 different choices - the Niemeier lattices.

3. Modular Invariance and the Heterotic String

The heterotic string is left-right asymmetric. The left-handers $X(\sigma, t)$ correspond to the bosonic string in 26 dimensions, 16 of which are compactified on a lattice defined by vectors p . The mass spectrum is given by

$$\frac{1}{4} m_L^2 = N_L - 1 + p^2/2 \quad (3.1)$$

where N_L is integer (the oscillator number) and p is a vector of the even self-dual lattice $E_8 \times E_8$ or $\text{spin}(32)/Z_2$. The right-handers correspond to the type I superstring, i.e., the GSO-projected NSR model. For the Neveu-Schwarz sector, we have

$$\frac{1}{4} m_R^2 = N_R - 1/2 ; N_R \in \mathbb{Z} + 1/2 \quad (3.2)$$

and in the Ramond sector

$$\frac{1}{4} m_R^2 = N_R ; N_R \in \mathbb{Z} \quad (3.3)$$

Consistency furthermore requires level matching, i.e.

$$m_L^2 = m_R^2 \quad (3.4)$$

These are a lot of rules and in the remainder of the chapter we want to analyze the special properties of the model, e.g., why does the gauge group have to be $E_8 \times E_8$ or $O(32)$ and what is the reason for the GSO projection?

A basic ingredient of string theory is the reparametrization invariance on the worldsheet swept by the string $X(\sigma, t)$. At the tree level of string scattering amplitudes, this worldsheet has the topology of the sphere and any σ - t reparametrization can be continuously connected to the identity. At higher level this is not true any longer. For example, at one loop the worldsheet has the topology of a torus

Figure 3

and certain reparametrizations, like the interchange of σ and τ , are disconnected from the identity. Reparametrization invariance of string theory, of course, requires invariance under these so-called modular transformations. The discussion is usually done in Euclidean space. We pass from (σ, τ) to the complex co-ordinate $\sigma = \sigma_1 + \tau\sigma_2$ and τ is called the modular parameter. The modular transformations as defined in (2.16) correspond exactly to the reparametrizations on the torus, that cannot be continuously connected with the identity. The torus can be defined by the lattice as shown in Fig. 4

Figure 4

The modular transformations are generated by $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$. The first one leaves σ_1 unchanged and transforms σ_2 to $\sigma_1 + \sigma_2$ while the second one essentially interchanges σ_1 and σ_2 . In calculating loop amplitudes, we have to integrate over τ , the modular invariant measure being $d^2\tau / (\text{Im}\tau)^2$. Because of (2.16) we cannot consider the full range of τ -

integration but "divide" by modular transformations and consider only a so-called fundamental domain [6]. One choice for such a domain is shown in Fig. 5

Figure 5

$-\frac{1}{2} \leq \text{Re}\tau \leq +\frac{1}{2}$, $\text{Im}\tau > 0$, $\tau^2 \geq 1$. The reader should check that all points in the upper half of the τ -plane can be reached by a modular transformation. This restriction of the τ -integration is very important for the discussion of finiteness of loop amplitudes in string theory.

The simplest loop diagram is the contribution to the vacuum energy. In field theory this gives usually a quartically divergent contribution. In string theory it corresponds to the τ -integration of the partition function

$$\int \frac{d^2\tau}{(\text{Im}\tau)^2} \frac{1}{(\text{Im}\tau)^4} Z(\tau) \bar{Z}(\bar{\tau}) \quad (3.5)$$

where Z is the contribution of the right-movers (a function of τ) and $\bar{Z}$ the contribution of left-movers (function of $\bar{\tau}$). Given (3.1) for the left-movers, we have

$$\bar{Z}(\bar{\tau}) = \bar{\eta}(\bar{\tau})^{-24} \sum_{\rho \in \Lambda_{16}} \bar{q}^{1/2} p^2 \quad (3.6)$$

We are now in a position to discuss the modular invariance of such an expression. Recall (2.15) to see that η^{24} is invariant under modular transformations [actually the $\tau \rightarrow -1/\tau$ transformation gives a prefactor but this cancels a factor arising in the transformation of the integration measure. Therefore, it would be better to say that η^{24} transforms covariantly. In the following we will just drop the $(-i\tau)^n$ factors]. The modular invariance of (3.6) requires therefore the invariance of the lattice sum, and we have discussed this question in the previous chapter. We have seen there that such transformations imply, in particular, a transition from a lattice to its dual lattice; invariance thus requires self-dual lattices. The restriction of the gauge group to $E_8 \times E_8$ or $O(32)$ comes from the requirement of modular invariance and is therefore a direct consequence of reparametrization invariance. For the lattice sum in (3.6), we have thus to insert either the square of (2.22) or (2.23). Observe also that in η^{24} it is crucial that the number of transverse oscillators is 24 by recalling the transformation of η under $\tau \rightarrow \tau+1$.

For the right-movers we have to discuss worldsheet fermions. In contrast to bosons, they can be either periodic or antiperiodic around the σ_1 -cycle ($0 \leq \sigma_1 \leq \pi$) of the torus. As a result one obtains two sectors: the Neveu-Schwarz (NS) sector (antiperiodic in σ_1) and the Ramond (R) sector (periodic in σ_1). In addition we can consider periodicity and antiperiodicity in the σ_2 -cycle. The partition function for a complex fermion with twisted boundary conditions is given by the θ -function in (2.11). From (2.13) we have $\theta, \phi = 0(\frac{1}{2})$ for antiperiodic (periodic) boundary conditions. Since the theory is defined in 8 transverse dimensions we have to consider as partition function

$$Z = \eta^{-12} \sum_{\theta_i, \phi_i = 0}^{1/2} a_{ij} \mathcal{D}^4 [\theta_i, \phi_j] \quad (3.7)$$

where 8 of the 12 η 's correspond to the transverse bosonic excitations. For NS, $\theta_i = 0$ while for R, $\theta_i = \frac{1}{2}$. In the construction of the superstring the coefficients a_{ij} were chosen to have special values to give the

desired result. We shall see now that modular invariance fixes the coefficients.

Let us start with the NS sector and consider antiperiodicity in σ_1 and σ_2 , i.e., $\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using (2.14) for $\tau \rightarrow \tau+1$, we obtain $\theta^4 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$. Geometrically this is obvious by looking at Fig. 6:

Figure 6

Under $\tau \rightarrow \tau+1$, η^{12} changes sign. A modular invariant partition function for the NS sector is therefore given by

$$Z_{NS} = \frac{1}{2} \eta^{-12} \left[\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \theta^4 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right] \quad (3.8)$$

The second term in (3.8) with the - sign gives the GSO projection in the NS sector. Following our discussion in the last chapter we see that (3.8) represents the vector conjugacy class of an $O(8)$ lattice. The allowed lattice vectors are $p = (n_1, \dots, n_4)$ with n_i integer and sum of n_i odd. This implies odd p^2 and this is exactly the GSO projection.

The last term in (3.8) corresponds to periodic boundary conditions in the σ_2 direction. A $\tau \rightarrow -1/\tau$ transformation exchanges σ_1 and σ_2 . Modular invariance teaches us that we cannot discuss the NS sector alone, it requires the existence of the R sector. Indeed, under $\tau \rightarrow -1/\tau$, $\theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ transforms to $\theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ following (2.14). Since η^{12} does not change sign in this transformation, the term inherits the - sign from the second term in (3.8). We have obtained

$$Z_R = -\frac{1}{2} \eta^{-12} \mathcal{D}^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \quad (3.9)$$

via this construction. Z_R still has to fulfil a non-trivial constraint from modular invariance: the $\tau \rightarrow \tau+1$ constraint $\theta^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ changes sign and cancels the sign obtained from the η^{-12} term, thus (3.9) is modular invariant. There are still some remarks to be made about (3.9):

- (i) the factor $\frac{1}{2}$ signifies the GSO projection in the R sector. This factor comes from modular invariance, once we have normalized the NS sector correctly. Using the fact that $\theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ vanishes, we can write (3.9) in a more symmetric form

$$Z_R = -\frac{1}{2} \eta^{-12} \left[\mathcal{D}^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \mathcal{D}^4 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right] \quad (3.10)$$

- (ii) Observe that $\tau \rightarrow \tau+1$ invariance works just for θ^4 (or θ^{8n+4}) corresponding to eight transverse dimensions.
- (iii) Equation (3.9) has an overall - sign. In view of (3.5) this signifies that the states in the R sector behave as space-time fermions. In the given model, modular invariance gives the correct relation between spin and statistics.
- (iv) Observe that the sum $Z + \bar{Z}$ vanishes. The model is supersymmetric and there is no one-loop contribution to the vacuum energy (3.5).

To close this chapter let us stress again the importance of modular invariance for the construction of consistent string models. In heterotic string theory, it implies $E_8 \times E_8$ or $O(32)$ from the left-handers and gives the superstring on the right-hand side. Actually so far we have required modular invariance for left- and right-handers separately. In principle, it would be sufficient to require it only for the full theory, in which case several non-supersymmetric modular invariant $d = 10$ theories can be found. We will not discuss them here separately, since our discussion of strings on orbifolds will include these cases.

4. The Geometry of Orbifolds

Ten space-time dimensions seem to be required for the superstring. What about strings in four extended space-time dimensions? The easiest

way to obtain such models is through torus compactification from $d = 10$ to $d = 4$. This construction is by now fully understood [7,8]. It can be characterized by an even integral self-dual Lorentzian lattice of signature $(22,6)$. $22 = 16+6$ for the left-movers and 6 for the right-movers. Such lattices [3] $\Lambda_{p,q}$ with signature (p,q) only exist if $p-q = 0 \pmod{8}$. In a given dimension such a lattice is unique up to Lorentz transformations $SO(p,q)$. Different models can be obtained through $SO(p,q)/SO(p) \times SO(q)$ transformations. For $d = 4$ this gives a 132 parameter family of consistent string models. The Lorentz transformations of the lattice are in one-to-one correspondence with background fields for the metric g_{ij} ($i,j = 1, \dots, 6$), the antisymmetric tensor b_{ij} and gauge potentials a_i^I ($I = 1, \dots, 16$) (corresponding to 132 degrees of freedom). The mass spectrum is given by

$$\begin{aligned} \frac{1}{4} m_L^2 &= N_L - 1 + \tilde{L}^2/2 \\ \frac{1}{4} m_R^2 &= N_R + \tilde{L}^2/2 \end{aligned} \quad (4.1)$$

where the six-dimensional vector $\tilde{L}$ is given by

$$\begin{aligned} \tilde{L} &= -n^i R_i e_i + \frac{m_i}{2R_i} e^{*i} - \frac{b_{ij}}{R_i} e^{*i} n^j + \\ &+ \frac{a_i^I}{2R_i} e^{*i} \left[p^I - \frac{1}{2} a_j^I n^j \right] \end{aligned} \quad (4.2)$$

where n, m are integers and e_i defines the six-dimensional torus $e_i \cdot e_j = g_{ij}$. As normalization we have chosen $e_i^2 = 2$ and R_i denote the radii. The 16-dimensional vectors p^I represent the lattice of either the $E_8 \times E_8$ or the $\text{spin}(32)/Z_2$ lattice. For the left-movers we have to consider a 22-dimensional vector L . Its first six components are given by $\tilde{L} + 2n^i R_i e_i$ and the last 16 by

$$L_{\text{extra}}^I = p^I - a_i^I n^i \quad (4.3)$$

The lattice $(L, \tilde{L})$ is even integral and self-dual. While in $d = 10$ we have gauge group $E_8 \times E_8$ or $O(32)$, we now obtain a gauge group of rank 22 from the left-movers. With suitable values of R_i , g_{ij} , b_{ij} any simply laced rank-six gauge group can be obtained. The Wilson lines a_i^I can further enhance the gauge group to, e.g., $O(44)$. For generic choices of the background fields, one obtains $U(1)^{22}$. Enhancement of the gauge group requires a careful choice of parameters. The potential has 132 flat directions. Given an enhanced gauge group, it can be broken smoothly by a variation of these parameters and a Higgs mechanism occurs.

Torus compactification of the heterotic string thus leads to a variety of $d = 4$ string models. Such a compactification, however, always leads to $N = 4$ supersymmetry (as in field theory) and thus to a non-chiral particle spectrum, not allowing a description of parity violation of the weak interactions.

To obtain chiral models we have to look for a more general construction principle, preferably one that is still simple enough to allow an exact treatment. Such models can be obtained via compactification on tori with twisted boundary conditions, so-called orbifolds. To construct these spaces one starts with a torus T^6 and then "divides" by a discrete group of symmetries P of the torus (the point group) to obtain $O^6 = T^6/P$. The geometrical object obtained in such a way is flat almost everywhere and therefore allows an exact treatment of string theory [1]. The curvature is concentrated on isolated points corresponding to the fixed points of P . Since T^6 is defined by a lattice Λ , we can equivalently define $O^6 = R^6/S$ where R^6 is flat space and the space group S contains P and the lattice shifts. To illustrate the concept let us discuss some two-dimensional examples. Start with an orthonormal lattice as shown in Fig. 7

Figure 7

with T^2 defined by e_1 and e_2 . We now divide by the group P of 180° rotations, i.e., we identify points connected by such a rotation. While the torus T^2 is given by the fundamental region $ABCD$ (with $AB \leftrightarrow CD$ and $AD \leftrightarrow BC$ identified), the orbifold $O^2 = T^2/P$ needs only a subset of this domain. Let us first determine the fixed points of transformation P . A , of course, is trivially found to be a fixed point. More fixed points can be found by consideration of half lattice vectors. Reflecting E at the origin gives a point at distance e_1 from E and therefore E remains fixed under P . The same holds for F and G leading to a total of four fixed points. Take now the square $AEGF$ and its reflection at the origin. By a shift $e_1 + e_2$ it is identified with $GHCI$. The same is true for $FGIB$ and $EDHG$. Thus the rectangle $AEIB$ is sufficient to describe the orbifold. Furthermore, AE has to be identified with BI , AF with FB and EG with GI : i.e., take $AEIB$, fold along FG and glue the edges. The resulting "ravioli-like" object is an orbifold with conical singularities at the four fixed points. One could repair these singularities by cutting off the neighbourhoods of A , E , G and F and gluing in smooth disks. Topologically, this space would then be equivalent to a sphere.

Let us give a second example of a Z_2 orbifold (Z_2 because P was the two-element group). Start with the $SU(3)$ root lattice as given in Fig. 8 by basis vectors e_1 and e_2

Figure 8

and identify points rotated by 180° . We have again four fixed points A, B, C and D. The reader should check that the orbifold looks as given in Fig. 9

Figure 9

Observe that the lattice in Fig. 8 has also a Z_3 symmetry (rotations by 120°). $O^2 = T^2/Z_3$ will be discussed later in detail.

For Z_N orbifolds with N a prime number, we only need to know the number of fixed points to know the number of singular points. If N is non-prime this identification is not so simple (a fact that also leads to

a subtlety discussing the massless spectrum of strings on such an orbifold). Let us see this in our final example. Start with the lattice of Fig. 7 and divide by $P = Z_4$, i.e., 90° rotations. Under those rotations only A and G are fixed points. But the group Z_4 also contains an element corresponding to 180° rotations with fixed points A, E, G and F. E and F have to be identified because of the 90° rotations. The square AEF G is enough to describe the Z_4 orbifold given in Fig. 10

Figure 10

i.e., fold along AG and glue the edges.

5. Strings on Orbifolds

Before we discuss the motion of closed strings on orbifolds, let us recall the situation in the case of torus compactification. One starts with closed strings on flat space R^n and then identifies points that differ by lattice vectors: as a result we can distinguish two types of closed strings:

- those already closed on R^n ;
- those not closed on R^n , but only on T^n ; i.e., the two ends differ by a lattice vector.

This second type of states are winding states on the torus. It is not possible to consider a theory without those winding states. An unwound state can split into winding states of opposite winding number. A theory excluding the winding states will in general not be modular invariant. The Hilbert space of string states of the torus thus consists of an unwound sector and the various winding sectors. In all these sectors one has to project on states invariant under the translation by

the lattice vectors defining the torus; the system has been put in a box, thus momenta in the compactified dimensions are quantized.

To discuss strings on orbifolds we start with a torus $T^n = R^n/\Lambda$ where Λ is a lattice generated by the basis vectors e_i^μ . Closed strings on the torus satisfy

$$X^\mu(\sigma=\pi) = X^\mu(\sigma=0) + 2\pi \ell^\mu \quad (5.1)$$

where $\ell^\mu = \sum m_i e_i^\mu$ ($m_i \in \mathbb{Z}$) is a vector in Λ . Configurations with $\ell^\mu \neq 0$ are the winding states. To obtain an orbifold $O^n = T^n/P$, one further divides the torus by a group P of discrete transformations. The action of P should be consistent with the symmetry of the lattice. Alternatively we can, of course, describe the orbifold as $O^n = R^n/S$ where the space group S contains P and the translations e_i^μ . Closed strings on O^n satisfy

$$X^\mu(\sigma=\pi) = q X^\mu(\sigma=0) + 2\pi \ell^\mu \quad (5.2)$$

for $q \in P$ an element of the point group. As a result we can distinguish two types of closed strings:

- untwisted. These are the states where q in (5.2) corresponds to the identity element of the point group, i.e., those states that correspond to strings already closed on the torus T^n . Not all of the states are kept, however, one has to project on the states invariant under P .

- twisted. Here q in (5.2) represents a non-trivial element of the point group P , and these states are only closed on O^n but not on T^n (it would be an open string on T^n whose endpoints will be identified via the action of P). Again we have to project on P -invariant states.

The states in the untwisted sector correspond to a subset of states on the torus. What do the twisted states look like? Since the space is flat almost everywhere we can start with the oscillator expansion

$$\begin{aligned}
X(\sigma, t) = & x_0 + p t + 2 \ell \sigma + \\
& + \frac{i}{2} \sum \left[\frac{\alpha_n}{n} e^{-2 i n(t+\sigma)} + \frac{\tilde{\alpha}_n}{n} e^{-2 i n(t-\sigma)} \right]
\end{aligned}
\tag{5.3}$$

Let us first consider the constraint through (5.2) for the centre-of-mass motion. Since (5.2) has to be true for all times t we obtain

$$X_0 + 2\pi \ell = q X_0 \tag{5.4}$$

This implies that X_0 has to coincide with qX_0 up to a lattice vector, i.e., X_0 must be a fixed point of P . The centre-of-mass of twisted sector string states is located at the orbifold singularities and the twisted sectors are characterized by the fixed points of the transformations in P .

Relation (5.2) also gives constraints for the oscillator modes in (5.3) leading to relations like

$$\alpha_n e^{-2 i n \pi} = q \alpha_n \tag{5.5}$$

For Z_N transformations, we can represent q by $\exp(2\pi i k/N)$ $0 \leq k \leq N-1$ and (5.5) then leads to fractional oscillator numbers for the different twisted sector states. To illustrate the string configurations in the various sectors, consider again a simple two-dimensional example, the Z_2 -orbifold from the $SU(3)$ root lattice (Fig. 11)

Figure 11

The solid line corresponds to a state in the untwisted sector, the dashed one to a winding state in the untwisted sector while the dotted line is a state in the twisted sector corresponding to fixed point A.

So much for the qualitative picture. Let us now discuss the spectrum of strings on orbifolds more explicitly and also formulate the constraints for consistent models. We have to satisfy the constraints of modular invariance and, actually, modular transformations can be used to construct a consistent string theory on an orbifold.

Let us again use complex co-ordinates as in Section 3, $\sigma = \sigma_1 + \tau\sigma_2$ where τ is the modular parameter. Consider the twisted string

$$X(\sigma_1 + \pi, \sigma_2) = h X(\sigma_1, \sigma_2)$$

$$X(\sigma_1, \sigma_2 + \pi) = g X(\sigma_1, \sigma_2) \quad (5.6)$$

where g and h denote elements of the point group P . The transition from $\sigma = 0$ to $\sigma = \pi(1+\tau)$ can be done in two ways and consistency requires therefore that g and h commute. We now have to consider a variety of sectors denoted by (g, h) and $h = 1$ denotes the untwisted sector. Observe the analogy with our discussion of the NSR string in Chapter 3 where we had a twist of order two on the fermions. The untwisted sector splits up into several subsectors $(g, 1)$ for $g \in P$. We have here to consider

$$\sum_{g \in P} (g, 1) \quad (5.7)$$

and the sum projects on the invariant states. In the NS model we had seen this to give the GSO projection. For a Z_N twist we have

$$\sum_{k=0}^{N-1} \exp(2\pi i k/N) = 0 \quad (5.8)$$

annihilating all the non-invariant states in the sum of (5.7). This concludes our discussion of the untwisted sector.

The twisted sectors can be obtained via modular transformations from the untwisted sector. Consider the sector $(g, 1)$ and recall that the transformation $\tau \rightarrow -1/\tau$ exchanges σ_1 and σ_2 , thus leading to a sector $(1, g)$. [Compare the discussion leading to formula (3.9) in Section 3.] This shows that a consistent string model on an orbifold has to include the twisted sectors. Other subsectors in the g -twisted sector can be obtained via a $\tau \rightarrow \tau+1$ transformation [see the transformation properties of θ -functions in (2.14) as well as in Fig. 6]. Starting from $(1, g)$ the transformation $\tau \rightarrow \tau+1$ will lead to (g, g) and repeated application of $\tau \rightarrow \tau+n$ to (g^n, g) in the twisted sector. For a twist of order N we would require the sector (g^N, g) to coincide with $(1, g)$ exactly and this is a restriction for the partition function of modular invariant models [9]. If this condition is satisfied, the model is one-loop modular invariant by construction. For the different twisted sectors we have to sum

$$\sum_h (h, g) \quad ; \quad hg = gh \quad (5.9)$$

only over elements h commuting with g . If P is non-Abelian, there might be new constraints from higher loop modular invariance [10]. Equation (5.9) again gives the correct projection on P -invariant states in the twisted sectors, as inherited from the correct projection in the untwisted sector.

This then gives the following construction scheme for string models on orbifolds. First choose a torus and a twist. Take the Hilbert space for strings on the torus and project on states invariant under the twist: this defines the untwisted sector. Then proceed with a modular transformation $\tau \rightarrow -1/\tau$ to obtain the twisted sectors. Check with $\tau \rightarrow \tau+n$ transformations whether each twisted sector closes under these transformations. We are now ready to discuss a special example in detail.

6. The Z -orbifold

Start with the $E_8 \times E_8$ heterotic string in $d = 10$. The mass formula is given by

$$\begin{aligned} m^2 &= 2m_R^2 = m_R^2 + m_L^2 \\ \frac{1}{4} m_R^2 &= N_R \\ \frac{1}{4} m_L^2 &= N_L - 1 + p^2/2 \end{aligned} \quad (6.1)$$

where N_L , N_R are integers and p is a vector of the $E_8 \times E_8$ lattice. (For the definition of this lattice see sections 2 and 3.) Compactify to $d = 4$ on a six torus given by the product of three $SU(3)$ root lattices and the twist P is chosen to be a simultaneous 120° rotation of these three lattices. On an individual $SU(3)$ root lattice this leaves three fixed points defining the weight lattice of $SU(3)$ (see Fig. 12); thus in total we have $3^3 = 27$ fixed points.

Figure 12

This completely specifies the twist on space-time bosons. We choose space-time fermions to be twisted in the same way, leading to a space-time supersymmetry.

In order to break the gauge group, one accompanies the space-time twist by a gauge transformation in $E_8 \times E_8$, i.e., we define a homomorphism of $P = Z_3$ in $G \subset E_8^2$. There are many ways to do this embedding of the twist in the gauge group. Let us here choose to represent the gauge transformation by a shift in the E_8 lattice (which in this case can be shown to be equivalent to a rotation [11]) $p \rightarrow p+v$. There are, of course, constraints on v . First of all, P is a twist of order three which implies that $3v$ should be an E_8 lattice vector. Moreover, the restriction from modular invariance discussed in the last section, (namely those from the $\tau \rightarrow \tau+1$ transformations in the twisted sectors) leave only a few possible choices. We choose here

$$v = \frac{1}{3} (1, 1, 2, 0, \dots, 0) \quad (6.2)$$

which is called the standard embedding (why it is called so will become clear later). The projection for strings on the Z -orbifold will now be on states invariant under the combined action of P and G . Going to a complex notation the eigenvalues are third roots of unity represented by $\alpha = e^{2\pi i/3}$. For massless states in the untwisted sector this gives

Right
8_v
8_s
Left
8_v
16
240
240'

 $\rightarrow$

1	α	α^2
2_v	3	3
2_s	3	3
2_v	3	3
16		
78	81	81
240'		

(6.3)

$8_{v,s}$ denote the vector (spinor) with respect to the "transverse" Lorentz group $O(8)$ in $d = 10$. Under the Z_3 twist 8_v splits up in an invariant $O(2)$ vector (the $d = 4$ transverse Lorentz group) and six scalars transforming as α or α^{-1} . 8_s gives rise to an invariant Weyl spinor and six non-invariant states. Here the P eigenvalue is linked with helicity.

Upon multiplying left and right states, we see that only one of the four possible gravitini remain in the spectrum, thus we have $N = 1$ supersymmetry. We also see here explicitly that the embedding of the twist in the gauge group is necessary for the breakdown of gauge symmetry. Here the eigenvalues are obtained by $\exp(2\pi i p v)$ giving invariant states in the adjoint of $SU(3) \times E_6$ and the 81 states with α correspond to a $(3, 27)$ representation of this group. The ones with α^{-1} correspond to $(\bar{3}, \bar{27})$ but to form invariant states they are combined with α states from the right-hand side having opposite helicity. We therefore obtain three chiral superfields transforming as $(3, 27)$. In addition we have nine singlet chiral superfields and of course the supergravity multiplet, as well as the gauge multiplet of $E_6 \times SU(3) \times E_8$. The unbroken gauge group is of course the one that commutes with the twist. The twist chosen here was the centre of the $SU(3)$. This concludes our discussion of massless states in the untwisted sector.

The twisted sector comes in 27 copies with identical spectrum associated with the 27 fixed points of Z_3 . We have to distinguish the sectors

twisted by 120° and by 240° . On the right-hand side these two sectors combine to give a chiral massless superfield at the lowest level (a fact which will become clear when we later derive the partition functions).

For the left-hand side we have six twisted oscillators: three with $N_L = Z+1/3$ and three with $Z+2/3$ and the mass formula

$$\frac{1}{4} m_L^2 = N_L - 2/3 + (p+v)^2/2 \quad (6.4)$$

where we have included the vacuum energy of twisted oscillators according to the formula [1]

$$-\frac{1}{24} + \frac{1}{4} \eta (1-\eta) \quad (6.5)$$

In the gauge group, we had represented the twist by a shift vector v . The shift in the vacuum energy is obtained here by the fact that $(p+v)^2 \geq 2/3$ has a minimal length. To obtain massless states one has two possibilities

$$- N_L = 1/3 \quad \text{with} \quad (p+v)^2 = 2/3$$

$$- N_L = 0 \quad \text{with} \quad (p+v)^2 = 4/3$$

Let us count the states using the representation of the E_8 lattice as discussed in Chapter 2. In the first case only the scalar conjugacy class of $O(16)$ contributes $(n_1, \dots, n_8) = p$ with $\sum n_i$ even. $(p+v)^2 = 2/3$ leads to three choices, corresponding to a $\bar{3}$ of $SU(3)$. At each fixed point we have $3(1, \bar{3})$ because of the three oscillators with $N_L = 1/3$.

The second case $(p+v)^2 = 4/3$ is more complicated. In the $O(16)$ scalar conjugacy class we have 11 solutions: $n_i = \pm 1$ for one i with $4 \leq i \leq 8$ and $n_3 = -1$, $n_1 = n_2 = 0$ gives ten possibilities corresponding to a 10 of $O(10)$ as a subgroup of $O(16)$ (compare Section 2) and the last one given by $n_1 = n_2 = -1$, $n_i = 0$ $i \geq 3$ a singlet of $O(10)$. In the spinor conjugacy class we find 16 states $(n_1+5/6, n_2+5/6, n_3+7/6, n_4+\frac{1}{2}, \dots, n_8+\frac{1}{2})$ with $\sum n_i$ even and $(\text{length})^2 = 4/3$ corresponding to a spinor of $O(10)$. These combine to a 27-representation of E_6 . In the twisted sectors we therefore find in total chiral supermultiplets transforming as $(1, \bar{3})$ with

multiplicity 81 and (27,1)) with multiplicity 27. Observe that gauge anomalies in untwisted and twisted sectors do not separately vanish, but only vanish in the complete theory. Let us also stress here that the obtained spectrum is chiral leading to 36 generations of E_6 . We have found this spectrum by a succession of educated guesses. The result can of course be rigorously obtained using the modular invariant construction presented in the last section. In the remainder of this section I shall sketch this construction explicitly for the case of the Z-orbifold. This necessarily will become quite technical and the reader should be acquainted with the material presented in Sections 2 and 3.

Let us discuss left- and right-handed partition functions separately. In the untwisted sector we have to consider the subsectors (1,1), (θ ,1) and (θ^2 ,1) where θ represents the 120° rotation. Let us concentrate on $P_{(\theta,1)}$ where we have the contribution from the E'_8 lattice

$$A = \frac{1}{2\eta^8} \left[\mathcal{D}^8 \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] + \mathcal{D}^8 \left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right] + \mathcal{D}^8 \left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix} \right] + \mathcal{D}^8 \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right] \right] \quad (6.6)$$

The E_8 lattice with phases

$$B = \frac{1}{2\eta^8} \left[\mathcal{D}^2 \left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix} \right] \mathcal{D} \left[\begin{smallmatrix} 0 \\ 2/3 \end{smallmatrix} \right] \mathcal{D}^5 \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] + \dots \right] \quad (6.7)$$

and the space-time oscillators

$$C^{-1} = q^{8/24} \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^n e^{2\pi i/3})^3 (1-q^n e^{-2\pi i/3})^3 \quad (6.8)$$

which, using θ -function identities, can be rewritten as

$$C = 3\sqrt{3} \eta \mathcal{D}^{-3} \left[\begin{smallmatrix} 1/2 \\ -1/6 \end{smallmatrix} \right] \quad (6.9)$$

The partition function $P_{(\theta,1)}$ in the (θ ,1) sector from the left-handers is then given by the product of A, B and C.

To obtain the twisted sectors, we have to perform a $\tau \rightarrow -1/\tau$ modular transformation. Except for the η 's A will remain invariant as a self-dual lattice. For B we shall obtain

$$B(-1/\tau) \cong \sum q^{1/2(p+v)^2} (1 + e^{2\pi i p \cdot w}) + \\ + \sum q^{1/2(p+w+v)^2} (1 + e^{2\pi i (p+w)w}) \quad (6.10)$$

where $w = \frac{1}{2}(1, 1, \dots, 1)$. Next we have to check the phases obtained after a $\tau \rightarrow \tau+3$ transformation to check for modular invariance. Here $\tau \rightarrow \tau+3$ gives $\exp(3i\pi(p+v)^2)$. One checks easily that in the case under consideration $3(p+v)^2 = 0$ modulo 2 as well as $3(p+w+v) = 0 \pmod{2}$. So for B there is no phase under the $\tau \rightarrow \tau+3$ transformation. In general this need not be the case. Only when all parts of the partition functions are multiplied does such a phase have to vanish. Next we have to discuss C including the Dedekind functions $C = \eta^{16} C'$. Dropping $(-i\tau)^{\frac{1}{2}}$ factors we obtain

$$C'(-1/\tau) = \frac{3\sqrt{3}}{4} \cdot \frac{\exp(-i\pi/2)}{\eta^{15} \vartheta^3[-1/6]} \quad (6.11)$$

Again $\tau \rightarrow \tau+3$ does not give rise to a phase. Observe the factor $3\sqrt{3}$ in (6.11). From the right-handed oscillators we will get a similar factor leading to 27 the number of fixed points. In a left-right asymmetric context one side gives an overall multiplicity corresponding to the square root of the number of fixed points [12]. Assuming that there is no additional phase from the right-handers (which we will show to be true later) we can now look at the projections in the twisted sectors. B comes with a phase $\exp(i\pi(p+v)^2)$ under $\tau \rightarrow -1/\tau$. For C' we obtain in the $(1, \theta)$ sector (dropping prefactors)

$$\eta^{-18} \prod (1 - q^{n-2/3})^3 (1 - q^{n-1/3})^3 \quad (6.12)$$

while in the (θ, θ) sector we have

$$e^{4\pi i/3} \eta^{-18} \prod (1 - q^{n-2/3} e^{-4\pi i/3}) (1 - q^{n-1/3} e^{4\pi i/3}) \quad (6.13)$$

Combining these results for C' and B we see that for the states with no

oscillator excitation, one projects on $(p+v)^2 - 4/3 = 0 \bmod 2$, while for the states with one $q^{1/3}$ excited oscillator one obtains $(p+v)^2 - 8/3 = 0 \bmod 2$, corresponding to the massless spectrum previously given.

There remains the discussion of the right-handed sector. The bosonic oscillators behave as C discussed in the left-handed sector and need no further elaboration. We can thus focus on the NSR model. In $d = 10$ this partition function can be written in terms of θ -functions as given in (3.8) and (3.10), and again we would like to represent the twist as a shift on the "lattice". The twists we consider are, in general, given in a complex basis as $\exp 2\pi i (aJ_{12} + bJ_{34} + cJ_{56})$ (here $a=b=1/3$, $c=2/3$). The eigenvalues of the space-time fermions are then given by $\exp i\pi(\pm a \pm b \pm c)$. This formula is useful for the discussion of remaining supersymmetries [1]. In the untwisted sector one now shifts the second argument of the θ -functions by (a,b,c) , e.g., in the NS sector, the $(\theta,1)$ sector is given by

$$\vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \vartheta^2\left[\begin{smallmatrix} 0 \\ 1/3 \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} 0 \\ 2/3 \end{smallmatrix}\right] - \vartheta^2\left[\begin{smallmatrix} 0 \\ 5/6 \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} 0 \\ 7/6 \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right] \quad (6.14)$$

The sum of the various subsectors then gives the projection on invariant states. In (6.14) one can explicitly see that of the eight ground states, two remain invariant while three have phase α and three phase α^{-1} . Twisted sector contributions are again obtained by a $\tau \rightarrow -1/\tau$ transformation. Typically this transformation exchanges the two components of the θ -functions and we have to consider contributions like

$$\vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \vartheta^2\left[\begin{smallmatrix} 1/3 \\ 0 \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} 2/3 \\ 0 \end{smallmatrix}\right] + \dots \quad (6.15)$$

Working out the ground state shows that only one helicity component is selected. The CPT conjugate can be found in the θ^2 twisted sector. This concludes our sketch of the partition functions and the modular invariance of the Z-orbifold. Formulae valid in more general cases can be found in Ref. [13].

7. Further Developments

We have discussed in detail the basic facts and techniques of orbifold compactification and the reader should now be able to follow the

original papers concerning further developments. These include the discussion of background fields [14], left-right asymmetric twists [12], methods to lower the rank of the gauge group [11] and many more technical details. Finally one would, of course, like to know whether such an approach can lead to a model that could realistically describe the spectrum and dynamics of quarks and leptons. Such attempts are reviewed in Ref. [15], where also a complete list of references can be found.

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