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MULTILOOP GHOST VERTICES AND THE DETERMINATION OF THE MULTILOOP MEASURE

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ABSTRACT

Using the group theoretic approach to string theory, the multiloop vertices previously computed are extended to include ghost oscillators. In accord with the approach, we show how demanding that zero norm physical states decouple leads to a set of first-order differential equations which uniquely determine the multiloop measure.

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Recently, in a desire to find a complete and transparent description of string theory, two new formulations have been given. One goes under the name of the "group theoretic approach" [1]-[5], while the other is based on the mathematics of Grassmannians [6]. Some connections between these two approaches can be found at the end of Ref. [7].

The group theoretic method calculates the scattering amplitude in two pieces, namely the vertex and the measure over which it is integrated. The parts of the vertex depending on α_n^μ oscillators have previously been computed. At the tree level these vertices [1] are in agreement, on-shell, with the old Lovelace-Olive vertices [8], while the multiloop vertices were given in Ref. [3]. For the tree-level Neveu-Schwarz vertices, see Ref. [4]. For an actual physical process, one need only apply positive norm physical states, and so only the α_n^μ oscillator piece of the vertex is relevant.

However, apart from specifying the conformal transformation properties of vertices which determine them, the group theoretic approach also demands the decoupling of zero norm physical states. It is this latter requirement that leads to first-order differential equations which determine the tree and loop measures. It has been possible to compute all tree and one-loop measures without including ghost oscillators [1-3]. At higher loops, however, without ghost oscillators the structure of zero norm physical states becomes complicated, and this makes the computation of the measure more problematic, although there may be a straightforward method. In the BRST formalism, zero norm physical states have a particularly simple form [9], namely $Q\Lambda$, where Q is the BRST charge, and as we shall see, this leads to a straightforward derivation of the measure.

Although it was clear that the group theoretic method could be applied to find the ghost oscillator piece of the vertices, this was only done [10,11] for the three-string open bosonic vertex and the fermion emission vertex. For very subsequent discussions of the former result, see Ref. [12]. Using the group method, the ghost extension to the tree vertices was found in Ref. [13], and in this paper we find the analogous result for the multiloop contributions.

Demanding that zero norm physical states decouple is equivalent to a condition that Q should pass through the vertex in a particular way. However, apart from the tree level three-string vertex [10], Q should not annihilate the vertex, as is advocated, for example, in the first two papers of Ref. [12], since then one obtains no condition on the measure. We find that the correct vertices have BRST properties which do indeed lead to first-order differential equations which uniquely determine the measure.

We suggest that the reader who wishes to learn the group theoretic approach to strings could read the review of Ref. [7], Refs. [3] and [5], and then the other papers. We now assume some familiarity with the method, which assumes the decoupling zero norm physical states mentioned above and the conformal properties of the vertex V . The latter can be written in terms of conformal transformations on the vertex V . There are such conditions involving transformations V^j , $j = 1, \dots, N$, associated with the external legs of the vertex, and for each loop there are conditions involving an arbitrary $SU(1,1)$ transformation P_n ($n = 1, \dots, g$), namely

$$V P_n^i = V \quad (1)$$

where $P_n^i = (V^i)^{-1} P_n V^i$. The vertex V depends on the moduli v_r ($r = 1, 2, \dots, 3(g-1)$, $g \geq 2$) and the Koba-Nielsen co-ordinates [14] z_j , $j = 1, \dots, N$. The moduli dependence comes from the three parameters in each P_n . The above refers to the Schottky representation, while in the Fuchsian representation we have two conditions like Eq. (1) corresponding to the A and B cycles. Such conditions on the vertex are embodied in overlap relations which relate two conformal operators acting on the vertex. For a conformal operator $R(z)$ of dimension d , these may be written in the form [1-5]

$$V \left\{ \frac{R^i(\xi^i)}{(\xi^i)^d} - \left[\frac{d\xi_j}{d\xi_i} \right]^d \frac{R^j(\xi^j)}{(\xi^j)^d} \right\} = 0 \quad (2)$$

where $\xi^i = (V^i)^{-1} z$ and $\xi^j = (V^j)^{-1} z$, and

$$V \left\{ \frac{R^i(z)}{(z)^d} - \left[\frac{d(P_n^i z)}{dz} \right]^d \frac{R^i(P_n^i z)}{(P_n^i z)^d} \right\} = 0 \quad (3)$$

The leg cycling transformations $(V^j)^{-1}(z)$ may be any analytic function of z which vanishes at z_j , but has non-zero derivative there [5]. The simplest choice, which we will use throughout this paper, is $(V^j)^{-1} z = z - z_j$. Equations (2) and (3) simply state that any two conformal operators, when conformally mapped to the same point on the surface, have the same effect on the vertex V .

Integrating these overlap identities we find that, for the special choice of cycling transformation above [1-5],

$$V \left\{ \sum_{j=1}^N \oint \frac{d\xi_j}{\xi_j} \frac{R^j(\xi_j)}{(\xi_j)^{d-1}} c(\xi_i) \right\} = 0 \quad (4)$$

where ϕ is a tensor of rank $-(d-1)$ [$\phi(dz)^{-(d-1)}$ is invariant] and $\xi_j + z_j = \xi_i + z_i$. The function ϕ must only have poles at Koba-Nielsen points z_j where the strings are emitted and must be defined on the relevant surface. For tree vertices, ϕ must be defined on the sphere (or Riemann sphere) and this leads to restrictions on ϕ which in turn restrict the type of identity of Eq. (4) which can be enforced [12]. For loops, ϕ must be defined on the corresponding Riemann surface or equivalent fundamental domain. Such a ϕ must then be real and satisfy [2,3]

$$\phi(P_n^i z) = \phi(z). \quad (5)$$

while for the Fuchsian case it should satisfy Eq. (5) and the reality condition becomes translated into another equation of the above type. An important exception [2,3] to this discussion is the case of $R = P^\mu(\xi)$, which results from the fact that $\xi(d/d\xi)Q^\mu = P^\mu$. There ϕ should satisfy Eq. (5), but can transform by a constant when we move by a rotation of 2π around an isometric circle.

We would like to note that Eq. (4) was used for the conformal operators $P^\mu(z)$, $L(z)$, $\beta(z)$ and $\bar{\beta}(z)$ in Ref. [10], and for superconformal ghosts in Ref. [11], to find the explicit form of the tree bosonic string vertex and the fermion emission vertex with ghosts. While in Ref. [15] the general theory behind integrated overlap equations [i.e., Eq. (4)] for operators of arbitrary conformal weight and their derivation from the more fundamental unintegrated overlaps [i.e., Eq. (2)] was given, it was also used there to derive the three-string bosonic and Neveu-Schwarz vertices. These equations were also discussed in the last paper in Ref. [6].

In this paper the conformal operators [16] of interest are

$$\beta(z) = \sum_n \beta_{-n} z^n, \quad \bar{\beta}(z) = \sum_n \bar{\beta}_{-n} z^n \quad (6)$$

which have conformal weight -1 and $+2$ respectively, and are more frequently referred to in the literature as $c(z)$ and $b(z)$. Consider then the identities for β and $\bar{\beta}$ for a multiloop vertex V ; unlike the case of P^μ , the function ϕ must be an automorphic tensor of rank $+2$ and -1 respectively. To find the oscillator vertex we simply substitute a basis of such functions in Eq. (4). Although ϕ can have poles at any of the Koba-Nielsen points, to evaluate the vertex it is simpler to take ϕ to have a pole at only one such point.

It follows from the Riemann-Roch theorem that if ϕ is analytic except at one point, then it must have a pole there of order at least $3(g-1)+1$. We may take as our basis of such functions

$$\phi_m(\bar{z}) = \frac{1}{\bar{z}^{3(g-1)+m+1}} + \sum_{p=-3(g-1)}^{\infty} a_{mp}(\bar{z}) \bar{z}^p \quad m=0,1,\dots \quad (7)$$

where a_{mn} are determined by $\phi_m(\xi)$ being automorphic. The construction of such functions has been given in the mathematical literature in terms of the prime form. If we have only one external string, the corresponding $\bar{\beta}$ identity is given by

$$V \left\{ \bar{\beta}_{-(3g-1)+m+2} + \sum_{p=-3(g-1)}^{\infty} a_{mp} \bar{\beta}_{p-1} \right\} = 0 \quad m=0,1,\dots \quad (8)$$

For many external strings we use the above ϕ_m , but with the argument $\xi_i = \xi_j + z_j - z_i$ inserted in Eq. (7) to yield

$$\phi_m(\bar{z}_j + z_j - z_i) = \begin{cases} \left(\frac{1}{\bar{z}_i} \right)^{3(g-1)+m+1} + \sum_{p=-3(g-1)}^{\infty} a_{mp}^{ii}(\bar{z}_i) \bar{z}_i^p & j=i \\ \sum_{n=0}^{\infty} a_{mn}^{ij}(\bar{z}_j) \bar{z}_j^n & j \neq i \end{cases} \quad m=0,1,\dots \quad (9)$$

We obtain the $\bar{\beta}$ identities

$$0 = V \left\{ \bar{\beta}_{-(3g-1)+m+2}^i + \sum_{p=-3(g-1)}^{\infty} a_{mp}^{ii} \bar{\beta}_{p-1}^i + \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{p=0}^{\infty} a_{mp}^{ij} \bar{\beta}_p^{ij} \right\} \quad (10)$$

For the β identities, on the other hand, ϕ is a two-tensor and the Riemann-Roch theorem tells us that there are $3(g-1)$ such analytic tensors which lead to $3(g-1)$ β identities.

The remaining set of β identities are found by substituting in (4) a basis of automorphic two-tensors. A suitable such basis is given by

$$\phi_n(\bar{z}) = \bar{z}^{-n} + \sum_{m=0}^{\infty} b_{nm}(\bar{z}) \bar{z}^{3(g-1)+m} \quad n=1,2,\dots \quad (11)$$

and so the identities are of the form

$$V \left\{ \beta_{-n+2} + \sum_{m=0}^{\infty} b_{nm} \beta_{3(g-1)+m+2} \right\} = 0 \quad (12)$$

for one external string.

Examining the above identities, we see that when solving for the vertex V it is natural to use the vacuum $\langle 0|$, which satisfies

$$\langle 0 | \bar{\beta}_m = 0 \quad m \leq -3(g-1)-2.$$

$$\langle 0 | \beta_n = 0 \quad n \leq 3(g-1)+1$$

(13)

From the $\bar{\beta}$ identities we derive the vertex to be

$$V = \left\{ \prod_{k=1}^N \langle 0 | \right\} \exp \left\{ \sum_{m=0}^{\infty} \beta_{3(g-1)+m+2}^i \left\{ \sum_{p=1}^{3(g-1)} a_{m,-p}^{i,i} \bar{\beta}_{-p-1}^i \right. \right. \\ \left. \left. + \sum_{p=0}^{\infty} \sum_{j=1}^N a_{m,p}^{i,j} \bar{\beta}_{p-1}^j \right\} \right\} \quad (14)$$

The β identities are identically satisfied. Indeed, one could deduce the form of the automorphic two-tensor by working backwards; one reads off the β identity from the vertex and then the corresponding two-tensor. For example, one easily realizes that b_{mn} and a_{mn} are simply related, one finds the form of the $3(g-1)$ holomorphic two-tensors, and that the two-tensor and vector field satisfy certain orthogonality relations. More generally, all the information about the Riemann surface can be extracted from the vertex.

We recall that the positive norm physical states, i.e., those which are annihilated by Q and are not zero norm states, are of the form [16,17] $\phi|-\rangle$, where $\beta_n|-\rangle = \beta_m|-\rangle = \bar{\beta}_0|-\rangle = 0$ $n, m \geq 1$. The vacuum $\langle + |$ defined by [16]

$$\langle + | \beta_0 = 0 \quad ; \quad \langle + | \beta_n = 0 = \langle + | \beta_m \\ n \leq 1 \quad , \quad m \leq 1. \quad (15)$$

satisfies $\langle + | - \rangle = 1$ and is related to the $\langle 0 |$ vacuum by

$$\langle 0 | = \langle + | \beta_1 \beta_2 \dots \beta_{3(g-1)+1} \quad (16)$$

Substituting one vacuum for the other, we find that

$$V = \langle + | \exp \left\{ \sum_{m=0}^{\infty} \sum_{p=-3(g-1)}^{\infty} \beta_{3(g-1)+m+2} a_{m,p} \bar{\beta}_{p-1} \right\} \\ \cdot \prod_{k=1}^{3(g-1)+1} \left\{ \beta_k + \sum_{m=0}^{\infty} \beta_{3(g-1)+m+2} a_{m,1-k} \right\} \quad (17)$$

Since the physical states are based on the $|-\rangle$ vacuum, we realize, as for the tree case, that V has the wrong ghost number and gives zero when used with physical states. This conclusion can also be seen as follows. Imagine building the vertex V using the vacuum $\langle \chi |$ which is annihilated by all β_m (i.e., $\langle \chi | \beta_m = 0$ m); then

V is simply given by the product of all the $\bar{\beta}$ factors of Eq. (8) on this vacuum. The $\langle + |$ vacuum, however, is related to $\langle \chi |$ by $\langle \chi | \bar{\beta}_{-\infty}, \dots, \bar{\beta}_{-2} \bar{\beta}_{-1} = \langle + |$, and hence, comparing the number of factors here with the $\langle + |$ vacuum, we find we are $3(g-1)+1$ factors of $\bar{\beta}$ short.

We now proceed as for the tree case [12], and consider a vertex U which satisfies the $\bar{\beta}$ identities of Eq. (8) as well as some additional $\bar{\beta}$ constraints which we now discuss. We may change the moduli of the vertex V by a conformal transformation [1-3]

$$V(\hat{\sigma}_r, \hat{z}_j) = V(v_r, z_j) \prod_{j=1}^N \mathcal{U}^j \quad (18)$$

where

$$\mathcal{U}^j = (V^j(z_r))^{-1} \mathcal{U} [V^j(\hat{z}_r)] \quad (19)$$

and

$$\mathcal{U} \hat{P}_n = P_n \mathcal{U}$$

where P_n and $\hat{P}_n$ are the loop cycling transformations corresponding to v_r and $\hat{v}_r$ respectively. For an infinitesimal transformation $\mathcal{U}(z) = z + \epsilon f(z)$, the latter equation becomes

$$P_n(z) + \epsilon f(z) \frac{dP_n}{dz} = \hat{P}_n(z) + \epsilon f(P_n z) \quad (20)$$

We note that if $\mathcal{U}^j = z + f^j(z)$, then $f^j(z) = f(z + z_j)$. For the infinitesimal change of the moduli, say $v_r \rightarrow \hat{v}_r = v_r + \epsilon h_r(v)$, we therefore have the equation

$$\sum_{r=1}^{3(g-1)} h_r(v) \frac{\partial V}{\partial v_r} = V \left\{ \sum_{j=1}^N \oint_{\bar{z}_j} \frac{f^j(\bar{z}_j)}{\bar{z}_j} f^j(\bar{z}_j) + c_f \right\} \quad (21)$$

In this equation, c_f is a constant which comes from the phase in Eq. (18) that we have not explicitly denoted. We will determine it shortly. We have one such equation for each change in the moduli, that is, $3(g-1)$ equations in all. We therefore take a basis of functions $f_m = 1, \dots, 3(g-1)$ which describes all the possible changes in the moduli. This set of basis functions can be written in the form

$$f_m(z) = \sum_{p=1}^{3(g-1)} \frac{e_{mp}}{(z)^p} + \sum_{n=0}^{\infty} d_{mn} z^n \quad (22)$$

We recall that any pole of higher degree than $3(g-1)$ corresponds to a moduli-preserving transformation and so may be removed.

In order to limit the number of indices in the equations, we now restrict ourselves to one external string. The extension to the multistring case is obvious with the replacement of $\xi_i = \xi_j + z_j - z_i$ to generate the appropriate coefficients, as for example between Eqs. (7) and (8) and Eqs. (9) and (10). Equation (21) now becomes

$$\sum_r h_r^m \frac{\partial V}{\partial \sigma_r} = V \left\{ \sum_{p=1}^{3(g-1)} e''_{mp} L_{p-1} + \sum_{p=0}^{\infty} d''_{mp} L_{p-1} + c_m \right\} \\ \equiv V \{ L \phi_m + c_m \} \quad (23)$$

We define a vertex U by imposing the following conditions

$$U \left\{ \sum_{j=1}^N \oint \frac{d\bar{z}}{\bar{z}} f'(\bar{z}) \frac{\bar{\beta}(\bar{z})}{\bar{z}} \right\} = 0 \quad (24)$$

When f is a moduli-preserving transformation we obtain Eq. (8), i.e., the previous $\bar{\beta}$ identities. However, when we take $f = f_m$, $m = 1, \dots, 3(g-1)$ we find $3(g-1)$ new constraints which are of the form

$$U \bar{\beta} \phi_m = U \left\{ \sum_{p=1}^{3(g-1)} e''_{mp} \bar{\beta}_{p-1} + \sum_{p=0}^{\infty} d''_{mp} \bar{\beta}_{p-1} \right\} = 0 \quad (25)$$

$m = 1, \dots, 3(g-1)$

Corresponding to the equation $\partial V / \partial z_1 = V L_{-1}$, we also have the constraint $U \bar{\beta}_{-1} = 0$. The vertex U is then determined to be

$$U = V \prod_{m=1}^{3(g-1)} \bar{\beta} \phi_m \bar{\beta}_{-1} \quad (26)$$

Clearly U has the correct ghost number to be a physical vertex.

Since the vertex V satisfies the integrated and unintegrated β , $\bar{\beta}$ and p^μ identities and Q is the integral of a conformal operator of weight one, it follows, from the general theory, that $VQ = 0$. As a result we find that

$$U Q = V \sum_{s=1}^{3(g-1)} \left\{ \bar{\beta} \phi_1 \dots L \phi_s \dots \bar{\beta} \phi_{3(g-1)} \right\} (-1)^{3(g-1)-s-1} \bar{\beta}_{-1} + V \prod_{m=1}^{3(g-1)} \bar{\beta} \phi_m L_{-1} \\ = \left[\sum_{s=1}^{3(g-1)} \left\{ \left(\sum_r h_r^s \frac{\partial V}{\partial \sigma_r} - V c_s \right) \bar{\beta} \phi_1 \dots \bar{\beta} \phi_s \dots \bar{\beta} \phi_{3(g-1)} \right\} \right]$$

$$\begin{aligned}
 & + \sum_{p=1}^{g-1} V \bar{\beta}_{\phi_1} \dots [\bar{\beta}_{\phi_p}, L_{\phi_s}] \dots \bar{\beta}_{\phi_s} \dots \bar{\beta}_{\phi_{2(g-1)}} \} (-1)^{2(g-1)-s-1} \bar{\beta}_{-1} \\
 & + \frac{\partial V}{\partial z_1} \prod_{m=1}^{2(g-1)} \bar{\beta}_{\phi_m} + \sum_{p=1}^{2(g-1)} V \bar{\beta}_{\phi_1} \dots \bar{\beta}_{\phi_{p-1}} [\bar{\beta}_{\phi_p}, L_{-1}] \dots \bar{\beta}_{\phi_{2(g-1)}} \quad (27)
 \end{aligned}$$

The actual scattering amplitude is given, as usual, by

$$W = \int \prod_{s=1}^{2(g-1)} d\sigma_s dz_1 U \bar{f}(\sigma, z_1)$$

and consequently we find, by applying the above equation, that

$$\begin{aligned}
 W\phi &= \int \prod_{\nu} d\sigma_{\nu} dz_1 \bar{f} V \left\{ \sum_{s=1}^{2(g-1)} \left[\left(+ \sum_{\nu} \frac{\partial}{\partial \sigma_{\nu}} (h_{\nu}^s \ln \bar{f}) + c_s \right) \bar{\beta}_{\phi_1} \dots \bar{\beta}_{\phi_s} \bar{\beta}_{\phi_{2(g-1)}} \right. \right. \\
 & + \sum_{p=1}^{2(g-1)} \bar{\beta}_{\phi_1} \dots \left(\sum_{\nu} h_{\nu}^s \frac{\partial \bar{\beta}_{\phi_p}}{\partial \sigma_{\nu}} - \sum_{\nu} h_{\nu}^p \frac{\partial \bar{\beta}_{\phi_s}}{\partial \sigma_{\nu}} - [\bar{\beta}_{\phi_p}, L_{\phi_s}] \right) \dots \bar{\beta}_{\phi_s} \bar{\beta}_{\phi_{2(g-1)}} (-1)^s \\
 & \cdot (-1)^{2(g-1)} \bar{\beta}_{-1} \} - \left\{ \frac{\partial \ln \bar{f}}{\partial z_1} \prod_{s=1}^{2(g-1)} \bar{\beta}_{\phi_s} + \sum_{p=0}^{2(g-1)} \bar{\beta}_{\phi_1} \dots \left(\frac{\partial \bar{\beta}_{\phi_p}}{\partial z_1} - [\bar{\beta}_{\phi_p}, L_{-1}] \right) \right. \\
 & \left. \left. \dots \bar{\beta}_{\phi_{2(g-1)}} \right\} \quad (28)
 \end{aligned}$$

To evaluate this expression further, we require the integrability condition for the moduli-changing equation (23). One finds the expression

$$\begin{aligned}
 0 &= V \left\{ \left[\sum_{\nu} h_{\nu}^m \frac{\partial L_{\phi_n}}{\partial \sigma_{\nu}} + L_{\phi_m} L_{\phi_n} + \sum_{\nu} h_{\nu}^m \frac{\partial c_n}{\partial \sigma_{\nu}} \right. \right. \\
 & \left. \left. - \sum_{r,s,p} (h_{\nu}^m \frac{\partial h_{\nu}^n}{\partial \sigma_{\nu}}) (h^{-1})_s^p (L_{\phi_p} + c_p) \right] - (n \leftrightarrow m) \right\} \quad (29)
 \end{aligned}$$

which can be identified with one of the moduli-preserving conformal transformations of V . However, such equations do not have constants, insofar as they can, by the action of Q , be derived from Eq. (8), where no constant is possible. Consequently, we must conclude that

$$\begin{aligned}
 0 &= V \left\{ \left[L_{\phi_m} L_{\phi_n} + \sum_{\nu} h_{\nu}^m \frac{\partial L_{\phi_n}}{\partial \sigma_{\nu}} \right. \right. \\
 & \left. \left. - \sum_{p=1}^{2(g-1)} \sum_{s,r} (h_s^m \frac{\partial h_r^n}{\partial \sigma_s}) (h^{-1})_r^p L_{\phi_p} \right] - (n \leftrightarrow m) \right\} \quad (30)
 \end{aligned}$$

and

$$\left[\sum_r h_r^m \frac{\partial C_n}{\partial \sigma^r} - \sum_p \sum_{r,s} (h_s^m \frac{\partial}{\partial \sigma^s} h_r^n) (h^{-1})_r^p c_p \right] - (n \leftrightarrow m) = 0 \quad (31)$$

Since $\bar{\beta}$ and L have the same moduli-preserving identities, by virtue of having essentially the same conformal weight, we may replace one of the L_ϕ 's in Eqs. (30) by $\bar{\beta}_\phi$. It is now straightforward to read off from Eq. (28) the $3(g-1)$ first-order differential equations for the measure by demanding $WQ = 0$. We find, taking for simplicity the basis f_m for which $h_r^m = \delta_r^m$, that

$$\frac{\partial \ln \bar{f}}{\partial \sigma^s} = -c_s \quad (32)$$

Equation (31) is none other than the integrability condition for this equation. By using a similar procedure, we find that the measure $\bar{f}$ is independent of z_1 .

The constant c_s is determined by applying a state of the correct ghost number, i.e.,

$$|\psi\rangle = \bar{\beta}_{-2g-1} \dots \bar{\beta}_{-1} |-\rangle \quad (33)$$

to Eq. (23).

The evaluation of this equation is straightforward, once the concrete form of the L_{ϕ_m} has been computed from Eq. (20). For the contribution, from the α_n^μ oscillators to the constant c , see Ref. [7]. In practice, it is probably simpler to take as our basis of functions f_m such that they have only one pole term in Eq. (22). Since the parameters from each loop enter on an equal footing, it should be straightforward to integrate these equations and to make contact with previous literature on the measure [18,19]. It may be particularly interesting to find the result in Schottky form.

One could make all the above formulae more elegant by rescaling V by the measure $\bar{f}$ (i.e., $\tilde{V} = \bar{f} V$). Then there is no constant in Eq. (21) when written in terms of $\tilde{V}$, and so it is this vertex that has the simplest conformal properties.

In order to get a better feel for the above discussion, we implement it at one loop to find explicitly the ghost oscillator vertex and the measure. We consider

the torus to be mapped to a rectangle of height $2\pi i$ and length $ln\omega$ (see Refs. [18] and [5] for a more explicit discussion). The automorphic function ϕ must satisfy

$$\phi(z + ln\omega) = \phi(z) \quad ; \quad (34)$$

$$\phi(z + 2\pi i) = \phi(z) \quad (35)$$

regardless of its rank, as (dz) itself is invariant under these transformations. Such functions can be taken to be the Weierstrass θ function and its derivatives. The $\bar{\beta}$ identities are

$$V \left\{ \bar{\beta}_{-n-1}^i + \sum_{j=1}^N \sum_{p=0}^{\infty} E_{np}^{ij} \bar{\beta}_{p-1}^j \right\} = 0 \quad n \geq 2 \quad (36)$$

and are found by substituting the vector field

$$\begin{aligned} \phi &= \frac{(-1)^n}{(n-2)!} \frac{d^{n-2}}{dz^{n-2}} \theta(\bar{z}_j + z_j + z) = \frac{1}{(\bar{z}_j + z_j - z_i)^n} + \sum_{p,q} \frac{1}{(\bar{z}_j + z_j - z_i + 2p\omega_1 + 2q\omega_2)^n} \\ &= \begin{cases} \sum_{p=0}^{\infty} E_{np}^{ij} (\bar{z}_j)^p & \text{for } j \neq i \\ \frac{1}{(\bar{z}_i)^n} + \sum_{p=0}^{\infty} E_{np}^{ii} (\bar{z}_i)^p & \text{for } j = i \end{cases} \end{aligned} \quad n \geq 2 \quad (37)$$

For a genus-one surface there exists one analytic tensor of any rank, which is in fact $\phi = \text{constant}$. Hence we have the additional $\bar{\beta}$ identity

$$V \sum_{j=1}^N \bar{\beta}_{-1}^j = 0 \quad (38)$$

Adopting a similar argument, we find the β identities to be

$$V \left\{ \beta_{-n+2}^i + \sum_{p=0}^{\infty} \sum_{j=1}^N E_{np}^{ij} \beta_{p+2}^j \right\} = 0 \quad n \geq 2 \quad (39)$$

and

$$V \sum_{j=1}^N \beta_2^j = 0 \quad (40)$$

Solving the $\bar{\beta}$ identities, we find the vertex is given by

$$V = \left\{ \prod_{i=1}^N \langle 0|_i \right\} \exp \left\{ \sum_{i,j=1}^N \sum_{n=2}^{\infty} \sum_{p=0}^{\infty} \beta_{n+1}^i E_{np}^{ij} \bar{\beta}_{p-1}^j \right\} \left\{ \sum_{j=1}^N \bar{\beta}_{-1}^j \right\} \quad (41)$$

where we have taken the $\langle 0|$ vacuum on each leg. In this case, Eq. (13) reads

$\langle 0 | \bar{\beta}_n = 0 = \langle 0 | \beta_m$, $n \leq -3$, $m \leq 2$. The β identities are automatically satisfied. The vertex V , however, has the wrong ghost number and so gives zero when used for physical states.

The vertex V , when supplemented by its α_n^μ oscillator piece [5], is annihilated by Q since it satisfies the integrated and unintegrated identities for P^μ , β and $\bar{\beta}$ and Q is the integral of an object of conformal weight one. The moduli-changing and preserving conformal transformations are discussed in detail in Refs. [1]-[5]. We recall that an infinitesimal preserving transformation is given by

$$V \sum_{j=1}^N \oint \frac{d\bar{z}_j}{\bar{z}_j} f^j(\bar{z}_j + z_j - z_i) \frac{L^j(\bar{z}_j)}{\bar{z}_j} = 0 \quad (42)$$

where f satisfies Eq. (36) and the moduli-changing transformation is given by [2,5]

$$\omega \frac{\partial V}{\partial \omega} = V \sum_{j=1}^N \left\{ \oint \frac{d\bar{z}_j}{\bar{z}_j} \bar{\zeta}(\bar{z}_j + z_j - z_i) \frac{L^j(\bar{z}_j)}{\bar{z}_j} + c \right\} \equiv V(L_{\phi_i} + c) \quad (43)$$

where c is a constant which we will determine shortly, and $\bar{\zeta}(z) = \zeta(z) - (\eta z / \pi i)$, ζ being the Weierstrass ζ function. The Koba-Nielsen variable-changing equations are [5]

$$\frac{\partial V}{\partial z_j} = V L_{-1}^j \quad (44)$$

We now construct a different vertex U which is subject to the above equations with the L replaced by $\bar{\beta}$ and the left-hand side of the second equation set to zero and dropping the constant. The first equation is none other than Eqs. (37) and (38) with V replaced by U , while the second equation is

$$u \bar{\beta}_{\phi_i} = u \left\{ \sum_{j=1}^N \oint \frac{d\bar{z}_j}{\bar{z}_j} \bar{\zeta}(\bar{z}_j + z_j - z_i) \frac{\bar{\beta}^j(\bar{z}_j)}{\bar{z}_j} \right\} = 0 \quad (45)$$

Corresponding to Eq. (45), we also impose

$$u \bar{\beta}_{-1}^j = 0 \quad (46)$$

for all j except one, say $j = i_1$. This exception corresponds to the existence of the one analytic vector field which leads to the constraint of Eq. (38). It is this z_{i_1} which we will not integrate over. As we will see shortly, the vertex does

indeed have the correct BRST properties and required ghost number. It is straightforward to see by inspection that the vertex U is given by

$$U = V \left\{ \prod_{\substack{i=1 \\ i \neq i_1}}^N \bar{\beta}_{-1}^i \right\} \left\{ \prod_{j=1}^N \bar{\beta}_{\phi_j} \right\} \quad (47)$$

In terms of the $\langle + |$ vacuum (recall $\langle 0 | = \langle + | \beta_2 \beta_1$), we find that U is given by

$$U = \prod_{i=1}^N \langle + | \exp \left\{ \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i,j=1}^N \beta_{n+1}^i E_{np}^{ij} \bar{\beta}_{p-1}^j \right. \\ \left. \sum_{i=1}^N \left(-\frac{\eta}{\pi i} \beta_2^i \bar{\beta}_0^i \right) \right\}$$

where

$$\mathcal{E}(\bar{z}_j + z_j - z_i) = \begin{cases} \frac{1}{\bar{z}_i} + \sum_{p=0}^{\infty} E_{1p}^{ii}(\bar{z}_i)^p & \text{for } j=i \\ \sum_{p=0}^{\infty} E_{1p}^{ij}(\bar{z}_j)^p & \text{for } j \neq i \end{cases} \quad (48)$$

In order that the trees don't obscure the wood, let us restrict ourselves to one external leg, in which case

$$U = V \bar{\beta}_{\phi} \quad (49)$$

Since $V Q = 0$, we find that

$$U Q = V L_{\phi} = \omega \frac{\partial V}{\partial \omega} - V c \quad (50)$$

The actual scattering amplitude is given by

$$W = \int d\omega \bar{f} U \quad (51)$$

and decouples zero norm physical states if

$$W Q = 0 = \int d\omega \bar{f} \left(\omega \frac{\partial}{\partial \omega} - c \right) V \quad (52)$$

or if

$$\frac{\partial}{\partial \omega} (\omega \ln \bar{f}) = c \quad (53)$$

To find the measure $\bar{f}$, it only remains to determine the constant c . This is achieved by putting the state $\bar{\beta}_{-2} |-, 0\rangle$ on each side of Eq. (50), where

$$\beta_0 |-, 0\rangle = \gamma^\mu |-, 0\rangle = 0 = \beta_n |-, 0\rangle = \bar{\beta}_m |-, 0\rangle \quad n \geq 1; m \geq 1 \quad (54)$$

that is

$$0 = V \cdot (L_\phi + c) \bar{\beta}_{-2} |-, 0\rangle = (u \phi + v c) \bar{\beta}_{-2} |-, 0\rangle \quad (55)$$

One can determine c from either of these equations. For example, on evaluating the first equation, we find

$$\begin{aligned} & V L_2 \bar{\beta}_{-2} |-, 0\rangle + V \left(-\frac{\gamma}{\pi i} \right) L_0 \bar{\beta}_{-2} |-, 0\rangle + c \\ & = + \left[+ D \left(\frac{1}{2 \ln \omega} - \frac{\gamma}{2 \pi i} \right) \right] + \left[-\frac{\gamma}{\pi i} \right] + c \end{aligned} \quad (56)$$

where we have used the relations

$$[L_m, \bar{\beta}_n] = (m-n) \bar{\beta}_{n+m}; \quad L_0 |-, 0\rangle = - |-, 0\rangle \quad (57)$$

and $\eta = \zeta(\pi i)$. Collecting the results, we have

$$c = - (D-2) \frac{\gamma}{\pi i} - \frac{D}{2 \ln \omega} \quad (58)$$

The evaluation of the first term of Eq. (56) is given in more detail in Refs. [2] and [5], and the final derivation of the measure is identical to these papers. One finds

$$\bar{f} = \frac{1}{\prod_{n=1}^{\infty} (1-\omega^n)^{D-2}} \cdot \frac{1}{(\ln \omega)^{D/2}} \cdot \frac{1}{\omega} \cdot \frac{1}{\omega^{D-2/4}} \quad (59)$$

which shows the well-known [20] reduction in the partition function by two powers.

We now return to the case of many external states. Acting with Q on Eq. (47), we find

$$\begin{aligned} u \sum_{j=1}^N \phi^j &= V \sum_{j=1}^N \left\{ \prod_{i \neq i_1} \bar{\beta}_{\phi_i} \right\} \bar{\beta}_{-1}^1 \dots L_1^j \dots \bar{\beta}_{-1}^N (-1)^{N-j} \\ &+ V \left\{ \sum_{m=1}^N \bar{\beta}_{\phi_1} \dots L_{\phi_m} \dots \bar{\beta}_{\phi_N} \right\} \prod_{i=1}^N \bar{\beta}_{-1}^i (-1)^m \end{aligned}$$

The actual scattering amplitude is

$$W = \int d\omega \prod_{\substack{i=1 \\ i \neq i_1}}^N dz_i \bar{f} U \quad (61)$$

Hence, acting with the BRST charge, we find that

$$\begin{aligned} W \sum_{j=1}^N Q^j = 0 &= \int d\omega \prod_{\substack{i=1 \\ i \neq i_1}}^N dz_i \bar{f} V \left\{ \left(-\frac{\partial}{\partial \omega} (\omega \ln \bar{f}) - c \right) \right. \\ &\sum_{m=1}^N \left(\prod_{\substack{i=1 \\ i \neq m}}^N \bar{\beta}_{\phi_i} \right) \prod_{\substack{j=1 \\ j \neq i_1}}^N \bar{\beta}_{-1}^j \sum_{\substack{j=1 \\ j \neq i_1}}^N \left(-\frac{\partial \ln \bar{f}}{\partial z_j} \right) \left\{ \prod_{i=1}^N \bar{\beta}_{\phi_i} \right\} \prod_{\substack{k=1 \\ k \neq i_1, j}}^N \bar{\beta}_{-1}^k (-1)^{N-j} \left. \right\} \end{aligned} \quad (62)$$

Consequently, Eq. (53) holds, and the measure is independent of z_j and so is given by Eq. (59). In deriving this result we have used certain identities which result from integrability conditions on the moduli-changing equations.

For the multiloop case with N external strings, we impose $N \cdot 3(g-1)$ conditions obtained from Eq. (24) by the variable change discussed before. The BRST invariance and derivation of the measure proceed as for the case of one external string with the appropriate scattering of i, j indices.

We have, of course, ignored the term which results from the integration by parts used between Eqs. (60) and (62) and could lead to a failure of BRST invariance. This indeed happens for all gauge groups except $SO(2^{13})$ [2,21] which were previously [22] noted to be the only divergent free open bosonic string theories.

The formulae in this paper apply to the open bosonic string. It is, however, a triviality to extend them to the closed bosonic string; one simply puts tildas every now and again and uses the relevant operators of a given conformal weight. There is also no difficulty in extending the results to the Neveu-Schwarz string. The form of the vertex U may seem unaesthetic; however, as will be reported elsewhere, it can be written in an elegant way, whose identities have a simple interpretation.

It would also be interesting to make contact with a number of other papers [6,23] which compute the correlation function for the ghost system.

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