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GAUGE INVARIANT FORMULATION OF NON-ABELIAN CHIRAL GAUGE THEORIES
IN TWO DIMENSIONS

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A B S T R A C T

We study the gauge invariant version of a chiral non-Abelian gauge theory. A local bosonic effective action is obtained and the covariant conservation of the gauge current is verified. A Hamiltonian analysis of the model and of its constraints is performed. We show that the constraints are first class and that no anomalous term appears in the commutators of the gauge group generators. The current algebra of the model is obtained and the gauge fixing is analyzed.

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1. - INTRODUCTION

Recently there has been considerable interest in the quantization of anomalous gauge theories, especially after the analysis of Jackiw and Rajaraman [1] who showed that the chiral Schwinger model [a $U(1)$ gauge field coupled to chiral fermions in two dimensions] can be quantized in a consistent, unitary and Lorentz invariant form, if it is regularized appropriately. In this case the theory is exactly solvable and it has been studied both in the Lagrangian [1] and the Hamiltonian [2,3] approach.

On the other hand, Faddeev and Shatashvili [4] have recently suggested the modification of the original action of an anomalous gauge theory by introducing a new chiral field with a Wess-Zumino action. This new field restores the gauge invariance of the theory and turns the second class [5] constraints into first class ones, which is necessary in order to keep gauge invariance at the quantum level. Subsequently it has been demonstrated [6-8] that this new field arises quite naturally in the path integral quantization. The basic observation leading to this result is that, in the path integral defining the generating functional of an anomalous gauge theory, it has no sense to restrict the integration over the gauge fields to gauge inequivalent configurations (gauge orbits) because two fields related by a gauge transformation give different contribution to the path integral. So we are forced to integrate over all gauge field configurations. In this way we end up with a generating functional which is obviously gauge invariant even if the bosonic effective action is not. In a sense this is just the opposite of the common statement concerning anomalies: here the theory has more symmetries than those of the effective action.

It is possible to adopt a formulation in which this gauge symmetry of the theory appears explicitly in the action [6]. The only thing one has to do is to fix the gauge by means of the Faddeev-Popov method. When one applies this method carefully to a chiral gauge theory and takes into account that the fermionic measure is not invariant under gauge transformations, the result is that the gauge group volume cannot be factorized away. In this way the group of gauge transformations, which is not a dynamical degree of freedom at the classical level, acquires the status of a physical field, giving rise to an effective action which is gauge invariant. This is exactly the same situation that appears in the Polyakov string [9] with the conformal factor and the Liouville action when we are not in the critical dimension of the theory. In the following this second formulation will be referred to as the gauge invariant formulation, whereas the first one will be called gauge non-invariant [10].

The proof of the gauge invariance of the generating functional in both formulations has been obtained by performing manipulations in the path integral and

thus it can be considered as a formal proof. Up to now only in the chiral Schwinger model has this fact been checked in detail [10-12]. The general non-Abelian case has been first studied in the gauge non-invariant formulation [13,14]. In the gauge invariant formulation, only partial results have been presented [15,16].

In this paper we derive from first principles a local effective bosonized action for a general two-dimensional non-Abelian chiral gauge theory in the gauge invariant formulation. This is done in Section 2, where the gauge invariance of the theory is explicitly shown and the equations of motion are obtained. It is demonstrated that the gauge current is covariantly conserved as a consequence of the field equations of motion. We want to perform an analysis within the canonical Hamiltonian formalism of the effective action derived in Section 2. In order to have all the needed formulas at our disposal we briefly review in Section 3 the canonical formalism for a Wess-Zumino-Witten non-linear sigma model, following the approach of Ref. [14]. In Section 4 we obtain the Hamiltonian for our system and we show that it has first class constraints. The algebra of the infinitesimal generators of the gauge group and the current algebra of the model are also analyzed in this section. In Section 5 we study the gauge fixing and the relation between the gauge invariant and gauge non-invariant approaches. Finally we summarize the results and conclusions in Section 6.

2. - BOSONIZED EFFECTIVE ACTION

Consider a theory of massless fermions with only one of its chiral components coupled to a SU(N) gauge field in two space-time dimensions. The Lagrangian density of such a model is given by:

$$\mathcal{L} = -\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{\partial} + e \not{A} (\frac{1-\gamma_5}{2})) \psi \quad (2.1)$$

where [17]:

$$A_{\mu} = A_{\mu}^a t^a \quad (2.2a)$$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ie [A_{\mu}, A_{\nu}] \quad (2.2b)$$

The bosonic effective action $W(A)$ is obtained by integrating the fermionic degrees of freedom in the generating functional:

$$W(A) = -\frac{1}{4} \int d^2x \text{tr}(F_{\mu\nu} F^{\mu\nu}) + \ln \det [i \not{\partial} + ie A \gamma_5] \quad (2.3)$$

In two space-time dimensions the fermionic determinant can be computed explicitly [18-23]. Let us represent the light-cone component A_- of the gauge field as:

$$A_- = \frac{i}{e} V \partial_- V^{-1} \quad (2.4)$$

where $V(x)$ is a group-valued field. With this representation, $W(A)$ is given by [23]:

$$W(A) = \Gamma(V) + \int d^2x \left[-\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{ae^2}{8\pi} \text{tr} A_\mu A^\mu \right] \quad (2.5)$$

where $\Gamma(V)$ is the functional of the Wess-Zumino-Witten (WZW) non-linear σ -model [24]:

$$\Gamma(V) = -\frac{1}{8\pi} \int d^2x \text{tr} (\partial_\mu V \partial^\mu V^{-1}) - \frac{1}{4\pi} \int_0^1 dt \int d^2x \epsilon^{\mu\nu} \text{tr} (V^{-1} \partial_\mu V V^{-1} \partial_\nu V) \quad (2.6)$$

In the second term in (2.6), V has to be understood as a smooth function of t and x , interpolating between 1 and $V(x)$: $V(x,0) = 1$, $V(x,1) = V(x)$. The term $(ae^2/8\pi) A_\mu A^\mu$ in $W(A)$ appears as a consequence of the regularization ambiguities in the calculation of the fermionic determinant. The parameter a is undetermined due to the lack of gauge invariance of the fermionic determinant.

The generating functional corresponding to the effective action $W(A)$ is:

$$Z = \int dA e^{iW(A)} \quad (2.7)$$

in which the integration is performed over all possible gauge field configurations (including those related by a gauge transformation). The action $W(A)$ appearing in (2.7) is not gauge invariant although Z obviously is. Following Ref. [6], let us introduce a gauge fixing condition $f(A) = 0$ by means of the Faddeev-Popov method [25]. First we define the Faddeev-Popov determinant $\Delta_f(A)$ by the equation:

$$\Delta_f(A) \int dG \delta(f(A^G)) = 1 \quad (2.8)$$

where dG is the invariant group measure and A_μ^G is given by:

$$A_\mu^G = G A_\mu G^{-1} + \frac{i}{e} G \partial_\mu G^{-1} \quad (2.9)$$

Inserting (2.8) into the definition of Z [Eq. (2.7)], relabelling the integration variable $A_\mu \rightarrow A_\mu^{G^{-1}}$ and taking into account that dA and $\Delta_f(A)$ are gauge invariant, we obtain:

$$Z = \int dA dG \delta(f(A)) \Delta_f(A) \exp[iW(A^{G^{-1}})] \quad (2.10)$$

As $W[A]$ is not gauge invariant, the group volume cannot be factorized as it is the case in the non-chiral gauge theories. We thus see that the gauge field G has acquired the status of a physical field and the relevant action is $W(A^{G^{-1}})$ rather than $W(A)$. We will verify in a while that the gauge invariance of the effective action is recovered.

Using the well-known property of the WZW functional [26]:

$$\Gamma(AB) = \Gamma(A) + \Gamma(B) - \frac{1}{4\pi} (\eta^{\mu\nu} + \epsilon^{\mu\nu}) \int d^2x \operatorname{tr}(A^{-1} \partial_\mu A B \partial_\nu B^{-1}) \quad (2.11)$$

$W(A^{G^{-1}})$ can easily be computed, with the result:

$$\begin{aligned} W(A^{G^{-1}}) = & \Gamma(G^{-1}) + \Gamma(V) + \int d^2x \left\{ \frac{\alpha}{8\pi} \operatorname{tr}(\partial_\mu G^{-1} \partial^\mu G) + \right. \\ & \left. + \frac{ie}{4\pi} [(1-\alpha)\eta^{\mu\nu} + \epsilon^{\mu\nu}] \operatorname{tr}(G \partial_\mu G^{-1} A_\nu) - \frac{1}{4} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{\alpha e^2}{8\pi} \operatorname{tr}(A_\mu A^\mu) \right\} \end{aligned} \quad (2.12)$$

The term $\Gamma(V)$ in (2.12) has a non-local dependence on the gauge field A_μ . It can be written in a local form by introducing an auxiliary field U and using the fact that:

$$e^{i\Gamma(V)} = \text{const.} \int dU e^{-i[\Gamma(UV) - \Gamma(V)]} \quad (2.13)$$

which is a consequence of the invariance of the Haar measure under group multiplication. In this way we obtain a bosonized version of our original fermionic theory. Using again Eq. (2.11), we get an effective bosonized action given by:

$$\begin{aligned} S_{\text{eff}}[A, U, G] = & \Gamma(G^{-1}) - \Gamma(U) + \int d^2x \left\{ \frac{a}{8n} \text{tr}(\partial_\mu G \partial^\mu G^{-1}) + \right. \\ & + \frac{ie}{4n} [(1-a)\eta^{\mu\nu} + \varepsilon^{\mu\nu}] \text{tr}(G \partial_\mu G^{-1} A_\nu) - \frac{ie}{4n} (\eta^{\mu\nu} + \varepsilon^{\mu\nu}) \text{tr}(U^{-1} \partial_\mu U A_\nu) \\ & \left. - \frac{1}{4} \text{tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{ae^2}{8n} \text{tr}(A_\mu A^\mu) \right\} \end{aligned} \quad (2.14)$$

It can easily be checked that this action is invariant under the group of gauge transformations:

$$A_\mu \rightarrow A_\mu^S \quad G \rightarrow SG \quad U \rightarrow US^{-1} \quad (2.15)$$

We want now to obtain the equations of motion corresponding to $S_{\text{eff}}[A, U, G]$. In order to do that, let us recall that the infinitesimal variation of the WZW functional is [24]:

$$\delta \Gamma(A) = -\frac{1}{4n} (\eta^{\mu\nu} - \varepsilon^{\mu\nu}) \int d^2x \text{tr}(A^{-1} \delta A \partial_\mu (A^{-1} \partial_\nu A)) \quad (2.16)$$

Using (2.16), it is very easy to obtain the following equations of motion:

$$\frac{i}{4n} (\eta^{\mu\nu} - \varepsilon^{\mu\nu}) D_\mu (U^{-1} \partial_\nu U) = \frac{e}{4n} (\eta^{\mu\nu} + \varepsilon^{\mu\nu}) \partial_\mu A_\nu \quad (2.17)$$

$$\frac{i}{4n} [(1-a)\eta^{\mu\nu} - \varepsilon^{\mu\nu}] D_\mu (G \partial_\nu G^{-1}) = \frac{e}{4n} [(1-a)\eta^{\mu\nu} + \varepsilon^{\mu\nu}] \partial_\mu A_\nu \quad (2.18)$$

$$D_\nu F^{\nu\mu} + e J^\mu = 0 \quad (2.19)$$

where the covariant derivative is defined by:

$$D_\mu = \partial_\mu - ie [A_\mu, \cdot] \quad (2.20)$$

and J^μ is the gauge current:

$$J^\mu = \frac{ae}{4\pi} A^\mu + \frac{i}{4\pi} [(1-a)\eta^{\mu\nu} - \varepsilon^{\mu\nu}] G \partial_\nu G^{-1} - \frac{i}{4\pi} (\eta^{\mu\nu} - \varepsilon^{\mu\nu}) U^{-1} \partial_\nu U \quad (2.21)$$

It is a trivial exercise to check that J^μ is covariantly conserved as a consequence of (2.17) and (2.18):

$$D_\mu J^\mu = 0 \quad (2.22)$$

This ensures that no contradiction appears when (2.19) is covariantly derived. The covariant conservation of the gauge current in this model was first checked in the gauge invariant formulation in Ref. [15], without introducing the auxiliary field U . In this method, one has to deal with the non-local expression which gives V in terms of A_μ , in contrast with our simple calculation. On the other hand, the introduction of this new field U is essential in order to perform a Hamiltonian analysis of the model, which is presented in the next sections of this paper.

3. - HAMILTONIAN ANALYSIS OF THE WESS-ZUMINO-WITTEN MODEL

Before developing a canonical Hamiltonian formalism for the action (2.14), let us recall the canonical approach for a WZW non-linear σ -model, following Ref. [14]. Let us consider the action:

$$S_{WZW}(g) = -\Gamma(g) = \frac{1}{8\pi} \int d^2x \operatorname{tr} (\partial_\mu g^{-1} \partial^\mu g) + \Gamma_{WZ}(g) \equiv \int d^2x \mathcal{L}_{WZW} \quad (3.1)$$

where $\Gamma_{WZ}(g)$ corresponds to the three-dimensional integral appearing in $-\Gamma(g)$ [see Eq. (2.6)]. We do not know the result of the t integration in Γ_{WZ} but we do know that it will be first order in time derivatives. So let us write [14]:

$$\Gamma_{WZ} = \frac{1}{4\pi} \int d^2x \operatorname{tr} (A(g) \partial_0 g) \quad (3.2)$$

where $A(g)$ is some unknown matrix-valued function of g . Actually we shall only need the antisymmetric tensor constructed by deriving A with respect to g :

$$F_{ij;kl}(g) = \frac{\partial A_{ij}(g)}{\partial g_{lk}} - \frac{\partial A_{kl}(g)}{\partial g_{ji}} \quad (3.3)$$

This tensor is easily calculated by computing the infinitesimal variation of $\Gamma_{WZ}(g)$, with the result:

$$F_{ij;kl}(g) = \partial_i g_{il}^{-1} g_{kl}^{-1} - g_{il}^{-1} \partial_i g_{kj}^{-1} \quad (3.4)$$

The canonical conjugate momentum $\tilde{\Pi}_{ij}(x)$ is defined as:

$$\tilde{\Pi}_{ij}(x) = \frac{\partial \mathcal{L}_{WZW}(g)}{\partial (\partial_0 g_{ij}(x))} = \frac{1}{4\pi} \partial_0 g_{ji}^{-1}(x) + \frac{1}{4\pi} A_{ji}(g) \quad (3.5)$$

and the non-vanishing canonical equal-time Poisson brackets are:

$$\{g_{ij}(x), \tilde{\Pi}_{kl}(y)\} = \delta_{ik} \delta_{jl} \delta(x^1 - y^1) \quad (3.6)$$

The Hamiltonian corresponding to S_{WZW} is:

$$\begin{aligned} H_{WZW} &= \int d^1x \left[\tilde{\Pi}_{ij}(x) \partial_0 g_{ij}(x) - \mathcal{L}_{WZW} \right] = \\ &= \int d^1x \left[-2\pi \text{tr}(n^T g n^T g) + \frac{1}{8\pi} \text{tr}(\partial_i g^{-1} \partial_i g) \right] \end{aligned} \quad (3.7)$$

where the superscript T denotes matrix transposition and:

$$\Pi_{ij}(x) = \tilde{\Pi}_{ij}(x) - \frac{1}{4\pi} A_{ji}(g) = \frac{1}{4\pi} \partial_0 g_{ji}^{-1}(x) \quad (3.8)$$

From (3.3), (3.4) and (3.6), it is immediate to prove that the $\Pi_{ij}(x)$ have the following equal-time Poisson brackets:

$$\{g_{ij}(x), \Pi_{kl}(y)\} = \delta_{ik} \delta_{jl} \delta(x^1 - y^1) \quad (3.9a)$$

$$\{ \Pi_{ij}(x), \Pi_{KL}(y) \} = -\frac{1}{4n} F_{ij;LK}(g) \delta(x^1 - y^1) \quad (3.9b)$$

Let us now define the left- and right-handed currents $j_{\pm}$ according to the Witten bosonization rules [24]:

$$j_+ = \frac{-i}{4n} g^{-1} \partial_+ g = i \Pi^T g - \frac{i}{4n} g^{-1} \partial_+ g \quad (3.10a)$$

$$j_- = \frac{-i}{4n} g \partial_- g^{-1} = -i g \Pi^T + \frac{i}{4n} g \partial_- g^{-1} \quad (3.10b)$$

with these definitions H_{WZW} can be written in the Sugawara [27] form:

$$H_{WZW} = n \int dx^1 (j_+^a(x) j_+^a(x) + j_-^a(x) j_-^a(x)) \quad (3.11)$$

where

$$j_{\pm}^a(x) = \text{tr} (j_{\pm}(x) t^a) \quad (3.12)$$

Using the equations (3.9) it is not difficult to check that the equal-time Poisson brackets of the currents $j_{\pm}^a$ with themselves correspond to those of a Kac-Moody algebra [28]:

$$\{ j_{\pm}^a(x), j_{\pm}^b(y) \} = f^{abc} j_{\pm}^c(x) \delta(x^1 - y^1) \pm \frac{\delta^{ab}}{2n} \delta'(x^1 - y^1) \quad (3.13a)$$

$$\{ j_+^a(x), j_-^b(y) \} = 0 \quad (3.13b)$$

The presence of the three-dimensional integral Γ_{WZ} in S_{WZW} , which is reflected in the fact that the Poisson bracket (3.9b) is non-vanishing, it is essential in order to obtain (3.13a) and (3.13b).

4. - HAMILTONIAN ANALYSIS OF A NON-ABELIAN CHIRAL GAUGE THEORY

We are going now to apply the method explained in the previous section in order to perform a canonical Hamiltonian analysis of the model described in Section 2, whose action is $S_{\text{eff}}(A, U, G)$. In particular we want to study the constraints of the system and check that they correspond to those of a gauge theory, namely they have to be first class and the Schwinger term in the commutator of the Gauss' law constraints has to vanish. Once this has been done the current algebra of the model can be obtained and in particular it will be possible to check that no Schwinger term appears in the charge density commutator.

First of all we calculate the canonical momenta corresponding to the fields U and G , and, after subtracting the contribution coming from the three-dimensional integrals [as we did in Eqs. (3.8)], we obtain:

$$\Pi_{ij}^U = \frac{1}{4\pi} \partial_0 U_{ji}^{-1} - \frac{ie}{4\pi} (A_- U^{-1})_{ji} \quad (4.1)$$

$$\Pi_{ij}^G = \frac{a-1}{4\pi} \partial_0 G_{ji}^{-1} - \frac{ie}{4\pi} (G^{-1} A_-)_{ji} + \frac{aie}{4\pi} (G^{-1} A_0)_{ji} \quad (4.2)$$

We immediately see from (4.2) that the case $a = 1$ is special [Π_{ij}^G does not depend on the "velocity" $\partial_0 G_{ji}^{-1}$ and (4.2) must be considered as a constraint]. From now on in this paper we shall restrict ourselves to study the case $a \neq 1$. The non-vanishing equal-time Poisson brackets involving Π^U and Π^G are:

$$\{U_{ij}(x), \Pi_{KL}^U(y)\} = \delta_{iK} \delta_{jL} \delta(x^1 - y^1) \quad (4.3a)$$

$$\{\Pi_{ij}^U(x), \Pi_{KL}^U(y)\} = -\frac{1}{4\pi} F_{ji;LK}(U) \delta(x^1 - y^1) \quad (4.3b)$$

$$\{G_{ij}(x), \Pi_{KL}^G(y)\} = \delta_{iK} \delta_{jL} \delta(x^1 - y^1) \quad (4.3c)$$

$$\{\Pi_{ij}^G(x), \Pi_{KL}^G(y)\} = -\frac{1}{4\pi} F_{ji;LK}(G) \delta(x^1 - y^1) \quad (4.3d)$$

For the gauge field, if we denote by Π_0^a and E^a the canonical conjugate momentum of A_0^a and A_1^a , respectively, we obtain:

$$\Pi_0^a \approx 0 \quad (4.4)$$

$$E^a = F_{01}^a \quad (4.5)$$

As is usually done when quantizing a gauge theory, Eq. (4.4) has to be considered as a primary constraint and it holds only in a "weak" sense. The canonical Poisson brackets in this case are the familiar ones:

$$\{A_0^a(x), \Pi_0^b(y)\} = \delta^{ab} \delta(x^1 - y^1) \quad (4.6a)$$

$$\{A_1^a(x), \Pi_1^b(y)\} = \delta^{ab} \delta(x^1 - y^1) \quad (4.6b)$$

In order to calculate the Hamiltonian we have to perform the Legendre transformation in the action $S_{\text{eff}}[A, U, G]$. As we have learned in the previous example, a simple expression is obtained when H is expressed in terms of the currents. In complete analogy with the previous case, let us define the following "currents":

$$J_+ = i(\Pi^U)^T U - \frac{i}{4\eta} U^{-1} \partial_1 U \quad (4.7a)$$

$$J_- = -iU(\Pi^U)^T + \frac{i}{4\eta} U \partial_1 U^{-1} \quad (4.7b)$$

$$\Pi_+ = \frac{i}{a-1} G(\Pi^G)^T + \frac{i}{4\eta} G \partial_1 G^{-1} \quad (4.8a)$$

$$\Pi_- = \frac{-i}{a-1} G(\Pi^G)^T + \frac{i}{4\eta} G \partial_1 G^{-1} \quad (4.8b)$$

In terms of $\rho_{\pm}$ and $\Pi_{\pm}$ the Hamiltonian density $\mathcal{H}$ can be put in the form:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} E^a E^a - A_0^a (D_1 E)^a + \Pi (\rho_+^a \rho_+^a + \rho_-^a \rho_-^a) - e \rho_+^a A_-^a + \\ & + \Pi (a-1) (\Pi_+^a \Pi_+^a + \Pi_-^a \Pi_-^a) - \frac{e}{2} [(2-a) \Pi_+^a A_-^a + a \Pi_-^a A_+^a] + \\ & + \frac{a^2 e^2}{8\pi(a-1)} A_1^a A_1^a + V^a \Pi_0^a \end{aligned} \quad (4.9)$$

where

$$\rho_{\pm}^a = \text{tr}(\rho_{\pm} t^a) \quad (4.10)$$

$$\Pi_{\pm}^a = \text{tr}(\Pi_{\pm} t^a) \quad (4.11)$$

In Eq. (4.9) we have added a term $V^a \Pi_0^a$ in order to take into account the primary constraint given by Eq. (4.4). V^a is a Lagrange multiplier, which is undetermined at this stage.

From the basic Poisson brackets given in Eqs. (4.3) and (4.6), it is possible to obtain the brackets of $\rho_{\pm}^a$ and $\Pi_{\pm}^a$ with themselves. The result is:

$$\left\{ \rho_{\pm}^a(x), \rho_{\pm}^b(y) \right\} = f^{abc} \rho_{\pm}^c(x) \delta(x^1 - y^1) \pm \frac{\delta^{ab}}{2\pi} \delta'(x^1 - y^1) \quad (4.12a)$$

$$\left\{ \rho_{+}^a(x), \rho_{-}^b(y) \right\} = 0 \quad (4.12b)$$

$$\begin{aligned} \left\{ \Pi_{+}^a(x), \Pi_{+}^b(y) \right\} = & \frac{-1}{2(a-1)} f^{abc} \left[(3a-2) \Pi_{+}^c(x) + a \Pi_{-}^c(x) \right] \delta(x^1 - y^1) \\ & + \frac{\delta^{ab}}{2\pi(a-1)} \delta'(x^1 - y^1) \end{aligned} \quad (4.13a)$$

$$\begin{aligned} \{ \Pi_{-}^a(x), \Pi_{-}^b(y) \} &= \frac{1}{2(a-1)} f^{abc} \left[(3a-4) \Pi_{-}^c(x) - (2-a) \Pi_{+}^c(x) \right] \delta(x^1 - y^1) \\ &\quad - \frac{\delta^{ab}}{2(a-1)} \delta^1(x^1 - y^1) \end{aligned} \quad (4.13b)$$

$$\{ \Pi_{+}^a(x), \Pi_{-}^b(y) \} = \frac{1}{2(a-1)^2} f^{abc} \left[a \Pi_{+}^c(x) + (2-a) \Pi_{-}^c(x) \right] \delta(x^1 - y^1) \quad (4.13c)$$

The Hamiltonian density $\mathcal{H}$ given in Eq. (4.9), together with the Poisson brackets appearing in (4.6), (4.12) and (4.13), define our Hamiltonian system. Using these equations it can be shown that the Hamiltonian equations of motion are just those written in (2.17), (2.18) and (2.19).

Let us now proceed with the constraint analysis. Our primary constraints are:

$$\Omega_1^a(x) \equiv \Pi_0^a(x) \approx 0 \quad (4.14)$$

The consistency of these constraints under time evolution requires that:

$$\begin{aligned} \Omega_2^a(x) &\equiv \{ \Pi_0^a(x), H \} = \\ &= (D_1 E)^a(x) + e f_{+}^a(x) + \frac{e}{2} \left[(2-a) \Pi_{+}^a(x) + a \Pi_{-}^a(x) \right] \approx 0 \end{aligned} \quad (4.15)$$

which are the generalizations of the usual Gauss' law constraints. As

$$\{ \Omega_2^a(x), H \} = -e f^{abc} A_0^b(x) \Omega_2^c(x) \quad (4.16)$$

no further constraints are needed. It can be checked that:

$$\{ \Omega_1^a(x), \Omega_2^b(y) \} = 0 \quad (4.17)$$

$$\{ \Omega_2^a(x), \Omega_2^b(y) \} = e f^{abc} \Omega_2^c(x) \delta(x^1 - y^1) \quad (4.18)$$

which means that the constraints are first class, as expected in a theory with gauge invariance. Equation (4.18) suggests that $1/e \Omega_2^a(x)$ are the generators of the infinitesimal time-independent gauge transformations. This is indeed the case as can be seen from the following Poisson brackets:

$$\{\Omega_2^a(x), A_1^b(y)\} = e f^{abc} A_1^c(x) \delta(x^1 - y^1) - \delta^{ab} \delta'(x^1 - y^1) \quad (4.19a)$$

$$\{\Omega_2^a(x), G_{ij}(y)\} = ie (t^a G)_{ij}(x) \delta(x^1 - y^1) \quad (4.19b)$$

$$\{\Omega_2^a(x), U_{ij}(y)\} = -ie (U t^a)_{ij}(x) \delta(x^1 - y^1) \quad (4.19c)$$

which can be easily calculated from Eqs. (4.3) and (4.6). We thus see that there are no anomalous terms in the Poisson brackets of the infinitesimal generators of the gauge group, which means that the theory can be quantized as an ordinary gauge theory and the Poisson brackets given above can be converted into commutators of the quantum theory by means of the usual substitution rule $\{A, B\} \rightarrow -i[A, B]$.

Using the above results, it is now easy to calculate the commutation relations among the different components of the gauge current. In order to do that, let us first write the components of $J_\mu(x)$, defined in Eq. (2.21), as a function of $\rho_\pm$, $\Pi_\pm$ and A_μ :

$$J_0(x) = \rho_+(x) + \frac{1}{2} [(2-a)\Pi_+(x) + a\Pi_-(x)] + \frac{e}{4\pi} A_- \quad (4.20)$$

$$J_1(x) = \rho_+(x) + \frac{1}{2} [(2-a)\Pi_+(x) - a\Pi_-(x)] + \frac{e}{4\pi} A_0 + \frac{e}{4\pi} (a + \frac{1}{a-1}) A_1 \quad (4.21)$$

One then finds for the commutators:

$$[J_0^a(x), J_0^b(y)] = i f^{abc} (J_0^c(x) - \frac{e}{4\pi} A_-^c(x)) \delta(x^1 - y^1) \quad (4.22)$$

$$\begin{aligned} [J_0^a(x), J_1^b(y)] = i f^{abc} \left[J_1^c(x) - \frac{e}{4\pi} A_0^c(x) - \frac{e}{4\pi(a-1)} (a^2 - a + 1) A_1^c(x) \right] \delta(x^1 - y^1) + \\ + \frac{ia^2}{4\pi(a-1)} \delta^{ab} \delta'(x^1 - y^1) \end{aligned} \quad (4.23)$$

$$\begin{aligned} [J_1^a(x), J_1^b(y)] = \frac{if^{abc}}{(a-1)^2} \left[2(1-a) J_1^c(x) - J_0^c(x) + a^2 J_+^c(x) + \frac{e}{4\pi} (2a-1) A_0^c(x) + \right. \\ \left. + \frac{e}{4\pi} (2a^2 - 2a + 1) A_1^c(x) \right] \delta(x^1 - y^1) \end{aligned} \quad (4.24)$$

Several comments are in order. First of all we observe that there is no Schwinger term in Eq. (4.22), confirming thus our previous results. Our second comment is related to the sign of the Schwinger term in Eq. (4.23). With our present conventions this term has to be positive in any consistent theory [29]. So we are forced to have $a > 1$, in agreement with a previous analysis [13]. The current algebra for the Abelian case was obtained in Refs. [11] and [12] and for this case their results coincide with ours.

5. - GAUGE FIXING

In order to analyze the physical content of any gauge theory, one has to fix a gauge. Only in this way one can determine the arbitrary constants that appear in the Hamiltonian. A possible method of doing this is to introduce the gauge fixing conditions as new constraints and study the persistence of these new constraints under time evolution [30]. For an admissible gauge fixing, the complete set of constraints has to be of the second class. The quantization of the model can be done by using the method of Dirac for a system with second-class constraints.

As we have said in the Introduction, there are two possible approaches for the quantization of the classical model defined by the Lagrangian (2.1). One is the approach that we have followed here in which, as we have checked in the previous sections, the gauge invariance of the theory is manifest. The starting point of this approach is Eq. (2.11) in which the Faddeev-Popov trick has been employed and, as a consequence, the field G appears as a dynamical field, and the gauge invariant action is $W(A^{G^{-1}})$. On the other hand, in the gauge non-invariant approach, the Faddeev-Popov procedure is not used and thus the starting point is Eq. (2.7). In this case the effective action is $W(A)$ and it is not gauge invariant.

The relation between these two formulations has been studied in Ref. [10] for the Abelian case. It was shown in this paper that the gauge non-invariant formulation can be understood as a gauge fixed version of the gauge invariant approach. We want to demonstrate here a similar statement for the more general non-Abelian case.

In order to establish this equivalence, a special gauge has to be chosen and the system has to be studied following the method described at the beginning of this section. A convenient gauge is that in which $\mathcal{G}_1 G^{-1}$ vanishes. This gauge is a direct generalization of that considered in Ref. [10]. So we introduce a new constraint:

$$\Omega_3^a(x) \equiv (G(x) \partial_1 G^{-1}(x))^a \approx 0 \quad (5.1)$$

The preservation of this constraint under time evolution implies that:

$$\{\Omega_3^a(x), H\} \approx 0 \quad (5.2)$$

This can be achieved if we impose an additional constraint

$$\Omega_4^a(x) = \Pi_-^a(x) - \Pi_+^a(x) - \frac{e}{2\pi} \left(A_0^a(x) + \frac{A_1^a(x)}{a-1} \right) \approx 0 \quad (5.3)$$

It is quite easy to calculate the Poisson brackets of the four constraints with themselves. The non-vanishing results are [together with (4.18)]:

$$\{\Omega_1^a(x), \Omega_4^b(y)\} = \frac{e}{2\pi} \delta^{ab} \delta(x^1 - y^1) \quad (5.4a)$$

$$\{\Omega_2^a(x), \Omega_3^b(y)\} = e f^{abc} (G \partial_1 G^{-1}) \delta(x^1 - y^1) + i e \delta^{ab} \delta'(x^1 - y^1) \quad (5.4b)$$

$$\{\Omega_2^a(x), \Omega_4^b(y)\} = e f^{abc} \left[\Pi_-^c - \Pi_+^c - \frac{A_1^c}{a-1} \right] \delta(x^1 - y^1) \quad (5.4c)$$

$$\{\Omega_3^a(x), \Omega_4^b(y)\} = \frac{e}{a-1} f^{abc} (G\partial, G^{-1})^c \delta(x^1 - y^1) + \frac{2i}{a-1} \delta'(x^1 - y^1)$$

(5.4d)

$$\{\Omega_4^a(x), \Omega_4^b(y)\} = \frac{e}{a-1} f^{abc} (\pi_-^c - \pi_+^c) \delta(x^1 - y^1) - \frac{i f^{abc}}{n(a-1)^2} (G\partial, G^{-1})^c \delta(x^1 - y^1)$$

(5.4e)

From these results, it can be shown by computing the determinant of the matrix $\{\Omega_i(x), \Omega_j(y)\}$ that we have a system of second-class constraints as we should. As we have explained above, the quantization of this gauge fixed system could be done by computing the Dirac brackets of our basic observables, which can be directly converted into quantum commutators. We shall not do that here. Instead we shall limit ourselves to obtain the gauge fixed Hamiltonian. We observe first that combining Eqs. (5.1) and (5.2) together with Eqs. (4.8) we obtain the following weak equalities:

$$\pi_-^a \approx -\pi_+^a \approx \frac{e}{4\pi} (A_0^a + \frac{A_1^a}{a-1})$$

(5.5)

Substituting this weak equalities in $\mathcal{H}$ [Eq. (4.9)], we obtain for the gauge-fixed Hamiltonian density:

$$\begin{aligned} \mathcal{H} \approx & \frac{1}{2} E^a E^a - A_0^a (D_1 E)^a + \pi (\rho_+^a \rho_+^a + \rho_-^a \rho_-^a) - e \rho_+^a A_-^a + \\ & + \frac{e^2}{8\pi} A_-^a A_-^a - \frac{ae^2}{8\pi} A_+^a A_-^a \end{aligned}$$

(5.6)

This is precisely the Hamiltonian density which is directly obtained from the effective action $W(A)$ and it was considered as the starting point in the canonical analysis of Refs. [13] and [14]. We therefore conclude that, as in the Abelian case, the formulation without gauge invariance is just equivalent to the gauge invariant approach in the special gauge given by Eq. (5.1). As has been done in Ref. [10], we could use the two remaining constraints (4.14) and (4.15) in order to obtain an unconstrained Hamiltonian. In fact, combining Eqs. (5.5) and (4.15), it is possible to show that:

$$A_0^a \approx \frac{4\pi}{e^2(a-1)} \left[-(D_1 E)^a - e \rho_+^a - \frac{e^2}{4\pi} A_1^a \right]$$

(5.7)

Substituting this result in Eq. (5.6), it is easy to see that the unconstrained Hamiltonian density can be put in the form:

$$\begin{aligned} \mathcal{H} \approx & \frac{1}{2} E^a E^a + \pi p_-^a p_-^a + \pi \left(p_+^a + \frac{e}{2\pi} A_1^a \right)^2 + \\ & + \frac{e^2(a-1)}{8\pi} (A_0^a A_0^a + A_1^a A_1^a) \end{aligned} \quad (5.8)$$

where A_0 has to be understood as a dependent quantity given by Eq. (5.7). As was first shown in Ref. [13], the Hamiltonian of Eq. (5.8) is unambiguous, Hermitian and for $a > 1$ positive. This strongly suggests that the resulting quantum theory is unitary and consistent only when this regularizing parameter a is chosen to be greater than one. Observe that we have obtained a similar conclusion from the sign of the Schwinger term in the current algebra [Eq. (4.23)].

6. - SUMMARY AND CONCLUSIONS

We have discussed the gauge invariant formulation of a two-dimensional non-Abelian gauge theory with chiral coupling to massless fermions. Our starting point is the result of the calculation of the fermionic determinant, from which we construct an effective bosonic action which is local and gauge invariant. We have checked that the gauge current is conserved covariantly as a consequence of the equations of motion. A canonical Hamiltonian approach has been developed and we have verified that, as expected in a gauge theory, the system has first-class constraints and there are no anomalous terms in the algebra of the infinitesimal generators of the gauge group. By fixing the gauge and eliminating the constraints, we have seen that a consistent unitary quantum theory is likely to exist.

In our approach, the bosonic fields (and in particular the gauge field) are treated classically, although the effective action that we have used contains quantum effects coming from the fermionic determinant. As we do not know how to solve exactly the resulting bosonic effective theory, it seems that perturbation theory has to be employed in order to study the effect of having quantum bosonic fields. A local gauge-invariant action like that given by Eq. (2.14) seems to be the best starting point for this purpose.

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$$\eta_{00} = -\eta_{11} = 1 \quad , \quad \varepsilon_{01} = -\varepsilon_{10} = 1$$

$$\gamma^0 = \sigma_1 \quad , \quad \gamma^1 = -i\sigma_2 \quad , \quad \gamma_5 = \gamma^0\gamma^1 = \sigma_3 \quad , \quad \varepsilon_{\mu\nu}\gamma^\nu = \gamma_\mu\gamma_5$$

$$[t^a, t^b] = i f^{abc} t^c \quad , \quad \text{tr}(t^a t^b) = \delta^{ab} \quad , \quad A_\pm = A_0 \pm A_1 \quad , \quad \partial_\pm = \partial_0 \pm \partial_1$$
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