CERN-TH.4905/87

THE KAON B-PARAMETER AND $K-\pi$ AND $K-\pi\pi$ TRANSITION AMPLITUDES ON THE LATTICE

M.B. Gavela ^{*)}
Dept. de Fisica Teorica
Univ. Autonoma de Madrid

L. Maiani, S. Petrarca and F. Rapuano
Istituto di Fisica "G. Marconi"
Università degli Studi di Roma "La Sapienza"
and INFN, Sezione di Roma

G. Martinelli
CERN - Geneva

O. Pene
LP THE - Orsay

and

C.T. Sachrajda
University of Southampton
Southampton SO9 5NH

A B S T R A C T

We present the results of a computation of the $K^0-\bar{K}^0$ and $\Delta I = 3/2$ $K-\pi$ and $K-\pi\pi$ amplitudes in lattice QCD. The numerical simulation has been done using the standard one-plaquette action and Wilson fermions, in the quenched approximation. Combining the results obtained at $\beta = 6$ on a $10^2 \times 20 \times 40$ lattice with 30 configurations with those at $\beta = 6.2$ on a $16^3 \times 48$ lattice with 15 configurations, we have obtained for the kaon B-parameter $B_{K^0-\bar{K}^0} = 0.65 \pm 0.15$. We have also evaluated the $\Delta I = 3/2$ $K-\pi$ and $K-\pi\pi$ weak Hamiltonian matrix elements. The relevant renormalized operator O_4 has been found to satisfy soft-pion theorems and chiral Lagrangian relations at the 30% level, and this makes us confident about our results. We find $(\langle K^+ | H_w | \pi^+ \pi^0 \rangle / m_K) = (7 \pm 2) 10^{-8}$ to be compared with the experimental value 3.7×10^{-8} . We suggest the possible physical origin of this difference.

^{*)} On leave of absence of LP THE, Orsay.

1. - INTRODUCTION

Lattice QCD offers the possibility of computing weak Hamiltonian matrix elements from first principles and with no free parameters [1,2]. The transcription of continuum operators on the lattice is complicated by a conflict between the lattice regularization and the chiral properties of the theory [3,4]. To obtain renormalized operators with finite matrix elements and belonging to definite chiral representations in the massless limit, it is necessary to go through a quite complicated renormalization procedure. In fact, because chiral symmetry is not a symmetry of the QCD action on the lattice, the bare operators can mix with other operators with different chiral properties and equal or lower dimension. In the latter case the mixing coefficients diverge as inverse powers of the lattice spacing a . To obtain renormalized, finite operators suitable subtractions must be applied.

A physically relevant and simple case is that of $\Delta I = 3/2$ transitions, where divergent subtractions are absent and the renormalized operator is safely given by perturbation theory. We focus on this case in the present paper.

In the continuum, the $K-\pi$ and $K-\pi\pi$ matrix elements are related by soft pion theorems. We find that the operators, renormalized on the lattice, with the general prescription of Refs. [3,4], satisfy quite well the expected soft pion relations confirming the validity of the theoretical analysis. For the physical $\Delta I = 3/2$ amplitude, we obtain [5]

$$\frac{\langle K^+ | H_W | \pi^+ \pi^0 \rangle}{m_K} = (7 \pm 2) \cdot 10^{-8} \quad (1.1)$$

to be compared with the experimental value

$$\frac{\langle K^+ | H_W | \pi^+ \pi^0 \rangle}{m_K} = 3.7 \cdot 10^{-8}$$

The computation of the $K^0-\bar{K}^0$ matrix element of the left-handed $\Delta S = 2$ operator is essentially the same as the computation of the $K-\pi$ matrix element of the operator O_4 . The relevant quantity for the kaon system is usually expressed through the ratio of the weak Hamiltonian matrix element to its value evaluated in the vacuum insertion approximation^{*)}:

$$B_{K^0\bar{K}^0} = \frac{\langle K^0 | \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma_\mu (1 - \gamma_5) d | \bar{K}^0 \rangle}{8/3 f_K^2 m_K^2} \quad (1.2)$$

*) This quantity is used to estimate the CP violating parameter ε .

We find:

$$B_{K^0 \bar{K}^0} = 0.65 \pm 0.15 \quad (1.3)$$

at the renormalization scale $\mu \approx 2.6 \div 3 \text{ GeV}^*$). This result can be compared with the result of Ref. [7], $B_{K^0 \bar{K}^0} = 0.33 \pm 0.20$, with the result obtained by using the sum rule techniques, $B_{K^0 \bar{K}^0} = 0.33 \pm 0.09$ [8] and a $1/N$ expansion $B_{K^0 \bar{K}^0} = 0.70 \pm 0.07$ [9]. In all our results, the main error, besides the statistical one, comes from the fact that the matrix elements have been evaluated in the so-called "quenched" approximation, i.e., in the limit $n_f/N_c \rightarrow 0$. Finally we have also computed f_K/f_π with the result [10]:

$$\frac{f_K}{f_\pi} = 1.17 \pm 0.10 \quad (1.4)$$

which, within the errors, coincides with the experimental value $(f_K/f_\pi)_{\text{exp}} \approx 1.24$. The rest of the paper is dedicated to a detailed discussion of the methods used to find the results presented above. In Section 2 we recall the main ingredients necessary for the renormalization on the lattice of the operators relevant for $\Delta S = 2$ and $\Delta I = 3/2$ transitions on the lattice. In Section 3 we give the parameters of our Monte Carlo simulation and describe how the matrix elements have been extracted from suitable combinations of quark propagators and Green functions. In Section 4 we present the results for $K^0 \bar{K}^0$ and $\Delta I = 3/2$ $K \rightarrow \pi\pi$ transitions. In Section 5 we give the results of the computation of the $K \rightarrow \pi\pi$ matrix elements and compare them with expectations based on chiral Lagrangian relations and soft pion theorems.

Finally, we propose a possible explanation to the difference between the result given in Eq. (1.1) and the corresponding experimental value. This explanation would modify the determination of $B_{K^0 \bar{K}^0}$ of Ref. [7].

2. - LATTICE RENORMALIZATION OF OPERATORS RELEVANT FOR $\Delta I = 3/2$ AND $\Delta S = 2$ TRANSITIONS

Composite operators on the lattice suffer from two different but related problems: severe ultra-violet divergences and bad chiral behaviour. Both problems arise because of the hard chiral symmetry breaking entailed by the Wilson term in

*) This result has to be multiplied by $(\alpha(\mu))^{-6/25}$ to obtain the renormalization group invariant B parameter defined in Ref. [6].

the lattice fermionic action. This term is necessary to avoid quark species doubling and control flavour symmetry. The general method to obtain renormalized operators with finite matrix elements has been discussed in detail in Refs. [3,4].

Here we recall how to proceed in the simplest case occurring in the computation of the weak Hamiltonian amplitudes, i.e., the case of the operators responsible for the $\Delta S = 2$ and $\Delta I = 3/2$ transitions:

$$O^{\Delta S=2} = \bar{s}\gamma^\mu(1-\gamma_5)d \bar{s}\gamma_\mu(1-\gamma_5)d \quad (2.1)$$

$$\begin{aligned} O_4 = & \bar{s}\gamma^\mu(1-\gamma_5)d \bar{u}\gamma_\mu(1-\gamma_5)u + \\ & \bar{s}\gamma^\mu(1-\gamma_5)u \bar{u}\gamma_\mu(1-\gamma_5)d - \\ & \bar{s}\gamma^\mu(1-\gamma_5)d \bar{d}\gamma_\mu(1-\gamma_5)d \end{aligned} \quad (2.2)$$

The operators in Eq. (2.1) and (2.2) can mix under renormalization only with other operators of dimension six and different chirality [11]. Since it is not possible to construct operators of dimension less than six with $\Delta S = 2$ or $\Delta I = 3/2$. To obtain a renormalized operator with the wanted transformation properties, we have to subtract the operators with the wrong chiral properties from the bare one. So, for example, the renormalized $\Delta S = 2$ operator has the form:

$$\hat{O}^{\Delta S=2}(\mu) = Z_{LATT}^{\Delta S=2}(\mu a, g_0(a)) [O^{\Delta S=2} + \sum_i c_i(g_0(a)) O_i^{\Delta S=2}] \quad (2.3)$$

$Z_{LATT}^{\Delta S=2}$ is completely determined, including finite terms, by the requirement that for the same value $\mu \ll a^{-1}$, $O_{(\mu)}^{\Delta S=2}$ obeys the same normalization conditions as the continuum operator, i.e., it matches the μ -dependence of the coefficient of the operator product expansion.

The coefficients c_i of the subtraction in Eq. (2.3) are finite, depend only on the lattice coupling constant g_0 and can be safely computed in perturbation theory. The corresponding operators have been classified in Refs. [11,12]. Typical operators occurring in the subtraction have the form:

$$\begin{aligned}
O_i &= \bar{s}d\bar{s}d - \bar{s}\gamma_5 d\bar{s}\gamma_5 d \\
&\quad \bar{s}\gamma^\mu d\bar{s}\gamma_\mu d - \bar{s}\gamma^\mu\gamma_5 d\bar{s}\gamma_\mu\gamma_5 d \\
&\quad \bar{s}\sigma^{\mu\nu} d\bar{s}\sigma_{\mu\nu} d
\end{aligned} \tag{2.4}$$

and are parity conserving.

It has been checked in Ref. [3] that, at lowest order in perturbation theory and up to terms of order a (a being the lattice spacing), the renormalized operators transform correctly under a chiral rotation. As a consequence, the $\Delta S = 2$ $K^0 - \bar{K}^0$ and $\Delta I = 3/2$ $K - \pi$ matrix elements should vanish in the chiral limit; for example:

$$\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle = \hat{\gamma} m_K^2 + O(m_K^4) \tag{2.5}$$

as $m_K \rightarrow 0$.

In practice, the subtraction from perturbation theory is imperfect because only the lowest order in g_0^2 has been computed and terms of order a are certainly present in a Monte Carlo simulation. As $m_K \rightarrow 0$, the matrix elements of operators like those in Eq. (2.4) do not vanish. As a consequence, as first noticed in Ref. [13], the behaviour of the operators in the chiral limit is not the expected one [cfr. Eq. (2.5)]:

$$\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle = \hat{\delta} + \hat{\gamma} m_K^2 + O(m_K^4) \tag{2.6}$$

It is difficult to evaluate a priori the systematic error in the physical amplitude implied by lattice effects and/or the imperfect subtraction. We will return on this point in Section 4. In the following, we shall take γ from Eq. (2.6) and use it to estimate the $K^+ \rightarrow \pi^+ \pi^0$ matrix element according to the soft-pion formula [7]:

$$\begin{aligned}
&\frac{\langle \pi^+ \pi^0 | H_W | K^+ \rangle}{m_K} = \\
&\frac{G_F \sin \theta_C \cos \theta_C}{2} \frac{3}{4} \frac{m_K^2 - m_\pi^2}{m_K} C_4(\mu) \left[\frac{\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle}{f_K m_K^2} \right]
\end{aligned} \tag{2.7}$$

Here we only notice that, for $\Delta I = 3/2$ transitions, one can improve the situation by computing directly $K \rightarrow 2\pi$ amplitudes [14]. In this case, since all the operators to be subtracted are parity conserving, only an overall renormalization of the bare operator is needed to evaluate the matrix element. This should automatically vanish in the chiral limit (up to corrections of order a). Moreover, a comparison of the $K - \pi$ and $K - \pi\pi$ matrix elements checks if the matrix elements of the operators on the lattice obey soft pion theorems, i.e., transform correctly under the chiral group.

3. - CALCULATION OF THE MATRIX ELEMENTS ON THE LATTICE

3.1 Generalities on the Monte Carlo simulation

The results of this paper have been obtained from two distinct Monte Carlo studies. The first have been performed on a $20 \times 10 \times 10 \times 40$ lattice at $\beta = 6$. We have generated 15 independent gauge field configurations, in the quenched approximation, on a $10^3 \times 20$ lattice which has been subsequently duplicated in the x and t directions. We estimate that the systematic error involved in the duplication procedure is much smaller than other statistical/systematic errors present in the computation of the various quantities of interest. For any of the configurations we have computed the quark propagator at three values of the Wilson hopping parameter $K = 0.1515, 0.1530$ and 0.1545 . For the computation of the ρ and π masses and of the kaon B -parameter, we have also used the quark propagators of an independent Monte Carlo simulation, at the same value of β , lattice volume and K 's, whose characteristics are described in detail in Ref. [15]. In the above cases, we have combined all the results at $\beta = 6$ which therefore correspond to a statistics of 30 configurations.

A second Monte Carlo simulation has been done on a $16^3 \times 48$ lattice at $\beta = 6.2$. We have generated 15 independent gauge field configurations on this lattice and evaluated the quark propagators at $K = 0.1500, 0.1510$ and 0.1520 . The volume has been scaled to approximately match the physical size of the previous lattices in Fermi [$a^{-1}(\beta=6.2)/a^{-1}(\beta=6) \approx 1.6$] [16]. In both cases, $\beta = 6$ and $\beta = 6.2$, the pseudoscalar masses are in the range ~ 600 - 1000 MeV.

An important point is the determination of the lattice spacing at $\beta = 6$ and $\beta = 6.2$. Different methods can be used. The first one is to evaluate the lattice spacing from the vector meson mass m_V at the critical point, corresponding to $m_{PS} = 0$, where m_{PS} is the pseudoscalar meson mass:

$$a^{-1} \simeq \frac{m_{\rho}^{exp}}{m_V a(m_{PS} = 0)} \quad (3.1)$$

$m_V a = m_V a(K)$ and $m_{PS} a = m_{PS} a(K)$ are only functions of the Wilson parameter K (i.e., the quark mass) at fixed g_0 and are measured on the lattice by looking at the exponential decay of the vector and pseudoscalar zero momentum correlation functions at large times:

$$G_{PS}(n_t) = \sum_{\vec{n}} \langle P_5(\vec{n}, n_t) P_5(0, 0) \rangle \rightarrow \frac{Z_5}{2m_{PS} a} e^{-(m_{PS} a) n_t} \quad (3.2)$$

$$G_V(n_t) = \sum_{\vec{n}} \langle V_k(\vec{n}, n_t) V_k(0, 0) \rangle \rightarrow \frac{Z_V}{2m_V a} e^{-(m_V a)n_t} \quad (3.3)$$

where $P_5 = \bar{\psi}\gamma_5\psi$ and $V_K = \bar{\psi}\gamma_K\psi$.

A second method, proposed in Ref. [16], consists in using the relation:

$$a^{-1} = A \frac{m_K^2 - m_\pi^2}{m_{K^*} - m_\rho} \quad (3.4)$$

where A is defined to be:

$$m_V a = A(m_{PS} a)^2 + B \quad (3.5)$$

Finally one can fix the scale from the proton mass at the critical K ($m_{PS}=0$):

$$a^{-1} = \frac{m_P^{exp}}{m_B a(m_{PS} = 0)} \quad (3.6)$$

where $m_B a$ is the dimensionless baryon mass. It is well known that m_P/m_ρ is systematically above the experimental value in the quenched approximation; hence the last method gives always a value smaller than the first one for a^{-1} . This is also true if one uses Eq. (3.4). So, for example, at $\beta = 6$ one finds:

$$\begin{aligned} a^{-1} &= (2.20 \pm 0.10) \text{GeV} \quad \text{method I} \\ a^{-1} &= (1.57 \pm 0.10) \text{GeV} \quad \text{method II} \\ a^{-1} &= (1.45 \pm 0.30) \text{GeV} \quad \text{method III} \end{aligned} \quad (3.7)$$

The difference among the different determinations of a^{-1} is due to systematic effects, such as the quenched approximation.

In this paper we have chosen to fix a^{-1} as in Eq. (3.1). This method gives a better agreement to the experimental dependence of the vector versus the pseudo-scalar mass as shown in Fig. 1. One has in any case to bear in mind a possible systematic error in the evaluation of a^{-1} of the order of 300-400 MeV. For this reason, in all the cases for which the value of a^{-1} has a high effect, we have reported separately the results obtained by taking a^{-1} from method I and by averaging the results obtained using methods I and II. Using Eq. (3.1) we find at $\beta = 6.2$:

$$a^{-1} = (3.00 \pm 0.20) \text{GeV} \quad (3.8)$$

To compute f_π and f_K we use the technique suggested in Ref. [17]. We compute the ratio:

$$\frac{G_{A_0}(n_t)}{G_{PS}(n_t)} = \frac{\sum_{\vec{n}} \langle A_0(\vec{n}, n_t) P_5(0, 0) \rangle}{\sum_{\vec{n}} \langle P_5(\vec{n}, n_t) P_5(0, 0) \rangle} \sim \frac{(f_{PS}a)(m_{PS}a)}{Z_A Z_5^{1/2}} \quad \text{at } n_t \text{ large} \quad (3.9)$$

where $A_0 = \bar{\psi}\gamma_0\gamma_5\psi$ and Z_A is the axial current renormalization constant on the lattice; Z_5 has been defined in Eq. (3.2). From Eq. (3.9) it is possible to evaluate f_{PS} . For the pion we find:

$$\begin{aligned} f_\pi &= (165 \pm 35) \text{MeV} \quad \text{method I} \\ f_\pi &= (135 \pm 30) \text{MeV} \quad \text{method I + II at } \beta = 6 \quad (\text{exp} \sim 132 \text{MeV}) \\ f_\pi &= (160 \pm 20) \text{MeV} \quad \text{method I} \\ f_\pi &= (139 \pm 17) \text{MeV} \quad \text{method I + II at } \beta = 6.2 \end{aligned} \quad (3.10)$$

Using the perturbative value of Z_A , $Z_A \approx 0.87$ [17]. Bearing in mind the error in the evaluation of Z_A at lowest order in perturbation theory [16] ($\sim 10\text{--}15\%$) and the error on a^{-1} , we claim good agreement between the results in Eqs. (3.10) and the experimental value.

To evaluate f_K , the best method is to proceed through the calculation of the ratio f_K/f_π , in such a way to cancel Z_A (which with very good accuracy is independent of the quark masses, for light quarks) and the dependence on a^{-1} . Combining the results at $\beta = 6$ and $\beta = 6.2$, we find:

$$\frac{f_K}{f_\pi} = 1.17 \pm 0.10 \rightarrow f_K = (158 \pm 13) \text{MeV} \quad (\text{exp} \simeq 164 \text{MeV}) \quad (3.11)$$

3.2 Three-point correlation functions

We now consider the more complicated case of three-point functions involving two pseudoscalar sources (i.e., $\bar{\psi}\gamma_5\psi$) and one of the operators whose matrix elements we want to compute.

*) In Ref. [10] the result was $f_\pi \sim 139 \text{ MeV}$ corresponding to the choice of $a^{-1}(\beta=6.2) = 2.6 \text{ GeV}$ has been presented.

For the B-parameter and of the K- π matrix elements of the weak Hamiltonian we study the three-point correlation:

$$G(n_{t_1}, n_{t_2}) = \sum_{\vec{n}_1, \vec{n}_2} \langle P_5(\vec{n}_1, n_{t_1}) \hat{O}(0, 0) P_5(\vec{n}_2, n_{t_2}) \rangle \quad (3.12)$$

where the operator $\hat{O}$ has the form $\hat{O} = (\bar{\psi}\Gamma\psi)(\bar{\psi}\Gamma\psi)$, Γ being one of the Dirac matrices.

Only the eight diagrams in Fig. 2 contribute to $\Delta S = 2$ and $\Delta I = 3/2$ K- π transitions. Since all the quark lines stem from the origin, we can use the quark propagators $S_{\alpha\beta}^{AB}(n, 0)$ ($A, B \equiv$ colour indices; $\alpha, \beta \equiv$ spin indices) used for the hadron spectroscopy to evaluate these diagrams [1,2]. On a finite, periodic lattice of length N_t in the time direction, we may have the two cases illustrated in Figs. 3a and 3b. In the first case $N_t - n_{t_1}$ and n_{t_2} are large, so that we isolate the lowest lying one-particle states, and $n_{t_1} - n_{t_2} \gg N_t - n_{t_1}$ and n_{t_2} . The correlation function in Eq. (3.12) is then dominated by the matrix element of the operator sandwiched between two pseudoscalar mesons, a kaon and a pion, for example (for simplicity, all the formulae in the rest of this section refer to the case $m_K = m_\pi = m$):

$$G(n_{t_1}, n_{t_2}) \rightarrow \frac{Z_5}{(2ma)^2} \langle K | \hat{O} | \pi \rangle e[-ma(N_t - n_{t_1} + n_{t_2})] \quad (3.13)$$

The second possibility is that n_{t_1} and n_{t_2} are both large but $N_t - n_{t_1} \gg n_{t_1}$ and $N_t - n_{t_2} \gg n_{t_2}$ (and moreover $n_{t_1} \neq n_{t_2}$). In this case $\pi\bar{K}$ intermediate state dominates and the correlation function is practically proportional to the matrix element $\langle 0 | \hat{O} | \pi\bar{K} \rangle$. Neglecting final state interaction [which gives a correction of order $O(m^4)$] and finite size effects^{*}:

$$G(n_{t_1}, n_{t_2}) \rightarrow \frac{Z_5}{(2ma)^2} \langle 0 | \hat{O} | \pi\bar{K} \rangle e[-ma(n_{t_1} + n_{t_2})] \quad (3.14)$$

The situation is clearly completely symmetric in the exchange $n_{t_1} \leftrightarrow n_{t_2}$.

In the intermediate region (n_{t_1} or $n_{t_2} \sim N_t/2$) we have a transition from the time interval where $G(t_1, t_2)$ is dominated by $\langle K | \hat{O} | \pi \rangle$ to the interval where $G(t_1, t_2)$ is dominated by $\langle 0 | \hat{O} | \pi\bar{K} \rangle$.

^{*}) Finite size effects are expected to be worse in this case since we have two mesons (instead of one as usual) in the same spatial volume. Final state interactions are by themselves of order m^2/Λ^2 , where Λ is some strong interaction scale and hence their effect is of order m^4 .

In both cases, considered in Eqs. (3.13) and (3.14), provided the time distances are all large enough, we have that the ratio:

$$R(n_{t_1}, n_{t_2}) = \frac{Z_5 G(n_{t_1}, n_{t_2})}{G_K(n_{t_1}) G_\pi(n_{t_2})} \rightarrow \text{constant} \quad \begin{array}{l} \nearrow < 0 | \hat{O} | \pi \bar{K} > \\ \searrow < K | \hat{O} | \pi > \end{array} \quad (3.15)$$

[$G_{K,\pi} = G_{PS}$ and remember that on a periodic lattice $G_{K,\pi}(n_t) = G_{K,\pi}(N_t - n_t)$]. So we expect to find two constant plateaus if we plot R as a function of n_{t_1} (at fixed n_{t_2}).

We show in Figs. 4a,b and 5a,b typical results for the operator O_4 [given in Eq. (2.2)] at $\beta = 6$, on the $20 \times 10 \times 10 \times 40$ lattice, for $K = 0.1515$ and 0.1545 and at $\beta = 6.2$, on the $16^3 \times 48$ lattice, for $K = 0.1500$ and 0.1520 . The signal worsen at low quark masses (larger K 's), but the two plateaus are always clearly visible. From the value of the plateaus it is possible to extract the matrix elements of the operator of interest.

3.3 Four-point correlation functions

To compute the $K \rightarrow 2\pi$ matrix elements we consider the four-point correlation function:

$$G(n_{t_1}, n_{t_2}, n_{t_3}) = \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3} < P_5(\vec{n}_1, n_{t_1}) \hat{O}(0, 0) P_5(\vec{n}_2, n_{t_2}) P_5(\vec{n}_3, n_{t_3}) > \quad (3.16)$$

The relevant Feynman diagrams for $K^0 \rightarrow \pi^+ \pi^-$ are shown in Figs. 6a-d. Contrary to the previous case, we have now quark propagators between two arbitrary points on the lattice, $S_{\alpha\beta}^{AB}(n_2, n_3) [n_2, 3 \equiv (\vec{n}_2, 3; n_{t_2, 3})]$. However, following Ref. [4], one can compute a set of generalized propagators to which we have inserted a pseudoscalar source (the π^- for example):

$$S_{\alpha\beta}^{(1)AB}(n_3, 0) = \sum_n S_{\alpha\gamma}^{AC}(n_3, n) (\gamma_5)_{\gamma\rho} S_{\rho\beta}^{CB}(n, 0) \delta_{n_t, n_{t_2}} \quad (3.17)$$

In terms of these $S^{(1)}$ propagators the diagrams in Figs. 6a-d are again reduced to the eight shaped diagrams, as shown in Fig. 7. In practice, since we have to compute a new $S^{(1)}$ for any value of n_{t_2} , the position of one of the pseudoscalar

sources (the π^- in our case), as well as the position of the operator, are always fixed. On the other side, we can vary at will the time position n_{t_1} and n_{t_2} . We have chosen $n_{t_2} \ll N_t - n_{t_1}$. At fixed n_{t_1} ($N_t - n_{t_1} \ll n_{t_1}$), by varying n_{t_3} , the correlation function will be dominated by two different matrix elements:

$$G(n_{t_1}, n_{t_2}, n_{t_3}) \rightarrow \frac{Z_5^{3/2}}{(2ma)^3} \langle K^0 \pi^- | \hat{O} | \pi^- \rangle e[-ma(2N_t - n_{t_1} - n_{t_3} + n_{t_2})] \quad (3.18)$$

for $N_t - n_{t_3} \ll n_{t_3}$ and:

$$G(n_{t_1}, n_{t_2}, n_{t_3}) \rightarrow \frac{Z_5^{3/2}}{(2ma)^3} \langle K^0 | \hat{O} | \pi^+ \pi^- \rangle e[-ma(2N_t - n_{t_1} + n_{t_2} + n_{t_3})] \quad (3.19)$$

for $n_{t_3} \ll N_t - n_{t_3}$.

As it was the case for the three-point correlation function, by dividing by suitable propagators, one should observe two plateaus corresponding to the two different matrix elements in Eqs. (3.18) and (3.19). This is shown in Fig. 8 where the ratio:

$$R(n_{t_1}, n_{t_2}, n_{t_3}) = Z_5^{3/2} \frac{G(n_{t_1}, n_{t_2}, n_{t_3})}{G_{K^0}(n_{t_1}) G_{\pi^-}(n_{t_2}) G_{\pi^+}(n_{t_3})} \quad n_{t_1} \text{ and } n_{t_2} \text{ fixed} \quad (3.20)$$

for the operator O_4 is reported at $\beta = 6$ and $K = 0.1515$.

4. - COMPUTATION OF THE $K^0 - \bar{K}^0$ B-PARAMETER AND OF THE $\Delta I = 3/2$ $K - \pi$ MATRIX ELEMENT

The computation of the $K^0 - \bar{K}^0$ and $K - \pi$ amplitudes is essentially the same and corresponds to the computation of the "eight" diagram in Fig. 2. As explained in Section 3.2, from the three-point correlation function we extract the two-matrix elements $\langle K^0 | O^{\Delta S=2} | \bar{K}^0 \rangle$ ($\langle K | O_4 | \pi \rangle$) and $\langle 0 | O^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle$ ($\langle 0 | O_4 | \pi \bar{K} \rangle$).

To lowest order in masses and momenta, one has:

$$\begin{aligned} \langle 0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle &= \alpha + \beta m^2 + \gamma p_{\bar{K}^0} \cdot p_{\bar{K}^0} + O(m^4) = \alpha + (\beta - \gamma) m^2 + O(m^4) \\ \langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle &= \alpha + (\beta + \gamma) m^2 + O(m^4) \end{aligned} \quad (4.1)$$

when the mesons are at rest.

For the truly renormalized operator $O_{\text{REN}}^{\Delta S=2}$, α and β vanish. Since, as discussed in Section 2, the subtraction has been computed only at lowest order in g_0^2 and terms of order a still play a rôle, the operator, renormalized in perturbation theory, has α and β different from zero. So we fit the matrix elements of $O^{\Delta S=2}$ to the expressions in Eq. (4.1) and take as our best estimate of the matrix element the coefficient γ . In so doing we are certainly left with a systematic error. In fact the residual unknown subtraction will not only cancel α and β but also modify γ ($\gamma_0 \rightarrow \gamma_{0,\text{REN}}$ only as $a \rightarrow 0$).

Actually, to evaluate the B-parameter, we found convenient to fit the matrix elements of the operator as a function of the squared matrix element of the axial current. The coefficient of the fit is the B-parameter (up to an overall renormalization constant $8/3Z_A^2$). In Fig. 8, the ratios $\langle 0 | O^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle / Z_5$ and $\langle K^0 | O^{\Delta S=2} | \bar{K}^0 \rangle / Z_5$ are reported as a function of $\frac{8}{3}(\langle 0 | A_0 | \bar{K}^0 \rangle)^2 / Z_5 = \frac{8}{3}(f_K^2 m_K^2 / Z_A^2 Z_5)$ at $\beta = 6.2$ (15 configurations). These combinations of matrix elements are particularly convenient to reduce the errors since they are directly computed by taking ratios of correlation functions:

$$\begin{aligned} \frac{\langle 0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle}{Z_5} &\rightarrow \frac{G(n_{t_1}, n_{t_2})}{G_K(n_{t_1}) G_K(n_{t_2})} \\ \frac{\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle}{Z_5} &\rightarrow \frac{8/3(\langle 0 | A_0 | \bar{K}^0 \rangle)^2}{Z_5} \rightarrow \frac{8}{3} \left(\frac{G_{A_0}(n_t)}{G_K(n_t)} \right)^2 \end{aligned} \quad (4.2)$$

at the plateaus

The value of the B-parameter obtained from a linear and a quadratic fit to the points in Figs. 8 and the analogous at $\beta = 6$ are reported in Table 1. We take as our best estimate for the B-parameter the results of the quadratic fits, given the fact that terms of order m^4 are still important in the range of masses considered. To combine the results at $\beta = 6$ and $\beta = 6.2$ we must take into account that the B-parameter is a function of the scale, and use the renormalization group evolution of the corresponding operator. Our final result reads:

$$B_{K^0-\bar{K}^0} = 0.65 \pm 0.15 \quad (4.3)$$

in the quenched approximation. Using Eq. (2.7), we obtain:

$$4.22 \cdot 10^{-7} C_4(\mu) \left[\frac{\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle}{f_K m_K^2} \right] \quad (4.4)$$

where $G_F M_P^2 = 1.023 \times 10^{-5}$ and $\sin\theta_c = 0.213$. Combining the results of a quadratic fit in m_K^2 at $\beta = 6$ and $\beta = 6.2$ of the quantity in square bracket, we obtain:

$$\begin{aligned} \left[\frac{\langle K^0 | \hat{O}^{\Delta S=2} | \bar{K}^0 \rangle}{f_K m_K^2} \right] &= (0.30 \pm 0.08) GeV \text{ at } \mu \simeq 3.0 GeV \text{ method I} \\ &= (0.26 \pm 0.07) GeV \text{ at } \mu \simeq 2.6 GeV \text{ method I + II} \end{aligned} \quad (4.5)$$

from which, using $C_4 \sim 0.55$:

$$\begin{aligned} \frac{\langle \pi^+ \pi^0 | H_W | K^+ \rangle}{m_K} &= (7 \pm 2) \cdot 10^{-8} \text{ method I} \\ &= (6 \pm 2) \cdot 10^{-8} \text{ method I + II} \end{aligned} \quad (4.6)$$

to be compared to the experimental result 3.7×10^{-8} .

5. - COMPUTATION OF THE K- $\pi\pi$ AMPLITUDES

As discussed in Section 3.3, we have also computed the K- $\pi\pi$ matrix elements of the operator O_4 , namely:

$$\langle K^0 \pi^- | \hat{O}_4 | \pi^- \rangle \text{ and } \langle K^0 | \hat{O}_4 | \pi^+ \pi^- \rangle \quad (5.1)$$

The chiral perturbation theory gives us the dependence of these matrix elements on the pion and kaon momenta, as well as their relation to the K- π matrix element:

$$\begin{aligned}
K^+ - \pi^+ & \quad \langle O \rangle = g_{3/2}(k \cdot q^+) + \dots \\
K^0 - \pi^+ - \pi^- & \quad \langle O \rangle = -\frac{i}{f} g_{3/2} [2k \cdot (q^- - q^+) - q^+ \cdot q^- - k \cdot q^+] + \dots
\end{aligned}
\tag{5.2}$$

where $k, q^\pm$ are the kaon and pion momenta respectively. Taking all the mesons at rest, with $m_K = m_\pi = m$ one finds:

$$\begin{aligned}
\langle K^+ | \hat{O}_4 | \pi^+ \rangle &= g_{3/2} m^2 + O(m^4) \\
\langle K^0 | \hat{O}_4 | \pi^+ \pi^- \rangle &= \frac{2i}{f} g_{3/2} m^2 + O(m^4) \\
\langle K^0 \pi^- | \hat{O}_4 | \pi^- \rangle &= -\frac{6i}{f} g_{3/2} m^2 + O(m^4)
\end{aligned}
\tag{5.3}$$

We first notice that the relation $\langle K^0 | \hat{O}_4 | \pi^+ \pi^- \rangle = -1/3 \langle K^0 \pi^- | \hat{O}_4 | \pi^- \rangle$ is quite well satisfied on the lattice as shown in Fig. 9. This implies that terms of order m^4 are not exceedingly large. In Table 2 we compare $g_{3/2}$ obtained from $\langle K^0 \pi^- | \hat{O}_4 | \pi^- \rangle$ and $g_{3/2}$ obtained from $\langle K^+ | \hat{O}_4 | \pi^+ \rangle$ at $\beta = 6$ and $\beta = 6.2$ as a check of soft-pion relations. The results for $-1/3 \langle K^0 \pi^- | \hat{O}_4 | \pi^- \rangle$ as a function of m^2 at $\beta = 6$ and $\beta = 6.2$ are reported in Figs. 10a and 10b respectively. Only a linear fit in m^2 on the points corresponding to the two largest values of the quark masses has been done. With the present accuracy, we cannot exclude sizeable corrections of order m^4 or effects of order a such that the matrix element does not really vanish as $m \rightarrow 0$. Nevertheless this last effect is probably unimportant since the range of masses is quite high.

Given that off-diagonal subtractions are only needed for $K-\pi$ amplitudes, and that meson masses are never very small, we consider this comparison an excellent test of the renormalization of our operators and of the approach to the chiral limit. From the Table we see that soft-pion relations are satisfied at the 30% level, as expected, and confirm, for the $\Delta I = 3/2$ the result given in Eq. (4.6). We think that the test of soft pion theorems is a very powerful way to control the renormalization of the relevant operators on the lattice and that the results in Table 2 confirm the analysis of Refs. [3,4]. A similar test should also be performed in the more complicated case of $\Delta I = 1/2$ transition amplitudes. We have already checked that similar relations are satisfied, within large errors, among the eight diagrams for $K-\pi$ and $K-\pi\pi$ transitions, in the case of the operators O_1 and O_2 which contributes to $\Delta I = 1/2$.

6. - CONCLUSIONS AND OUTLOOK

In this paper we have presented a calculation of the B-parameter for the kaon system and of the $K\pi$ and $K\pi\pi$ $\Delta I = 3/2$ matrix elements. In the latter case, we have found good agreement with expectations based on soft pion theorems and chiral Lagrangian predictions. However, as in other calculations of the same quantity [19], the rate that we have calculated for $\Delta I = 3/2$ transitions is larger than the observed one. Bearing in mind the uncertainty on the determination of a^{-1} ($\approx 20\%$) and the fact that all the results have been obtained in the so-called quenched approximation, the discrepancy is not very important.

It has been recently proposed [20] that electro-penguin can give a sizeable contribution in reducing the $\Delta I = 3/2$ amplitude. The coefficient of the electro-penguin is rather small ($\sim \alpha_{em}/\alpha_s$) and it is also suppressed by charge factors and Clebsch-Gordan coefficients. However, its matrix elements are large and have a sign opposite to that of the matrix element of O_4 ^{*)}. A reduction of about $20\pm 30\%$ to the rate predicted using only O_4 is expected because of the presence of the electro-penguin. This brings practically in agreement our theoretical prediction with the experimental value. In view of the electro-penguin contribution to the $\Delta I = 3/2$ transition, the determination of the B-parameter in Ref. [7] should be reconsidered.

ACKNOWLEDGEMENTS

We thank all the members of the Computer Centres of Rutherford Appleton Laboratory, of the CCVR of the Ecole Polytechnique and of CINECA for continuous help and assistance in developing and running the programs necessary to carry out these calculations. The support from CNRS, INFN and SERC is gratefully acknowledged. We have benefited during this work from continuous discussions with G.C. Rossi and M. Testa. We thank M.P. Paciello and B. Taglienti for their calculation of the lattice spacing from the proton mass.

*) The first computation of the "eight diagrams" for left-right operators, which are relevant for the electropenguin, were presented in Refs. [1,2]; for a recent determination see, for example, Ref. [14] and references therein.

B-parameter		
Linear fit	Quadratic fit	
0.81 ± 0.16	0.75 ± 0.20	$\beta = 6$ 30 configurations
0.88 ± 0.20	0.60 ± 0.20	$\beta = 6.2$ 15 configurations

TABLE 1

Results for the B-parameter obtained from a linear and a quadratic fit to the matrix elements of $0^{\Delta S=2}$. We have used the formula

$$\langle 0^{\Delta S=2} \rangle / Z_5 = \alpha + \beta x^2 + \gamma y^2 + \delta x^2 y^2 + \epsilon x^4$$

where $x = 8/3 [(\langle 0 | A_0 | \bar{K}^0 \rangle)^2 / Z_5]$; $y^2 = -x^2$ for the matrix element $\langle 0 | 0^{\Delta S=2} | K^0 \bar{K}^0 \rangle$ and $y^2 = x^2$ for $\langle K^0 | 0^{\Delta S=2} | \bar{K}^0 \rangle$. In the linear fit δ and ϵ are put equal to zero. The B-parameter is then given by the relation $B = \gamma Z_A^{-2}$.

$g_{3/2} a^2$		
$K^+-\pi^+$	$K^0-\pi^+\pi^0$	
$(2.0 \pm 0.8)10^{-2}$	$(1.8 \pm 0.4)10^{-2}$	$\beta = 6$ 15 configurations
$(5.9 \pm 1.2)10^{-3}$	$(5.0 \pm 1.2)10^{-3}$	$\beta = 6.2$ 15 configurations

TABLE 2

The coupling $g_{3/2}$ defined in Eqs. (5.3) as computed from the $K-\pi$ and $K-\pi\pi$ matrix elements. Notice the good agreement between the two values both at $\beta = 6$ and $\beta = 6.2$. At $\beta = 6$ we have used a set of 15 configurations, for which the $S^{(1)}$'s were available. From these configurations, one finds $(\langle K^+ | H_w | \pi^+\pi^0 \rangle / m_K) \approx 11 \div 12 \times 10^{-8}$ using Eq. (2.7). Moreover, only a linear fit has been performed since only three values in K were available. For this reason the values reported in this Table would give a larger $\Delta I = 3/2$ transition amplitude than the one reported as final result in Eq. (4.6).

REFERENCES

- [1] N. Cabibbo, G. Martinelli and R. Petronzio - Nucl.Phys. B244 (1984) 381.
- [2] R.C. Brower, B.M. Gavela, R. Gupta and G. Maturana - Phys.Rev.Lett. 53 (1984) 1318.
- [3] M. Bochicchio et al. - Nucl.Phys. B262 (1985) 331.
- [4] L. Maiani, G. Martinelli, G.C. Rossi and M. Testa - Phys.Lett. 176B (1986) 445; Nucl.Phys. B289 (1987) 505.
- [5] A first determination was given by:
L. Maiani and G. Martinelli - Phys.Lett. 181B (1986) 344.
- [6] A.J. Buras, W. Slominsky and H. Steger, Nucl. Phys. B238 (1984) 529.
- [7] J.F. Donoghue, E. Golowich and B.R. Holstein - Phys.Lett. 119B (1982) 412.
- [8] A. Pich and E. de Rafael - Phys.Lett. 158B (1985) 477.
- [9] W.A. Bardeen, A.J. Buras and J.M. Gerard - MPI Report MPI/PAE/PTH 22/87.
- [10] Preliminary results on $B_{K^0-\bar{K}^0}$ and f_{K^0}/f_{π} have been presented by O. Pène at the EPS Conference on High Energy Physics, Uppsala, June 15 - July 1, 1987.
- [11] G. Martinelli - Phys.Lett. 141B (1984) 395.
- [12] C. Bernard et al. - UCLA/87/TEP/18 (May 1987).
- [13] C. Bernard et al. - Phys.Rev.Lett. 55 (1985) 2770.
- [14] C. Bernard et al. - Talk given at the Int. Symposium on Field Theory on the lattice, Seillac (France), Sept.28-Oct. 3 (1987).
- [15] G. Martinelli and C.T. Sachrajda - Phys.Lett. 196B (1987) 184.
- [16] This is compatible with the measurements of the string tension in the same range of β ; see for example:
D. Barkai, K.J. Moriarty and C. Rebbi - Phys.Rev. D30 (1984) 1293.
- [17] L. Maiani and G. Martinelli - Phys.Lett. 178B (1986) 265.
- [18] B. Meyer and C. Smith - Phys.Lett. 123B (1983) 62;
G. Martinelli and Z.C. Zhang - Phys.Lett. 123B (1983) 433;
R. Groot, J. Hoek and J. Smith - Nucl.Phys. B237 (1984) 111.
- [19] M.A. Shifman, A.I. Vainshtein and V.J. Zakharov - Sov.Phys. JETP 45 (1977) 670.
- [20] L. Maiani and G. Martinelli - in preparation.

FIGURE CAPTIONS

- Fig. 1 $m_V a$ is reported against $(m_{PS} a)^2$ ( ); the points have been obtained from 30 configurations at $\beta = 6$ on a $20 \times 10 \times 10 \times 40$ lattice. The line is a fit to this points of the form:

$$(m_V a) = A(m_{PS} a)^2 + B$$
From $m_V a|_{m_{PS} a=0}$ we find $a^{-1} \approx 2.2$ GeV. Full dots and squares refer to the experimental values for the following combinations of masses (m_ρ, m_π^2) , (m_{K^*}, m_K^2) and $(m_\phi, m_{\eta_\phi}^2 = 2m_K^2 - m_\pi^2)$ assuming $a^{-1} = 2.2$ GeV and $a^{-1} = 1.8$ GeV respectively. Bearing in mind possible effects due to the quenched approximation, a larger value of a^{-1} is neatly preferred.
- Fig. 2 Typical "eight" diagram; the four-fermion operator O is at the origin and the two pseudoscalar sources P_5 create (annihilate) pseudoscalar mesons at different points on the lattice.
- Fig. 3a,b The circle represents the time on a periodic lattice. The operator is at the origin and with the time position shown in the Figure, the three-point correlation function of Eq. (3.12) is dominated by the matrix element of the operator sandwiched between a kaon and a pion (a), or by the matrix element $\langle 0|O|\pi\bar{K}\rangle$ (b).
- Fig. 4a,b The ratio $R = Z_5 G(n_{t_1}, n_{t_2}) / (G_K(n_{t_1}) G_\pi(n_{t_2}))$ as a function of n_{t_1} , at $\beta = 6$ and $K = 0.1515$ (4a) and $K = 0.1545$ (4b); $t_\pi = m_{t_2} = 7$.
- Fig. 5a,b R (as in Figs. 4a,b) as a function of n_{t_1} , at $\beta = 6.2$ at $K = 0.1500$ (5a) and $K = 0.1520$ (5b), $t_\pi = n_{t_2} = 12$.
- Fig. 6 a),b),c),d) Relevant Feynman diagrams for the computation of the $K-2\pi$ matrix elements of O_4 .
- Fig. 7 The use of $S^{(1)}$ reduces the diagrams in Fig. 6a-d to "eight diagrams" where one of the usual quark propagator S is replaced by $S^{(1)}$ (indicated in the Figure by a double line).

Fig. 8 $\langle 0 \rangle / Z_5$ as a function of $8/3[(\langle 0 | A_0 | \bar{K}^0 \rangle)^2 / Z_5] = 8/3(f_{K\bar{K}}^2 m_K^2 / Z_A^2 Z_5)$ at $\beta = 6.2$ on the $16^3 \times 48$ lattice. The points correspond to the matrix elements $\langle 0 | 0^{\Delta S=2} | \bar{K}^0 \bar{K}^0 \rangle$, i.e. $p_K \cdot p_K = -m_K^2$ on the left-hand side, and $\langle K^0 | 0^{\Delta S=2} | \bar{K}^0 \rangle$, i.e., $p_K \cdot p_K = m_K^2$ on the right-hand side. Six different combinations of quark masses corresponding to $K = 0.1500, 0.1510$ and 0.1520 have been used.

Fig. 9 $K-\pi\pi$: the ratio R in Eq. (3.20) is given as a function of t_{π^+} . The two plateaus satisfy approximately the relation $\langle K^0 \pi^+ | 0_4 | \pi^- \rangle \approx -1/3 \langle K^0 \pi^- | 0_4 | \pi^- \rangle$. The plateau corresponding to the last case is always better at all values of K and β . This plateau has been used to check soft-pion relations.

Fig. 10a,b The quantity $g_{3/2} m^2 / f$, defined in Eqs. (5.3) and derived from the matrix element $-1/3 \langle K^0 \pi^- | 0_4 | \pi^- \rangle$ is reported at $\beta = 6$ and $\beta = 6.2$ as a function of the pseudoscalar meson mass $m_K^2 = m_\pi^2 = m^2$.

Fig. 1

Fig. 2

Fig. 3a

Fig. 3b

Fig. 4a

Fig. 4b

Fig. 5a

Fig. 5b

Fig. 6a

Fig. 6b

Fig. 6c

Fig. d

Fig. 7

Fig. 8

Fig. 9

Fig. 10a

Fig. 10b