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COMPOSITE GAUGE FIELDS AND HIGHER-DERIVATIVE COUPLINGS
IN $N = 2$ SUPERSYMMETRY

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ABSTRACT

An intimate connection is revealed between the harmonic superfield potentials of $N = 2$ matter and $N = 2$ gauge superfields. Every hyper-Kähler or quaternionic potential (in the flat or curved cases respectively) gives rise to a composite $SU(2)$ gauge superfield, the hyper-Kähler or quaternionic invariances being realized as gauge transformations of the latter. Possible implications for construction of higher-derivative matter couplings in $N = 2$ supersymmetry are discussed.

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1. In $N = 1$ supersymmetry (SUSY), there exists an interesting analogy between the matter Kähler potential $K(\phi, \bar{\phi})$ (ϕ are chiral $N = 1$ superfields) and the Maxwell gauge superfield $\mathcal{V}$. The Kähler transformations

$$K'(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}) \quad (1.1)$$

look very similar to the gauge transformations of $\mathcal{V}$. So $K(\phi, \bar{\phi})$ can be regarded as a composite Maxwell superfield and the Lagrangian of rigid $N = 1$ matter as the corresponding Fayet-Iliopoulos term.

This analogy has recently been exploited in Ref. [1] for the construction of higher-derivative self-couplings of matter in $N = 1$ supergravity (SG). Such couplings are predicted by the superstring models in the four-dimensional point-field limit; it is therefore extremely important to know their general structure. As has been shown in Ref. [1], the requirement of Kähler invariance severely restricts the possibilities. The Kähler potential may enter either into a fixed combination with the $N = 1$ SG compensators and/or via its own Maxwell field strength.

It is interesting to see what happens in the $N = 2$ case, where the analogues of the Kähler potential and transformation (1.1) can be defined. These are, respectively, the hyper-Kähler [2,3] and quaternionic [4] potentials and gauge transformations, depending on which case, flat or curved, is considered [5,6]. One may expect that the invariances under these transformations also restrict the structure of higher-derivative interactions in $N = 2$ SUSY and have a non-trivial origin in the appropriate superstring models.

As a first step in attacking this problem, we demonstrate in the present note that the $N = 1$ analogy between the Kähler potential and the Maxwell gauge superfield can be carried over to the $N = 2$ case. For any hyper-Kähler or quaternionic potential, one may construct a composite $SU(2)$ Yang-Mills (YM) superfield, so that the relevant invariances are realized by some restricted $SU(2)$ gauge transformations. This opens up a way to analyze the problem of higher-derivative couplings in $N = 2$ SUSY on the same grounds as in the $N = 1$ case.

2. We use the harmonic superspace approach [7] as the most adequate to $N = 2$ SUSY, and start with the rigid case. The most general off-shell $N = 2$ matter action containing no more than two derivatives on the physical fields can be written as the integral over the $N = 2$ analytic harmonic superspace $\{\mathcal{Z}^M, u\} = \{x_A^m, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm}\}$ [2,3]:

$$S_Q = \frac{1}{2\xi^2} \int d^3z^{(+4)} du \mathcal{L}^{(+4)}(Q^+, D^{++}Q^+, u^\pm), [\xi] = cm^2 \quad (2.1)$$

$$\mathcal{L}^{(+4)} = H^{+a}(Q^+, u) D^{++}Q^+_a + V^{(+4)}(Q^+, u), a=1, \dots, n. \quad (2.2)$$

Here $Q^{+a} = Q^{+a}(u)$ are analytic hypermultiplet superfields, $D^{++} = u^+(\partial/\partial u^-) - 2i\theta^+ \not\partial \bar{\theta}^+$ is the harmonic derivative and $H^{+a}, V^{(+4)}$ are arbitrary functions of their arguments, including the explicit harmonics $u^{\pm i}$ (for more details see Refs. [7], [2] and [3]).

The object $\mathcal{L}^{(+4)}$ is called the hyper-Kähler potential [2,3] and codes all the information about the relevant hyper-Kähler manifold of physical bosons. It is defined up to a transformation

$$\mathcal{L}^{(+4)'} = \mathcal{L}^{(+4)} + D^{++}\Lambda^{++}(Q^+, u) \quad (2.3)$$

which is the direct analogue of (1.1) and is called the hyper-Kähler transformation. The geometric meaning of the latter was revealed in Ref. [3].

We wish to show that (2.3) can be represented as a restricted SU(2) YM transformation. To this end, we define the object

$$\mathcal{V}^{(+2)ij} = u^{-i}u^{-j}\mathcal{L}^{(+4)} - u^{+i}u^{+j} \quad (2.4)$$

(the minus sign before the second term is absolutely essential!). Making use of the basic properties of the harmonics $u^{\pm i}u^{\pm j}_i = 0$, $u^{+i}u^{-}_i = 1$, it is a simple exercise to check that $\mathcal{V}^{(+2)ij}$ transforms under (2.3) as the standard N = 2 YM superfield

$$\begin{aligned} \delta \mathcal{V}^{(+2)ij} &= D^{++}\Lambda^{ij} - \Lambda^{ik}\mathcal{V}^{(+2)}_{k\ j} - \Lambda^{jk}\mathcal{V}^{(+2)}_{\ k\ i}, \\ \Lambda^{ij} &\equiv u^{-i}u^{-j}\Lambda^{++}. \end{aligned} \quad (2.5)$$

This particular choice of gauge parameters corresponds to an Abelian subgroup of the full SU(2) gauge group; the Lie bracket of the two transformations (2.5) vanishes. This explains why the action (2.1) can be written as the N = 2 Fayet-Iliopoulos term

in complete analogy with the $N = 1$ case.

3. Let us turn to the curved case. Our consideration will be based on the version of Einstein $N = 2$ SG with the analytic q^+ compensator [8,9]. This version involves an infinite number of auxiliary fields, and is the only one to allow the most general off-shell coupling of $N = 2$ matter.

The action (2.1) is replaced by [9]:

$$S_Q = -\frac{1}{2\kappa^2} \int d\tilde{z}^{(+4)} du \omega^2 \left[\tilde{H}^{(+4)}(z, u) - \frac{\kappa^2}{\xi^2} \mathcal{I}^{(+4)}(Q^+, D^{++} Q^+, w^\pm) \right]; \quad (3.1)$$

$$\begin{aligned} \mathcal{I}^{(+4)} &= H^{+a}(Q, w) \mathcal{D}_w^{++} Q_a^+ + V^{(+4)}(Q^+, w), \\ \tilde{H}^{(+4)} &= H^{(+4)} - \mathcal{D}_u^{++} N^{++} - (N^{++})^2, \quad w^{-i} = u^{-i}, \quad w^{+i} = u^{+i} - N^{++} u^{-i}, \\ \mathcal{D}_w^{++} &= \mathcal{D}_u^{++} + N^{++} \mathcal{D}^0, \end{aligned} \quad (3.2)$$

where κ is the Einstein constant. The meaning of the entities in these equations was explained in Refs. [9] and [4]. We recall that ω and N^{++} are the $u^\pm$ -projections of the q^+ compensator and are designed to compensate the co-ordinate transformations of the analytic superspace measure and the local $SU(2)$ transformations respectively. The quantities Q^{+a} , H^{+a} , $V^{(+4)}$ have zero superconformal weights, and the harmonics $w^{\pm i}$ and the derivative $\mathcal{D}_w^{++}$ are inert under the $N = 2$ SG group. This ensures manifest invariance of the expression within the square brackets in (3.1).

The object $\mathcal{I}^{(+4)}$ has the meaning of a quaternionic potential [4], since the physical bosons of $N = 2$ matter in the curved space parametrize quaternionic manifolds [6]. The curved analogue of the hyper-Kähler transformation (2.3), as well as the transformation properties of the different entities in (3.1) and (3.2) under arbitrary $w^\pm$ -dependent reparametrizations of Q^{+a} , were defined in Ref. [4]. A characteristic feature of these transformations is that they also act on the compensator superfields. For our purpose, we need to know the transformations of $\tilde{H}^{(+4)}$, $\mathcal{I}^{(+4)}$ and the harmonics $w^{\pm i}$:

$$\delta_\varphi \tilde{H}^{(+4)} = 0, \quad \delta_\varphi w^{\pm i} = 0, \quad \delta_\varphi \mathcal{I}^{(+4)} = (H^{+a} \mathcal{D}_w^{--} \varphi_a^+) \left[\tilde{H}^{(+4)} - \frac{\kappa^2}{\xi^2} \mathcal{I}^{(+4)} \right] \quad (3.3)$$

$$\delta_\Lambda \tilde{H}^{(+4)} = \frac{\kappa^2}{\xi^2} \mathcal{D}_w^{++} \Lambda^{++}, \quad \delta_\Lambda \mathcal{Z}^{(+4)} = \mathcal{D}_w^{++} \Lambda^{++} - (\partial_w^- \Lambda^{++}) [\tilde{H}^{(+4)} - \frac{\kappa^2}{\xi^2} \mathcal{Z}^{(+4)}]$$

$$\delta_\Lambda w^{-i} = 0, \quad \delta_\Lambda w^{+i} = \frac{\kappa^2}{\xi^2} \Lambda^{++} w^{-i}. \quad (3.4)$$

The parameters $\rho^{+a}(Q^+, w)$ correspond to reparametrizations, while $\Lambda^{++}(Q^+, w)$ correspond to the quaternionic transformations. Note that in these formulae $\partial_w^- = w^{-i}(\partial/\partial w^{+i})$ acts only on the harmonics $w^{\pm i}$ appearing explicitly.

To generalize the construction of Section 2, we need to make one preliminary step. Let us notice that the compensator ω appears only in the matter action and has no kinetic term, so one can vary with respect to it. The result is

$$\tilde{H}^{(+4)} = \frac{\kappa^2}{\xi^2} \mathcal{Z}^{(+4)}. \quad (3.5)$$

This equation is covariant under all the symmetries present in (3.1), and still places no restrictions on the matter superfields Q^{+a} which, as before, are off-shell and unconstrained.

Upon inserting (3.5) into the action (3.1), the latter vanishes. The matter self-couplings now appear from the pure SG action because $H^{(+4)}$ is the $++$ component of the harmonic vielbein and it is expressed in terms of matter superfields. It is instructive to give $H^{(+4)}$ explicitly (in the gauge $N^{++} = 0$):

$$H^{(+4)} = \frac{\kappa^2}{\xi^2} \left(1 - \frac{\kappa^2}{\xi^2} H^{+a} \partial_u^- Q_a^+\right)^{-1} [H^{+a} \hat{\mathcal{D}}_u^{++} Q_a^+ + V^{(+4)}] \quad (3.6)$$

where $\hat{\mathcal{D}}_u^{++}$ stands for the part of $\mathcal{D}_u^{++}$ including no $H^{(+4)}$. This mechanism of elimination of $H^{(+4)}$ is a generalization of the procedure of compensation by a non-linear multiplet [10]. Indeed, in the absence of matter, Eq. (3.5) becomes the definition of the non-linear multiplet in harmonic superspace [9].

Taking into account Eq. (3.5), the transformation laws (3.3) and (3.4) are radically simplified:

$$\delta_\rho \tilde{H}^{(+4)} = \delta_\rho \mathcal{Z}^{(+4)} = 0, \quad \delta_\Lambda \tilde{H}^{(+4)} = \frac{\kappa^2}{\xi^2} \delta_\Lambda \mathcal{Z}^{(+4)} = \frac{\kappa^2}{\xi^2} \mathcal{D}_w^{++} \Lambda^{++} \quad (3.7)$$

and we are prepared to write down the curved space analogues of Eqs. (2.4) and (2.5). These are

$$v^{(+2)ij} = u^{-i} u^{-j} \mathcal{L}^{(+4)} - c w^{+i} w^{+j}, \quad c = \left(1 - \frac{\kappa^2}{\xi^2}\right)^{-1}; \quad (3.8)$$

$$\delta_\Lambda v^{(+2)ij} = \partial_u^{++} \Lambda^{ij} - \Lambda^{ik} v^{(+2)}_{k j} - \Lambda^{jk} v^{(+2)}_{k i}, \quad \Lambda^{ik} \equiv u^{-i} u^{-k} \Lambda^{++}; \quad (3.9)$$

$$\delta_\theta v^{(+2)ij} = \delta_{SG} v^{(+2)ij} = 0. \quad (3.10)$$

The appearance of a non-trivial numerical coefficient in Eq. (3.8) is related to the fact that the harmonics w^{+i} are transformed under the Λ^{++} gauge group. The same property accounts for the non-Abelian character of the transformation (3.9) in the curved case. The Lie bracket of two such transformations is again of the form (3.9) with

$$\Lambda^{ik}_{bz} = \frac{\kappa^2}{\xi^2} u^{-i} u^{-j} \left(\Lambda^{++}_2 \partial_w^{--} \Lambda^{++}_1 - \Lambda^{++}_1 \partial_w^{--} \Lambda^{++}_2 \right).$$

4. Now we dwell on the question of how to construct covariant field strengths for the composite YM superfields (2.4) and (3.8). Surprisingly, already at the rigid level one encounters trouble when trying to apply the standard techniques of Refs. [7] and [11].

The YM field strength in the harmonic superspace approach is expressed through the gauge superfield $v^{(-2)ij}$ [7]:

$$W^{ij} = (D^{+\alpha} D^+_\alpha) v^{(-2)ij}, \quad D^{+\alpha} = u^{+i} D^+_\alpha; \quad (4.1)$$

while $v^{(-2)ij}$ is itself expressed through the analytic connection $v^{(+2)ij}$ from the covariant equation [11]

$$D^{++} v^{(-2)ij} - D^{--} v^{(+2)ij} + v^{(+2)ik} v^{(-2)}_{k j} + v^{(+2)jk} v^{(+2)}_{k i} = 0 \quad (4.2)$$

Specifying $v^{(+2)ij}$ as in (2.4), and representing $v^{(-2)ij}$ by its $u^\pm$ -projections,

$$\mathcal{V}^{(-2)ij} = u^{-i}u^{-j}\mathcal{V} + (u^{+i}u^{-j} + u^{-i}u^{+j})\mathcal{V}^{(-2)} + u^{+i}u^{+j}\mathcal{V}^{(-4)} \quad (4.3)$$

one rewrites Eq. (4.2) as the set of equations

$$D^{++}\mathcal{V}^{(-4)} = 0 \quad (4.4a)$$

$$D^{++}\mathcal{V}^{(-2)} - \mathcal{I}^{(+4)}\mathcal{V}^{(-4)} + 1 = 0 \quad (4.4b)$$

$$D^{++}\mathcal{V} - 2\mathcal{I}^{(+4)}\mathcal{V}^{(-2)} - D^{--}\mathcal{I}^{(+4)} = 0 \quad (4.4c)$$

This set has no solution in terms of functions decomposable in usual series in the harmonics $u^{\pm i}$. Indeed, for such functions Eq. (4.4a) yields $\mathcal{V}^{(-4)} = 0$ and Eq. (4.4b) becomes

$$D^{++}\mathcal{V}^{(-2)} + 1 = 0 \quad (4.5)$$

which evidently possesses no regular solution.

The situation becomes less hopeless when recalling that analogous equations emerge in the search for the multi-instanton solutions of the self-dual YM problem in the harmonic space approach [12]^{*)}. There they are solved in terms of harmonic functions having singularities on the sphere $SU(2)/U(1)$. One may also try to proceed in a similar way in the present case. It turns out that in doing so, one gets a wide class of solutions to Eq. (4.5). For definiteness we quote the simplest particular solution,

$$\mathcal{V}_0^{(-2)} = -2 \frac{c^{--}c^{+-}}{1 + 2(c^{+-})^2} = - \frac{c^{+-}}{c^{++}} \quad (4.6)$$

where $c^{\pm\pm} = c^{ij}u_i^{\pm}u_j^{\pm}$, $c^{+-} = c^{ij}u_i^{+}u_j^{-}$ and c^{ij} is a constant $SU(2)$ vector with unit norm

$$c^{ij}c_{ij} = 1.$$

For any solution of Eq. (4.4b) and $\mathcal{V}^{(-4)} = 0$, the remaining Eq. (4.4c) can be solved (at least formally) using the harmonic distributions [7]:

*) I am indebted to Emery Sokatchev who called my attention to this possibility.

$$v = \int du_1 \frac{(u^+ u_1^-)}{(u^+ u_1^+)} \left[2 \mathcal{Z}^{(+4)}(1) v_o^{(-2)}(1) + D_1^- \mathcal{Z}^{(+4)}(1) \right]. \quad (4.7)$$

This expression correctly reproduces the hyper-Kähler transformation properties of v . When $\mathcal{Z}^{(+4)}$ transforms as in Eq. (2.3), v transforms according to the law

$$\delta v = 2 \Lambda^{++} v_o^{(-2)} + D^{--} \Lambda^{++}.$$

Nevertheless, the situation remains rather puzzling, because the covariant field strength is not expressed entirely in terms of $v^{(+2)ij}$ (contrary to the dogma tacitly confessed in Refs. [7] and [11]), but involves extra constants emerging in the process of solving Eq. (4.2). It is worth mentioning that a similar phenomenon occurs in the theory of the improved tensor $N = 2$ multiplet [13], where a triplet of extra constants appears too. These have been shown to parametrize singularities of the Dirac string type. It remains to find out whether a similar topological interpretation is possible for the constants arising in the present problem, and if one can give them any physical meaning.

All these subtleties persist when passing to the SG case. We close this section by giving the curved-space version of Eqs. (4.4) (in the gauge $N^{++} = 0$):

$$\begin{aligned} \left(\mathcal{D}_u^{++} + \frac{\kappa^2}{\xi^2} \mathcal{D}_u^{--} \mathcal{Z}^{(+4)} \right) v^{(-4)} - 2 \frac{\kappa^2}{\xi^2} c v^{(-2)} &= 0 \\ \left(\mathcal{D}_u^{++} + \frac{\kappa^2}{\xi^2} \mathcal{D}_u^{--} \mathcal{Z}^{(+4)} \right) v^{(-2)} - c^{-1} \mathcal{Z}^{(+4)} v^{(-4)} + c \left(1 - \frac{\kappa^2}{\xi^2} v \right) &= 0 \\ \left(\mathcal{D}_u^{++} + \frac{\kappa^2}{\xi^2} \mathcal{D}_u^{--} \mathcal{Z}^{(+4)} \right) v - 2 c^{-1} \mathcal{Z}^{(+4)} v^{(-2)} - \mathcal{D}_u^{--} \mathcal{Z}^{(+4)} &= 0. \end{aligned} \quad (4.4')$$

5. We have shown that the remarkable $N = 1$ analogy between the Kähler potential and the Maxwell gauge superfield [1] is generalized to the $N = 2$ case, where a correspondence exists between the hyper-Kähler or quaternionic potentials on the one hand, and the $SU(2)$ YM superfield on the other. However, the derivation of the covariant field strength for the relevant composite YM superfield, and hence the construction of higher-derivative matter self-coupling in a way consistent with the hyper-Kähler or quaternionic gauge invariances, are somewhat obscure in the $N = 2$ case. The situation could be clarified by a direct component calculation, at least for the simple example given in the previous section.

We end with two comments. Firstly, in $N = 2$ SG there is another possible origin of higher-derivative matter self-couplings respecting quaternionic gauge invariance. These may appear from the powers of the Weyl multiplet field strength $W_{(\alpha\beta)}$ in view of the relation (3.5) matching supergravity and matter fields. This is a new option as compared to the $N = 1$ case [1]. Secondly, any construction which exploits Eq. (3.5) and the YM superfield (3.8) will manifestly preserve any isometry of any particular $\mathcal{Z}^{(+4)}$. As has been shown in Ref. [4], any such isometry reduces to a specific combination of reparametrizations and quaternionic gauge transformations.

Finally, we would like to point out that the ultimate aim of this study is to establish a link with the superstring models which give rise to an $N = 2$ supersymmetric effective theory in the 4D point-field limit, e.g., of the kind discussed in Ref. [14].

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