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WEYL-GROUPS, SUPERCURRENTS AND COVARIANT LATTICES

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ABSTRACT

A systematic investigation of an orbifold inspired construction of supercurrents for string theories based on covariant lattices is presented. Using mathematical results on conjugacy classes of Weyl groups we classify the supercurrents that can be obtained, and discuss their properties. This also clarifies the intimate relation between rank 22 orbifolds and covariant lattices.

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1. Introduction

One of the unresolved issues in the covariant lattice formulation [1] of four-dimensional heterotic strings is the general construction of a supercurrent for the world-sheet supersymmetry of the right-moving sector (or both sectors of type II strings). Such a supercurrent is essential for the construction of the Lorentz-generators. Indeed, for the class of lattices built out of D_n -factors considered in [1], imposing Lorentz-invariance on the spectrum of chiral theories leads naturally to the construction of a certain supercurrent. The presence of this particular supercurrent corresponds to the “triplet constraint” of [2]. It was observed in [3,4] that the most general bosonic form of the supercurrent is

$$T_F(z) = \sum_{\vec{t}} A(\vec{t}) e^{i\vec{t} \cdot \vec{X}(z)} + i \sum_{\vec{l}} B(\vec{l}) \cdot \partial \vec{X}(z) e^{i\vec{l} \cdot \vec{X}(z)}, \quad (1.1)$$

where $\vec{t}^2 = 3, \vec{l}^2 = 1$ and $\vec{l} \cdot \vec{B}(\vec{l}) = 0$. This is only sufficient to guarantee that T_F has conformal weight $\frac{3}{2}$. However A and B have to be further constrained for T_F to have the appropriate operator product with itself. The physically interesting subset of solutions to these conditions are those that allow space-time chirality. This necessarily requires that $B = 0$ (but of course this is not a sufficient condition for chirality).

Most of the known supercurrents of the form (1.1) are obtained by bosonizing the free fermion supercurrent of [5]. This is however not the most general solution to the conditions on T_F . An example that is not of this form was given in [3] and [4]. In the first of these papers it was furthermore pointed out that this example is related to Z_3 asymmetric orbifolds [6]. This is a line of thought which we will pursue further in this paper.

We will give a systematic construction of a large class of new solutions of the form (1.1). Rather than borrowing from fermionic constructions [2], [7] we will tap the apparently richer vein of asymmetric orbifolds. In fact, one is naturally led to our construction if one tries to re-express asymmetric orbifolds in terms of covariant lattices.

An asymmetric orbifold is specified by giving an even self-dual lattice $\Gamma_{22;6}$ [8] plus a set of twists (and/or shifts). A covariant lattice theory is defined by giving an even self-dual lattice $\Gamma_{22;14}$ plus a supercurrent of the form (1.1). Our aim is to translate these two languages into each other. In general, this cannot be done, because the gauge group of a covariant lattice theory is necessarily of rank 22. We will therefore restrict our attention

to orbifolds defined by twists in the Weyl-group of the underlying Narain lattice. Such twists are guaranteed not to change the rank. Moreover, the twisted sectors of the orbifold construction can be represented by shifts in the lattice formulation, which can be used to change one self-dual lattice into another. This will be discussed in section 2.

In the formulation of [1] the untwisted sector of the asymmetric orbifold corresponds to a covariant lattice of the form $\Gamma_{22;6} \times E_8$, where the first factor is the Narain lattice, and the second represents the bosonized ghosts and fermions. The supercurrent is $\lambda^i \partial Y^i$ (where λ^i are six Majorana-Weyl fermions and Y^i represent six bosons), and can be bosonized to an expression of the form (1.1) with $A = 0, B \neq 0$. The three bosons replacing the six Majorana-Weyl fermions λ^i belong to the maximal torus of E_8 . Note that without the twisted sectors this theory is not chiral, because $B \neq 0$.

To represent a chiral orbifold theory by a covariant lattice we must replace the supercurrent by one that has $A \neq 0, B = 0$. This clearly requires a “rebosonization” of ∂Y^i . To obtain the correct conformal weight and the required form of the supercurrent these operators must be written as linear combinations of terms of the form $e^{i\vec{\alpha} \cdot \vec{X}}$, where $\vec{\alpha}^2 = 2$. The form of these rebosonization formulas is dictated by the relationship between Weyl twists and shifts [9], which in turn dictates how the lattice $\Gamma_{22;6} \times E_8$ is transformed into $\Gamma_{22;14}$. This will all be discussed in more detail in section 3.

To classify the supercurrents that can be obtained in this way, one is led to study the conjugacy classes of the Weyl groups of simple Lie groups. A lot is known about this, and we summarize the relevant material in section 4. We also prove some very useful properties of a special kind of element which we will call *non-degenerate*. These are elements that leave no direction on the torus fixed, and which will turn out to be the only ones one needs to consider for chiral theories. We prove that for such elements of the Weyl-group all “lifts” to the group are conjugate to each other and that their orbit structure is identical to that in the Weyl-group. We determine the order of such elements when acting on arbitrary representations, and the shift vector that represents them.

In section 5 we discuss how one can obtain the rebosonization formulas corresponding to these conjugacy classes. This turns out to be especially simple for so-called Coxeter elements.* We also discuss the uniqueness of this procedure. Some illustrative examples are given in section 6.

* Orbifolds with Coxeter twists have also been studied in [10] and [11].

Although our main goal is to use the asymmetric orbifold philosophy to construct new supercurrents for covariant lattices, we can also draw some conclusions regarding the relation between the two approaches. Our (perhaps slightly biased) conclusion is that working with lattices is considerably simpler than dealing with the rather complex set of written and unwritten rules that define asymmetric orbifolds. Of course the orbifold rules are complicated because they are designed for a more general situation, allowing also for gauge groups of rank less than 22. But there seems to be a clear advantage to working with lattices whenever possible. The lattice point of view is in fact helpful in clarifying some of these orbifold rules. This is discussed in section 7, where we also discuss under what conditions we are able to write covariant lattice theories in terms of (asymmetric) orbifolds. We do not have a proof that this is always possible, nor do we know of a counterexample. The relationship between orbifolds and covariant lattices has also been discussed recently in [12], with conclusions similar to ours.

Finally, in section 8 we formulate some remaining questions, in particular regarding the overlap between covariant lattices and fermionic constructions.

2. Covariant lattices and shift vectors

As explained in [1],[4], [13], the class of four-dimensional string theories we consider is determined by a lattice $\Gamma_{22;14}$ (with vectors $(\vec{v}_L, \vec{v}_R)$, where $\vec{v}_L$ has 22 components and $\vec{v}_R$ 14) and a supercurrent $T_F(z)$ of the form (1.1). They must satisfy the following conditions

1. $\Gamma_{22;14}$ must be even and self-dual with respect to a metric $(-)^{22}(+)^{14}$.
2. The last five components of every lattice vector $(\vec{v}_L, \vec{v}_R)$ must belong to one of the conjugacy classes $(0), (v), (s)$ or (c) of D_5 .
3. For every vector $\vec{t}$ or $\vec{l}$ appearing in (1.1), the lattice must contain vectors $(\vec{v}_L, \vec{v}_R)$ with $\vec{v}_L = 0$ and $\vec{v}_R = (\vec{t}, (v))$ or $(\vec{l}, (v))$.

We remind the reader that the lattice describes the momenta of all world-sheet degrees of freedom except the four uncompactified bosons X^μ . We have specialized to four dimensions, but the generalization to other dimensions is straightforward.

There are many known ways of making such lattices [1],[8],[13], [14-17]. In particular one way of constructing a new self-dual lattice Λ' from another one (denoted Λ) is to employ

a shift vector. That is, one finds a vector $\vec{s}$ so that $n\vec{s} \in \Lambda$, but $k\vec{s} \notin \Lambda$ for $1 \leq k < n$, and defines a sublattice $\Lambda_0 \subset \Lambda$ by

$$\Lambda_0 = \{\vec{\lambda} \in \Lambda \mid \vec{\lambda} \cdot \vec{s} \in Z\}. \quad (2.1)$$

By self-duality of Λ , it is easy to show that there exists a vector $\vec{r} \in \Lambda$ such that $\vec{r} \cdot \vec{s} = \frac{1}{n} \bmod 1$. We can now make a coset decomposition of Λ with respect to Λ_0 :

$$\Lambda = \sum_{k=0}^{n-1} (\Lambda_0 + k\vec{r}). \quad (2.2)$$

Suppose now that $\vec{s}^2 = 0 \bmod 2$, and consider the lattice

$$\Lambda' = \sum_{k=0}^{n-1} (\Lambda_0 + k\vec{s}). \quad (2.3)$$

Clearly Λ' is integral and even. Furthermore, since it has the same number of cosets with respect to Λ_0 as Λ , it is unimodular, and hence self-dual.

If $\vec{s}^2 \neq 0 \bmod 2$ we will assume that at least $\vec{s}^2 = \frac{2k}{n}, k \in Z$. (note that because $n\vec{s} \in \Lambda$, we know that $\vec{s}^2 = 0 \bmod \frac{2}{n^2}$). It follows that we can replace $\vec{s}$ in (2.3) by $\vec{s}_0 = \vec{s} - k\vec{r}$, so that $(\vec{s}_0)^2 = 0 \bmod 2$. In this way we again obtain an even self-dual lattice. Of course we might as well have taken $\vec{s}_0$ directly as our shift vector, but we consider the more general situation because of its closer resemblance to orbifolds.* We note in passing that if we replace $\vec{s}$ by $\vec{s} + \vec{\lambda}$, with $\vec{\lambda} \in \Lambda$, the result will be the same.

This method applies equally well to Lorentzian as to Euclidean lattices, and thus may be used to change one heterotic string theory into another, provided we respect conditions 2. and 3. listed above. One can guarantee that these conditions continue to be satisfied by restricting the shift vector as follows

- Its last five components should be weights of D_5
- Its dot-product with the constraint vectors $(\vec{t}, (v))$ and $(\vec{l}, (v))$ should be integer.

* For comparison with orbifolds one should furthermore note that for *odd* n it is sufficient that $\vec{s}^2 = 0 \bmod \frac{1}{n}$ since $\vec{s}$ can then be replaced by the equivalent shift vector $(n-1)\vec{s}$, which *does* satisfy $\vec{s}^2 = \frac{2k}{n}$.

The second condition guarantees that the constraint vectors lie on Λ_0 , and hence on Λ' . Therefore one can keep the same supercurrent. If the shift vector violates these conditions one may still be able to construct a new supercurrent for Λ' , but we know of no general procedure for doing this.

It is interesting to observe that our condition on $\vec{s}^2$ is precisely the analogue of the level matching conditions on orbifolds. Indeed, given a general shift vector $\vec{s}$, we can define a map on the conjugacy classes of Λ_0^*/Λ (where Λ_0^* is the dual of Λ_0) by

$$p_1(k\vec{s} + \Lambda) = \frac{1}{2}(k\vec{s})^2 \bmod \frac{1}{n}. \quad (2.4)$$

It is elementary to see that this is a well-defined map from Z_n to Z_n . It is required to be trivial in order to define the new lattice Λ' in Λ_0^* . This map is the lattice analogue of the first Pontryagin class.

3. Orbifold inspired covariant lattices

Rather than trying to describe the relationship between lattices and orbifolds in any detail, our purpose here is to give instead an simple “orbifold inspired” recipe for new lattice theories. The precise relation with orbifolds will in fact become evident in the remaining sections. Suppose we are given a Narain lattice $\Gamma_{22,6}$ such that the right-moving sector contains the root lattice of a rank 6 group G . In addition suppose we are given an element σ of the Weyl group of G , defined relative to a maximal torus $\partial Y^i(z)$. One could use this as the starting point for the construction of an asymmetric orbifold, in which case one may have to include further twists and satisfy level matching conditions. It is however simpler not to follow this course, but to construct directly a set of covariant lattice theories that will therefore be manifestly modular invariant [1], [18], [19].

The Weyl twist σ can always be realized as an element[†] $\tilde{\sigma}$ of G . In the Lie algebra one can define a new basis such that $\tilde{\sigma}$ lies in a maximal torus. This allows us to define a new set of bosons $\partial X^i(z)$ that are linear combinations of the ∂Y^i and vertex operators of

[†] This lift $\tilde{\sigma}$ of σ is in general not unique. This will be discussed in detail in section 4

the form $e^{i\vec{\alpha}\cdot\vec{Y}}$, in such a way that

$$\tilde{\sigma} = \exp(2\pi i \oint \frac{dz}{2\pi i z} \tilde{\sigma} \cdot \partial \vec{X}(z)). \quad (3.1)$$

The basis transformation that maps the X^i 's to the Y^i 's is generated by conjugation on the Lie algebra, and preserves all operator products. Clearly the inverse of this transformation of the Lie algebra yields an expression for the ∂Y^i in terms of the ∂X^i and vertex operators of the form $e^{i\vec{\alpha}\cdot\vec{X}}$. Manifestly from (3.1), $\tilde{\sigma}$ leaves ∂X^i fixed and performs an orthogonal rotation on the ∂Y^i 's.

We will restrict our attention to those elements of the Weyl group that have no eigenvalues equal to 1 when acting upon the Cartan subalgebra (As we will see, this is not really a restriction, but is required to get a chiral theory). Such elements will henceforth be called *non-degenerate*. For non-degenerate elements the expression for ∂Y^i in terms of X^i cannot contain any of the currents ∂X^i .

We choose a complex eigen-basis $(\partial Y^A, \partial Y^{\bar{A}})$ ($A=1,2,3$) for the ∂Y^i , such that $\tilde{\sigma}$ has eigenvalue $e^{2\pi i \theta_A}$ on ∂Y^A . In this basis ∂Y^A may be written as

$$\partial Y^A = \sum_{\vec{\alpha}} a_{\vec{\alpha}}^A e^{i\vec{\alpha}\cdot X}, \quad (3.2)$$

where $\vec{\alpha}\cdot\vec{\sigma} = \theta_A \bmod 1$ when $a_{\vec{\alpha}}^A \neq 0$.

The supercurrent, which is necessarily real, has the form

$$T_F(z) = \lambda^A \partial Y^{\bar{A}} + \lambda^{\bar{A}} \partial Y^A \quad (3.3)$$

and trivially bosonizing the fermions yields a supercurrent of the form (1.1) with $\vec{B}(\vec{l}) = 0$. By construction T_F satisfies the appropriate operator products.

Suppose we now view the momenta of the bosons $X(z)$ as lying on the original Narain lattice $\Gamma_{22;6}$. If we now define the covariant lattice $\Lambda = \Gamma_{22;6} \times E_8$ in the usual manner, then obviously the constraint vectors $(\vec{t}, (v))$ obtained from the supercurrent defined above have the form $(\vec{\alpha}, \vec{w}, (v))$, where $\vec{\alpha}$ is a root of $\mathcal{G}$, $\vec{w}$ is a vector weight of D_3 and hence $(\vec{w}, (v))$ are roots of E_8 . Therefore $(\vec{t}, (v))$ lies on Λ , and we have defined a good supercurrent for this theory.

In spite of having a supercurrent of the appropriate form, this theory is not chiral due to the presence of roots extending D_5 to $D_8 \subset E_8$. If we could find a shift vector that projects out all such roots while preserving the constraint vectors $(\vec{t}, (v))$, then we have a chance of constructing a chiral theory. Such a vector is given to us by the foregoing construction. We may choose a basis so that the vectors $\vec{t}$ are precisely

$$\pm(\vec{\alpha}_1, 1, 0, 0), \pm(\vec{\alpha}_2, 0, 1, 0), \pm(\vec{\alpha}_3, 0, 0, 1), \quad (3.4)$$

where the $\vec{\alpha}_A$ denote the sets of roots appearing in ∂Y^A . Define the right-moving part of the shift vector to be

$$\vec{s}_R = (\vec{\sigma}, -\theta_1, -\theta_2, -\theta_3, (0)^5). \quad (3.5)$$

By construction this has integer dot product with the vectors $(\vec{t}, (v))$. Moreover, the dot-product of $\vec{s}_R$ with the $((v), (v))$ conjugacy class of $D_3 \times D_5$ picks out one of the θ_A 's, and this is non-integral because σ was assumed to be non-degenerate. Thus the unwanted conjugacy classes are projected out^{*}. The shift defined by (3.5) is probably the simplest, but obviously not the most general vector that has this desirable effect.

Let N be the order of the non-degenerate Weyl twist σ . It turns out that the cyclic group generated by σ may lift to (at worst) a double cover in G , so that if $\bar{\sigma}$ is a lift of σ then $\bar{\sigma}^{2N} = 1$ on every representation (this will be discussed in detail in section 4). Therefore from (3.1) $2N\vec{\sigma}$ is a root of G , and hence lies on $\Gamma_{22;6}$. Furthermore $2N\theta_A \equiv 0 \pmod{2}$ and thus lies on the E_8 lattice. As a result there is a finite smallest k such that $k\vec{s}_R$ lies on the lattice $\Gamma_{22;6} \times E_8$ (clearly k is either N or $2N$).

In general $\vec{s}_R$ will be paired with a vector $\vec{s}_L$ on the left lattice, chosen so that a finite multiple of it lies on the lattice. The complete shift vector $\vec{s} = (\vec{s}_L, \vec{s}_R)$ must be chosen so that it satisfies the condition on $\frac{1}{2}(\vec{s})^2$ given in the previous section. A trivial solution to these conditions can be obtained if the left part of the lattice contains a copy of G and an E_8 factor. This is automatically the case if one makes an Englert-Neveu [15] type compactification of the $E_8 \times E_8$ heterotic string. This corresponds to a Narain lattice of

* If σ is degenerate at least one of the θ 's is integral, and therefore at least one of the $((v), (v))$ conjugacy classes is not projected out, so that the theory cannot be chiral. This is true independent of the absence of a ∂X in the supercurrent. We note that if there is an *odd* number of eigenvalues $+1$, then the θ_A are not strictly defined, but nevertheless an easy generalization of this argument yields the same conclusions regarding chirality.

the form $(E_8 \times E_8 \times G)_L(G)_R$, with the conjugacy classes of the weight-lattice of G paired between left and right. In this case one can simply take on the left lattice a shift vector which is identical to (3.5), with the entries θ_A embedded in one of the two E_8 factors. The resulting shift vector has lorentzian length zero, and this construction corresponds to a symmetric orbifold, at least in spirit.[†] Notice the intriguing interplay between the E_8 factors on the left and the right lattice, one of which represents gauge symmetries and the other space-time symmetries. Notice also that if the right E_8 is broken down to E_6 (which is a necessary and sufficient condition for $N = 1$ space-time supersymmetry [1]), then so is the one on the left, giving rise to the familiar $N = 1, E_6$ vacua.

Clearly this is not the only possibility. Since there is an enormous number of ways of choosing $\vec{s}_L$, one can expect to generate a very large number of new theories in this way.

4. The Weyl group and its conjugacy classes

To describe and classify all the new covariant lattice theories that can be generated by the techniques described in the previous section would be an enormous undertaking. It is however within our compass to classify all the right-moving supercurrents that can be obtained in this way. To accomplish this one must first describe all the conjugacy classes of the Weyl group. Fortunately such a description exists already in the mathematics literature [20-24]. In this section we will review the known facts, and supplement them where necessary.

4.1 THE WEYL GROUP AND ITS LIFTS

Let G denote any simple, simply laced Lie group of rank l and dimension d (many of the conclusions of this section remain valid for non-simply laced groups, but for our purposes such groups are not needed). Let $\mathcal{G}$ and $\mathcal{H}$ denote a Cartan subalgebra (CSA), and T denote the torus generated by $\mathcal{H}$ in G . Choose a set of simple roots $\vec{\alpha}_i, i = 1, 2, \dots, l$. The Weyl group W of $\mathcal{G}$ is generated by the Weyl reflections

$$r_i(\vec{\lambda}) = \vec{\lambda} - \left(\frac{2\vec{\alpha}_i \cdot \vec{\lambda}}{\vec{\alpha}_i \cdot \vec{\alpha}_i} \right) \vec{\alpha}_i. \quad (4.1)$$

Given a representation of G , there is a natural lift of W to a finite subgroup $\tilde{W}$ of G . The

[†] There does not seem to exist a generally accepted definition of symmetric orbifolds as opposed to asymmetric ones. Although we will only use the asymmetric construction, the resulting theories may very well be symmetric orbifolds according to some definitions

generators of $\tilde{W}$ are defined by

$$\tilde{r}_i = \exp\left(\frac{i\pi}{2}(E_{\alpha_i} + E_{-\alpha_i})\right) \quad (4.2),$$

where E_α denotes the usual root space basis of $\mathcal{G}$. Let H_i denote an orthonormal basis for $\mathcal{H}$. It is elementary to see that

$$\tilde{r}_i(\vec{\lambda} \cdot \vec{H})\tilde{r}_i^{-1} = (r_i(\vec{\lambda})) \cdot \vec{H} \quad (4.3)$$

and also

$$\tilde{r}_i E_\beta \tilde{r}_i^{-1} = c_i(\beta) E_{r_i(\beta)}, \quad (4.4)$$

where c_i is a complex phase. It is important to note that using the lift $\tilde{w}$ defined via (4.2) one cannot always choose the E_β so that these phases are equal to 1.

The group $\tilde{W}$ has an abelian normal subgroup D generated by the $(\tilde{r}_i)^2$, such that $W \cong \tilde{W}/D$ [25]. Since $(\tilde{r}_i)^4 = 1$ on all representations of G , and because D is abelian, it follows that D is isomorphic to some product of Z_2 factors. Moreover, if $w \in W$ satisfies $w^k = 1$, then any cover $\tilde{w} \in \tilde{W}$ of w satisfies $\tilde{w}^k \in D$ and hence $\tilde{w}^{2k} = 1$. In particular one should note that one cannot guarantee that w and $\tilde{w}$ have the same order even upon the adjoint representation of G .

The simplest illustration of this difficulty is the Weyl reflection with respect to a root $\vec{\alpha}$ of $SU(3)$. In weight space this is an element of order 2 on all roots. However it is easy to check that its lift (4.2) satisfies

$$\tilde{r}_\alpha^2 E_\beta \tilde{r}_\alpha^{-2} = -E_\beta, \quad (4.5)$$

where $\vec{\beta}$ is a root with $\vec{\alpha} \cdot \vec{\beta} = -1$.

The lift $\tilde{W}$ of W to G defined by (4.2) is by no means the only possible one. Indeed, since $W \cong \frac{N(T)}{T}$ (where $N(T)$ is the normalizer of T in G), the generators can in general be taken to be

$$\tilde{r}'_j = \tilde{r}_j \exp(2\pi i \vec{v}_j \cdot \vec{H}) \quad (4.6)$$

for any set of vectors $\vec{v}_j$. This manifestly preserves the form of (4.3) and (4.4). One should note that such a lift need not even be of finite order in G , since $\vec{v}_j$ might be incommensurate with the torus.

From the point of view of string theory the lift $\tilde{w}$ defined by (4.2) is the natural one since the alternative, (4.6), involves the string momentum generators $\vec{H}$ and thus first translates or shifts the state by $\vec{v}_j$ before twisting it with $\tilde{r}_j$. The lift $\tilde{w}$ is generated by pure twists, and we will therefore call it the *canonical* or *shiftless* lift.

In this paper we will only be interested in the lift of a single element of W , and furthermore only in non-degenerate elements. We will now show that this property reduces these subtleties to a considerable degree. Namely, if w is a non-degenerate element of order n then

1. All lifts are conjugate to each other, and hence have the same order
2. All lifts have order n on the adjoint representation. Moreover the lifts respect *all* orbits of w , in the following sense:

$$\begin{aligned} \text{if for some root } \vec{\alpha} : \quad & w^k(\vec{\alpha}) = \vec{\alpha} \\ \text{then :} \quad & \tilde{w}^k E_{\alpha} \tilde{w}^{-k} = E_{\alpha} \end{aligned} \tag{4.7}$$

3. A lift has order n on a representation characterized by a weight $\vec{w}$ if and only if

$$\frac{1}{2}\vec{w}^2 = 0 \bmod \frac{1}{n} \tag{4.8}$$

and order $2n$ otherwise.

To prove 1., consider two lifts $\tilde{w}$ and $\tilde{w}'$ of $w \in W$. Since $W \cong \frac{N(T)}{T}$, it follows that $\tilde{w}' = \tilde{w} \exp(i\vec{x} \cdot \vec{H})$. Consider a conjugation $\tilde{v} = \exp(i\vec{y} \cdot \vec{H}) \tilde{w} \exp(-i\vec{y} \cdot \vec{H})$. By definition of $\tilde{w}$, we have $\tilde{v} = \tilde{w} \exp(i\vec{z} \cdot \vec{H})$, where $\vec{z} = (w^{-1} - 1)\vec{y}$. If w is non-degenerate, $w^{-1} - 1$ is invertible, so that we can always choose $\vec{y}$ in such a way that $\vec{z} = \vec{x}$, or $\tilde{v} = \tilde{w}'$.

To prove 2. we shall construct a “phaseless” lift $\tilde{w}_0$, which satisfies

$$\tilde{w}_0 E_{\alpha} \tilde{w}_0^{-1} = E_{w(\alpha)} \tag{4.9}$$

Since by 1. every lift is conjugate to $\tilde{w}_0$, its existence would clearly prove 2. Given any lift $\tilde{w}$, we can always find a $t \in T$ so that $\tilde{w}_0 \equiv \tilde{w}t$ satisfies (4.9) on the simple roots α_i . In the

standard Cartan-Weyl basis the commutators of the generators are

$$\begin{aligned}
 [H_i, H_j] &= 0 \\
 [H_i, E_\alpha] &= \alpha_i E_\alpha \\
 [E_\alpha, E_{-\alpha}] &= \alpha^i H_i \\
 [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta} \text{ if } \vec{\alpha} \cdot \vec{\beta} = -1
 \end{aligned}
 \tag{4.10}$$

Using the fact that *any* root generator can be obtained as a string of commutators of the *simple* root generators, (4.9) would follow provided that the coefficients $N_{\alpha,\beta}$ were Weyl invariant, *i.e.*

$$N_{w(\alpha),w(\beta)} = N_{\alpha,\beta} \tag{4.11}$$

It can be proven inductively that the E_α 's can always be chosen so that this is the case. Indeed, that standard way of interrelating the $N_{\alpha,\beta}$'s is repeated use of the Jacobi-identities, and the evaluation of coefficients using the first three commutators in (4.10). These three commutators are manifestly Weyl invariant. Wherever a phase is not fixed by this procedure, one chooses them in such a way that (4.11) is respected.

Since every element of G can be conjugated into the maximal torus T , it is convenient to take an element of T as the conjugacy class representative. Such an element has the form

$$t = e^{2\pi i \vec{\sigma} \cdot \vec{H}} \tag{4.12}$$

There are (at least) two kinds of ambiguities in the definition of $\vec{\sigma}$. First of all one can change it by any conjugation which maps the torus into itself. In general, for arbitrary vectors $\vec{\sigma}$ this conjugation must be by an element of the Weyl group. Thus in any case $\vec{\sigma}$ is only determined up to Weyl rotations. One should note that there are other elements of G which keep t within the torus, namely those which commute with t . If t is a *regular* element of G then (by definition) the subgroup formed by these elements is isomorphic to the torus T . for some choices of w the lift $\tilde{w}$ may not be a regular element of G . As a result the subgroup of G commuting with w (or, equivalently, with t) may be larger than T . While this causes no ambiguity for the choice of $\vec{\sigma}$, such considerations play an important role in the construction of a supercurrent. This summarizes all the obvious ambiguities arising from conjugation within the torus. In fact it seems plausible that these are all the conjugations mapping the torus into the torus, but we have a rigorous proof only for regular elements t .

The second source of ambiguity in the shift vector is the fact that t is a torus element, so that $\vec{\sigma}$ is only determined up to shifts by lattice vectors which define the torus. Clearly, shifting $\vec{\sigma}$ by a *root* does not affect the action of t on any representation, so that this is always allowed. Furthermore, shifting $\vec{\sigma}$ by a *weight* $\vec{w}$ does not affect its action on the adjoint representation, because $\exp(i\pi\vec{w}\cdot\vec{H})$ is an element of the center of G . On the other hand such a shift introduces phases into its action on representations belonging to different conjugacy classes.

However, multiplication of t by an element of the center leads to an element of T conjugate to it, for the following reason. Let g be the element of G which conjugates $\tilde{w}$ into the torus, *i.e.*

$$\tilde{w} = g t g^{-1} \quad (4.13)$$

If we multiply t with an element $c = \exp 2\pi i \vec{w} \cdot \vec{H}$ of the center, the result of the analogous conjugation of tc is $\tilde{w}c$, because the center commutes with g . But because of 1. above, $\tilde{w}c$, which is a lift of w , is conjugate to $\tilde{w}$. Hence tc must be conjugate to t .

Thus there must exist some element $f \in G$ so that

$$f t f^{-1} = t c,$$

or, more explicitly,

$$\exp(2\pi i \vec{\sigma} \cdot f \vec{H} f^{-1}) = \exp(2\pi i (\vec{\sigma} + \vec{w}) \cdot \vec{H}).$$

Therefore

$$\vec{\sigma} \cdot f \vec{H} f^{-1} = (\vec{\sigma} + \vec{w} + \vec{r}) \cdot \vec{H},$$

where $\vec{r}$ is some vector on the root lattice. We can square both sides of this identity and take traces. Using the Cartan-Killing form and its invariance under conjugation we find then

$$\vec{\sigma} \cdot \vec{w} = -\frac{1}{2} \vec{w}^2 - \vec{r} \cdot \vec{\sigma} \pmod{1} \quad (4.14)$$

Because $\tilde{w}$ and hence t are of order n on the adjoint representation, it follows that $\vec{r} \cdot \vec{\sigma} = 0 \pmod{\frac{1}{n}}$. We can now use (4.14) to determine its order on all other representations, which can be calculated from $x = \vec{w} \cdot \vec{\sigma}$. The order n_w on a representation with weight $\vec{w}$ is the

smallest positive integer with $n_w x \in Z$. Since we know from previous arguments that n_w is either equal to n or $2n$ there are just two possibilities

- $\frac{1}{2}\vec{w}^2 = \frac{p}{n}, p \in Z$, or equivalently $n_w = n$
- $\frac{1}{2}\vec{w}^2 = \frac{p}{2n}, p \in Z, p \text{ odd}$, or equivalently $n_w = 2n$

This establishes the third claim we made above. Note that because multiplication by an element of the center is (for non-degenerate elements) equivalent to conjugation, the order of a shift vector on any representation must remain the same if a weight is added to it. Hence the order must be divisible by m_c , where m_c is the maximal value of the order of the elements of the center. For A_n this is equal to $n + 1$; for D_{2k} it is 2; for D_{2k+1} it is four and for E_6, E_7, E_8 it is 3, 2, 1 respectively.

Putting this all together we find that for the simply laced algebras the possibilities are as follows

- A_n . In this case there is only one conjugacy class of non-degenerate elements, namely the Coxeter class (see the discussion of Weyl conjugacy classes below). Its order is $n + 1$, and the norms of the fundamental weights (corresponding to tensors of rank i) are given by $\frac{i(n+1-i)}{n+1}, i = 0, \dots, n$. For *even* n , we see that on all representations the Coxeter element has order $n + 1$. For *odd* n , the order is $n + 1$ on even-rank tensors, and $2(n + 1)$ on odd-rank tensors.
- D_{4k} . Non-degenerate Weyl group elements must be of even order n and have the same order on all representations.
- D_{4k+2} . The order of all non-degenerate elements must be even. If the order is a multiple of four, then it has the same order on all representations. Otherwise it doubles on the spinors.
- D_{2k+1} . Every non-degenerate element must have an order divisible by four to be consistent with the condition on weight lengths. If it is a multiple of eight, it is the same on all representations. Otherwise it doubles on the spinors.
- E_6 . The order is always a multiple of three, and is the same on all representations.
- E_7 . The order is even. If it is a multiple of four, there is no doubling on the (56); otherwise there is.
- E_8 . The order is not restricted by these conditions, and there is no doubling .

If a conjecture we made earlier holds, we can make a slightly stronger statement about the addition of weights to shift vectors. Assuming that the possibilities for conjugation of the torus element t (conjugate to $\tilde{w}$) within the torus are just Weyl rotations, it follows that if a weight is added to a shift vector, the result must be equal to some Weyl rotation γ of that shift vector modulo the root lattice, *i.e.*

$$\vec{\sigma} + \vec{w} = \gamma(\vec{\sigma}) + \vec{r}, \quad (4.15)$$

where r denotes some vector on the root lattice of $\mathcal{G}$

Most of this does not hold for degenerate elements. For example, the most obvious degenerate element is the identity, with shift vector $\vec{\sigma} = 0$. Alternatively, one could represent it by any weight of $\mathcal{G}$, but such shift vectors are clearly not equivalent, since they have a different order.

Degenerate elements can however always be thought of as non-degenerate elements of some regular subalgebra. In that case all the above remains valid as long as one considers only shifts by weights of that subalgebra.

4.2 CONJUGACY CLASSES AND SHIFT VECTORS

A complete description of all conjugacy classes in the Weyl group of any simple Lie algebra has been given by Carter [20]. The results are summarized below

Every element of W can be expressed as a product of reflections (4.1). For a given element w denote by $l(w)$ the minimal number of reflections one needs to represent it. Then $l(w)$ is invariant under conjugation, and is less than or equal to the rank l of the Lie algebra. It is not difficult to show that the non-degenerate elements are precisely those for which $l(w) = l$.

An element of order 2 is called an *involution*. One can now prove the following theorems

- Every involution r can be expressed as a product of $l(r)$ reflections corresponding to mutually orthogonal roots
- Every element w of W can be expressed as the product of two involutions: $w = w_1 w_2$, $w_1^2 = w_2^2 = 1$, with $l(w) = l(w_1) + l(w_2)$ and $V_{-1}(w_1) \cap V_{-1}(w_2) = 0$, where $V_{-1}(w_i)$ is the set of eigenvectors of w_i with eigenvalue -1 .

In a way very similar to Dynkin diagrams, one can associate a graph with such a decomposition. Every root β appearing in the decomposition of $w = w_1 w_2$ in terms of basic reflections r_β is represented by a node. The nodes representing two roots β and γ are connected by an n -fold bond, where $n = \frac{4(\vec{\beta} \cdot \vec{\gamma})^2}{\beta^2 \gamma^2}$. In general n can be 0, 1, 2 or 3, but if the Lie algebra is simply laced (which we will assume to be the case from here on), n is either 0 or 1.

For a Lie algebra of rank l , we must consider graphs with at most l nodes. It follows from the above that these nodes must correspond to *linearly independent* roots (note that they do not have to be simple). Furthermore it must be possible to decompose this set of roots into two sets, so that the roots *within* each set are mutually orthogonal. Such a graph is called an *admissible diagram*. A new feature in comparison with Dynkin diagrams is that a graph may contain *cycles*, *i.e.* closed loops formed by nodes and bonds. Only diagrams for which each cycle contains an *even* number of nodes are admissible, since otherwise one cannot divide the roots into two sets of mutually orthogonal ones.

In [20] all admissible diagrams are constructed for every simple Lie algebra. A subclass consists of all the Dynkin diagrams of all the regular subalgebras of $\mathcal{G}$. All other diagrams contain at least one cycle. The Dynkin diagram of $\mathcal{G}$ itself can be realized by taking the product of reflections with respect to all the simple roots. This is called the *Coxeter element* of $\mathcal{G}$. Its order is equal to the Coxeter number ($n + 1$ for A_n , $2(n - 1)$ for D_n and 12, 18, 30 for E_6, E_7, E_8), which in turn is equal to the eigenvalue of the quadratic Casimir operator in the adjoint representation (or half of it, depending on normalization) and to the maximal degree of the Casimir operators of $\mathcal{G}$. The Carter diagrams that are Dynkin diagrams of regular subalgebras correspond to the Coxeter elements of those subalgebras (for semi-simple subalgebras one takes the product of the Coxeter elements of the factors.)

In general, to obtain a Weyl group element corresponding to a Carter diagram, one must first find a set of independent roots which “realize” the diagram. One decomposes these roots into two sets $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_m)$, with $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$ if $i \neq j$, and similarly for second set. Then one writes a product of reflections of the form

$$w = (r_{\alpha_1} \dots r_{\alpha_n})(r_{\beta_1} \dots r_{\beta_m}) \quad .$$

This is a Weyl group element corresponding to the diagram (the degree of uniqueness of this procedure will be discussed below). Note that the ordering of the reflections is important,

except of course within the parentheses. Coxeter elements form an exception to this rule, since it can be shown that they are conjugate to each other for any ordering.

Since all admissible diagrams of a regular subalgebra of $\mathcal{G}$ are automatically admissible in $\mathcal{G}$, it is sufficient to classify all diagrams that cannot be realized in a subalgebra. In particular it can be shown that one only has to consider connected diagrams, because all others can be realized in some subalgebra.

In [20] a list of all admissible diagrams not belonging to regular subalgebras is given for all simple Lie algebras. For A_n this list consist only of the Dynkin-diagram of A_n . For all other simply laced Lie algebras there are additional diagrams with cycles. To obtain the *complete* list of admissible diagrams, one must first of all find all the regular subalgebras of $\mathcal{G}$, which can be done by means of the extended Dynkin-diagram. In the resulting diagrams one must then consider all possible ways of replacing a connected component by a cycle-diagram of the corresponding Lie algebra.

The question naturally arises to what extent the admissible diagrams describe the conjugacy classes of $W(\mathcal{G})$. It turns out that the correspondence is not one-to-one, but it is not too far from being so, and the exceptions are listed in [20]. If w_1 is conjugate to w_2 and w_1 can be represented by a graph Γ , then so can w_2 . However, it may happen that two different graphs represent the same conjugacy class. Then every element in that class can be written in two essentially different ways as a product of reflections. It may also happen that one graph describes two conjugacy classes, *i.e.* that it has two non-conjugate realizations. Fortunately the exceptions are always degenerate elements, so that we will not encounter them.

In Figs 1. and 2. we have collected the Carter diagrams with cycles for all simply laced simple Lie algebras (of the non-simply laced algebras only F_4 has a cycle diagram). Where there are two admissible diagrams for a conjugacy class we give only one of them.

To illustrate the use of the diagrams we will consider the cycle diagrams for D_n as an example. They can be realized as follows. First remove one node, so that the cycle disappears, but the diagram is not cut. The resulting string of $n - 1$ nodes can realized by $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$, where we use the standard basis for D_n . To realize $D_n(a_{i-1})$ one adds to this set the independent root $e_i + e_{i+1}$. The resulting Weyl group element acts as follows. From the structure of the diagram one can easily see that on all roots of the form $\pm e_m \pm e_n, m, n \leq i$ it acts as the Coxeter element of $D_{i+1} \subset D_n$. Likewise, on all roots

of the form $\pm e_m \pm e_n, m, n \geq i+1$ it acts as the Coxeter element of $D_{n-i+1} \subset D_n$. On these roots the Coxeter element of D_{i+1} produces $i-1$ orbits of length $2i$, and that of D_{n-i+1} produces $n-i-1$ orbits of length $2(n-i)$ (see section 6 for further details on Coxeter elements of D_n).

To understand the action on the remaining roots, one has to work out the effect on vector weights. One finds that the vectors $e_k (k = 1, \dots, i)$ are cyclically permuted with an odd number of sign changes, so that they go through a cycle of order $2i$. Similarly, the vectors e_k with $i+1 \leq k \leq n$ go through a cycle of order $2(n-i)$. The orbits of roots of the form $\pm e_k \pm e_l (k \leq i, l \geq i+1)$ depend now on whether or not i and $n-i$ are commensurate. If they have no common factors, there are two orbits of length $2i(n-i)$, so that the total number of orbits is equal to n , the rank of D_n . If they do have factors in common, the orbits are shortened, and hence there will be more of them. For the other cycle diagrams the idea is the same, and we have worked out their orbit structure in a similar fashion. There are always at least as many orbits as the rank of $\mathcal{G}$, for reasons that should be obvious.

Because of the properties of lifts of non-degenerate elements we derived earlier, we know now also all the orbits for all lifts of such elements to G . Furthermore, since we know explicitly the rotation matrix in weight space, we know also all the eigenvalues on the CSA (the characteristic polynomials of which they are roots are in fact listed in [20]). The eigenvalues on the root generators can be obtained in the following way. For an orbit of length k generated from E_α one constructs the following linear combinations of generators

$$V_\alpha(l) = \sum_{i=0}^{k-1} \theta^{il} \tilde{w}^i E_\alpha \tilde{w}^{-i} \quad (4.16)$$

where $\theta = \exp(\frac{2\pi i}{k})$ and $\tilde{w}$ is any lift of w . Clearly these linear combinations are twist eigenvalues

$$\tilde{w} V_\alpha(l) \tilde{w}^{-1} = \theta^l V_\alpha(l)$$

Knowledge of all these eigenvalues, both on the CSA and on the root generators turns out to be sufficient to determine the shift vector $\vec{\sigma}$ belonging to w . On the adjoint representation one can obtain equivalent shift vectors by making Weyl rotations and adding weight vectors to it. This enables us to minimize the length of any shift vector by subtracting weights, and Weyl-rotate it into the positive Weyl chamber. Furthermore, if $w^n = 1$, $n\vec{\sigma}$ must lie on the

weight lattice. Thus we can specify such a “minimal” shift vector by giving its dot-products σ_i with the simple roots. These dot-products must be non-negative rational numbers of the form $\frac{p}{n}$, for some $p \in \mathbb{Z}$. Thus we have

$$\begin{aligned}\sigma_i &\equiv \vec{\sigma} \cdot \vec{\alpha}_i \\ \sigma_i &\geq 0 \\ \sigma_i &= \frac{p_i}{n}, p_i \in \mathbb{Z} \\ \sigma_i G_{ij} a_j &\leq \frac{1}{2} a_i G_{ij} a_j,\end{aligned}\tag{4.17}$$

where α_i are the simple roots, a_i the Dynkin-labels of some weight and G_{ij} the inverse Cartan matrix. The last inequality follows from the requirement that adding a weight should not diminish the length of $\vec{\sigma}$, and it is easily seen to limit σ_i to values below $\frac{1}{2}$.

One can now try to find $\vec{\sigma}$ by checking the finite set of possible choices for σ_i , by comparing their dot-products with all the roots with the twist eigenvalues determined above. This method turns out to produce a unique answer in all simple algebras with rank not larger than eight (which are the only ones we have checked), so that the resulting vector must be the shift vector. We have tabulated the result, together with other information about non-degenerate elements in table I. The shift vectors for E_6, E_7 , and E_8 have also been obtained in [24]. Their results agree with ours where they are directly comparable.

It is instructive to compute the length of the shift vector. By explicit calculation for all cases we have studied we find that the following identity holds^{*}

$$\frac{1}{2}(\vec{\sigma})^2 = \frac{1}{4} \sum_{i=1}^l \tau_i(1 - \tau_i),\tag{4.18}$$

where τ_i is related to the eigenvalues of w on the torus; they are rational numbers, chosen so that $0 < \tau_i < 1$, in terms of which the torus eigenvalues of w are $e^{2\pi i \tau_i}$ (obviously $n\tau_i \in \mathbb{Z}$). This is what one might have expected, since in physical terms this identity states that the zero point energy for the “shifted” theory is equal to the zero point energy for the “twisted” theory. Note that (4.18) is an exact equality, not just modulo $\frac{1}{n}$. This is a consequence of the fact that both sides of the equation represent the exact minimum of the energy. Replacing

^{*} Note that we count an eigenvalue as well as its complex conjugate, so as to be careful with unpaired -1 eigenvalues.

$\vec{\sigma}$ by $\vec{\sigma} + \vec{w}$ leads to a completely equivalent shift vector, if $\vec{w}$ is a weight of the algebra in which w is non-degenerate. For such a shift vector (4.18) holds only up to multiples of $\frac{1}{n}$. If $\vec{\sigma}$ is part of a shift vector which satisfies level matching, then adding a weight to it will clearly not upset the matching. The fact that the matching of the zero point energy is no longer exact simply means that there must exist another lattice vector on the same lattice for which (4.18) is satisfied. This is in fact explicitly true if, as we expect, (4.15) holds. The true vacuum in the twisted/shifted sector is then always obtained by subtracting root vectors from the shift vector.

We find it somewhat unsatisfactory that we had to obtain (4.18) by explicit calculation. In the next subsection we show how one can derive it in a purely group theoretical way for Coxeter elements. We suspect that a similar proof exists for the other conjugacy classes as well, and that finding it might enhance ones insight in the relation between twists and shifts.

For future purposes it will be useful to know the *invariant subgroup* K_w belonging to a non-degenerate Weyl group element. This, by definition, is the group of elements of G which commute with $\bar{w}$:

$$K_w = \{g \in G | g\bar{w}g^{-1} = \bar{w}\}. \quad (4.19)$$

When one chooses a conjugacy class representative w expressed in terms of basic reflections (4.1), then one sees clearly that the dimension of the Lie algebra $\mathcal{K}_w$ of K_w is equal to the number of orbits of w on the root system (at least if w is non-degenerate). When we conjugate w into a torus element t we can identify the generators E_α of $\mathcal{K}_w$ by finding those roots $\vec{\alpha}$ which have integer dot product with $\vec{\sigma}$. This set includes in any case all simple roots $\vec{\alpha}_i$ with $\sigma_i = 0$, and all roots that are linear combinations of them. In the Dynkin diagram of $\mathcal{G}$ one identifies this subalgebra of $\mathcal{K}_w$ by keeping all nodes with $\sigma_i = 0$ and the bonds connecting them, and discarding all other nodes. Then $\mathcal{K}_w$ is the non-abelian Lie algebra whose Dynkin diagram one obtains, multiplied with as many $U(1)$ factors as are needed to get $\text{rank}(\mathcal{K}_w) = \text{rank}(\mathcal{G})$. Although *a priori* one is not certain that this is all of $\mathcal{K}_w$, it turns out (by checking its dimension with the number of orbits) that in all cases it is.

If the roots of $\mathcal{G}$ appear on a lattice as a subset of a larger root system, then the invariance algebra is in general enlarged. To find the enlarged invariance algebra, one can write the minimal shift vector in terms of the simple roots of the algebra in which $\mathcal{G}$ is embedded, and apply the procedure explained above.

4.3 COXETER ELEMENTS

Up to now we have considered all conjugacy classes of non-degenerate elements in the Weyl group. A lot more is explicitly known about Coxeter elements, and we will review some of the relevant facts below (we do not assume that G is simply laced here). The majority of these facts are contained in a very illuminating paper by Kostant [21]. First we need to establish some notation.

Let Δ^+ denote the set of positive roots of $\mathcal{G}$. Define

$$\vec{\delta} = \frac{1}{2} \sum_{\alpha \in \Delta^+} \vec{\alpha}. \quad (4.20)$$

Let $\vec{\lambda}_j$ denote the fundamental weights, *i.e.*

$$\vec{\lambda}_j \cdot \vec{\alpha}_i = \delta_{ij}. \quad (4.21)$$

Then it is known that [26]

$$\vec{\delta} = \sum_{j=1}^l \vec{\lambda}_j. \quad (4.22)$$

Let $\vec{\psi}$ denote the highest root of $\mathcal{G}$, and define integers q_j such that

$$\vec{\psi} = \sum_{j=1}^l q_j \vec{\alpha}_j. \quad (4.23)$$

Recall that the quadratic Casimir on the adjoint is given by

$$\frac{1}{2} \vec{\psi} \cdot (\vec{\psi} + 2\vec{\delta}) = 1 + \sum_{i=1}^l q_i = h, \quad (4.24)$$

where h denotes the Coxeter number. Given any root $\vec{\alpha}$ of $\mathcal{G}$ its level (or height) is defined by the integer

$$L(\vec{\alpha}) = \vec{\alpha} \cdot \vec{\delta} = \sum_{i=1}^l c_i, \quad (4.25)$$

where c_i are the integers such that $\vec{\alpha} = \sum c_i \vec{\alpha}_i$

Let b_k denote the number of roots at level k . Finally we define the exponents $m_i, i = 1, 2, \dots, l$ of $\mathcal{G}$ to be those integers with $m_i \leq m_{i+1}$ such that $m_i + 1$ are the degrees of all the Casimir operators of G . In particular $m_1 = 1$ and $m_l = h - 1$ (the values of $m - i$ are well-known, and can be found for example in [26]). The following facts are then established in [21].

1. Let $\omega = e^{\frac{2\pi i}{h}}$. Then the eigenvalues of a Coxeter element γ on $\mathcal{H}$ are precisely ω^{m_i} . As a consequence of reality this implies that

$$m_i = h - m_{l-i+1} \quad (4.26).$$

(Note that, as expected, γ does not have any eigenvalue equal to 1 on $\mathcal{H}$.)

2. We can decompose the roots $\vec{\alpha}$ of $\mathcal{G}$ into orbits Γ under γ . There are precisely l such orbits, denoted Γ_i , and all of these orbits have length h . As an immediate consequence we see that for any Lie algebra

$$d = l(h + 1). \quad (4.27)$$

3. There are precisely l roots $\vec{\phi}_i$ such that $\vec{\phi}_i \in \Delta^+$ but $\gamma(\vec{\phi}_i) \notin \Delta^+$. These roots generate the l orbits Γ_i described above.
4. Using (4.16) and combining this with point 1. listed above we know now explicitly all the eigenvalues of the lift $\tilde{\gamma}$ on $\mathcal{G}$. Indeed, on $\mathcal{G}$, γ has eigenvalues $\omega^j, j = 0, 1, \dots, h - 1$ with degeneracy $l + n_j$, where n_j is the number of m_i 's equal to j . For the simple Lie algebras $n_j = 0$ or 1, except that $n_{2k} = 2$ for D_{2k} .
5. The Coxeter element γ is conjugate to $\exp(\frac{2\pi i}{h}\vec{\delta} \cdot \vec{H})$, or in other words $\frac{\vec{\delta}}{h}$ is a shift vector for γ . In fact it will turn out to be the minimal shift vector.
6. The invariant subgroup corresponding to γ is isomorphic to the torus T .
7. Because $\vec{\delta}$ is a weight and $2\vec{\delta}$ is always a root the fact that $\tilde{\gamma}^h = 1$ on the adjoint and that $\tilde{\gamma}^{2h}$ on all representation is now manifest. Furthermore, if $\vec{\delta}$ should happen to be a root then one has $\tilde{\gamma}^h = 1$ on *any* representation. It is elementary to check that this is the case for $A_{2n}, D_{4n}, D_{4n+1}, F_4, E_6$ and E_8 .

The inner products $\vec{\delta} \cdot \vec{\alpha}$ taken modulo h take the values $k = 1, 2, \dots, h - 1$ with degeneracy $l + n_k$, where n_k is defined above. This can be refined further. Recall the definition of $L(\alpha)$

and the integers b_k . Then $b_1 = l$, since at level 1 we have precisely the simple roots. Moreover $b_{h-1} = 1$ and $b_h = 0$ by uniqueness of the highest root. However, we also have [21]:

$$b_k - b_{k+1} = n_k. \quad (4.28)$$

Reality of $\mathcal{G}$ implies that $b_k = b_{-k}$, and using this with (4.26) one deduces

$$b_{h-k} + b_k = l + n_k. \quad (4.29)$$

These results give us sufficiently precise information about the relation between the exponents m_i of $\mathcal{G}$ and the inner products of $\vec{\delta}$ with the roots that we may obtain $\vec{\delta}^2$ and sums of various powers of m_i in terms of h and l . The derivations are given in appendix A, and here we simply state the results

$$\sum_{i=1}^l m_i = \sum_{k=1}^{h-1} b_k = \frac{hl}{2} \quad (4.30)$$

$$\sum_{i=1}^l m_i^2 = \frac{1}{6}h(2h-1)l \quad (4.31)$$

$$\sum_{i=1}^l m_i^3 = \frac{1}{5}h^2(h-1)l \quad (4.32)$$

$$\vec{\delta}^2 = \frac{1}{12}h(h+1)l = \frac{1}{12}hd \quad (4.33)$$

In particular one can derive (4.18) .

5. Construction of supercurrents

Before describing how one can construct the supercurrents belonging to the Weyl conjugacy classes discussed above, we should point out that for many purposes they are not needed. To study the spectrum, and even to calculate Yukawa couplings, all one needs to know is the lattice. The only information about world-sheet supersymmetry that is needed is that it is consistent with the lattice. If the lattice is constructed by means of the shift vectors calculated above, then that is automatically guaranteed, but one may also construct such lattices directly.

For calculations of most scattering amplitudes, one does however need the supercurrent, and we will now describe in some more detail how one actually obtains it, and what the ambiguities are in this procedure. It might be useful to start with a summary of the problem.

We are given a CSA $\mathcal{H} \subset \mathcal{G}$ with a corresponding torus T , and a lifted non-degenerate Weyl group element $\tilde{w}$. Associated with $\tilde{w}$ there is a Lie algebra $\mathcal{K}_w^I$ invariant under it

$$\mathcal{K}_w^I = \{x \in \mathcal{G} | \tilde{w}x\tilde{w}^{-1} = x\}$$

The generators of $\mathcal{K}_w^I$ are the orbits of the root generators E_α of $\mathcal{G}$, which we will denote as $O_I(E_\alpha)$, $I = 1, \dots, \dim(\mathcal{K}_w^I)$. There is always a linear combination of these orbits which forms a new CSA $\mathcal{H}'$ with generators H_i^I :

$$H_i^I = \sum_I c_i^I O_I(E_\alpha) \quad (5.1)$$

There exists a conjugation by some element $g \in G$ which takes $\tilde{w}$ into $t \in T$, and $\mathcal{H}'$ into $\mathcal{H}$. It maps the invariance algebra $\mathcal{K}_w^I$ into the invariance algebra $\mathcal{K}_w$ of t . We can define a set of root generators E'_α with respect to $\mathcal{H}'$ which are conjugated into E_α . Thus we have

$$\begin{aligned} g\tilde{w}g^{-1} &= t \\ g\tilde{H}_i^I g^{-1} &= H_i \\ g\tilde{E}'_\alpha g^{-1} &= E_\alpha \\ g\tilde{\mathcal{K}}_w^I g^{-1} &= \mathcal{K}_w \end{aligned}$$

What we are interested in is the effect of the same conjugation on the torus generators H_i , on which $\tilde{w}$ acts non-trivially. The result is some linear combination of the root generators E_α

$$gH_i g^{-1} = P_i(E_\alpha) \quad (5.2)$$

This defines a set of functions P_i which we will call the *inverse orbits*. These are the counterparts of the orbits O_I defined above, or more precisely, of the linear combinations $c_i^I O_I$. To make the symmetry between the orbits and the inverse orbits more manifest, we

can apply an inverse conjugation to (5.2) , to get

$$H_i = P_i(E'_\alpha) \tag{5.3}$$

These inverse orbits $P(E'_\alpha)$ are the quantities appearing on the right hand side of the rebosonization identity (3.2) , in terms of which the supercurrent (3.3) can be written down immediately. Manifestly from the conjugations given above t acts on the inverse orbits exactly like $\tilde{w}$ acts on $\mathcal{H}$.

There are several potential sources of ambiguity in this procedure. First of all there is the choice of the lift $\tilde{w}$. Fortunately we know from the previous section that all such lifts are conjugate to each other, and therefore they can all be mapped to the same torus element t . Then there is the fact that t itself is not unique, but only defined up to Weyl transformations. If one chooses a Weyl rotated element one must simply apply the same Weyl rotation to all the roots appearing in the expression (5.3) for H_i . The most important source of ambiguity stems from conjugation by elements in the invariance group K_w (whose Lie algebra is $\mathcal{K}_w$). This group leaves t invariant, but it changes the functions P_i . In general, there is always the possibility of conjugation by elements of H , which does nothing more drastic than changing the phases between the terms on the right-hand side of (5.2) .

The more complicated case is $H \subset K_w$, but $K_w \neq H$. In this case there is a freedom of choice of the linear combinations of root generators that define P_i . This freedom of choice is of course nothing but a “mirror image” of the freedom we had previously to write H'_i in terms of orbits, *i.e.* the choice of the coefficients c_i^I in (5.1) .

In orbifold-like constructions these ambiguities are all irrelevant precisely because they are the result of conjugations. They amount merely to a change of basis in the Hilbert space of states, so that the supercurrent looks different in the new basis, but the spectrum and all physical processes remain unchanged. Torus compactifications provide a nice example. If a torus compactified theory admits, in addition to the standard choice (3.3) also a bosonic supercurrent (1.1) obtainable by Weyl rotation, then this does not mean that one can obtain two different string theories by choosing either one. The two theories are completely equivalent to each other.

The situation changes drastically if one adds shift vectors. Clearly, the two supercurrents admit different shift vectors: the standard supercurrent does not allow any shifts leading to a chiral theory, whereas, in general, (1.1) does (if $B=0$).

This is why the ambiguity in determining the inverse orbits (and hence the supercurrent) is important. If one uses the shift vector representing the twist from which the supercurrent was obtained, then a subgroup of G isomorphic to K_w remains unbroken, and one can relate different choices of the supercurrent by conjugation by elements of this group. One may however not be interested in correctly representing the Weyl twist, and simply consider any shift vector which is allowed by the supercurrent. In that case it is clearly important which of the possible expressions for P_i one chooses, since they may allow entirely different covariant lattices.

In many cases, and in particular for Coxeter elements, $K_w = \mathcal{H}$. Then one may change some of the phases in P_i , but not the terms appearing in it. Therefore all these conjugate P_i 's, and the resulting supercurrents, generate world-sheet supersymmetries for the same covariant lattices. Thus ambiguities arising from $\mathcal{H}$ are always irrelevant. One is in this situation if and only if the number of orbits equals the rank.

In this situation one can get a long way towards determining the supercurrent by means of the following argument. Consider two inverse orbits P_A and $P_{\bar{A}}$ which have complex conjugate shift eigenvalues, not equal to -1 (we have written them in a complex basis here, as in section 3). This means that the roots appearing in P_A are minus those in $P_{\bar{A}}$. We will assume that one wants to obtain a solution so that P_A and $P_{\bar{A}}$ are each others' complex conjugates. The rebosonization formula corresponding to P_A has the form

$$\partial Y^A = \sum_{\alpha} a_{\alpha}^A e^{i\vec{\alpha} \cdot X}.$$

If one calculates the operator product with $\partial Y^{\bar{A}}$, one will in general obtain terms proportional to ∂X , which should cancel. This leads to the condition

$$\sum_{\alpha} |a_{\alpha}^A|^2 \vec{\alpha} = 0, \tag{5.4}$$

where the sum is over all roots having the same dot product mod 1 with the shift vector. For Coxeter elements in a rank l group (and for any element that has precisely l orbits) there are always $l + 1$ such roots in the sum, if the corresponding torus eigenvalue has multiplicity one (this is always the case for Coxeter elements, except for that of D_{2n} , which has one torus eigenvalue with multiplicity two). If, as is usually the case, there is only one

linear dependency among these $l + 1$ roots, then the absolute values of the coefficients are determined (apart from the overall scale).

The $l + 1$ phases of the coefficients are not determined by this procedure. For one ∂Y^A ($A = 1$, say), l of these phases are irrelevant because of conjugation by the $U(1)$'s in $\mathcal{K}_w$, as discussed above. Furthermore an overall phase can always be absorbed in the world-sheet fermion fields multiplying ∂Y^A in the supercurrent. For the other values of A one has to fix the phases using the operator products with ∂Y^1 . If one is less lucky (in particular if the number of orbits exceeds the rank) (5.4) only gives a set of linear relations among the squares of the coefficients. Then more extensive calculations of the operator products of the ∂Y are needed to fix the coefficients (apart from the ambiguities we mentioned).

For Coxeter elements we can immediately write down two of the inverse orbits, using again [21]. The shift vector $\frac{1}{h}\vec{\delta}$ has the same inner product $\frac{1}{h}$ with all the simple roots. If these were all the roots with that inner product, we could never satisfy (5.4), because the simple roots are linearly independent. However there is one additional root, $-\vec{\psi}$, which has the same inner product modulo 1. Let $a_i, i = 1, 2, \dots, l$ be any non-zero complex numbers, and define the following elements of $\mathcal{G}$

$$\begin{aligned} z_1 &= \sum_{i=1}^l a_i E_{\alpha_i} + E_{-\psi} \\ z_l &= \sum_{i=1}^l \left(\frac{q_i}{a_i}\right) E_{-\alpha_i} + E_{\psi} \end{aligned} \tag{5.5}$$

Then z_1 and z_l are two of the inverse orbits. Moreover, the set of elements of $\mathcal{G}$ that commute with z_1 (or alternatively z_l) comprise precisely the complete set of inverse orbits. One should observe that requiring a *real* basis for the new CSA formed by the inverse orbits means that we must choose $a_i = \sqrt{q_i}$ in (5.5). In that case one observes that the coefficients satisfy (5.4), as they should. For completeness sake we have included further information about Coxeter elements in appendix B.

So far we have discussed how to obtain the rebosonization identities and the ambiguities in this procedure. There may be some further freedom in constructing the supercurrent from it. If two or more of the complex eigenvalues of the non-degenerate Weyl-twists are identical, one has the freedom of making rotations in their eigenspace before multiplying ∂Y^A with the world-sheet fermions λ^A . This freedom is very similar to the $\mathcal{K}_w$ rotations described

earlier. Indeed, if two or more eigenvalues are identical, the canonical shift vector (3.5) does not break the E_8 factor of the right lattice to D_{8-n} (for strings in $10 - 2n$ dimensions) but to $D_{8-n} \times \mathcal{K}_F$, where $\mathcal{K}_F \subset D_3$ is the subalgebra not broken by the shift vector (the subscript F indicates that it originates from the lattice factor representing the world-sheet fermions). Since in bosonic language all internal right-moving fields are bosons on equal footing with each other, it is natural to define the complete invariance algebra $\mathcal{K}_{tot} = \mathcal{K}_w \times \mathcal{K}_F$. Any supercurrent can be modified by conjugations by elements of the corresponding group. This group is at least as large as $U(1)^{3n}$, but these $U(1)$ transformations only introduce a few harmless phases in the supercurrent. If there are non-abelian factors however, the form of the supercurrent can be modified more significantly. If one transforms the supercurrent by an arbitrary element of $\mathcal{K}_{tot}$, all terms that can appear in it will indeed appear, but for special choices of the free parameters in the choice of the supercurrent, one can usually get a simpler form with fewer terms. This has the advantage that there will be fewer constraint vectors, and hence more possibilities for adding shift vectors. In particular, one can usually by adding further shifts break $\mathcal{K}_{tot}$ to $U(1)^{3n}$. There is never an advantage to writing the supercurrent in the more complicated form, since the lattices which admit such a supercurrent admit the simpler one (related by conjugation) as well.

The supercurrents for four-dimensional strings that one can obtain using this method are listed in Table 2. Enumerating them is simply a matter of listing all simply-laced rank 6 algebras, and constructing their conjugacy classes using Table 1. Non-degenerate elements of subalgebras are not separately listed in the table. For example, the Coxeter element of $D_4 D_2$ is also a non-degenerate element of D_6 . In these cases one can simply regard the root lattice of the larger algebra as that of its subalgebra with some additional weights of length 2. In this situation the invariance group of the Weyl group element will in general be enlarged.

We have not yet discussed space-time supersymmetry. In general, not much can be said about it, because one has to know the lattice. However, one can check if the shift vector preserves some of the 4 supersymmetries one already has on the Narain lattice, that is if one or more of the original gravitini remains in the invariant lattice Λ_0 (the “untwisted sector”). If not, then there is still a chance of getting gravitini from the twisted sectors, but one does not have enough control over this to give a general rule.

The condition that a gravitino (*i.e.* an E_8 spinor) is not projected out is obviously

$$\theta_1 \pm \theta_2 \pm \theta_3 = 0 \bmod 2, \quad (5.6)$$

for one of the four possible sign choices (fermions in four dimensions appear always in CPT conjugate pairs, so that a lattice vector is equivalent to its reflection through the origin). Since the twist angles θ_A are only defined modulo 1, it is in fact sufficient that the right hand side of (5.6) is an odd integer, because one can shift one of the θ 's by 1. In table 2. we have indicated whether or not that is the case (the Coxeter twists that preserve supersymmetry have also been listed in [10]). Thus for entries in the table labelled “N=1”, there exists a choice of signs in (5.6) for which the right hand side is integer. Then there are two *inequivalent* shift vectors, one which preserves space-time supersymmetry, and one that doesn't. One can easily check that a non-degenerate element can at most preserve one gravitino (which does not exclude the possibility that additional ones might come from the twisted sectors). Note that the “standard” supercurrent $D_2 D_2 D_2$ corresponding to the triplet constraint projects out all gravitini. Nevertheless many supersymmetric examples are known for this choice, illustrating the role of twisted sectors.

The reader may have observed that if we would have used the *orbits* instead of the *inverse orbits*, we would also have obtained a CSA, and hence a perfectly acceptable supercurrent. The difference is that for the latter we know a shift vector by construction, whereas for the former we do not. We will show in section 7 that for *chiral* theories, the condition that supercurrents of this form admit a non-trivial shift vector is satisfied if and only if they were obtained as the inverse orbits of some Weyl transformation. In other words, although one can construct good supercurrents from the orbits, in general one will not be able to find a chiral covariant lattice theory supporting such a supercurrent. As we will see in the next section, for $SU(N)$ the orbits and the inverse orbits are actually the same, but that is not true in general.

6. Examples

Probably the simplest re-bosonization identity can be obtained from the Coxeter element γ of A_{n-1} . The root system of A_{n-1} can be represented by $n^2 - n$ vectors $e_i - e_j, i, j = 1, \dots, n$, for which a system of simple roots is $\alpha_i = e_i - e_{i+1}, i = 1, \dots, n-1$. The highest root $\psi = e_1 - e_n$, and the Coxeter element acts by cyclic permutation of the e_i :

$$\gamma : e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow \dots \rightarrow e_n \rightarrow e_1. \quad (6.1)$$

Hence the CSA $\mathcal{H}'$ is defined by

$$H'_j = \sum_{k=1}^n (E_{e_k - e_{k+j}}), \quad j = 1, \dots, n-1, \quad (6.2)$$

where we adopt the convention that $e_i = e_{i-n}$ for $i > n$. The shift vector $\frac{\vec{\delta}}{n}$ is explicitly given by

$$\frac{1}{2n}[(n-1)e_1 + (n-3)e_2 + \dots + (1-n)e_n]. \quad (6.3)$$

Because the degrees of the Casimirs of $SU(n)$ are $2, 3, \dots, n$ the eigenvalues of γ on weight space are simply $e^{2\pi i \frac{k}{n}}$.

To manufacture a new supercurrent we need the CSA $\mathcal{H}$ in terms of the root vectors E'_α that correspond to $\mathcal{H}'$. The method for accomplishing this was described in section 5. For $SU(n)$ the coefficients q_i defined in (4.23) are all equal to one. We see therefore that two elements of $\mathcal{H}$ are given by

$$\begin{aligned} H_1 &= \sum_{j=1}^{n-1} E'_{e_j - e_{j+1}} + E'_{e_n - e_1} \\ H_{n-1} &= \sum_{j=1}^{n-1} E'_{e_{j+1} - e_j} + E'_{e_1 - e_n}. \end{aligned} \quad (6.4)$$

The remainder of the CSA consists of all elements of the Lie algebra commuting with H_1 . A moments thought shows that (6.4) is essentially the same as (6.2), but with primes in different places. It follows that the entire CSA is given by (6.2) for all values of j but with the primes moved from H_j to E_α . Indeed it is elementary to verify that $e_k - e_{k+j}$ has dot product $j \bmod n$ with $\vec{\delta}$ and hence conjugation of H_j with $\exp(\frac{2\pi i}{n} \vec{\delta} \cdot \vec{H}')$ yields a phase $\exp(\frac{2\pi i j}{n})$.

The roots of D_n can be taken to be $\pm e_i \pm e_j, i, j = 1, \dots, n$, with all choices of signs. A simple root system is $\alpha_i = e_i - e_{i+1}, i = 1, \dots, n-1$, and $\alpha_n = e_{n-1} + e_n$. The Coxeter element has order $2(n-1)$ and permutes the vectors e_i according to

$$\begin{aligned} \gamma : e_1 &\rightarrow e_2 \rightarrow \dots \rightarrow e_{n-1} \rightarrow -e_1 \rightarrow -e_2 \rightarrow \dots \rightarrow -e_{n-1} \rightarrow e_1 \\ \gamma : e_n &\rightarrow -e_n \end{aligned} \quad (6.5)$$

The shift vector is $\frac{\vec{\delta}}{2(n-1)}$, where

$$\vec{\delta} = \sum_{j=1}^{n-1} (n-j)e_j. \quad (6.6)$$

The basis H'_j of the CSA $\mathcal{H}$ can be generated by constructing the orbits of γ on the root vectors $E_{e_j+e_{n-1}}, j = 1, \dots, n-2$, and $E_{e_1 \pm e_n}$.

To derive the rebosonization identity we need the expression for H_j in terms of E'_α . In the present case $\psi = e_1 + e_2$ and $q_1 = q_{n-1} = q_n = 1$, while all other q_i 's are equal to 2. Using (5.5) with $a_i = \sqrt{q_i}$ we get then

$$\begin{aligned} H_1 &= E'_{e_1-e_2} + E'_{e_{n-1}-e_n} + E'_{e_{n-1}+e_n} + \sqrt{2} \sum_{i=2}^{n-2} E'_{e_i-e_{i+1}} \\ H_n &= H_1^*. \end{aligned} \quad (6.7)$$

To compute the elements of D_n that commute with H_1 is somewhat more complicated. Note however that for the elements with complex shift eigenvalues only the phases between the terms have to be determined.

The result for D_4 is to supplement H_1 and H_4 with

$$\begin{aligned} H_2 &= i(E'_{e_1-e_4} - E'_{e_1+e_4} - E'_{-e_1+e_4} + E'_{-e_1-e_4}) \\ H_3 &= E'_{e_1+e_4} + E'_{-e_1-e_4} + E'_{e_1-e_4} + E'_{-e_1+e_4} - 2E'_{e_2+e_3} - 2E'_{-e_2-e_3} \end{aligned} \quad (6.8)$$

7. From Covariant Lattices to Orbifolds

There is obviously a close parallel between the constructions described in the previous sections and asymmetric orbifolds. Clearly, using the procedure described before, we can associate with every Weyl transformation a covariant lattice plus a supercurrent. Because of the equivalence between twists and shifts, and the equality of the ground state energies, it follows that these two approaches describe the same theories. In this section we will reverse this philosophy and consider how to convert a covariant lattice theory into an asymmetric orbifold.

We start with a lattice $\Lambda = \Gamma_{22;14}$, and we will denote the right-moving part (which is general is not a lattice) as $\Gamma_9 \times D_5$. To turn this into an orbifold we must choose three orthonormal vectors $\vec{v}_A$ which define the fermionization of three bosonic degrees of freedom, according to

$$\begin{aligned}\lambda_A &= e^{i\vec{v}_A \cdot \vec{X}} \\ \lambda_{\bar{A}} &= e^{-i\vec{v}_A \cdot \vec{X}}\end{aligned}\tag{7.1}$$

The vectors $\vec{v}_A$ have to lie inside the nine-dimensional vector space spanned by Γ_9 , but of course they do not lie on the lattice Λ . We will require however that there is an integer N so that $N\vec{v}_A \in \Lambda$ for $A = 1, 2, 3$. This guarantees that we get an orbifold interpretation which involves only twists of finite order.

In addition to these fairly natural requirements, we have to impose two further conditions on the three vectors. Let $\{\vec{t}\}$ denote the set of vectors appearing in the supercurrent. Then we need

1. For each vector $\vec{t}$ there is precisely one $\vec{v}_A$ for which $\vec{t} \cdot \vec{v}_A = \pm 1$, while the remaining $\vec{v}_B$ are orthogonal to $\vec{t}$.
2. for each $\vec{v}_A$ there must be some $\vec{t}$ for which $\vec{v}_j \cdot \vec{t} \neq 0$

These are necessary and sufficient conditions to be able to write the supercurrent (1.1) (with $\vec{B} = 0$) in the form

$$T_F(z) = \lambda^A O^{\bar{A}} + \lambda^{\bar{A}} O^A,\tag{7.2}$$

where

$$O^A = \sum_{\alpha} a_{\alpha}^A e^{i\vec{\alpha} \cdot \vec{X}}.\tag{7.3}$$

The vectors $\vec{\alpha}$ appearing here are vectors of norm 2, obtained by adding or subtracting from each $\vec{t}$ the vector $\vec{v}_A$ with which it has non-vanishing inner product. The coefficients in (7.3) follow trivially from the coefficients $A(\vec{t})$ in (1.1) .

Condition 2 above is always satisfied as a consequence of requiring that the operator product expansion of T_F with itself produce the complete energy momentum tensor for all the nine bosons of Γ_9 . The first condition is however, as far as we can tell, only a consequence of attempting to write the supercurrent in the orbifold form. Unlike the second condition it does not seem to follow from the requirement that T_F be a good supercurrent. It is therefore by no means obvious that it is possible to satisfy this condition for a general covariant lattice theory and its supercurrent. Thus there might exist covariant lattice theories that cannot be expressed as orbifolds, but we know of no examples.

Define vectors $\vec{w}_A = (\vec{v}_A, (v))$, where (v) is a D_5 conjugacy class. They satisfy $\vec{w}_A^2 = 2$, and so may be used as in section 2 to define a new lattice. These vectors will extend D_5 to D_8 . The successive application of shifts $\vec{w}_A$ as in (2.3) for each value of A reproduces an even self-dual lattice of the form $\Gamma_{22,6} \times D_8$, where (as usual) “ $\times$ ” is not meant to indicate a direct product. Such a lattice corresponds to a Narain-type torus compactification of a non-supersymmetric ten-dimensional heterotic string. In general, there may be several ways to pick the vectors $\vec{v}_A$, and one may end up with different compactified ten-dimensional theories depending on the choice one makes. Some of these may in fact be supersymmetric, but even if D_8 is not automatically extended to E_8 , one can use a further shift by a spinor of D_8 to accomplish this. Thus under the aforementioned conditions there exists a set of shifts that turn the covariant lattice into a lattice $\Lambda' = \Gamma_{22,6} \times E_8$. Conversely, there must then be a set of even vectors $\vec{s}_A \notin \Lambda'$ that can be used to convert Λ' back into Λ .

By converting the lattice Λ to the Narain lattice Λ' by adding the vectors $\vec{w}_A$ one automatically introduces all the vectors $\vec{\alpha}$ appearing in (7.3). These roots extend the $U(1)^6$ generated by the operators ∂X to a rank 6 Lie algebra $\mathcal{G}$. From the fact that the operator product of T_F with itself must produce the complete energy momentum tensor, it follows that the roots $\vec{\alpha}$ must span the six-dimensional space, *i.e.* the algebra has no abelian factors. Moreover the operator products of $T_F(z)$ require that the $O^A(z)$ and $O^{\vec{A}}(z)$ define a new CSA for the Kac-Moody algebra based on $\mathcal{G}$.

Consider now the shift vectors $\vec{s}_A$ defined above. For each such vector define the following

element of the group G (with Lie algebra $\mathcal{G}$)

$$R(\vec{s}) = \exp(2\pi i \vec{s} \oint \frac{dz}{2\pi i z} \partial \vec{X}(z)). \quad (7.4)$$

Note that the inner product of $\vec{s}_A$ with the constraint vectors of the supercurrent $(\vec{t}, (v))$ is equal to 0 mod 1 for all $\vec{t}$ in the supercurrent. It follows that

$$R(\vec{s}_A) O_A R(\vec{s}_A)^{-1} = e^{2\pi i \phi(\vec{s}_A)} O_A, \quad (7.5)$$

where the phase $\phi(\vec{s}_A)$ compensates the phase that λ_A picks up under this transformation, so that the supercurrent is invariant. Therefore $R(\vec{s}_A)$ performs an orthogonal rotation of the CSA defined by O_A , so that it defines an element of the Weyl-group of $\mathcal{G}$, lifted to G . Furthermore all these Weyl-transformations commute.

Let us now assume that the covariant lattice theory defined by Λ is chiral. Then we can show that a product of some or all of the Weyl-transformations $R(\vec{s}_A)$ must be non-degenerate.

The main complication we have to deal with is the fact that we are using successive shifts, rather than one single shift. In general, if one applies the shift vector method successively with different shift vectors, the result depends on the order in which the shifts are made. This is obvious once one realizes that the shift vectors need not have integer inner products, in which case they cannot co-exist on the same lattice. Fortunately we are in a simpler situation here, because the shift vectors $\vec{w}_A$ *do* have integer inner products. Using this fact, one can show that the “inverse” shift vectors $\vec{s}_A$ can also be chosen to have integer mutual inner products (they can also be chosen differently, but this merely obscures the discussion). Then the shift vectors $\vec{w}_A$ lie all on Λ' , and the inverse shift vectors $\vec{s}_A$ lie all on Λ . The self-dual lattices that one obtains as intermediate steps in the successive $\vec{w}_A$ shifts contain the shift vectors $\vec{w}_A$ that one has already taken into account, but they must also contain the *inverse* shift vectors $\vec{s}_A$ corresponding to the shifts $\vec{w}_A$ still to be performed. Since these vectors appear on the same self-dual lattice, we conclude that we can choose the inverse shift vectors $\vec{s}_A$ so that

$$\vec{w}_A \cdot \vec{s}_B = 0 \text{ mod } 1 \quad \text{if } A < B. \quad (7.6)$$

Furthermore the inverse shift vectors $\vec{s}_A$ must project out (*i.e.* have non-integer inner product with) the shift vectors $\vec{w}_A$. This follows from the fact that the original covariant lattice

Λ described a chiral theory, and hence cannot contain any of the vectors $\vec{w}_A$. That means that for each A there must exist some $B \geq A$ so that $\vec{w}_A \cdot \vec{s}_B \neq 0 \bmod 1$.

Although one cannot determine all the inner-products of $\vec{w}_A$ and $\vec{s}_B$ for $A > B$ by general arguments, we do have now sufficient information to construct a shift vector $\vec{s}$ which projects out all vectors $\vec{w}_A$ simultaneously. This can be done iteratively, as follows. First choose $\vec{s} = \vec{s}_1$. This vector projects out $\vec{w}_1$, but we don't know whether it projects out $\vec{w}_2$. If it does, there is no need to change it. If it does not, define $\vec{s} = \vec{s}_1 + \vec{s}_2$, which manifestly projects out both $\vec{w}_1$ and $\vec{w}_2$. In the case of interest here (where one has at most three shift vectors) we need just one more such step to construct the vector $\vec{s}$.

Consider now for this $\vec{s}$ the corresponding shift operator (7.4). On the six-dimensional part of the lattice spanned by the roots $\vec{\alpha}$ it represents a Weyl transformation acting on the operators O_A as in (7.5), but with three non-trivial phases $0 < \phi(\vec{s}_A) < 1$. Therefore it is a non-degenerate element of the Weyl group. The supercurrent (7.2) must thus be one of those classified in this paper, listed in table 2.

In general, the shift vector $\vec{s}$ need not be sufficient by itself to recover the original lattices Λ (it is in fact sufficient if the orders of the three $\vec{w}_A$'s are incommensurate), but this is not the point. To recover the original lattice, one can always add further shifts, if necessary. The existence of a $\vec{s}$ as defined above implies that there is a non-degenerate Weyl-transformation that acts on the CSA formed by the operators O_A and $O_{\bar{A}}$ by pure shifts. Thus, by the arguments of section 4, there exists than a conjugation of $\mathcal{G}$ by some element of G which maps this CSA to the canonical CSA formed by the operators ∂X , and at the same time takes (7.4) into the canonical lift of a Weyl-group element, expressible in terms of basic reflections of the form (4.2). It follows that the operators O_A and $O_{\bar{A}}$ can be obtained from the canonical CSA by the inverse of this conjugation, *i.e.* they are inverse orbits corresponding to a non-degenerate Weyl group element.

There now remains one subtlety, that of the order n of a twist or the order m of the shift vector $\vec{\sigma}$. These orders are defined, respectively, as the smallest power of the twist eigenvalues on the torus ω_i such that $\omega_i^n = 1$ for all i , and the smallest integer such that $m\vec{\sigma}$ is on the lattice Λ . From section 4. we know

$$\begin{aligned} n\vec{\sigma} &\in \Lambda_W \\ 2n\vec{\sigma} &\in \Lambda_R \end{aligned}, \tag{7.7}$$

where Λ_W and Λ_R denote the weight lattice and root lattice of $\mathcal{G}$. It is always true that Λ

contains Λ_R , but in general we can not expect that all weights are present on the $\mathcal{G}$ subspace of Λ . Therefore we may not have $m = n$, but only $m = 2n$. Thus to construct the Hilbert space of the lattice Λ from twisted fields one may have to double the naive order of the twist in the orbifold construction and project with the properly lifted Weyl twist operator $\tilde{\sigma}$. One may avoid this problem in a different way by requiring that the self-dual lattice Λ contains none of the “dangerous” representations that double the naive order of σ . This is, in fact, precisely the content of the mod 2 condition imposed in [6].

Suppose part of $\Gamma_{22;6}$ corresponds to a Lie algebra, and that the only twist is by an element σ of the Weyl group of that algebra. In [6] one considers the invariant sublattice I and the normal sublattice N , of vectors in $\Gamma_{22;6}$ orthogonal to I . The momentum states of the new theory are drawn from the duals I^* and N^* of these lattices. These dual lattices are equal to the orthogonal projection of $\Gamma_{22;6}$ onto the spaces defined by I and N , and given any vector $\vec{v} \in \Gamma_{22;6}$ the orthogonal projection onto I is given by

$$\vec{u}_1 = \frac{1}{n} \sum_{j=0}^{n-1} \sigma^j(\vec{v}), \quad (7.8)$$

Let $\vec{u}_2$ be the orthogonal projection of $\vec{v}$ onto N , so that $\vec{u}_2 = \vec{v} - \vec{u}_1$.

Using $\vec{v} \cdot \sigma^{n-j}(\vec{v}) = \vec{v} \cdot \sigma^j(\vec{v})$ one can easily show that for twists of *odd* order n

$$\vec{u}_2^2 = 0 \bmod \frac{1}{n} \quad (7.9)$$

For *even* order one finds that the same identity holds if and only if

$$\vec{v} \cdot \sigma^{n/2}(\vec{v}) \equiv 0 \bmod 2 \quad (7.10)$$

However, we observe that N must contain the root lattice some subalgebra $\mathcal{G}'$ of $\mathcal{G}$, for which σ is a non-degenerate element of the Weyl group of $\mathcal{G}'$. The vector $\vec{u}_2$ defined above must then be a weight of $\mathcal{G}'$, because it must have integer inner product with all vectors in N . Comparing the conditions for the doubling of the order of the lift on arbitrary representations (see (4.14) and the discussion following it) with (7.9), we conclude that the shift vector $\vec{\sigma}$ corresponding to the twist σ is of order n (rather than $2n$) on the lattice if and only if (7.10)

holds. A second mod 2 condition given in [6], involving the twist eigenvalues $\exp(2\pi i \frac{\tilde{r}_i}{n})$ on the right lattice is

$$\sum \tilde{r}_i = 0 \bmod 2 \quad (7.11).$$

In bosonized form this condition is equivalent to requiring that in the E_8 factor of the right lattice representing the world-sheet fermions the shift vector must be of order n rather than $2n$. Thus it is simply the counterpart of (7.10) in the E_8 factor of the lattice.

In [6] these mod 2 conditions follow from the level matching conditions, adapted to asymmetric orbifolds. Presumably the related procedure of “taking the square root of the Hilbert space” cannot be formulated without these conditions. From our point of view there is however absolutely no problem with defining sensible string theories for such “illegal” twists. We can simply consider the complete shift vector $\vec{s}$ as a vector of order $m = 2n$ on the lattice, and proceed as before. As long as this vector satisfies the condition formulated in section 2,

$$\vec{s}^2 = \frac{2k}{m},$$

one can use it to construct a new (though not necessarily different) self-dual lattice, and hence we are certain to obtain a modular invariant theory.

It would appear that using covariant lattices we can construct new string theories that are explicitly eliminated by the rules formulated in [6]. A trivial example is the class of theories obtained with the $(D_2)^3$ supercurrent in table 2. It is easy to see that the corresponding Weyl-twist (of order 2 on the root lattice) has $\tilde{r}_i = 1$, $i = 1, 2, 3$ in the notation of [6], and clearly does not satisfy (7.11). Nevertheless, this supercurrent is the canonical and most intensively studied one, and we know that there are huge numbers of corresponding consistent string theories. Whether these theories can be obtained by orbifold constructions is only a matter of interpretation of the rules. Although they do not seem to allow some theories when taken literally, the shift vector construction we have given here is still very much in the spirit of orbifold constructions. (Observe however that the usual formulation of orbifold constructions strictly speaking does not allow us to represent the twists in the right-moving sector by shifts, because of the way world-sheet supersymmetry is preserved in orbifold constructions).

8. Conclusions

Asymmetric orbifolds obtained by twists by Weyl group elements (plus pure shifts commuting with these twists) can be written in terms of covariant lattices with a certain choice for the supercurrent. Using mathematical results on the conjugacy classes in the Weyl groups of the simple Lie algebras, we have classified all supercurrents relevant for chiral theories that can be obtained in this way. For each such supercurrent we expect a large number of covariant lattices that admit the corresponding world-sheet supersymmetry, each yielding a consistent fermionic string theory. The classification problem of these theories is still very far from being completed, and even for the most thoroughly researched choices of supercurrent ($(D_2)^3$ and $(A_2)^3$ in four dimensions) only a small tithe of the theories have been explored so far. Using the shift vectors provided in table 1 one can construct many examples of lattices for the other supercurrents.

Although our method was orbifold-inspired, our main goal was to extend the scope of the covariant lattice construction, not too study the overlap with orbifold constructions. Therefore in most of this paper we have not attempted to follow the orbifold construction very closely, since the consistency conditions for covariant lattices are (in our opinion) much easier to deal with than those of orbifolds. In particular we have not discussed in much detail how the orbifold method of taking the square root of the left-right Hilbert space relates to the shift vector method for constructing new self-dual lattices, although it should not be very difficult to make this relation more precise. Since both methods yield a consistent fermionic string theory for a given Weyl twist, we are however confident that those fermionic string theories are in fact identical. It is clear that there is a large, and perhaps complete overlap between the two approaches for the theories with maximal rank gauge groups (see also [12]).

Although all asymmetric orbifolds with Weyl-twists can be written in terms of covariant lattices, the opposite is not so obvious. We have succeeded in writing covariant lattice theories as orbifolds only if the supercurrent satisfies a condition formulated in section 7. It is not clear whether this condition holds for all supercurrents. If it does, then the present paper provides a complete classification of all bosonic supercurrents that can in principle yield chiral string theories. If it does not hold, then it appears that there are covariant lattice theories that cannot be obtained by the asymmetric orbifold construction. Even for the known supercurrents, one has to use a rather liberal interpretation of the orbifold rules

to obtain certain lattice theories. In particular the somewhat mysterious “mod 2” conditions for twists of even order are clarified considerably by adopting a lattice point of view, and turn out to be unnecessarily restrictive (at least for Weyl-twists).

Most of the supercurrents of table 2 cannot be written in terms of free fermion trilinears, which have been used in fermionic constructions of four-dimensional strings [2],[7],[5]. This should not be confused with writing the Hilbert spaces in terms of fermionic oscillators. Indeed, the arguments of [27] show that bosonic lattice theories can be written in terms of complex free fermions (albeit with complicated boundary conditions) and vice-versa. The fermionic theory will undoubtedly have world-sheet supersymmetry if the bosonic theory did, but this does not mean that the supercurrent can be written as a product of three free fermion fields. Explicit examples where this is not the case are known [3], [4]. One encounters a very similar problem if one tries to write the “spinor” root generators of $D_8 \subset E_8$ in terms of fermion bilinears [28].

At present, any maximal rank fermionic string theory that we know of, in any dimension, can be written in terms of a covariant lattice. It would be very interesting if this could be proved in general. We would then have a completely general and moreover very simple description for at least one large class of string theories, which perhaps could be used as a starting point for exploring the space of all vacua.

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APPENDIX A

Our purpose here is to derive (4.30)—(4.33). The first of these is very simple, since

$$\sum_{k=1}^{h-1} b_k$$

is the number of positive roots and this is equal to $\frac{1}{2}(d-l) = \frac{hl}{2}$ by (4.27). The second part

of the identity is a trivial consequence of

$$\sum_{k=1}^l m_i = \sum_{i=1}^l (h - m_i)$$

which follows directly from (4.26) . The remaining identities are obtained by writing $\vec{\delta}^2$ in two different ways. First because

$$\vec{\delta}^2 = \frac{1}{2} \sum_{\alpha \in \Delta^+} (\vec{\alpha} \cdot \vec{\delta})$$

and using (4.28) we obtain

$$\begin{aligned} \vec{\delta}^2 &= \frac{1}{2} \sum_{k=1}^{h-1} k b_k \\ &= \frac{1}{2} \sum_{j=0}^{l-1} [(l-j) \sum_{k=m_j+1}^{m_{j+1}} k], \end{aligned} \tag{A1}$$

with the convention that $m_0 = 0$. Performing the sum yields

$$\vec{\delta}^2 = \frac{1}{4} \sum_{j=1}^l m_j (m_j + 1) \tag{A2}$$

Alternatively one can use the well-known equality

$$h \delta_{\mu\nu} = \sum_{\alpha \in \Delta^+} \alpha_\mu \alpha_\nu \tag{A3}$$

and hence

$$h \vec{\delta}^2 = \sum_{\alpha \in \Delta^+} (\vec{\delta} \cdot \vec{\alpha})^2$$

Proceeding in an identical manner to above yields

$$\begin{aligned} h \vec{\delta}^2 &= \sum_{k=1}^{h-1} b_k k^2 \\ &= \frac{1}{6} \sum_{i=1}^l m_i (m_i + 1) (2m_i + 1) \end{aligned} \tag{A4}$$

On the other hand multiplying (4.29) by k^2 and summing over k one obtains

$$\begin{aligned} l \sum_{k=1}^{h-1} k^2 + \sum_{i=1}^l m_i^2 &= \sum_{k=1}^{h-1} (b_{h-k} + b_k) k^2 \\ &= \sum_{k=1}^{h-1} b_k [k^2 + (h-k)^2] \end{aligned} \tag{A5}$$

One now employs (4.30) (A 2) and (A 4) to re-express the sums on the right hand side in terms of h and $\vec{\delta}^2$, and then (4.30) and (A 2) to re-express the left hand side, again in terms of h and $\vec{\delta}^2$. The result is the identity (4.33). The remaining identities are then obtained from (A 2) and (A 4).

APPENDIX B

Coxeter Elements and Principle Three Dimensional Subalgebras

One would not realize from our discussion in the main body of this paper that Kostant's primary motivation in [21] appears to have been the study of a special class of three dimensional subalgebras in $\mathcal{G}$. We therefore rectify the omission and review some further facts relating Coxeter elements to three dimensional subalgebras (TDS's) of $\mathcal{G}$.

At the end of section 5 we were considering $x_0 = \vec{\delta} \cdot \vec{H}'$ in a CSA $\mathcal{H}'$ of $\mathcal{G}$, and also the group element $\tilde{\gamma} = \exp(\frac{2\pi i}{h} x_0)$. Let a_i be any non-zero complex number, and let r_i be the real numbers such that $\vec{\delta} = \sum r_i \vec{\alpha}_i$. Define

$$\begin{aligned} e_0 &= \sum_{i=1}^l a_i E_{\alpha_i} \\ f_0 &= \sum_{i=1}^l \frac{r_i}{a_i} E_{-\alpha_i} \end{aligned} \tag{B1}$$

Then $\{x_0, e_0, f_0\}$ define a canonically normalized $SU(2)$ subalgebra, called a *principal* three dimensional subalgebra by Kostant, because of the choice of x_0 . Denote this algebra by $\mathcal{A}$. If one decomposes $\mathcal{G}$ into irreducible representations of $\mathcal{A}$, then there are precisely l of them, and their dimensions are precisely $d_i = 2m_i + 1$. Conversely, those TDS's under which $\mathcal{G}$ decomposes into exactly l irreducible representations are, up to conjugacy in G , principal TDS's.

The CSA $\mathcal{H}'$ is of course the set of states with zero charge under x_0 . The new CSA, $\mathcal{H}$, is almost obtained by looking at those states with maximum x_0 -charge in each irreducible representation. More precisely, consider the set $\mathcal{U}$ of elements in $\mathcal{G}$ that commute with e_0 . Since these elements are annihilated by e_0 it follows that $\mathcal{U}$ decomposes into l eigenspaces of x_0 with eigenvalues $m_1, m_2, \dots, m_l$. Moreover, the m_1 eigenvector is e_0 , and the $m_l = h - 1$ eigenvector is E_ψ . In addition to this it may be shown [21] that $\mathcal{U}$ is a commuting subalgebra. However it is not necessarily a CSA as its elements may be nilpotent, *i.e.* non-diagonalizable. To get the CSA we need the complex conjugates.

Define therefore a conjugate principal TDS by taking

$$\begin{aligned}\tilde{e}_0 &= \sum_{i=1}^l \frac{q_i}{a_i} E_{-\alpha_i} \\ \tilde{f}_0 &= \sum_{i=1}^l \frac{a_i r_i}{q_i} E_{\alpha_i}\end{aligned}\tag{B2}$$

As before, $\{x_0, \tilde{e}_0, \tilde{f}_0\}$ define a principal TDS, $\mathcal{A}$. (Note that if we took $a_i = \sqrt{q_i}$ then this is the complex conjugate of $\mathcal{A}$). Define $\tilde{\mathcal{U}}$ to be those elements commuting with $\tilde{e}_0$. Then $\tilde{\mathcal{U}}$ is an l -dimensional commutative subalgebra that decomposes into x_0 -eigenspaces with eigenvalues $-m_1, \dots, -m_l$. Choose a basis u_i of $\mathcal{U}$ and $\tilde{u}_i$ of $\tilde{\mathcal{U}}$ with eigenvalues m_i and $m_i - h$ respectively. Define^{*}

$$z_i = u_i + \tilde{u}_i\tag{B3}$$

then by construction this has $\tilde{\gamma}$ eigenvalues $\exp(\frac{2\pi i m_i}{h})$. Moreover it defines a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. It is also elementary to see that u_i and $\tilde{u}_i$ can be chosen so that z_1 and z_2 are given by (5.5) .

* there is an ambiguity in the pairing of u_i and $\tilde{u}_i$ if the eigenspaces are not one-dimensional. This only occurs for the eigenvalues $m_n = m_{n+1} = 2n - 1$ of D_{2n} and is trivially resolved. See the D_4 example in section 6

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TABLE CAPTIONS

— *Table 1*

Non-degenerate conjugacy classes in the Weyl group for simply laced algebras of rank not larger than 8. For algebras not appearing in the table the only non-degenerate elements are Coxeter elements, and the relevant information can be found in section (4.3). Column 1 defines the Carter-diagram, and corresponds to Figs 1. and 2. Column 2 gives the orbit structure, with $(M)^N$ denoting N orbits of length M . The order n is equal to the length of the longest orbit. Column 3. gives the integers k_i so that $\exp(2\pi i(k_i/n))$ are the eigenvalues on the torus. The last column gives the shift vector: $(\dots, m_i, \dots)$ indicates that the shift vector has inner product $\frac{m_i}{n}$ with the simple root i . These roots are ordered as in [29].

— *Table 2*

Supercurrents for four-dimensional strings. Listed are all non-degenerate Weyl group elements of rank 6 simply laced algebras. The second column indicates whether the shift vector can be chosen to preserve supersymmetry. A particular model could have more supersymmetry than shown in the table because extra gravitini can come from the twisted/shifted sectors.

FIGURE CAPTIONS

— *Figure 1*

Carter diagrams for D_n , E_6 and E_7

— *Figure 2*

Carter diagrams for E_8

TABLE 1

Graph	Orbit structure	Torus eigenvalues	shift vector
D_4	$(6)^4$	1,3,3,5	(1,1,1,1)
$D_4(a1)$	$(4)^6$	1,1,3,3	(1,0,1,1)
D_5	$(8)^5$	1,3,4,5,7	(1,1,1,1,1)
$D_5(a1)$	$(12)^2(6)^2(4)^1$	2,3,6,9,10	(2,1,1,2,2)
D_6	$(10)^6$	1,3,5,5,7,9	(1,1,1,1,1,1)
$D_6(a1)$	$(8)^7(4)^1$	1,2,3,5,6,7	(1,1,0,1,1,1)
$D_6(a2)$	$(6)^{10}$	1,1,3,3,5,5	(1,0,1,0,1,1)
D_7	$(12)^7$	1,3,5,6,7,9,11	(1,1,1,1,1,1,1)
$D_7(a1)$	$(20)^2(10)^4(4)^1$	2,5,6,10,14,15,18	(2,2,1,1,2,2,2)
$D_7(a2)$	$(24)^2(8)^3(6)^2$	3,4,9,12,15,20,21	(3,1,2,2,1,3,3)
D_8	$(14)^8$	1,3,5,7,7,9,11,13	(1,1,1,1,1,1,1,1)
$D_8(a1)$	$(12)^9(4)^1$	1,3,3,5,7,9,9,11	(1,1,1,0,1,1,1,1)
$D_8(a2)$	$(30)^2(10)^4(6)^2$	3,5,9,15,15,21,25,27	(3,2,1,3,1,2,3,3)
$D_8(a3)$	$(8)^{14}$	1,1,3,3,5,5,7,7	(1,0,1,0,1,0,1,1)
E_6	$(12)^6$	1,4,5,7,8,11	(1,1,1,1,1,1)
$E_6(a1)$	$(9)^8$	1,2,4,5,7,8	(1,1,0,1,1,1)
$E_6(a2)$	$(6)^{12}$	1,1,2,4,5,5	(1,0,1,0,1,0)
E_7	$(18)^7$	1,5,7,9,11,13,17	(1,1,1,1,1,1,1)
$E_7(a1)$	$(14)^9$	1,3,5,7,9,11,13	(1,1,0,1,1,1,1)
$E_7(a2)$	$(12)^{10}6^1$	1,2,5,6,7,10,11	(1,0,1,0,1,1,1)
$E_7(a3)$	$(30)^3(10)^3(6)^1$	3,5,9,15,21,25,27	(2,1,2,1,2,3,1)
$E_7(a4)$	$(6)^{21}$	1,1,1,3,5,5,5	(0,0,1,0,0,1,0)
E_8	$(30)^8$	1,7,11,13,17,19,23,29	(1,1,1,1,1,1,1,1)
$E_8(a1)$	$(24)^{10}$	1,5,7,11,13,17,19,23	(1,1,0,1,1,1,1,1)
$E_8(a2)$	$(20)^{12}$	1,3,7,9,11,13,17,19	(1,1,0,1,0,1,1,1)
$E_8(a3)$	$(12)^{20}$	1,1,5,5,7,7,11,11	(1,0,1,0,0,1,0,0)
$E_8(a4)$	$(18)^{13}(6)^1$	1,3,5,7,11,13,15,17	(0,1,0,1,0,1,1,1)
$E_8(a5)$	$(15)^{16}$	1,2,4,7,8,11,13,14	(1,0,1,0,1,0,1,0)
$E_8(a6)$	$(10)^{24}$	1,1,3,3,7,7,9,9	(0,0,1,0,0,1,0,0)
$E_8(a7)$	$(12)^{18}(6)^4$	1,2,2,5,7,10,10,11	(0,1,0,1,0,0,1,0)
$E_8(a8)$	$(6)^{40}$	1,1,1,1,5,5,5,5	(0,0,0,1,0,0,0,0)

TABLE 2

Supercurrent	Space-time Supersymmetry	Supercurrent	Space-time Supersymmetry
A_6	$N = 1$	$A_5 A_1$	$N = 1$
$A_4 A_2$	$N = 0$	$A_3 A_3$	$N = 1$
$A_4 D_2$	$N = 0$	$A_3 A_2 A_1$	$N = 0$
$A_3 D_2 A_1$	$N = 0$	$A_2 A_2 A_2$	$N = 1$
$A_2 A_2 D_2$	$N = 0$	$A_2 D_2 D_2$	$N = 0$
$D_2 D_2 D_2$	$N = 0$	D_6	$N = 0$
$D_4 D_2$	$N = 0$	$D_5 A_1$	$N = 1$
$D_4 A_2$	$N = 1$	$D_6(a_1)$	$N = 1$
$D_6(a_2)$	$N = 0$	$D_5(a_1) A_1$	$N = 0$
$D_4(a_1) D_2$	$N = 1$	$D_4(a_1) A_2$	$N = 0$
E_6	$N = 1$	$E_6(a_1)$	$N = 0$
$E_6(a_2)$	$N = 1$

Figure 1

Figure 2