CERN-TH.4959/88

GLUON PRODUCTION AMPLITUDES FOR $q\bar{q}$ HIGH-ENERGY BACKWARD SCATTERING^{*)}

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ABSTRACT

A factorization method is used for the investigation of gluon bremsstrahlung in the quark-antiquark backward scattering. The differential equations for scattering amplitudes are obtained in a double-logarithmic approximation. These equations are solved for some cases of interest.

^{*)} While this paper was finalized during L. Lipatov's visit to CERN and is circulated as a CERN preprint, it is here reproduced as it was typed in Leningrad.

1. Introduction

Due to asymptotic freedom [1] it is possible to apply perturbative quantum chromodynamics (QCD) to the theoretical description of hadron reactions at small distances when both energy and momentum transfer are large enough. Here the parton-parton scattering appears as an elementary sub-process [2]. Asymptotics of scattering amplitudes at high energy and fixed momentum transfer for the hard sub-processes can be calculated in the leading-logarithmic approximation.

In ^{the} framework of the complex angular momenta approach [3] the high energy asymptotics is determined by singularities of the cross channel partial waves. In the case of quark-antiquark forward scattering (the t-channel exchange of Pomeron pole with vacuum quantum numbers) the amplitudes were discussed in Ref [4]. Similarly, the quark-antiquark backward scattering involves non-vacuum exchanges. The elastic amplitudes for this process were discussed in Ref [5].

Obviously, the backward quark-antiquark scattering is accompanied by considerable gluon bremsstrahlung of colliding objects. In a way, the situation is alike to $e^+ e^-$ - backward scattering that was examined in Ref [6] in the case when produced photons were emitted in certain multi-Regge kinematics. In the last reference a Bethe-Salpeter-type equation for inelastic amplitudes was constructed. In principle, ^{an} analogous equation can be constructed for quark-antiquark backward scattering amplitudes also, but such approach would entail great technical difficulties.

Instead of that, we suggest a new factorization method to construct simple equations for the gluon bremsstrah-

lung amplitudes with external particles on their mass shell. The method is a generalization of the approach of Ref [5] to inelastic scattering. It has been applied recently to derive the inelastic form-factor for e^+e^- - annihilation into quarks and gluons [7]. The expression for it was guessed in Ref [8] (see also [9]). Besides that, by using this method we have obtained the inelastic e^+e^- - backward scattering amplitudes when emitted photons were in arbitrary multi-Regge kinematics [10].

Our object in this paper is to construct the differential equations for the quark-antiquark scattering amplitudes with the gluon bremsstrahlung taken into account. To a great extent, we will discuss multi-Regge kinematics. The explicit formulae for the amplitudes are given in some cases of interest.

Throughout the paper we use double-logarithmic approximation (DLA) but it is possible to generalize the method in order to take into account single logarithmic contributions.

In Sec.2 we briefly review the factorization method in reference to an especially simple case of the elastic quark-antiquark backward scattering [5].

In Sec.3 the method is applied to the case of the single gluon emission in the $q\bar{q}$ -backward scattering. We discuss also the cross - channel colour structure of the production amplitude.

In Sec.4 we derive the differential equations for inelastic amplitudes for an arbitrary number of emitted gluons in the multi-Regge kinematics. They take an especially simple form in terms of the u -channel partial waves. Solutions of these equations for some cases of interest are given.

In Sec.5 we briefly discuss non-multi-Regge kinematics of emitted gluons.

2. The elastic $q\bar{q}$ -backward scattering

Let quark Q^{i_1} and antiquark $\bar{Q}_{i_2}$ convert after a hard collision into quark Q'^{j_1} and antiquark $\bar{Q}'_{j_2}$. Their momenta being P_1, P_2 for the initial particles and P'_1, P'_2 for the final particles correspondingly (see Fig.1). The indices $i_{1,2}, j_{1,2}$ label the colour states of the particles. We will discuss the Regge kinematical region for backward scattering

$$|u| \ll S \quad (1)$$

where

$$S = (P_1 + P_2)^2, \quad u = (P'_2 - P_1)^2. \quad (2)$$

Let us consider this process in the cross-channel where quark states $|Q^{i_1} Q'^{j_1}\rangle$ and $|Q^{i_2} Q'^{j_2}\rangle$ play a part of initial and final states correspondingly. It is convenient to expand the amplitude $M_{i_2 j_2}^{i_1 j_1}(s, u)$ into a sum of invariant amplitudes $M_T(s, u)$ irreducible in colour

$$M_{i_2 j_2}^{i_1 j_1} = \sum_{T=3,6} P_T^{i_1 j_1}_{i_2 j_2} M_T \quad (3)$$

where

$P_T^{i_1 j_1}_{i_2 j_2}$ is the projecting operator into the irreducible

SU(3) - representation having dimension T (T= 3,6),

$$P_T^{i_1 j_1}_{i_2 j_2} = \frac{1}{2} \left(\delta_{i_2}^{i_1} \delta_{j_2}^{j_1} + \zeta_T \delta_{j_2}^{i_1} \delta_{i_2}^{j_1} \right) \quad (4)$$

and

$$\zeta_T = 1 \quad \text{corresponds to } T = 6, \quad (5)$$

$$\zeta_T = -1 \quad \text{corresponds to } T = 3.$$

To calculate M_T we have to take into account all appropriate Feynman diagrams. A number of such diagrams have an infrared divergence. To deal with this problem, we suggest a procedure for ^{their}infrared regularization. We introduce an infrared cut-off μ for all virtual particles momenta $K_{i\perp}$ transversal to the plane containing P_1 and P_2 :

$$|K'_{i\perp}| \gg \mu \quad (6)$$

where K'_i are momenta of virtual particles. Under condition (6) we can neglect masses of virtual particles and put them equal to zero. It was shown in Ref.[7] that, besides being infrared regulator, μ takes place of a characteristic mass for invariant amplitudes (3). Therefore we write

$$M_T = M_T \left(\frac{s}{\mu^2}, \frac{u}{\mu^2} \right) \quad (7)$$

and define u -channel partial waves $f_T(\omega, u/\mu^2)$ by the Mellin transformation:

$$M_T = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{\mu^2} \right)^\omega f_T(\omega, u/\mu^2). \quad (8)$$

Naturally, the value of μ must be large enough to apply the perturbative approach.

As far as we use DLA, $K'_{i\perp}$ satisfy one of the conditions ($|K'_{i_1\perp}| \ll |K'_{i_2\perp}| \ll \dots$). We denote $K'_\perp \equiv K'_{i_1\perp}$ the minimal transversal momentum of the virtual particles and name i_1 as the soft particle (though its energy may be large). Apparently, the integration region over $|K'_\perp|$ may have μ as the lower limit unlike other loop transversal momenta. In turn, $|K'_\perp|$ is the lower limit for the integration over remaining K'_{i_1} .

The next step is to specify soft^{est} particles. There are two possibilities:

- (i) The soft^{est} particle is a gluon;
- (ii) The soft^{est} particle is a quark. In this case the DL-contribution is possible only if there is another soft quark. (Fig.3). Thereby the amplitude factories into two pieces, every one being the same amplitude but with μ replaced by $|K'_\perp|$, while S is replaced by $2 P_1 K'$ for the left blob and by $2 P_2 K'$ for the right one. Both blobs are on-shell.

As for the integration region over $|K'_\perp|$ there is an essential difference between two above discussed situations. For situation (i) the region is

$$|K'^2_\perp| \gg \mu^2 \quad (9)$$

while for situation (ii) it is

$$|K'^2_\perp| \gg \max\{\mu^2, |u|\}. \quad (10)$$

We shall call by the infrared region the region with the infrared cut-off equal to μ . Hereafter, soft quarks have

DL - contributions in the infrared region only if

$$|u| \lesssim \mu^2. \quad (11)$$

We could obtain an integral equation for M_T by summing the Born term and both μ -dependent and independent terms from Figs. 2, 3. Instead of that, we prefer to construct a simple differential equation for M_T which involves μ -dependent terms only.

The equation is

$$\frac{\partial}{\partial \mu} M_T = \frac{\partial}{\partial \mu} \left[M_T^{(\text{gluon})} + M_T^{(\text{quark})} \right] \quad (12)$$

where $M_T^{(\text{gluon})}$ corresponds to the graphs in Fig. 2 (soft gluon contribution) and $M_T^{(\text{quark})}$ corresponds to the graph in Fig. 3. There are two different cases to discuss:

$$|u| \lesssim \mu^2, \quad (13a)$$

$$|u| \gg \mu^2. \quad (13b)$$

We shall keep the notation M_T for the case (13a) while the amplitude for the case (13b) is denoted by $\tilde{M}_T$. Let f_T and $\tilde{f}_T$ be the partial waves for cases (13a) and (13b) correspondingly. Then by combining (8) and (12) one can obtain the equations (cf. Ref[5])

$$\frac{d}{dx} f_T(x) - x f_T(x) + \frac{4\pi^2 \lambda}{2+N_{\Sigma_T}} - \frac{1}{8\pi^2 \lambda} f_T^2(x) = 0, \quad (14a)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{z}} \right) \tilde{f}_T(x, \bar{z}) = (x + 2\delta \bar{z}) \tilde{f}_T(x, \bar{z}). \quad (14b)$$

Here g is the QCD coupling constant and $x = \lambda^{-1} \omega$,

$$Z = -\lambda \ln(|u|/\mu^2), \quad \gamma = 1/2 (N_{\Sigma_T} + 2), \quad N = 3, \quad (15)$$

$$\lambda = \left[g^2 (N^2 + N_{\Sigma_T} - 2) / 8\pi^2 N \right]^{1/2}.$$

It isn't difficult to solve eq.(14a) (the details see in Ref.[5])

$$f_T = -8\pi^2 \lambda \frac{d}{dx} \ln(d(x)), \quad (16)$$

$$d(x) \equiv \frac{1}{\Gamma(-\rho)} \int_0^\infty dy y^{-(1+\rho)} \exp(-xy - y^2/2) = D_\rho(x) \exp(x^2/4), \quad (17)$$

$$\rho = -\gamma,$$

where D_ρ is the parabolic cylinder function [11]. As for eq. (14b), we can easily obtain its first integral in the form

$$\tilde{f}_T(x, z) = \Phi_T(x-z) \exp(x^2/2 + \gamma z^2). \quad (18)$$

Here Φ_T is a smooth function.. To define Φ_T one can use the boundary condition

$$\tilde{M}_T\left(\frac{s}{\mu^2}, \frac{u}{\mu^2}\right) \Big|_{|u| \sim \mu^2} = M_T\left(\frac{s}{\mu^2}\right) \quad (19)$$

where $M_T(s/\mu^2)$ corresponds to the case (13a). It is given by (8), (16) and therefore we have

$$\Phi_T = -8\pi^2 \lambda \frac{d'(x-z)}{d(x-z)} \exp\left[-(x-z)^2/2\right], \quad (20)$$

3. Emission of a single gluon.

Let us discuss the quark-antiquark backward scattering provided that the gluon G_b^a is emitted. The indices a, b denote its colour. The gluon momentum is K_1 (See Fig.4). It is convenient to introduce momentum transfers q_1, q_2 :

$$q_1 = p_1 - p_2', \quad q_2 = q_1 - K_1 = p_1' - p_2. \quad (21)$$

We shall discuss the multi-Regge kinematics

$$|u_{1,2}| \ll s_1, s_2 \ll s \quad (22)$$

where

$$u_1 = q_1^2, \quad u_2 = q_2^2, \quad (23)$$

$$s_1 = (p_2' + K_1)^2, \quad s_2 = (K_1 + p_1')^2.$$

Again let us consider the process in ^{the} cross-channel where both initial and final quark states are the same as ones in Sec.2. Then the amplitude $M_{i_2 j_2 b}^{i_1 j_1 a}$ can be expanded into the sum of the invariant amplitudes $M_{T_1, T_2}^{(1)}$ by

$$M_{i_2 j_2 b}^{i_1 j_1 a} = \sum_{T_1, T_2} P_{T_1, T_2}^{i_1 j_1 a} M_{T_1, T_2}^{(1)} \quad (24)$$

where projecting operators $P_{T_1, T_2}^{i_1 j_1 a}$ take the form:

$$P_{T_1, T_2}^{i_1 j_1 a} = \frac{1}{4} \left(\delta_{\epsilon}^{i_1} \delta_{\epsilon}^{j_1} + \delta_{T_1} \delta_{\epsilon}^{i_1} \delta_{\epsilon}^{j_1} \right).$$

¹⁾As it is known [6] in DLA the spin structure of inelastic amplitudes coincides with that in the Born approximation. So, in the following discussion we omit in the amplitudes these Born factors.

$$\cdot \left(\delta_{i_2}^z \delta_{j_2}^a + \{ T_2 \delta_{i_2}^a \delta_{j_2}^z \} \right) - \frac{1}{6} \delta_{T_1, T_2} \delta_b^a \left(\delta_{i_2}^{i_1} \delta_{j_2}^{j_1} + \{ T_1 \delta_{i_2}^{i_1} \delta_{j_2}^{j_1} \} \right). \quad (25)$$

Here T_1 (T_2) = 1 if the initial (the final) diquark state belongs to the symmetrical colour SU(3) - representation, while

T_1 (T_2) = -1 corresponds to antisymmetrical one. In DLA, the amplitudes M_{T_1, T_2} are functions of both $u_{1,2}$ and $s_{1,2}$.

It is useful to have in mind that

$$s_1 s_2 = s |K_{1\perp}^2| \quad (26)$$

as far as we concern with the multi-Regge kinematics (22).

In the first place, we shall discuss the diagonal amplitudes M_T for which $T_1 = T_2 \equiv T$. If $u_1 \sim u_2$ the amplitudes M_T coincide with the elastic ones. In DLA there remains for examination the region

$$|u_1| \gg |u_2| \quad (27)$$

and the reversal one. In the inelastic scattering, the condition (6) for infrared regulator must be supplemented with the restriction

$$\mu \ll |K_{1\perp}|. \quad (28)$$

In inelastic scattering, the definition of the soft virtual particle is more complicated as compared to the elastic case (see [7]). The definition consists of the following steps:

To begin with, for every virtual particle i with the momentum K'_i one can define $K'_{i\perp}(A, B)$ as the component transversal with respect to the plane formed by the momenta A and B ($A \neq B$). After this, one can define K'_{it} by

$$|K'_{it}| = \min_{A,B} \left\{ |K'_{i\perp}(A,B)| \right\} \quad (29)$$

where A, B run over $p_1, p'_1, p_2, p'_2, K_1$.

Then one has to take into consideration all virtual particles to extract the minimal value of $|K'_{it}|$ which we label $|K'_{i,t}| \equiv |K'_t|$. At last, one must compare $|K'_t|$ with $|K_{1\perp}|$ ($K_{1\perp}$ is the emitted gluon momentum transversal to the plane formed by the momenta p_1 and p_2). We define the soft virtual particle as particle i_1 provided that

$$|K'_t| \ll |K_{1\perp}|. \quad (30)$$

The soft virtual gluon emitters are the external particles both quarks and the gluon G_a^6 . The integration over other loop momenta gives on-shell $M_T^{(1)}$ in which is substituted by $|K'_t|$. The soft virtual quark situation is similar to the one in Sec.2.

One can use the factorization to yield a differential equation for $M_T^{(1)}$. Naturally, the equation takes especially simple form in the terms of the partial waves which can be introduced by the Mellin transformation

$$M_T^{(1)}\left(\frac{s_1}{\mu^2}, \frac{s_2}{\mu^2}, \frac{u_1}{\mu^2}, \frac{u_2}{\mu^2}\right) = \int_{-i\infty}^{i\infty} \frac{d\omega_1}{2\pi i} \frac{d\omega_2}{2\pi i} \left(\frac{s_1}{\mu^2}\right)^{\omega_1} \left(\frac{s_2}{\mu^2}\right)^{\omega_2} f_T^{(1)}\left(\omega_1, \omega_2, \frac{u_1}{\mu^2}, \frac{u_2}{\mu^2}\right). \quad (31)$$

In the case $|u_2| \gg \mu^2$ this equation is

$$\begin{aligned} -(\omega_1 + \omega_2 + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}) f_T^{(1)} = & -\frac{g^2}{8\pi^2} R_T \left[\ln\left(\frac{|u_1|}{\mu^2}\right) + \right. \\ & \left. + \ln\left(\frac{|u_2|}{\mu^2}\right) \right] f_T^{(1)} - \frac{g^2}{8\pi^2} \frac{C_V}{2} \ln\left(\frac{|K_{1\perp}^2|}{\mu^2}\right) f_T^{(1)} - \frac{g^2}{8\pi^2} \\ & 2C_T \left(\frac{\partial}{\partial \omega_1} + \frac{\partial}{\partial \omega_2} + \ln\left(\frac{|K_{1\perp}^2|}{\mu^2}\right) \right) f_T^{(1)} \end{aligned} \quad (32)$$

where

$$C_V = N, \quad C_T = (N^2 - 2 + \zeta_T N) / 2N, \quad (33)$$

$$R_T = \frac{N^2 - 1}{2N} - C_T.$$

The left - (right-) hand part of equation (32) corresponds to the left- (right-) hand part of the equation in Fig.5.

Let us explain the meaning of the terms in eq.(32). We shall consider non-convariant quantities in the cms frame. The terms in the left-hand part in (32) (just *the first* one in Fig.5) correspond to the derivative of $M_T^{(n)}$ with respect to μ . The terms in the right-hand part of eq.(32) (and es. in Fig.5) correspond to the μ -dependent DL soft gluon contributions. There are several angular regions which make this contributions. The first term in the right hand part of eq (32) (the first square bracket in Fig.5) represents the kinematics in which the soft gluon is emitted into a narrow angular region close to one of external quark momentum directions. The next term in (32) corresponds to the emission of the soft virtual gluon into a similar region around the external gluon momentum. The last term in (32) corresponds to the coherent emission of the soft virtual gluon by the initial (final) diquark state. Then C_T is the square of the colour diquark charge of the state T.

For the case

$$|u_2| \lesssim \mu^2 \quad (34)$$

u_2 -dependent terms in eq (32) are absent but the soft virtual quark term $Q_T^{(q)}$ must be added to the equation:

$$Q_T^{(1)} = \frac{1}{8\pi^2} f_T^{(1)}(x_1, x_2, u_1/\mu^2) f_T(x_2) \quad (35)$$

where f_T is given by (17).

The solution of eq (32) is (cf. [6]):

$$\begin{aligned} M_T^{(1)} = & -8\pi^2 \lambda^2 \int_{-i\infty}^{i\infty} \frac{d\ell_1}{2\pi i} \frac{d\ell_2}{2\pi i} \left(\frac{s_1}{|u_1|} \right)^{\lambda \ell_1} \left(\frac{s_2}{|u_1|} \right)^{\lambda \ell_2} \frac{1}{(\ell_2 - \ell_1 + z_2 - z_1)} \cdot \\ & \cdot \left[\frac{d'(\ell_1)}{d(\ell_1)} - \frac{d'(\ell_2 + z_2 - z_1)}{d(\ell_2 + z_2 - z_1)} \right] \frac{d(\ell_2 + z_2 - z_1)}{d(\ell_2)} \cdot \\ & \cdot \exp \left\{ \lambda z_1 \ln(s_1/|u_1|) + \lambda z_2 \ln(s_2/|u_1|) - z_1^2/2 + \gamma(z_1^2 + z_2^2)/2 - \right. \\ & \left. - (g^2/16\pi^2)(C_V/2) \ln^2(|K_{1\perp}|/\mu^2) \right\}. \end{aligned} \quad (36)$$

The variables $\ell_{1,2}$ are related to the complex angular momenta $j_{1,2}$ by

$$j_i = \lambda \ell_i = \omega_i - \lambda z_i \quad (i=1,2) \quad (37)$$

while

$$z_{1,2} = -\lambda \ln(|u_{1,2}|/\mu^2). \quad (38)$$

To derive (36) from (32) we used the boundary conditions

$$M_T^{(1)}\left(\frac{s_1}{\mu^2}, \frac{s_2}{\mu^2}, \frac{u_1}{\mu^2}, \frac{u_2}{\mu^2}\right) \Big|_{|u_2| \sim \mu^2} = \widetilde{M}_T^{(1)}\left(\frac{s_1}{\mu^2}, \frac{s_2}{\mu^2}, \frac{u_1}{\mu^2}\right), \quad (39)$$

$$\widetilde{M}_T^{(1)}\left(\frac{s_1}{\mu^2}, \frac{s_2}{\mu^2}, \frac{u_1}{\mu^2}\right) \Big|_{|u_1| \sim \mu^2} = M_T\left(\frac{s}{\mu^2}\right), \quad (40)$$

where $\tilde{M}_T^{(1)}(s_1/\mu^2, s_2/\mu^2, u_1/\mu^2)$ is the amplitude for the case (34). Let us return to the non-diagonal amplitudes $M_{T_1, T_2}^{(1)}$ for which $T_1 \neq T_2$. One can show that they are absent on DLA: Indeed if $u_1 \rightarrow u_2$ then $M_{T_1, T_2}^{(1)}$ transform themselves into the elastic amplitudes $\tilde{M}_{T_1, T_2}$ which are diagonal in colour i.e. $\tilde{M}_{T_1, T_2} = \delta_{T_1, T_2} \tilde{M}_{T_1}$ (see Sec.2) because of the colour invariance.

4. n gluons emitted.

Let the total number of emitted gluons be n , their momenta being $K_1, K_2, \dots, K_n$. One can introduce q_i (see Fig.6) to label momenta of the intermediate crossing-channel states

$$q_1 = p_1 - p'_2, q_2 = q_1 - K_1, \dots, q_{n+1} = q_n - K_n = p'_1 - p_2. \quad (40)$$

We fix multi-Regge kinematics:

$$p_1 q_1 \ll p_1 q_2 \ll \dots \ll p_1 q_{n+1}, p_2 q_1 \gg p_2 q_2 \gg \dots \gg p_2 q_{n+1}, \quad (41)$$

$$|q_{i_1}^2| \lesssim |q_{i_2}^2| \lesssim \dots \lesssim |q_{i_{n+1}}^2| \quad (i_1, \dots, i_{n+1} = 1, \dots, n+1). \quad (42)$$

As well as ⁱⁿ the case $n=1$ we can proceed to the invariant amplitudes $M_{T_1}^{(n)}, \dots, T_{n+1}$. They are functions of s_i and u_i where

$$s_1 = (p'_2 + K_1)^2, s_2 = (K_1 + K_2)^2, \dots, s_{n+1} = (K_n + p'_1)^2, \quad (43)$$

$$u_1 = q_1^2, u_2 = q_2^2, \dots, u_{n+1} = q_{n+1}^2. \quad (44)$$

It is not difficult to generalize the soft virtual particle definitions to the present situation (see [7]). We can define $|K_{it}|$ as the minimal value of the gluon i momentum

transversal to the plane formed by any pair of the external particle momenta $\{p_{1,2}, p'_{1,2}, K_1, \dots, K_n\}$ (the pairs don't involve K_i). After this, one must select from them

$$|K_{i_1 t}| = \min \{ |K_{i t}| \} \equiv |K_t|. \quad (46)$$

By adding to the external momentum set in (29) the gluon momenta $K_2, \dots, K_n$ it is possible to construct $|K'_t|$ for the virtual particles.

Then μ is the infrared cut-off for these momenta:

$$\mu \ll |K_t|, |K'_t| \quad (47)$$

and the virtual particle i'_1 is the soft particle if

$$|K'_{i'_1 t}| \equiv |K'_t| \ll |K_t|. \quad (48)$$

Also it is possible to pass to the invariant amplitudes

$M^{(n)}_{T_1, \dots, T_{n+1}} \left(\frac{s_1}{\mu^2}, \dots, \frac{s_{n+1}}{\mu^2}, \frac{u_1}{\mu^2}, \dots, \frac{u_{n+1}}{\mu^2} \right)$ and further to the partial waves $f^{(n)}_{T_1, \dots, T_{n+1}}$ according to the Mellin transformation

$$M^{(n)}_{T_1, \dots, T_{n+1}} = \int_{-i\infty}^{i\infty} \frac{d\omega_1}{2\pi i} \dots \frac{d\omega_{n+1}}{2\pi i} \left(\frac{s_1}{\mu^2} \right)^{\omega_1} \dots \left(\frac{s_{n+1}}{\mu^2} \right)^{\omega_{n+1}} \quad (49)$$

$$f^{(n)}_{T_1, \dots, T_{n+1}} \left(\omega_1, \dots, \omega_{n+1}, u_1/\mu^2, \dots, u_{n+1}/\mu^2 \right).$$

Again we shall discuss the diagonal amplitudes $M^{(n)}_T$ in the first place. Let $T_1 = T_2 = \dots = T_{n+1} \equiv T$. Then the factorization of the soft particle makes it possible to sum DL contributions in the infrared region and to construct the equation for $f^{(n)}_T$.

This equation is (cf.(32))

$$\begin{aligned}
 & - \left(\omega_1 + \dots + \omega_{n+1} + \sum_{i=1}^{n+1} u_i \frac{\partial}{\partial u_i} \right) f_T^{(n)} = \\
 & - \frac{g^2}{8\pi^2} R_T \left[\ln\left(\frac{|u_1|}{\mu^2}\right) + \ln\left(\frac{|u_{n+1}|}{\mu^2}\right) \right] f_T^{(n)} - \\
 & - \frac{g^2}{8\pi^2} 2 C_T \left[\frac{\partial}{\partial \omega_1} + \dots + \frac{\partial}{\partial \omega_{n+1}} + \sum_{i=1}^n \ln\left(\frac{|K_{i\perp}^2|}{\mu^2}\right) \right] f_T^{(n)} \quad (50) \\
 & - \frac{g^2}{8\pi^2} \frac{C_V}{2} \left[\sum_{i=1}^n \ln\left(\frac{|K_{i\perp}^2|}{\mu^2}\right) \right] f_T^{(n)} + Q_T^{(n)} .
 \end{aligned}$$

The left-hand part of the equation (50) corresponds to the derivative of $M_T^{(n)}$ with respect to μ . The terms in the right-hand part of (50) (except $Q_T^{(n)}$ -term) correspond to the factorization of the soft gluon. The $Q_T^{(n)}$ is the contribution of the soft quarks. It appears if there are u_{i_k} in (42), (44) for which

$$|u_{i_k}| \lesssim \mu^2 \quad \text{For example, if}$$

$$|u_{i_1}| \sim |u_{i_2}| \sim \dots \sim |u_{i_g}| \sim \mu^2 \quad (51)$$

and

$$i_{r_1} \neq i_{r_2} \pm 1 \quad (52)$$

for any $r_1, r_2 < b$

then

$$Q_T^{(n)} = \frac{1}{8\pi^2} \sum_{\ell=i_1}^{i_8} f_T^{(\ell-1)} \left(\omega_1, \dots, \omega_\ell, u_1/\mu^2, \dots, u_{\ell-1}/\mu^2 \right) \cdot$$

$$f_T^{(n+1-\ell)} \left(\omega_\ell, \dots, \omega_{n+1}, u_{\ell+1}/\mu^2, \dots, u_{n+1}/\mu^2 \right). \quad (53)$$

Now we proceed to the general case when T_i are not necessarily equal. It can be shown that in DLA and in the multi-Regge kinematics such amplitudes vanish. We give a sketch of the proof:

- i) One may solve eq (50) by iteration in the coupling constant.
- (ii) In the Born approximation, the production amplitudes in the multi-Regge kinematics are diagonal in T_i .
- (iii) The soft virtual particle contributions do not change colour of $M_{T_1, \dots, T_{n+1}}^{(n)}$ on the right of eq (50).

So, the emission of gluons does not change the n -channel colour.

Eq (50) can be solved for any specific set of momenta $q_{i_K}^2$ in (42) if we impose appropriate boundary conditions on $M_T^{(n)}$. One can use the recursion conditions. The amplitude $M_T^{(n)}$ resolves itself into M_T^{n-1} or $M^{(n-2)}$ when the restriction (52) breaks (see [10]). Thus it is possible to pass from $M_T^{(n)}$ to M_T by increasing in turns number of $q_{i_K}^2$ in (51).

We present solutions of this equation for some particular cases. In the first place we consider the multi-Regge kinematics with strongly-ordered transfer momenta for which (42) takes the form

$$i_z = n+2-z \quad (z=1, \dots, n+1). \quad (54)$$

The "ordered" amplitude $\tilde{M}_T^{(n)}$ is given by

$$\tilde{M}_T^{(n)} = -8\pi^2 \lambda^2 \int \frac{d\ell_1}{2\pi i} \dots \frac{d\ell_{n+1}}{2\pi i} \left(\frac{s_1}{|u_1|} \right)^{\lambda \ell_1} \frac{d'(\ell_1)}{d(\ell_1)}. \quad (55)$$

$$\cdot \prod_{i=2}^{n+1} \left\{ \frac{1}{(\ell_{i+1} - \ell_i + z_{i+1} - z_i)} \left(\frac{s_i}{|u_{i-1}|} \right)^{\lambda \ell_i} \frac{d(\ell_i + z_i - z_{i-1})}{d(\ell_i)} \right\} \cdot F_n \cdot$$

$$\cdot \exp \left[\lambda z_1 \ln \left(\frac{s_1}{|u_1|} \right) + \sum_{i=2}^{n+1} \lambda z_i \ln \left(\frac{s_i}{|u_{i-1}|} \right) - \frac{z_1^2}{2} + \frac{\gamma}{2} (z_1^2 + z_{n+1}^2) \right],$$

$$F_n = \prod_{i=1}^n \exp \left\{ -\frac{g^2}{16\pi^2} \frac{C_V}{2} \ln^2 \left(\frac{|K_{i+1}|}{\mu^2} \right) \right\}. \quad (56)$$

The expression (55) implies the following restrictions on integration contour positions

$$Re(\ell_1 + z_1) < Re(\ell_2 + z_2) < \dots < Re(\ell_{n+1} + z_{n+1}). \quad (57)$$

The variables ℓ_i in (55) are related to the complex angular momenta j_i by the formula

$$j_i = \lambda \ell_i = \omega_i - \lambda z_i \quad (i=1, \dots, n+1) \quad (58)$$

while

$$z_i = -\lambda \ln(|u_i|/\mu^2) \quad (i=1, \dots, n+1). \quad (59)$$

Notations (58), (59) are used for the other cases considered below.

Let us dwell upon ^{the} $n=2$ case. Besides the "ordered" kinematics, there are two following independent multi-Regge kinematics:

$$\mu^2 \ll |u_1|, |u_3| \ll |u_2| \quad (60)$$

and

$$\mu^2 \ll |u_2| \ll |u_3| \ll |u_1|. \quad (61)$$

In the case (60) the amplitude $M_T^{(2)}$ is given by

$$\begin{aligned} M_T^{(2)} = & 8\pi^2 \lambda^2 \int \frac{d\ell_1}{2\pi i} \frac{d\ell_2}{2\pi i} \frac{d\ell_3}{2\pi i} \left(\frac{s_1}{|u_2|} \right)^{\lambda \ell_1} \left(\frac{s_2}{|u_2|} \right)^{\lambda \ell_2} \cdot \\ & \cdot \left(\frac{s_3}{|u_2|} \right)^{\lambda \ell_3} \frac{1}{(\ell_3 - \ell_2 + z_3 - z_2)} \frac{1}{(\ell_2 - \ell_1 + z_2 - z_1)} \frac{d(\ell_1 + z_1 - z_2)}{d(\ell_1)} \cdot \\ & \cdot \frac{d(\ell_3 + z_3 - z_2)}{d(\ell_3)} \frac{d'(\ell_2)}{d(\ell_2)} \cdot F_2 \cdot \exp \left[\lambda \sum_{i=1}^3 z_i \ln \left(\frac{s_i}{|u_2|} \right) - \frac{z_2^2}{2} + \right. \\ & \left. + \gamma (z_1^2 + z_3^2)/2 \right]. \end{aligned} \quad (62)$$

It is assumed in (62) that the following disposition of contours in ℓ_i - planes takes place.

$$\text{Re}(\ell_2 + z_2) < \text{Re}(\ell_1 + z_1) < \text{Re}(\ell_3 + z_3). \quad (63)$$

In the case (61) the corresponding expression for the amplitude $\tilde{M}_T^{(2)}$ is more complicated

$$\tilde{M}_T^{(2)} = M' + M'' \quad (64)$$

where

$$M' = -8\pi^2 \lambda^2 \int \frac{d\ell_1}{2\pi i} \frac{d\ell_3}{2\pi i} \left(\frac{s_1}{|u_1|} \right)^{\lambda \ell_1} \left(\frac{s}{s_1} \right)^{\lambda \ell_3} \frac{1}{(\ell_3 - \ell_1 + z_1 - z_2)} \frac{d'(\ell_1)}{d(\ell_1)} \frac{d(\ell_3 + z_3 - z_2)}{d(\ell_3)} \cdot F_2 \cdot \exp \left[\lambda z_1 \ln \left(\frac{s_1}{|u_1|} \right) + \lambda z_3 \ln \left(\frac{s}{s_1} \right) - \frac{z_1^2}{2} + \frac{\gamma}{2} (z_1^2 + z_3^2) \right], \quad (65)$$

$$M'' = 8\pi^2 \lambda^2 \int \frac{d\ell_1}{2\pi i} \frac{d\ell_2}{2\pi i} \frac{d\ell_3}{2\pi i} \left(\frac{s_1}{|u_1|} \right)^{\lambda \ell_1} \left(\frac{s_2 |u_2|}{|u_1 \cdot u_3|} \right)^{\lambda \ell_2} \cdot \left(\frac{s_3}{|u_3|} \right)^{\lambda \ell_3} \frac{1}{(\ell_3 - \ell_2 + z_3 - z_2)} \frac{1}{(\ell_2 - \ell_1 + z_2 - z_1)} \frac{d'(\ell_1)}{d(\ell_1)} \cdot \frac{d'(\ell_3)}{d(\ell_3)} d(\ell_2 + z_2 - z_1) d(\ell_2 + z_2 - z_3) \cdot \exp(\ell_2^2/2) \cdot [J(\ell_2) - J(\ell_2 + z_2 - z_3)] \cdot F_2 \cdot \exp [z_2(z_1 + z_3) + \frac{(z_1^2 + z_3^2 - z_2^2)}{2} + \frac{\gamma(z_1^2 + z_3^2)}{2} + \sum_{i=1}^3 \lambda z_i \ln \left(\frac{s_i}{\mu^2} \right)] \cdot \quad (66)$$

J(x) in (66) is given by

$$J(x) = \int_{-\infty}^x \frac{dt}{D_p^2(t)} = \frac{\sqrt{\kappa}}{\Gamma(-p)} \frac{D_p(-x)}{D_p(x)} \quad (67)$$

where we have used the following relation for the parabolic cylinder functions D_p

$$\frac{d}{dt} (D_p(-t)) D_p(t) - D_p(-t) \frac{d}{dt} D_p(t) = \frac{\Gamma(-p)}{\sqrt{\kappa}} \quad (67a)$$

The disposition of l_i - contours in (65), (66) is

$$\operatorname{Re}(\ell_1 + z_1) < \operatorname{Re}(\ell_3 + z_3) < \operatorname{Re}(\ell_2 + z_2). \quad (68)$$

It is not difficult to see, that the singularities of the partial waves in all above cases are the Regge-poles. They are in d -functions zeroes.

5. Non-multi-Regge kinematics

In spite of the fact that we discussed only the amplitudes with the multi-Regge kinematics of emitted gluons it turns out that the virtual non-multi-Regge-gluons have been taken into account. They give exponential contributions in the formulae for the scattering amplitudes as well as the multi-Regge virtual gluons. This point was discussed in Ref. [6]. As for non-multi-Regge kinematics for the emitted real gluons, there is an important case of the cascade gluon emission [8].

Let the multi-Regge kinematics conditions (41) involve $n - n_1$ emitted gluons while the gluon cascade involves n_1 other gluons. We shall discuss the case in which these n_1 gluons are in the cascade induced by the gluon γ .

Let us assume that both energies ω_i and emission angles $\theta_{i,j}$ of these gluons are strongly ordered by

$$\omega_\gamma \gg \omega_1 \gg \omega_2 \gg \dots \gg \omega_{n_1} \quad (69)$$

$$\theta_{1,\gamma} \gg \theta_{2,1} \gg \theta_{3,2} \gg \dots \gg \theta_{n_1, n_1-1}$$

where $\theta_{A,B}$ is the angle between the gluon momenta $\vec{k}_A$ and $\vec{k}_B$. The conditions (69) mean that the gluon 1 is emitted by gluon γ , the gluon 2 is emitted by gluon 1 etc. It is

possible to show that the colour structure of the amplitude coincides with the Born colour structure (see [8]). The invariant amplitude $M^{(n)}$ is given by

$$M^{(n)} = M_T^{(n-n_1)} \exp \left\{ -\frac{g^2}{16\pi^2} \frac{C_V}{2} \left[\ln^2 \left(\frac{\omega_1^2 \theta_{1,2}^2}{\mu^2} \right) + \sum_{i=2}^{n_1} \ln^2 \left(\omega_i^2 \theta_{i,i-1}^2 / \mu^2 \right) \right] \right\} \quad (70)$$

where $M_T^{(n-n_1)}$ is the multi-Regge amplitude. Expression (70) may be generalized to more complicated cascades (cf. Refs. [7,8]).

Acknowledgements

We are thankful to A.P. Bukh^yostov, V.S. Fadin, V.A. Khoze, M. Krawczyk, E.A. Kuraev and M.M. Terekhov for useful discussions.

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FIGURE CAPTIONS

- Fig.1. Elastic $q\bar{q}$ -backward scattering.
- Fig.2. DL-soft gluon contribution. $|K'_\perp|$ is the infrared cut-off of the on-shell scattering amplitudes, K' is the soft gluon momentum.
- Fig.3. DL-soft quark contribution. $|K'_\perp|$ is the cut-off for both amplitudes. K' is the soft quark momentum.
- Fig.4. Single gluon emission in $q\bar{q}$ -backward scattering
- Fig.5. Equation for $M_T^{(1)}$ in the case $|u_1|, |u_2| \gg \mu^2$.
The symbols inside the blobs denote the cut-off for on-shell amplitudes.
- Fig.6. The soft quark contribution in the equation for $M_T^{(1)}$ in the case $|u_1| \gg |u_2| \sim \mu^2$. $|K'_t|$ is the cut-off for both on-shell amplitudes.
- Fig.7. Emission of n bremsstrahlung gluons in the multi-Regge kinematics.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

$$\begin{aligned}
\frac{\partial}{\partial \mu} \left[\text{Diagram 1} \right] &= \frac{\partial}{\partial \mu} \left[\text{Diagram 2} + \text{Diagram 3} \right] + \\
&+ \frac{\partial}{\partial \mu} \left[\text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \right] + \\
&+ \frac{\partial}{\partial \mu} \left[\text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} \right].
\end{aligned}$$

Fig. 5

$$\frac{\partial}{\partial \mu} \left[\text{Diagram 12} \right]$$

Fig. 6

Fig. 7