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QUARK SCATTERING AMPLITUDES WITH QUASI-ELASTIC UNITARITY

L.N. Lipatov^{*)}
CERN -- Geneva

ABSTRACT

We consider quark-quark scattering at high energies and fixed momentum transfer. In a model where in the s- and u-channel intermediate states only gluons with sufficiently small transverse momenta are emitted, the scattering amplitudes are expressed in terms of the S-matrix elements for exactly soluble two-dimensional field theories.

^{*)} Permanent address: Leningrad Nuclear Physics Institute, Academy of Science of the USSR, 188350 Gatchina USSR.

1. - INTRODUCTION

The well-known property of asymptotic freedom for the effective coupling constant g in QCD allows a perturbative calculation of the Regge trajectory of the bare Pomeron, at large momentum transfers $q = \sqrt{-t} \gg \Lambda_{\text{QCD}}$ [1]. The leading Regge pole is situated to the right of the point $j = 1$ with intercept $\Delta \equiv j-1 > 0.3$. These results follow from the explicit solution [1] of the Bethe-Salpeter type equation for the scattering amplitudes at high energies $\sqrt{s}$ obtained earlier in the leading logarithmic approximation (LLA) [2]. The LLA corresponds to summing the asymptotic contributions $\sim (g^2 \ln s)^n$ of the Feynman diagrams in QCD. In LLA the total cross-section rapidly grows with energy $\sigma_{\text{TOT}} \sim s^\Delta$ in contradiction with the Froissart bound $\sigma_{\text{TOT}} < c \ln^2 s$, which is a consequence of the fact that in this approximation, the scattering amplitudes do not satisfy s- and u-channel unitarity.

We need a new, more accurate approximation in which at each step of the iteration procedure of Ref. [2], based on dispersion relations and s- and u-channel unitarity, both real and imaginary parts of inelastic amplitudes are taken into consideration on equal footing. This modified LLA was called in Ref. [3] the $i\pi$ approximation because it corresponds to summing some non-leading contributions in which a number of large factors $\sim \ln s$ are substituted by $i\pi$. The scattering amplitudes in the $i\pi$ approximation satisfy the multi-Regge unitarity with summation over the intermediate states in which there are gluons with fixed transverse momenta and large relative energies.

Having in mind to work out the $i\pi$ approximation in detail in the future, we solve in this paper the more simple problem of constructing the scattering amplitudes with quasi-elastic unitarity (QEU). QEU corresponds to taking into account only soft gluons with small transverse momenta $k_\perp$ in the intermediate states of the s- and u-channels:

$$|k_\perp| < M. \quad (1)$$

The mass parameter M is chosen sufficiently small in order to conserve the elastic kinematics of the quark-quark scattering. In principle, M may be a function of the energy. It is reasonable to assume that gluons with $k_\perp > M$ produce hadronic jets having large invariant masses. Therefore, Eq. (1) may be considered as an upper restriction on the energy at which only minijets exist. It is important to remember that in the approximation of QEU, not all leading logarithmic terms of Refs. [1,2] are included as would be the case of the $i\pi$ -approximation.

In the next section we construct a differential evolution equation for scattering amplitudes which allows us to sum all contributions from real and virtual gluons with small transverse momenta (1). For this purpose the generalized version of the Gribov theorem [4] is used (cf. Ref. [5]). A similar approach was developed in Ref. [6] but the authors of this reference did not examine the unitarity properties of the obtained amplitudes.

In the third section we discuss the problem of taking into account contributions of hard virtual gluons with $k_{\perp} \gtrsim M$, which turns out to be equivalent to building the S-matrix with elastic unitarity in the s and u channels. We find the Feynman diagrams having two particle intermediate states and calculate explicitly S in several orders of perturbation theory.

In the fourth section the general properties of the elastic S-matrix are formulated. It is shown that the result depends only on a definite linear combination of $\ln s$ and a function of the coupling constant. This is a consequence of the renormalizability of the corresponding two-dimensional field theory. Asymptotic freedom in this theory makes the asymptotic behaviour of the S-matrix stable with respect to variations of subtraction terms.

In the fifth section we consider two simple examples of S-matrices with elastic or quasi-elastic unitarity. It is shown that the coincidence of the S-matrix with perturbation theory does not fix it in the unique way because of the possible CDD poles.

In conclusion we discuss the results obtained and the problems left for further investigation.

2. - REAL AND VIRTUAL GLUONS WITH SMALL TRANSVERSE MOMENTA

Due to the asymptotic freedom in QCD, the perturbative approach is valid only in the region of large momentum transfers:

$$q \gg \Lambda_{QCD} . \quad (2)$$

In this section we consider M in Eq. (1) to be smaller than q to neglect the influence of emitted gluons on the quark scattering:

$$q > M \gg \Lambda_{QCD} . \quad (3)$$

Given (2), the hadron-hadron scattering amplitude in the impulse approximation can be expressed in terms of qq scattering amplitudes. In the QCD amplitudes we have

to introduce an infra-red cut-off λ for the transverse momenta $k_{\perp}$ of virtual gluons which is of the order of transverse momenta of the quarks inside the hadron:

$$|k_{\perp}| \gtrsim \lambda \quad (4)$$

Indeed, the soft gluons with $|k_{\perp}| \ll \lambda$, due to the quantum coherence phenomenon, are emitted by the hadron as a pointlike object. The probability of this process is small because of the colour screening.

By comparison of Eqs. (1) and (4) we see that there is a large region for momenta of real and virtual gluons

$$\lambda \ll |k_{\perp}| \ll M, \quad (5)$$

in which one can use the logarithmic approximation with summation of contributions of order $(g^2 \ln(M^2/\Lambda^2))^n$. (For simplicity we neglect here the fact that g is a running coupling constant.)

For further applications it is important to note that coefficients of the leading contributions contain both $\ln$ s and π terms and for our purpose of constructing the scattering amplitudes with quasi-elastic unitarity, we must sum them simultaneously. Symbolically, we obtain for the region of applicability of this approximation

$$g^2 \ln \frac{S}{M^2} \ln \frac{M^2}{\lambda^2} \sim g^2 \pi \ln \frac{M^2}{\lambda^2} \sim 1, \quad g^2 \ll 1. \quad (6)$$

The general method of deriving evolution equations in field theory consists of differentiating the physical quantities in the ultra-violet [7] or infra-red [5] cut-offs with subsequent factorization of the result into a product of the quantities of the same nature. We use here the second approach. The contribution of the virtual gluon with the smallest transverse momentum $k_{\perp}$ can be factorized from the same amplitude but with the substitution $\lambda \rightarrow |k_{\perp}|$ for the infra-red cut-off [4,5,8] (see Fig. 1):

$$\frac{\partial A(s,t)}{\partial \ln \frac{M^2}{\lambda^2}} = \frac{g^2}{8\pi^2} \left[- \left(0, \frac{N^2-1}{N} \right) \ln \left(-\frac{S}{M^2} \right) + \left(0, \frac{N^2-1}{N} \right) \ln \frac{S}{M^2} \right] A. \quad (7)$$

Here we consider $q\bar{q}$ scattering in the Yang-Mills theory with the $SU(N)$ gauge group. On the right-hand side of Eq. (7) we neglected the terms of order $g^2 \ln(M^2/\lambda^2)$ which arise in particular from the diagrams of Fig. 1e,f. $A(s,t)$ represents a two-dimensional vector consisting of two invariant amplitudes $A^{(0)}$ and $A^{(8)}$ corresponding to

t-channel exchanges of the states with quantum numbers of the vacuum and the gluon:

$$A_{i_1 i_2' i_1' i_2} = A^{(0)} \frac{1}{N} \delta_{i_1 i_1'} \delta_{i_2 i_2'} + A^{(8)} \frac{1}{2} \lambda_{i_1 i_1'}^a \lambda_{i_2 i_2'}^a, \frac{1}{2} t_2 \lambda_{i_1 i_1'}^a \lambda_{i_2 i_2'}^a = \delta_{i_1 i_1'} \delta_{i_2 i_2'},$$

$$A \equiv \begin{pmatrix} A^{(0)} \\ A^{(8)} \end{pmatrix}.$$
(8)

The positive or negative signs in front of s in the arguments of logarithms in Eq. (7) are in accordance with analytic properties of the Feynman diagrams in Fig. 1.

Equation (7) can be written in another form

$$\frac{\partial A(s, t)}{\partial \ln \frac{M^2}{\lambda^2}} = -\frac{g^2}{4\pi^2} \begin{pmatrix} 0 & \frac{N^2-1}{N} L_- \\ \frac{1}{N} L_- & \frac{N}{2} L_+ + \frac{N^2-4}{2N} L_- \end{pmatrix} A(s, t),$$

$$L_+ = \frac{1}{2} \left(\ln\left(-\frac{s}{M^2}\right) + \ln \frac{s}{M^2} \right) = \ln \frac{|s|}{M^2} - i \frac{\pi}{2},$$

$$L_- = \frac{1}{2} \left(\ln\left(-\frac{s}{M^2}\right) - \ln \frac{s}{M^2} \right) = -i \frac{\pi}{2} \operatorname{sign} s.$$
(9)

Two independent solutions of Eq. (9) are

$$A^\pm\left(\frac{s}{M^2}, \frac{M^2}{\lambda^2}\right) = \left(\frac{M^2}{\lambda^2}\right)^{\frac{g^2}{4\pi^2} a^\pm} \begin{pmatrix} -\frac{N^2-1}{N} L_- \\ a_\pm \end{pmatrix},$$
(10)

where the eigenvalues of the stationary equation corresponding to (9) are

$$a^\pm = -\frac{1}{2} \left(\frac{N}{2} L_+ + \frac{N^2-4}{2N} L_- \pm \sqrt{\frac{1}{4} \left(\frac{N}{2} L_+ + \frac{N^2-4}{2N} L_- \right)^2 + \frac{N^2-1}{N^2} L_-^2} \right).$$
(11)

At high energies $\ln(s/\mu^2) \gg \pi$ we have

$$a^+ \simeq -\frac{\pi^2}{2} \frac{N^2-1}{N^3} \frac{1}{\ln \frac{s}{M^2}}, \quad a^- \simeq -\frac{N}{2} \ln \frac{s}{M^2}$$
(12)

and therefore $A^\pm$ (10) are asymptotically determined by the vacuum and gluon quantum number exchanges with Regge-type behaviour.

The initial conditions at $\lambda = M$ for Eq. (9) fix the coefficients in the linear combination of solutions $A^\pm$. In particular, if we put A at $\lambda^2 = M^2$ equal to the Born term the result turns out to be

$$A(s, t) = \frac{1}{a^+ - a^-} \left(A^+ \left(\frac{s}{q^2}, \frac{\vec{q}}{\lambda^2} \right) - A^- \left(\frac{s}{q^2}, \frac{\vec{q}}{\lambda^2} \right) \right) / g^2 \frac{s}{q^2}, \quad (13)$$

$$t = -\vec{q}^2,$$

where we put $M^2 = \vec{q}^2$ because the scattering amplitude in the Born approximation does not depend on M^2 . It is convenient, further, to work in the impact parameter representation

$$A(s, t) = 2s \int d^2 \vec{b} e^{i \vec{q} \cdot \vec{b}} f(s, \vec{b}) \quad (14)$$

and introduce for each fixed p the S-matrix

$$S = I + i f. \quad (15)$$

Here I is the unit operator $\delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'}$, in the s-channel and in the component representation (8)

$$I = N \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (16)$$

The S-matrix satisfies the evolution equation [cf. (9)]

$$\frac{\partial S}{\partial G(\frac{M}{\lambda})} = - \begin{pmatrix} 0 & , & 2 \frac{N^2 - 1}{N} L_- \\ \frac{2}{N} L_- & , & N L_+ + \frac{N^2 - 4}{N} L_- \end{pmatrix} S, \quad (17)$$

where we have introduced a new variable:

$$G\left(\frac{M}{\lambda}\right) = \frac{g^2}{8\pi^2} \ln \frac{M^2}{\lambda^2}. \quad (18)$$

It is interesting to note that Eq. (17) determines S including the Born approximation for the initial conditions (16) corresponding to the free theory:

$$S\left(\frac{s}{M^2}, \vec{q}\right) = \frac{\pi i}{2} \frac{1}{a^+ - a^-} \left(\frac{1}{a^+} A^+ \left(\frac{s}{M^2}, \frac{1}{g^2 \lambda^2} \right) - \frac{1}{a^-} A^- \left(\frac{s}{M^2}, \frac{1}{g^2 \lambda^2} \right) \right) \quad (19)$$

Sometimes, it is more convenient to introduce instead of $A^{(0)}$ and $A^{(8)}$ in Eq. (8) the invariant amplitudes in the s-channel

$$A_{i_1 i_2}^{i_1' i_2'} = A_S^{(0)} \frac{1}{N} \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'} + A_S^{(8)} \frac{1}{2} \lambda_{i_1}^{i_1'} \lambda_{i_2}^{i_2'},$$

$$A_S = \begin{pmatrix} A_S^{(0)} \\ A_S^{(8)} \end{pmatrix}. \quad (20)$$

The relation between two discussed invariant sets is given below

$$\begin{pmatrix} S^{(0)} \\ S^{(8)} \end{pmatrix} = \begin{pmatrix} \frac{1}{N} & \frac{N^2-1}{N} \\ \frac{1}{N} & -\frac{1}{N} \end{pmatrix} \begin{pmatrix} S_S^{(0)} \\ S_S^{(8)} \end{pmatrix}, \quad \begin{pmatrix} S_S^{(0)} \\ S_S^{(8)} \end{pmatrix} = \begin{pmatrix} \frac{1}{N} & \frac{N^2-1}{N} \\ \frac{1}{N} & -\frac{1}{N} \end{pmatrix} \begin{pmatrix} S^{(0)} \\ S^{(8)} \end{pmatrix}. \quad (21)$$

Expressions (13) and (9) are also correct in the s representation after the substitutions:

$$S \rightarrow \begin{pmatrix} S_S^{(0)} \\ S_S^{(8)} \end{pmatrix}, \quad A^\pm \rightarrow e^{2GA^\pm} \begin{pmatrix} -\frac{N^2-1}{N^2} L_- + \frac{N^2-1}{N} a^\pm \\ -\frac{N^2-1}{N^2} L_- - \frac{1}{N} a^\pm \end{pmatrix} \quad (22)$$

and Eq. (17) is rewritten in the form

$$\frac{\partial S}{\partial G / \left(\frac{M^2}{\lambda^2} \right)} = \begin{pmatrix} -\frac{N^2-1}{N} \ln \left(-\frac{S}{M^2} \right), & \frac{N^2-1}{N} \ln \frac{S}{M^2} \\ \frac{1}{N} \ln \frac{S}{M^2}, & \frac{1}{N} \ln \left(-\frac{S}{M^2} \right) - \frac{2}{N} \ln \frac{S}{M^2} \end{pmatrix}. \quad (23)$$

In the soft-gluon approximation the gluon bremsstrahlung amplitudes can be expressed through the elastic amplitude by the known factorization formulae [5,8]. For these inelastic amplitudes we can write down equations similar to Eq. (23) but on the right-hand side of them now appears a sum of terms corresponding to all possible ways to attach the virtual gluon line to external particle legs (cf. [5]). These equations are applicable in the region where the infra-red cut-off λ is lower than the transverse momenta of all real gluons; in the opposite case the softest real gluon with $|k_\perp| < \lambda$ is emitted by free external particles.

The above set of equations determines the S -matrix with quasi-elastic unitarity including intermediate states with an arbitrary number of soft gluons provided that the initial conditions for these equations satisfy the unitarity demands but with comparatively hard real and virtual gluons having $|k_\perp| > M$.

Indeed, if we differentiate the expression SS^+ with respect to $\ln(M^2/\lambda^2)$ and use the evolution equation of the type of Eq. (23), then the real part of the operator on the right-hand side of this equation turns out to be zero due to cancellations between the virtual and real gluon emission and its imaginary part is cancelled between the terms corresponding to differentiating S and S^+ . This means that SS^+ is a constant equal to unity as initially.

Thus, summing up the contributions of real and virtual soft gluons does not violate S-matrix unitarity if it existed without these gluons. In the next sections we build S-matrices satisfying the elastic unitarity and crossing relations and use them as initial conditions for the above evolution equations.

3. - qq SCATTERING AMPLITUDES WITH ELASTIC UNITARITY

As was argued in the introduction, a reasonable approximation consists in imposing on the transverse momenta of produced particles an upper bound [see Eq. (1)]. Then the initial condition for Eq. (23) at $\lambda = M$ must satisfy elastic unitarity in the s- and u-channels:

$$\begin{aligned} T_{i_1 i_2}^{i_1' i_2'} (T_{i_1 i_2}^{i_1' i_2'})^* &= \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'}, \\ U_{i_1 i_2}^{i_1' i_2'} (U_{i_1 i_2}^{i_1' i_2'})^* &= \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'}. \end{aligned} \quad (24)$$

Here $T_{i_1 i_2}^{i_1' i_2'}$ the S-matrix for $q\bar{q}$ scattering ($i_1 \bar{i}_2 \rightarrow i_1' \bar{i}_2'$) and the S-matrix $U_{i_1 i_2}^{i_1' i_2'}$ describes qq scattering ($i_1 i_2 \rightarrow i_1' i_2'$). The colour structure of T and U can be simplified due to colour invariance

$$\begin{aligned} T_{i_1 i_2}^{i_1' i_2'} &= \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'} t_1 + \delta_{i_1}^{i_2'} \delta_{i_2}^{i_1'} t_2, \\ U_{i_1 i_2}^{i_1' i_2'} &= \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'} u_1 + \delta_{i_1}^{i_2'} \delta_{i_2}^{i_1'} u_2, \end{aligned} \quad (25)$$

where t_i, u_i depend on the impact parameter ρ only through the Fourier-transformed gluonic propagator [cf. (18)]

$$G(MS) = \frac{g^2}{(2\pi)^3} \int \frac{d^2 \vec{q}}{|\vec{q}|^2} e^{-i\vec{q}\vec{s}} = \frac{g^2}{8\pi^2} \ln \frac{1}{s^2 M^2}, \quad (26)$$

$|\vec{q}|^2 \gg M^2,$

which means that we neglect transverse momenta at high energies in the quark propagators.

T and U are expressed in terms of s- and u-channel partial waves $f_{s,u}$:

$$\begin{aligned} T &= I + i f_s \\ U &= I + i f_u. \end{aligned} \quad (27)$$

The matrices $f_{s,u}$ have eigenvalues with positive imaginary parts due to Eq. (24):

$$2 \operatorname{Im} f_{s,u} = |f_{s,u}|^2, \quad (28)$$

where matrix multiplication is implied.

The scattering amplitudes in the q representation are related to $f_{s,u}$ by equations [see (14)]

$$\begin{aligned} A(s, t) &= 2s \int d^2 \vec{q} e^{i\vec{q}\vec{s}} f_s = 2u \int d^2 \vec{q} e^{i\vec{q}\vec{s}} f_u, \\ u &\simeq -s \end{aligned} \quad (29)$$

and therefore f_u can be obtained as an analytic continuation of f_s :

$$f_u(u, G) = -f_s(s, G). \quad (30)$$

The continuation can be performed along path "a" in Fig. 2 with the subsequent complex conjugation of f_s :

$$f_u(u, G) = -f_s^*(|s|e^{i\pi}, G). \quad (31)$$

Another possibility is to go between the right and left cuts (see the path "b" in Fig. 2). In the high energy region where the two cuts effectively merge the second continuation looks dubious.

However, it will be shown later that $f(s, G)$ depends only on one parameter θ

$$\theta = \ln \frac{s}{\Lambda} + \varphi(G), \quad \varphi(G) = \frac{C_0}{G} + C_1 G + \dots \quad (32)$$

Here $\varphi(G)$ is an odd function of G and $\sqrt{\Lambda}$ is a mass parameter which does not

necessarily coincide with M . On the other hand, θ turns out to be the rapidity in the corresponding field model

$$\theta = \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}}, \quad (33)$$

where m is the mass of the colliding particles. Therefore, by comparing Eqs. (32) and (33) at large $s \gg m^2$, we obtain the relation well-known in asymptotically-free models

$$m^2 = \Lambda^2 e^{-\frac{c_0}{G}} (1 - c_1 G + \dots) \quad (34)$$

This means that as a result of summing the perturbative terms, a finite gap between the left and the right cuts arises. So, the analytic continuation along the path "b" in Fig. 2 is possible and can be carried out in two states. The first stage consists in moving around the right cut on its lower edge. The second stage is performed along the semi-circle in Fig. 2. The resulting transformation in the θ -plane is the following [see (33)]

$$\theta \rightarrow i\pi - \theta. \quad (35)$$

Therefore, the analytic continuation along the path "b" in Fig. 2 gives the relation [see (32)]

$$f_u(u, G) = f_s(s, G) \Big|_{\substack{G \rightarrow -G \\ \ln \frac{s}{\Lambda^2} \rightarrow i\pi - \ln \frac{|s|}{\Lambda^2}}} \quad (36)$$

The signs of the right-hand sides of Eqs. (33) and (36) are opposite because the transformation $G \rightarrow -G$ changes the sign of the Born term.

The relations (31) and (36) do not touch the colour indices and can be written for each S-matrix element of T and U (25):

$$U_{ab}^{cd}(u, G) = \left(T_{ab}^{cd}(|s| e^{i\pi}, G) \right)^*, \quad (37a)$$

$$U_{ab}^{cd}(u, G) = T_{ab}^{cd}(s, -G) \Big|_{\ln \frac{s}{\Lambda^2} \rightarrow i\pi - \ln \frac{|s|}{\Lambda^2}} \quad (37b)$$

On the other hand, in the formula for charge symmetry relation

$$U_{ab}^{cd}(u, G) = \left(T_{ab}^{cd}(u, -G) \right)' \quad (38)$$

there is the reverse permutation of all λ -matrices on one quark line (denoted by a prime) with both quantities U and T depending on u .

From Eqs. (37) and (38) two relations for T can be obtained:

$$(T_{ab}^{cd}(S, G))^* = T_{ab}^{cd}(S, G) \Big|_{\ln \frac{S}{\Lambda} \rightarrow -\ln \frac{S}{\Lambda}}, \quad (39a)$$

$$(T_{ab}^{cd}(S, -G))' \Big|_{S \leftrightarrow u} = (T_{ab}^{cd}(S, G))^* \Big|_{G \rightarrow -G}. \quad (39b)$$

Now we pass to the perturbative solution of Eqs. (24). For convenience the colour structure of each Feynman diagram is denoted further by the same graph, for example:

$$\begin{aligned} \begin{array}{c} i_1' \\ \uparrow \\ i_1 \end{array} \begin{array}{c} i_2' \\ \downarrow \\ i_2 \end{array} &= \frac{1}{2} (\lambda^a)_{i_1}^{i_1'} \frac{1}{2} (\lambda^a)_{i_2'}^{i_2}, \\ \begin{array}{c} i_1' \\ \uparrow \\ i_1'' \\ \uparrow \\ i_1 \end{array} \begin{array}{c} i_2' \\ \downarrow \\ i_2'' \\ \downarrow \\ i_2 \end{array} &= \frac{1}{2} (\lambda^a)_{i_1''}^{i_1'} \frac{1}{2} (\lambda^b)_{i_1}^{i_2''} \frac{1}{2} (\lambda^b)_{i_2''}^{i_2'} \frac{1}{2} (\lambda^a)_{i_2}^{i_1'}. \end{aligned} \quad (40)$$

Then the result of calculating the S-matrix in arbitrary order of perturbation theory can be presented as a sum of colour diagrams with some coefficients which are polynomials in $\ln(\Lambda/-s)$ and $\ln(\Lambda/-u)$. The S-matrix for $q\bar{q}$ scattering can be written in the form:

$$\begin{aligned} T &= I + i f, \\ f &= f_R \left(\ln \frac{\Lambda}{-s} \right) + f_L \left(\ln \frac{\Lambda}{s} \right), \\ f_R \left(\ln \frac{\Lambda}{-s} \right) &= 2\pi \sum_{n=1}^{\infty} G^n P_{n-1} \left(\ln \frac{\Lambda}{-s} \right), \\ f_L \left(\ln \frac{\Lambda}{-u} \right) &= 2\pi \sum_{n=1}^{\infty} G^n Q_{n-1} \left(\ln \frac{\Lambda}{-u} \right). \end{aligned} \quad (41)$$

Temporarily we include the colour factors in the polynomials P_{n-1} and Q_{n-1} . The functions f_R and f_L can be considered as the dispersion contributions of right and left cuts in the S-plane:

$$f_R = \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s'-s} \text{Im}_{s'} f(s'), \quad f_L = \frac{1}{\pi} \int_{-\infty}^0 \frac{du'}{u'+s} \text{Im}_{u'} f(u'). \quad (42)$$

Therefore f_R and f_L are real functions of their arguments. Using this fact, we

conclude from Eq. (39a) that

$$\begin{aligned} P_n(x) &= (-1)^n P_n(-x), \\ Q_n(x) &= (-1)^n Q_n(-x). \end{aligned} \quad (43)$$

Analogously from Eq. (39b), one obtains

$$Q_n(x) = (-1)^n (P_n(x))' \quad (44)$$

where the prime means again the reverse transmutation of λ -matrices on one quark line.

The leading terms for $x \rightarrow \infty$ of the polynomials P_n and Q_n were calculated in Ref. [9] but for our purpose of achieving explicit elastic unitarity we must sum also all subleading terms [see Eq. (6)]. In other cases such a procedure could not be carried out due to its complexity. Moreover, in the considered case of the amplitude with elastic unitarity the solution of this program is not unique because of the arbitrariness of subtraction terms in Eq. (42).

Below we suggest a definite procedure for fixing these subtraction terms. The final result for the S-matrix corresponds to a two-dimensional field theory with asymptotic freedom. Therefore, it is very plausible that the functional form of the obtained results is stable and subtraction constants discussed above may change only the function $\phi(G)$ in Eq. (32). This mechanism inspires us to assume the analogous universality for the case of the S-matrix with multi-Regge unitarity (the π approximation).

To begin with, we write down the S-matrix in the Born approximation

$$T = \uparrow\downarrow + 2\pi i G \uparrow\downarrow. \quad (45)$$

Using the unitarity equations (28) we obtain in particular

$$\text{Im}_S f^{(2)} = 2\pi^2 G^2 \uparrow\downarrow, \quad \text{Im}_u f^{(2)} = -2\pi^2 G^2 \uparrow\downarrow, \quad (46)$$

where we took into account the crossing relation (30). The general solution of Eqs. (46) is

$$f^{(2)} = 2\pi G \left[\uparrow\downarrow \left(\ln \frac{1}{s} + c_1 \right) - \uparrow\downarrow \left(\ln \frac{1}{s} + c_2 \right) \right], \quad (47)$$

where c_1 and c_2 are arbitrary constants; but using Eqs. (43) and (44) we obtain

$$C_1 = C_2 = 0 \quad (48)$$

and therefore

$$T = \uparrow\downarrow + 2\pi i \left[G |H| + G^2 \left(|H| \ln \frac{\Lambda}{-s} - |H| \ln \frac{\Lambda}{s} \right) \right] \quad (49)$$

Analogously we can find $\text{Im}_{s,u} f^{(3)}$ but here and in all odd orders of perturbation theory the relations (43) and (44) do not allow us to determine the subtraction constants in an unique way.

It is useful to have a diagrammatic representation of T . The Feynman diagrams leading to the scattering amplitudes having elastic thresholds are of the eikonal type (see Fig. 3) namely only those of them which can be represented in the s - or u -channels as a chain of links attached to each other by a gluon line or a blob consisting of an analogous chain of links in another channel (see Fig. 4a,b). The integration over the transverse momenta of the virtual gluons in the p -representation is factorized into an effective four-particle coupling constant G (26). The quark propagators $D_{\pm}$ depend only on longitudinal momenta k_0, k_3 and, taking into account their numerators, can be written in the form

$$D_{-} = \frac{1}{k_{-} + \frac{i\varepsilon}{k_{+}}}, \quad D_{+} = \frac{1}{k_{+} + \frac{i\varepsilon}{k_{-}}}, \quad k_{\pm} = k_0 \pm k_3 \quad (50)$$

for the first and second quarks correspondingly.

In the product of the quark propagators for each Feynman diagram, we must integrate over all independent two-dimensional momenta k^i and multiply the result by the factor

$$2\pi G \left(\frac{i}{\pi} G \right)^{n-1} C \quad (51)$$

where C is the corresponding colour structure. For example, for the box diagram in the u -channel we have (see Fig. 5)

$$\begin{aligned} f_H^{(2)} &= 2\pi G^2 \frac{i}{\pi} \int d^2 k \frac{1}{(p_1^- - k^- - \frac{i\varepsilon}{k^+})(-p_2^+ + k^+ + \frac{i\varepsilon}{k^-})} |H| = \\ &= -2\pi G^2 |H| \ln \left(\frac{\Lambda}{-u - i\varepsilon} \right). \end{aligned} \quad (52a)$$

Here Λ is an ultra-violet cut-off [cf. (49)].

Analogously, the box diagram in the s-channel gives

$$\begin{aligned} f_H^{(e)} &= 2\pi G^2 \frac{i}{\pi} \int d^2 k \frac{1}{(p_2^- - k^- - \frac{i\varepsilon}{k^+})(-p_2^+ - k^+ + \frac{i\varepsilon}{(-k^-)})} H = \\ &= 2\pi G^2 H \ln\left(\frac{\Lambda}{-s-i\varepsilon}\right). \end{aligned} \quad (52b)$$

The gluon propagators in the two-dimensional effective theory are absent and instead of them the four particle vertices G arise. Therefore, the contribution of the diagrams of Fig. 4a with contracted gluon lines is factorized (see Fig. 6) and we need to consider only the chains in which all conjunctions are non-trivial blobs (see Fig. 4b). For these irreducible chains we use the dispersive method by calculating all possible discontinuities of them across the elastic thresholds and after that we restore the real parts of the amplitudes with the help of Eq. (42). The ultra-violet cut-off Λ is introduced directly in the dispersion integrals (cf. (52)). It is our way of fixing the subtraction terms.

In general, it is enough to know only the dispersive integral of a power of $\ln(\Lambda/s)$:

$$R_{n+1}(\ln s) = \int_0^\Lambda \frac{ds'}{s' - s - i\varepsilon} \left(\ln \frac{\Lambda}{s'}\right)^n = \lim_{a \rightarrow 0} \left(\frac{d}{da}\right)^n \int_0^\Lambda \frac{ds'}{s' - s - i\varepsilon} \left(\frac{\Lambda}{s'}\right)^a. \quad (53)$$

The last integral can be calculated easily

$$\begin{aligned} \int_0^\Lambda \frac{ds'}{s' - s - i\varepsilon} \left(\frac{\Lambda}{s'}\right)^a \Big|_{s \ll \Lambda} &= \frac{\pi}{\sinh \pi a} \left(\frac{\Lambda}{-s}\right)^a + \int_0^\Lambda \frac{ds'}{s'} \left(\frac{\Lambda}{s'}\right)^a = \\ &= \frac{\pi}{\sinh \pi a} \left(\frac{\Lambda}{-s}\right)^a - \frac{1}{a}. \end{aligned} \quad (54)$$

Substituting the integral in Eq. (53) by Eq. (54) we obtain in some particular cases

$$\begin{aligned}
 \int_0^{\Lambda} \frac{ds'}{s' - s - i\varepsilon} &= \ln \frac{\Lambda}{-s}, \\
 \int_0^{\Lambda} \frac{ds'}{s' - s - i\varepsilon} \ln \frac{\Lambda}{s'} &= \frac{1}{2} \ln^2 \frac{\Lambda}{-s} + \frac{\pi^2}{6}, \\
 \int_0^{\Lambda} \frac{ds'}{s' - s - i\varepsilon} \ln^2 \frac{\Lambda}{s'} &= \frac{1}{3} \ln^3 \frac{\Lambda}{-s} + \frac{\pi^2}{3} \ln \frac{\Lambda}{-s}, \\
 \int_0^{\Lambda} \frac{ds'}{s' - s - i\varepsilon} \ln^3 \frac{\Lambda}{s'} &= \frac{1}{4} \ln^4 \frac{\Lambda}{-s} + \frac{\pi^2}{2} \ln^2 \frac{\Lambda}{-s} + \frac{7}{60} \pi^4
 \end{aligned} \tag{55}$$

and so on. Using Eqs. (55) we calculate five orders of perturbation theory for T (49):

$$T = \uparrow \downarrow + 2\pi i \sum_{h=1}^{\infty} G^h \left(P_{h-1} \left(\ln \frac{\Lambda}{-s} \right) + Q_{h-1} \left(\ln \frac{\Lambda}{s} \right) \right), \tag{56}$$

where

$$\begin{aligned}
 P_0 &= Q_0 = \frac{1}{2} \mathbb{H}, \quad x = \ln \frac{\Lambda}{-s}, \quad y = \ln \frac{\Lambda}{s}, \\
 P_1(x) &= \mathbb{H} x, \quad Q_1(y) = -P_1'(y) = \mathbb{H}(-y), \\
 P_2(x) &= \mathbb{H} x^2 + (\mathbb{H} + \mathbb{H}) \left(-\frac{1}{2} x^2 - \frac{\pi^2}{6} \right), \\
 Q_2(y) &= (P_2(y))' = \mathbb{H} y^2 + (\mathbb{H} + \mathbb{H}) \left(-\frac{1}{2} y^2 - \frac{\pi^2}{6} \right), \\
 P_3(x) &= \mathbb{H} x^3 + (\mathbb{H} + \mathbb{H}) \left(-\frac{1}{2} x^3 - \frac{\pi^2}{6} x \right) + \mathbb{H} \left(-\frac{1}{3} x^3 + \frac{\pi^2}{3} x \right) + \\
 &+ (\mathbb{H} + \mathbb{H} + \mathbb{H} + \mathbb{H}) \left(-\frac{1}{6} x^3 - \frac{\pi^2}{3} x \right) + (\mathbb{H} + \mathbb{H} + \mathbb{H}) \left(\frac{1}{3} x^3 + \frac{\pi^2}{3} x \right), \\
 Q_3(y) &= -P_3'(y), \\
 P_4(x) &= \mathbb{H} x^4 + (\mathbb{H} + \mathbb{H}) \left(-\frac{x^4}{2} - \frac{\pi^2}{6} x^2 \right) + (\mathbb{H} + \mathbb{H}) \left(-\frac{x^4}{3} + \frac{\pi^2}{3} x^2 \right) + \\
 &+ \mathbb{H} \left(\frac{1}{2} x^4 + \frac{\pi^2}{6} x^2 \right) + (\mathbb{H} + \mathbb{H}) \left(\frac{5}{24} x^4 - \frac{\pi^2}{12} x^2 - \frac{5}{72} \pi^4 \right) + \\
 &+ (\mathbb{H} + \mathbb{H} + \mathbb{H} + \mathbb{H}) \left(-\frac{1}{6} x^4 - \frac{\pi^2}{3} x^2 \right) + (\mathbb{H} + \mathbb{H}) \left(\frac{x^4}{3} + \frac{\pi^2}{3} x^2 \right) +
 \end{aligned} \tag{57}$$

$$\begin{aligned}
 & + \left(\text{diagram 1} + \text{diagram 2} \right) \left(-\frac{x^4}{12} + \frac{\pi^2}{60} \right) + \text{diagram 3} \left(\frac{1}{6} x^4 - \frac{\pi^2}{3} x^2 - \frac{13}{90} \pi^4 \right) + \\
 & + \left(\text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} + \text{diagram 8} + \text{diagram 9} \right) \left(-\frac{1}{12} x^4 - \frac{\pi^2}{3} x^2 - \frac{17}{180} \pi^4 \right) + \\
 & + \left(\text{diagram 10} + \text{diagram 11} + \text{diagram 12} + \text{diagram 13} + \text{diagram 14} + \text{diagram 15} + \text{diagram 16} + \text{diagram 17} \right) \left(-\frac{x^4}{24} + \frac{\pi^2}{4} x^2 + \frac{3}{40} \pi^4 \right) + \\
 & + \left(\text{diagram 18} + \text{diagram 19} + \text{diagram 20} + \text{diagram 21} + \text{diagram 22} + \text{diagram 23} + \text{diagram 24} + \text{diagram 25} \right) \left(\frac{x^4}{8} + \frac{\pi^2}{3} x^2 + \frac{31}{360} \right) + \text{cont.} \quad (57) \\
 & + \left(\text{diagram 26} + \text{diagram 27} + \text{diagram 28} + \text{diagram 29} \right) \left(-\frac{x^4}{4} - \frac{\pi^2}{2} x^2 - \frac{7}{60} \pi^4 \right) + \left(\text{diagram 30} + \text{diagram 31} \right) \left(\frac{x^4}{12} - \frac{\pi^2}{60} \right),
 \end{aligned}$$

$$Q_4(y) = P_4'(y).$$

4. - GENERAL PROPERTIES OF THE S-MATRIX WITH ELASTIC UNITARITY

Here we discuss some important properties of the series (56) which allow us to find exact solutions in some particular cases. Let us begin with the renormalizability of the S-matrix (56). It turns out that we can consider G as a bare coupling constant and fix its dependence of $\ln(\Lambda/\mu^2)$ where μ^2 is a normalization point in such a way that Eq. (56) ceases to depend on Λ

$$\frac{dS}{d \ln \Lambda} = \frac{\partial S}{\partial \ln \Lambda} + \frac{\partial S}{\partial G} \psi(G) = 0, \quad (58a)$$

where $\psi(G)$ is the Gell-Mann-Low function:

$$\frac{dG}{d \ln \Lambda} = \psi(G). \quad (58b)$$

We consider temporarily a general case where the quark-antiquark scattering in the Born approximation is an arbitrary matrix

$$f_{i_1 i_2}^{(1) i_1' i_2'} = 2\pi G_{i_1 i_2}^{i_1' i_2'} = 2\pi \text{diagram} \quad (59)$$

and its matrix elements are coupling constants. Note that we include G in the colour diagrams of Eqs. (57). In the general case, the second term in Eq. (58a) is understood in such a way that after removing one of the G 's from the colour diagrams (57) as a result of differentiating them, we must replace it by the matrix ψ .

It can be verified that Eq. (58a) is valid for the perturbative expansion (57) if

$$\begin{aligned} \psi(G) = & \text{[diagram 1]} - \text{[diagram 2]} + \frac{\pi^2}{3} (\text{[diagram 3]} - \text{[diagram 4]}) + \\ & + \frac{\pi^2}{6} (\text{[diagram 5]} + \text{[diagram 6]} + \text{[diagram 7]} + \text{[diagram 8]} - \text{[diagram 9]} - \text{[diagram 10]} - \text{[diagram 11]} - \text{[diagram 12]}) + \dots \end{aligned} \quad (60)$$

We see in particular that $\psi(V)$ is an even function of G . Thus, for each solution of Eq. (58b) with $\psi(G)$ given by (60) the S-matrix (56) is renormalizable. It would be interesting to find all solutions of Eq. (58b) because they seem to classify a number of exactly soluble two-dimensional models [10], but here we consider only two highly symmetric examples of such solutions.

In the first case the gauge group is $O(N)$ and colliding particles belong to its vector representation. Then the Born structure of the scattering amplitude is

$$G_{i_1 i_2, i_1' i_2'}^{i_1' i_2'} = G \left(\delta_{i_1 i_2} \delta_{i_1' i_2'} - \delta_{i_1 i_2'} \delta_{i_2 i_1'} \right). \quad (61)$$

Putting this ansatz in Eqs. (58) and (60) we obtain the Gell-Mann-Low equation for the coupling constant G in this model

$$\frac{dG}{d \ln \frac{\Lambda}{\mu_0}} = \psi(G) = -(N-2)G^2 - \frac{\pi^2}{3}(N-2)(N-6)G^4 + O(G^6), \quad (62)$$

which reveals the asymptotic freedom at large energies. Moreover, the S-matrix depends only on one definite combination of $\ln(s/\Lambda)$ and a function of G :

$$\Theta = \ln \frac{s}{\Lambda} + \int \frac{dG}{\psi(G)} = \ln \frac{s}{\Lambda} + \frac{1}{(N-2)G} + \frac{\pi^2}{3}(N-6)G^3 + \dots \quad (63)$$

The second case is $q\bar{q}$ scattering in $SU(N)$ model. Here

$$G_{i_1 i_2, i_1' i_2'}^{i_1' i_2'} = G \frac{1}{4} (\lambda^a)_{i_1}^{i_1'} (\lambda^a)_{i_2}^{i_2'} = \frac{1}{2} G \left(\delta_{i_1 i_2} \delta_{i_1' i_2'} - \frac{1}{N} \delta_{i_1}^{i_1'} \delta_{i_2}^{i_2'} \right). \quad (64)$$

From relations (58a) and (60) we find the Gell-Mann-Low equation for G [cf. (62)]

$$\frac{dG}{d \ln \frac{\Lambda}{\mu_0}} = \psi(G) = -\frac{N}{2} G^2 + \frac{\pi^2}{12} N G^4 + O(G^6). \quad (65)$$

Again the model turns out to be asymptotically free and the S-matrix depends only on one variable

$$\theta = \ln \frac{s}{\Lambda^2} + \int^G \frac{dG}{4(G)} = \ln \frac{s}{\Lambda^2} + \frac{2}{N} \frac{1}{G} - \frac{1}{N} \frac{\pi^2}{3} G + \dots \quad (66)$$

Furthermore, θ in these two models may be considered as the rapidity because asymptotically Eq. (33) has a logarithmic behaviour and small corrections to the asymptotics are of order $m^2/s \sim e^{-a/g}/s$ being beyond perturbation theory. It is important also that Eqs. (56) and (57) satisfy relations (43) derived in particular with the use of the rapidity transformation (35).

Equation (39a) in terms of θ can be rewritten in a more simple form

$$T^*(\theta) = T^{-1}(\theta) = T(-\theta) \quad (67)$$

because of the unitarity conditions (24). The S-matrix is a meromorphic function of θ and real on the imaginary axis:

$$\text{Im } T(\theta) \Big|_{\theta=iy} = 0 \quad (68)$$

with the exception of possible poles arising as compound states of colliding objects.

Due to the asymptotic freedom in the limit of large θ the S-matrix must coincide with its perturbative expansion (45) where $G(\theta)$ is the effective coupling constant

$$T(\theta) \Big|_{\theta \rightarrow \infty} = I + 2\pi i G(\theta) \mathbb{1} + \dots \quad (69)$$

$$G(\theta) \simeq \frac{C_0}{\theta} \quad (70)$$

Here C_0 is the coefficient in front of the singular term $1/G$ in Eqs. (63) and (66) [see (32)].

The important property of S-matrices with elastic unitarity is that they satisfy triangle equations of the type (see for details Refs. [10])

$$(71)$$

where blobs represent S-matrices depending on corresponding differences $\theta_i - \theta_j$ and the summing over all possible particles a, b, c is implied. The non-linear equations (71) can be solved for many cases [10] (see the next section).

Concluding this section we discuss shortly an especially simple case of an Abelian gauge theory where all colour factors in Eqs. (57) equal unit and the S-matrix (56) simplifies drastically:

$$S = 1 + 2\pi i G + \frac{(2\pi i G)^2}{2!} + \frac{(2\pi i G)^3}{3!} + \frac{(2\pi i G)^4}{4!} + G^5 \cdot 0 + O(G^6)_{(72)}$$

We see that all logarithmic terms are cancelled and we have the perturbative expansion of the eikonal expression

$$S = \left(\frac{1}{g^2 M^2} \right)^{i\alpha}, \quad \alpha = \frac{g^2}{4\pi} \quad (73)$$

up to fifth order in perturbation theory. In the order $\sim G^5$ the calculated coefficient in Eq. (72) turns out to be zero instead of $i 4/15\pi^5$ as follows from Eq. (73). Of course, S (72) is also a pure phase $\sim \exp i\delta$ with δ -depending on G in a non-trivial way in comparison with Eq. (73) but Eq. (73) is proved to be valid in QED. This shortcoming of expressions (56) and (57) is a consequence of our subtraction procedure in which only diagrams having elastic intermediate states were taken into account. It is possible that there is another more clever approach in which the eikonal formula (73) is obtained in the Abelian limit.

5. - S-MATRICES FOR $O(N)$ and $SU(N)$ GROUPS

Factorizable S-matrices satisfying the triangle relations (71) were examined very intensively in connection with their relation with the exactly soluble models of quantum field theories and statistical mechanics [11]. In particular the

S-matrix with $O(N)$ symmetry was constructed by A.B. Zamolodchikov and Al.B. Zamolodchikov [12] in the form

$$S_{i_1 i_2}^{i_3 i_4}(\theta) = \sigma_1(\theta) \delta_{i_1 i_2} \delta_{i_3 i_4} + \sigma_2(\theta) \delta_{i_1 i_3} \delta_{i_2 i_4} + \sigma_3(\theta) \delta_{i_1 i_4} \delta_{i_2 i_3} \quad (74)$$

where

$$\sigma_1 = -\frac{i\lambda}{i\pi - \theta} \sigma_2, \quad \sigma_3 = -\frac{i\lambda}{\theta} \sigma_2, \quad (75)$$

$$\sigma_2^{\pm} = \frac{\Gamma(-\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}) \Gamma(\frac{1}{2} - i\frac{\theta}{2\pi}) \Gamma(\frac{1-\lambda}{2} + i\frac{\theta}{2\pi}) \Gamma(1 + i\frac{\theta}{2\pi})}{\Gamma(\frac{1-\lambda}{2} + \frac{\theta}{2\pi} - i\frac{\theta}{2\pi}) \Gamma(-\frac{i\theta}{2\pi}) \Gamma(1 + i\frac{\lambda}{2\pi} + i\frac{\theta}{2\pi}) \Gamma(\frac{1}{2} + i\frac{\theta}{2\pi})}$$

The upper sign in the expression for σ_2 leads to the S-matrix for the well-known non-linear σ -model, the lower sign gives the S-matrix for the Gross-Neveu model [12]. The ratio of $\sigma_2^{\pm}$ is

$$\frac{\sigma_2^+}{\sigma_2^-} = \frac{\tanh(\frac{\theta}{2} + i\frac{\lambda}{2})}{\tanh(\frac{\theta}{2} - i\frac{\lambda}{2})} \quad (76)$$

which represents a particular case of the CDD factor

$$\chi(\theta) = \prod_z \frac{\sinh \frac{\theta + \theta_z}{2}}{\sinh \frac{\theta - \theta_z}{2}} \quad (77)$$

with θ_z purely imaginary. We can multiply the S-matrix by this factor without violating its unitarity and the properties of analyticity including relation (67). Nevertheless, the new S-matrix contains as a rule new poles and corresponds to another theory.

In the region of large θ we have

$$\chi(\theta) \Big|_{\theta \rightarrow \infty} = \prod_z e^{\theta_z} \quad (78)$$

and therefore if $\Sigma \theta_z = 2\pi i n$, where n is integer, $\chi(\theta)$ cannot be determined from perturbation theory (56) and (57). In particular,

$$\left. \frac{\sigma_2^+}{\sigma_2^-} \right|_{\theta \rightarrow \infty} = 1 + 2e^{-\theta} (e^{i\lambda} - e^{-i\lambda}) + \dots \quad (79)$$

and due to Eq. (63) the difference of σ_2^+ and σ_2^- turns out to be of order $e^{-c/G}$ and does not manifest itself in perturbation theory. This means that solutions of the unitarity equations (24) contain non-perturbative effects which have influence on their analytic properties.

An especially simple form of S-matrix occurs for the O(3) non-linear σ -model:

$$\begin{aligned} \theta_1 &= \frac{2\pi i}{\theta} \frac{1}{(1 - \frac{2\pi i}{\theta})(1 + \frac{\pi i}{\theta})} \simeq \frac{2\pi i}{\theta} (1 + \frac{\pi i}{\theta} - \frac{3\pi^2}{\theta^2} + \dots), \\ \theta_2 &= 1 - \frac{1}{(1 + i\frac{\theta}{2\pi})(1 - i\frac{\theta}{2\pi})} \simeq 1 - \frac{2\pi^2}{\theta^2} (1 + \frac{\pi i}{\theta}) + \dots \quad (80) \\ \theta_3 &= -i \frac{2\pi}{\theta} \frac{1 - \frac{\pi i}{\theta}}{(1 - \frac{2\pi i}{\theta})(1 + \frac{\pi i}{\theta})} \simeq -\frac{2\pi i}{\theta} (1 - \frac{2\pi^2}{\theta^2}) + \dots \end{aligned}$$

Putting here

$$\theta = \ln \frac{s}{\Lambda} + \frac{1}{G} - \pi^2 G + \dots \quad (81)$$

from Eq. (63) we obtain

$$\begin{aligned} \theta_1 &\simeq 2\pi i G (1 + G (\ln \frac{\Lambda}{s} + \pi i) + G^2 (\ln^2 \frac{\Lambda}{s} + 2\pi i \ln \frac{\Lambda}{s} - 2\pi^2) + \dots), \\ \theta_2 &\simeq 1 - 2\pi^2 G^2 (1 + G (2 \ln \frac{\Lambda}{s} + \pi i) + \dots), \\ \theta_3 &\simeq -2\pi i G (1 + G \ln \frac{\Lambda}{s} + G^2 (\ln^2 \frac{\Lambda}{s} - \pi^2) + \dots) \end{aligned} \quad (82)$$

in agreement with Eqs. (56) and (57).

Factorizable S-matrices for the SU(N) model were obtained in Ref. [13]. For $\bar{q}q$ and qq scattering, in our case only one of these solutions is suitable [see (25)]:

$$\begin{aligned}
 t_1 &= \chi(\theta) \frac{\Gamma(\frac{1}{2} + \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\lambda}{2} - \frac{\theta}{2\pi i})}{\Gamma(\frac{1}{2} - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\lambda}{2} + \frac{\theta}{2\pi i})}, \\
 t_2 &= \chi(\theta) (-\frac{\lambda}{2}) \frac{\Gamma(\frac{1}{2} + \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\lambda}{2} - \frac{\theta}{2\pi i})}{\Gamma(\frac{3}{2} - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2} + \frac{\lambda}{2} + \frac{\theta}{2\pi i})}, \quad (83)
 \end{aligned}$$

$$\begin{aligned}
 u_1(\theta) &= \chi(i\pi - \theta) \frac{\Gamma(1 - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2}\lambda + \frac{\theta}{2\pi i})}{\Gamma(\frac{\theta}{2\pi i}) \Gamma(1 + \frac{1}{2}\lambda - \frac{\theta}{2\pi i})}, \\
 u_2(\theta) &= \chi(i\pi - \theta) (-\frac{\lambda}{2}) \frac{\Gamma(1 - \frac{\theta}{2\pi i}) \Gamma(\frac{1}{2}\lambda + \frac{\theta}{2\pi i})}{\Gamma(1 + \frac{\theta}{2\pi i}) \Gamma(1 + \frac{1}{2}\lambda - \frac{\theta}{2\pi i})}, \quad (84)
 \end{aligned}$$

where $\chi(\theta)$ is a possible CDD factor (76). In Eqs. (83) and (84) the parameter λ is

$$\lambda = \frac{\ell}{N}. \quad (85)$$

Let us show that the factor $\chi(\theta)$ is necessary to combine Eqs. (83) and (84) with the perturbation theory (56) and (57). For large θ we have

$$\begin{aligned}
 t_1(\theta) \Big|_{\theta \rightarrow \infty} &= \chi(\theta) \left(\frac{\theta}{2\pi i}\right)^{-\frac{1}{2}\lambda} \left(-\frac{\theta}{2\pi i}\right)^{\frac{1}{2}\lambda} = e^{i\frac{\pi}{2}\lambda} \chi(\theta), \\
 t_2(\theta) \Big|_{\theta \rightarrow \infty} &= \chi(\theta) \left(-\frac{\lambda}{2}\right) \left(\frac{\theta}{2\pi i}\right)^{-\frac{1}{2}\lambda} \left(-\frac{\theta}{2\pi i}\right)^{\frac{\lambda}{2}-1} = \frac{\lambda}{2} \frac{2\pi i}{\theta} e^{i\frac{\pi}{2}\lambda} \chi(\theta). \quad (86)
 \end{aligned}$$

Therefore to obtain the perturbative result (69) and (64), we need to include a factor $\chi(\theta)$ with the asymptotic behaviour

$$\chi(\theta) \Big|_{\theta \rightarrow \infty} = e^{-i\frac{\pi}{2}\lambda}. \quad (87)$$

The simplest example of such a function is [see (77)]

$$\chi(\theta) = - \frac{\text{Sh} \frac{1}{2} (\theta + i\pi - i\frac{\pi}{2}\lambda)}{\text{Sh} \frac{1}{2} (\theta - i\pi + i\frac{\pi}{2}\lambda)} . \quad (88)$$

The S-matrix with the CDD factor (88) has a compound state in the $q\bar{q}$ channel (83) at

$$\theta = i\pi(1 - \frac{\lambda}{2}) \quad (89)$$

and one in the qq channel (84) at

$$\theta = i\pi \frac{\lambda}{2} . \quad (90)$$

For $N = 2$ we have $\lambda = 1$ due to (85) and the q and $\bar{q}$ representations coincide, which results in relations

$$t_i(\theta) = u_i(\theta) . \quad (91)$$

Using expression (66) for θ , Eq. (89) can be written in the form

$$\ln \frac{s}{\Lambda} + \frac{2}{N} \frac{1}{G} - \frac{1}{N} \frac{\pi^2}{3} G + \dots = i\pi(1 - \frac{\lambda}{2}) . \quad (92)$$

We recall that G is a function of p due to Eq. (26) and

$$G = \frac{\ell}{\sqrt{s}} \quad (93)$$

where ℓ is the s-channel angular momentum. Therefore (92) can be considered as an equation on the Regge trajectory

$$\ell = \ell(s) \quad (94)$$

in the s-channel! Remarkably, in the non-Abelian gauge theory we obtain $q\bar{q}$ compound states for arbitrary large energies.

The expression (83) can be expanded in a series in $1/\theta$:

$$\begin{aligned} t_1(\theta) &= 1 - \frac{2\pi i}{N} \frac{1}{\theta} + \dots , \\ t_2(\theta) &= \frac{2\pi i}{N} \frac{1}{\theta} + \dots . \end{aligned} \quad (95)$$

By the use of Eqs. (66) we rearrange (95) in a series in G :

$$t_1(\theta) = 1 + 2\pi i \left[-\frac{1}{2N} G + G^2 \left(\frac{1}{2} \ln \frac{\Lambda}{S} + \frac{1}{2} i\pi \left(1 + \frac{1}{N^2} \right) \right) + \dots \right], \quad (96)$$

$$t_2(\theta) = 2\pi i \left[\frac{1}{2} G + G^2 \left(-\frac{1}{2} N \ln \frac{\Lambda}{S} - \frac{\pi i}{2N} \right) + \dots \right].$$

It is not difficult to verify that (96) coincides with the perturbative expansion (56) for the SU(N) symmetry.

Now we return to the problem of building the S matrix with quasi-elastic unitarity (see Section 2). For simplicity we discuss here only the case of the O(3) model where the S-matrix with s and u-channel elastic unitarity is especially simple [see (80)].

It is convenient to write three invariant amplitudes σ_i (74) in one vector

$$S = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \quad (97)$$

Then we obtain for this model the following evolution equation [cf. (23)]:

$$\frac{\partial S}{\partial G \left(\frac{\mu^2}{\lambda^2} \right)} = \begin{pmatrix} -4 \ln \frac{S}{\Lambda} + 4\pi i, & 2\pi i, & -2 \ln \frac{S}{\Lambda} + 2\pi i \\ 2 \ln \frac{S}{\Lambda}, & 0, & 2 \ln \frac{S}{\Lambda} - 2\pi i \\ -2 \ln \frac{S}{\Lambda}, & -2\pi i, & -4 \ln \frac{S}{\Lambda} \end{pmatrix} S, \quad (98)$$

which sums the contributions of soft virtual gluons.

If we use as an initial condition for Eq. (98) at $\lambda^2 = M^2$ the free theory result $S = 1$ then the perturbative expansion of its solution is

$$\sigma_1 = 2\pi i G (1 + G (\ln \frac{\Lambda}{s} + \pi i)) + G^2 \left(\frac{2}{3} \ln^2 \frac{\Lambda}{s} + 2\pi i \ln \frac{\Lambda}{s} - 2\pi^2 \right),$$

$$\sigma_2 = 1 - 2\pi^2 G^2 (1 + O.G + \dots), \quad (99)$$

$$\sigma_3 = -2\pi i G (1 + G \ln \frac{\Lambda}{s} + G^2 \left(\frac{2}{3} \ln^2 \frac{\Lambda}{s} - \frac{2\pi i}{3} \ln \frac{\Lambda}{s} - \frac{2\pi^2}{3} \right) + \dots).$$

Let us compare these formulas with that for the elastic unitarity case Eq. (82). We discover in particular that the imaginary part of the scattering amplitude f_2

$$f_2 = \frac{\sigma_2 - 1}{i} \quad (100)$$

proportional to the total cross-section at fixed p contains in the elastic case a negative contribution $\sim -2\ln(s/\Lambda)G^3$ related to the gluon reggeization in the Yang-Mills theory, but this suppression is absent in the quasi-elastic case due to the cancellations of contributions of real and virtual soft gluons. This means that the total cross-section in the quasi-elastic case is larger than that in the elastic case.

At very large energies the behaviour of S (97) is determined by a maximal eigenvalue of the eigenvalue equation corresponding to (98)

$$\lambda \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 0, & \frac{8}{3}\pi i, & 0 \\ 2\pi i, & -2\ln \frac{s}{\Lambda} + \pi i, & \frac{5}{3}\pi i \\ 0, & \pi i, & -6\ln \frac{s}{\Lambda} + 3\pi i \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} \quad (101)$$

where we have introduced the invariant amplitudes S_T describing the scattering with the fixed value T of the t -channel isospin:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}, & 1, & \frac{1}{3} \\ \frac{1}{2}, & 0, & -\frac{1}{2} \\ \frac{1}{2}, & 0, & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \quad (102)$$

To determine the solution of Eq. (101) we use an iterative procedure. At large energies the matrix in the right-hand side of Eq. (101) is diagonalized and we obtain two approximate eigenfunctions:

$$\begin{aligned} S^{(1)} &\simeq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda^{(1)} \simeq -2 \ln \frac{s}{\Lambda}, \\ S^{(2)} &\simeq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda^{(2)} \simeq -6 \ln \frac{s}{\Lambda}, \end{aligned} \quad (103)$$

which describe reggeized amplitudes with the t-channel quantum numbers $T = 1, 2$.

To find the third eigenfunction and its eigenvalue we assume that $\lambda^{(3)}$ is small:

$$|\lambda^{(3)}| \ll 1. \quad (104)$$

Then λ can be neglected in all equations except the first one in (101) and we calculate iteratively:

$$\begin{aligned} S_2 &\simeq \frac{\pi i}{6 \ln \frac{s}{\Lambda}} S_1, \\ S_1 &\simeq \frac{\pi i}{\ln \frac{s}{\Lambda}} S_0, \\ \lambda^{(3)} S_0 &= \frac{8}{3} \pi i S_1 \simeq -\frac{8}{3} \frac{\pi^2}{\ln \frac{s}{\Lambda}} S_0 \end{aligned} \quad (105)$$

and therefore

$$S^{(3)} \simeq \begin{pmatrix} 1 \\ \pi i / \ln \frac{s}{\Lambda} \\ -\frac{\pi^2}{6 \ln^2 \frac{s}{\Lambda}} \end{pmatrix}, \quad \lambda^{(3)} \simeq -\frac{8}{3} \frac{\pi^2}{\ln \frac{s}{\Lambda}}, \quad (106)$$

which gives the amplitude with vacuum t-channel quantum numbers at high energies.

In the case where the initial conditions for Eqs. (98) are trivial ($S=1$), we obtain in the large energy asymptotics

$$S \simeq S^{(3)} l^{G\lambda^{(3)}} - \frac{\pi i}{\ln \frac{S}{\Lambda}} S^{(4)} l^{G\lambda^{(4)}} + \frac{\pi^2}{6 \ln^2 \frac{S}{\Lambda}} S^{(5)} l^{G\lambda^{(5)}} \quad (107)$$

and therefore the imaginary part of f_2 (100) decreases logarithmically:

$$\text{Im } f_2 /_{S \rightarrow \infty} \simeq -G\lambda^{(3)} \simeq \frac{8}{3} \frac{\pi^2}{\ln \frac{S}{\Lambda}} G \quad (108)$$

We can compare it with the elastic case (80) where

$$\text{Im } f_2 /_{S \rightarrow \infty} \simeq \frac{2\pi^2}{\ln^2 \frac{S}{\Lambda}} \quad (109)$$

Both asymptotics (108) and (109) agree with the unitarity restriction $\text{Im} f_2 < 1$ but in the quasi-elastic case (108) the partial cross-sections $\sigma(\rho)$ falls not as rapidly as in the elastic case. We assume that in the multi-Regge unitarity regime $\sigma(\rho)$ tends to a constant in agreement with the Froissart-like picture.

Now we briefly discuss the situation when the elastic S-matrix (80) is used as an initial condition for the evolution equation (98) at $\lambda = M$:

$$\left(\begin{array}{c} S_0 \\ S_1 \\ S_2 \end{array} \right) \bigg|_{\substack{\lambda \rightarrow M \\ S \rightarrow \infty}} \simeq \left(\begin{array}{c} 1 \\ 2\pi i / \ln \frac{S}{\Lambda} \\ -\pi^2 / \ln^2 \frac{S}{\Lambda} \end{array} \right) \quad (110)$$

This means that instead of Eq. (107) we have

$$S \simeq S^{(3)} l^{G\lambda^{(3)}} + \frac{\pi i}{\ln \frac{S}{\Lambda}} S^{(4)} l^{G\lambda^{(4)}} - \frac{5\pi^2}{6 \ln^2 \frac{S}{\Lambda}} S^{(5)} l^{G\lambda^{(5)}} \quad (111)$$

and the result (109) is also valid in this case.

6. - CONCLUSION

In this paper we built examples of S-matrices in the non-Abelian gauge theory satisfying elastic and quasi-elastic unitarity. In these approximations, the s-channel partial waves $f(s,\rho)$ fall logarithmically with growing energy obeying the Froissart bound. The next step would be to build an S-matrix with multi-Regge unitarity. In this case the approach related to the solution of the evolution equations of the type of (98) may be helpful but we must give up the leading logarithmic approximation and take into account non-leading terms of order $(g^2)^n \ln(q^2/\lambda^2)$. After that the equation analogous to Eq. (98) will relate the amplitudes with a different number of produced particles. We may hope that the corresponding eigenvalue equation can be solved iteratively in a way similar to the one used in the last section [see Eqs. (105) and (106)].

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6