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BEAM-LIKE GRAVITATIONAL WAVES AND THEIR GEODESICS

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A B S T R A C T

Exact solutions for the gravitational wave produced by an impulsive, massless beam of arbitrary energy profile are constructed in any number of space-time dimensions. Geodesics can be explicitly computed and, for homogeneous, axisymmetric, finite-size beams, they exactly focus at the location of the curvature singularity found in the infinite shell collision.

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1. Introduction. In 1971 Aichelburg and Sexl (AS) [1] found an exact solution of the classical Einstein equations, describing the plane fronted, impulsive wave generated by a massless particle of given energy E ($p_z = p = E$).

Recently, Dray and 't Hooft, using new techniques, were able to rederive [2] the AS metric and to extend it [3] to the case of a planar shell of null matter with constant energy density along its wavefront. They also constructed [3] the exact solution corresponding to the head-on collision of two such waves, exhibiting singularities analogous to those found earlier [4,5] in the collision of sourceless plane waves.

In this note we shall first extend, almost trivially, the work of Dray and 't Hooft to the case of waves produced by a massless beam of arbitrary energy distribution on the wavefront (hyper) plane. Having in mind further studies of string-theoretic corrections, we shall also extend our calculations to any number of space-time dimensions.

Then, as a preliminary to the study of the fully-fledged wave collision problem, we consider the shape of null geodesics for the general axisymmetric case. We are able to find explicit solutions for the geodesics and to discuss some of their amusing properties. In particular, we find that the wave associated with a beam of finite size and constant energy density ρ acts as a perfect converging lens for massless (null) incident test particles (geodesics) with a focal distance given by:

$$t_F = -z_F = (D/2 - 1) (8\pi G \rho)^{-1} \quad (1)$$

where G is Newton's constant and $c=1$ throughout this paper.

This expression, which agrees completely with a recent low energy density result by one of us [6], also coincides, for any D^* , with the location of the curvature singularity found in refs.[4,5,3], reinforcing the belief [7] that focusing is a crucial ingredient in the generation of such singularities.

*)The results of ref. 3 for the location of the singularities can be extended to arbitrary D in a straightforward way.

2. Beam-shaped waves. Following Dray and 't Hooft [2] we look for a metric of the form:

$$ds^2 = -dudv + f(x,y)\delta(u) du^2 + dx^2 + dy^2 \quad (2)$$

corresponding to an impulsive wave moving along the positive z axis. Here $u=t-z$, $v=t+z$ and the wave-front is $u=0$. For the D -dimensional case we generalize the ansatz (2) as:

$$ds^2 = -dudv + f(\underline{x})\delta(u) du^2 + d\underline{x}^2 \quad (2')$$

where $\underline{x}$ is the set of $(D-2)$ transverse coordinates.

Following again ref.[2], one finds the only non vanishing component of the Ricci tensor to be:

$$R_{uu} = -1/2 \Delta f \delta(u) \quad (3)$$

where Δ is the d'Alembertian in $(D-2)$ dimensions. Einstein's equations then require:

$$\Delta f \delta(u) = -16\pi G T_{uu} = -16\pi G \rho(\underline{x}) \delta(u) \quad (4)$$

where we have used the known expression of the stress-energy tensor for an impulsive null source having energy density $\rho(\underline{x})$ on the wave-front. Eq.(4) is immediately solved for arbitrary $\rho(\underline{x})$, since one knows the Green's functions of the problem:

$$\begin{aligned} \Delta \log r &= 2\pi \delta^{(2)}(x,y) \\ \Delta r^{4-D}/(4-D) &= \Omega_{D-2} \delta^{(D-2)}(\underline{x}) \end{aligned} \quad (5)$$

in which $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the solid angle in d -dimensions and $r^2 = \underline{x}^2$.

For the two particular cases of a single particle (AS) and of an infinite, homogeneous shell (Dray-'t Hooft) we obtain, respectively:

$$\rho(\underline{x}) = \rho \delta^{(D-2)}(\underline{x}) \quad : \quad f = -16\pi G \rho r^{4-D} / \Omega_{D-2} (4-D) \quad (6a)$$

$$\rho(\underline{x}) = \rho = \text{constant} \quad : \quad f = -8\pi G \rho r^2 / (D-2) \quad (6b)$$

From now on we shall concentrate on the axisymmetric case, the z axis being identified with the beam axis. Since $\rho(\underline{x}) = \rho(r)$ implies $f(\underline{x}) = f(r)$, it is straightforward to check that:

$$f'(r) = -16\pi G E(r) r^{3-D} / \Omega_{D-2} \xrightarrow[D \rightarrow 4]{} -8G E(r)/r \quad (7)$$

where $E(r)$ is the energy located on the wavefront within a distance r from the axis. The quantity $f'(r)/r$ will control the magnitude of the deflection angles for geodesics, as we shall now discuss.

3. Null Geodesics. Consider a generic test particle of zero mass travelling, in the far past, in the direction opposite to that of the wave and at distance b (say in the $x=x_1$ direction) from the beam axis. The corresponding geodesic will obviously lie in the $x-z$ plane. In order to determine it, we note that the only non-vanishing components of the Christoffel symbols are:

$$\begin{aligned} \Gamma_{uu}^v &= -f(r) d/du \delta(u) \\ \Gamma_{ru}^v &= \Gamma_{ur}^v = -f'(r) \delta(u) \\ \Gamma_{uu}^r &= -f'(r) \delta(u)/2 \end{aligned} \quad (8)$$

where the prime denotes derivative with respect to r . The geodesic equations are easily solved to give (discontinuous, broken) straight lines parametrized by:

$$\begin{aligned} x &= b + f'(b) u \theta(u)/2 \\ v &= f'(b)^2 u \theta(u)/4 + f(b) \theta(u) \end{aligned} \quad (9)$$

with $\theta(u)$ the usual step function. Note that, at $u=0$, the particle (the geodesic) suffers a shift [2] in space and in time given by:

$$\Delta t = \Delta z = \Delta v/2 = f(b)/2 \quad (10)$$

This shift, which is instantaneous due to the impulsive nature of the wave, will play an important role in improving the focusing process, as discussed below. It is amusing to notice that Δt and Δz of eq. (10), after use of eq.(6a) and insertion into the quantum plane wave phase $p \cdot x/\hbar$, yield precisely the phase shift found in ref.[8] (see also ref. [9]), when quantum string and higher order classical effects are neglected.

After the impact with the wave, the test particle hits the z -axis ($x=0$) at:

$$\begin{aligned} u = u_F &= -2b/f'(b) \xrightarrow{D \rightarrow 4} b^2/(4G E(b)) \\ v = v_F &= -bf'(b)/2 + f(b) \xrightarrow{D \rightarrow 4} 4G E(b) + f(b) \end{aligned} \quad (11)$$

which are immediately convertible into a z_F and a t_F . Thus a ring of null geodesics lying at a distance b from the axis, converges, after a time t_F to the point z_F , but, in general, z_F and a t_F depend upon b . For example, in the AS metric in $D=4$,

$$\begin{aligned} t_F &= 2Gp + b^2/8G E(b) - 4Gp \ln b/b_0 \\ z_F &= 2Gp - b^2/8G E(b) - 4Gp \ln b/b_0 \end{aligned} \quad (12)$$

where b_0 is an irrelevant scale [2]. Depending on whether the ratio $b/4GE(b)$ is larger or smaller than 1, the particle is either "refracted" or "reflected" by the wave. It is captured by it only when this ratio goes to zero, this being presumably the equivalent of a particle's capture by a black hole.

Can we make both z_F and a t_F independent of b ? This looks a priori hard since one has to solve two conditions, (see eqs.(11)), in terms of the single function $f(b)$. Surprisingly, however, and thanks to the shift term $f(b)$, there is a solution,

$$f'(b) = \text{const. } b \quad (13)$$

yielding $v_F = 0$ ($z_F = -t_F$) and, by eq.(7),

$$E(r) \propto r^{D-2} \quad (14)$$

This solution precisely corresponds to a beam of constant energy density. Inserting the expression (6b), valid in this case, gives:

$$u_F/2 = -z_F = t_F = (D-2) (16\pi G \rho)^{-1} \quad (15)$$

i.e. the result anticipated in eq.(1).

If the beam has constant energy density over a radius R and vanishing (as in ref.[6]) or just decreasing density outside, perfect focusing at the position (15) still occurs for all geodesics having $b < R$, but not, of course, for the others, which diverge from each other as b is increased.

The above results can be easily extended to the case of geodesics coming out of the z -axis at some $u_0 < 0$ (and, say, at $v_0=0$) with a non-vanishing angle θ_1 in the x - u plane. They will hit the wave-front at $u=0$ and $x = b = -u_0 \tan \theta_1$.

It is again straightforward (if a little more tedious) to compute the corresponding geodesics. We shall omit their explicit form here, and just quote the result for the "focus" i.e. for the point in which the geodesic corresponding to an angle θ_1 and an impact parameter b intersects again, possibly, the z -axis:

$$v_F = -bf'(b)/2 + f(b) \quad (16a)$$

$$u_F = -2b/(f'(b) + 2\tan\theta_1) > 0 \quad \text{---->} \quad \tan\theta_1 < -f'(b)/2 \quad (16b)$$

Note that only u_F is affected by the value of θ_1 !

Defining $\text{tg}\theta_2 = b/u_F$ we can rewrite eq. (16b) as:

$$\begin{aligned} 1/u_0 + 1/u_F &= -f'(b)/2b \\ \text{tg}\theta_1 + \text{tg}\theta_2 &= -f'(b)/2 \end{aligned} \quad (17)$$

which agrees with ref.[2] in the AS case and looks precisely as the equation of conjugate points of a thin anastigmatic lense of focus F provided that, again:

$$-f'(b)/2b = \text{const} = F \quad (18)$$

i.e. that we have an homogeneous disc! Thus, even for non parallel incident particles, an homogeneous, axisymmetric, finite size beam acts as a perfect converging lens.

4. Outlook. What are the implications of our results? We can see a few both in terms of practical applications and in terms of conceptual, even fundamental questions.

a) By going away from the (transverse) plane symmetric situation (infinite shell) discussed so far in the literature, our results could be of astrophysical interest, since they could apply to the case of realistically produced gravitational waves. Focusing, in some lucky circumstance, could increase the flux of gravitational energy impinging on earth, improving chances of detection.

b) The singularity encountered in the plane symmetric situation is often attributed to such a symmetry (see e.g. Tipler, ref.[5]). Our results would naively imply that an infinite energy density is formed even in the collision of finite-fronted waves, provided one of them, at least, is homogeneous (constant ρ). The only difference would be that the singularity is at a focal point rather than on a focal plane.

c) The remark just made brings up the question of how faithfully the geodesics simulate the actual collision's properties. The geodesics, by definition, do not interact with each other. In the actual collision, however, the gravitons interact with each other after being deflected. Such interactions are small as long as the spacial separation of the gravitons exceeds the gravitational radius associated with their invariant energy, but become strong and non-linear afterwards. As discussed in ref. [6], one can only argue that as much an energy gets focused in as small a region as to

imply, at least, the formation of a collapsed object, if not of a singularity. We plan to study soon the full collision problem, possibly in some limit, in order to clarify this issue.

d) Suppose that further work will establish that a singularity does form. It would be then extremely important [6] to study whether or not string corrections can avoid it. In this connection we can only make, for the time being, a simple-minded observation. We have seen (eqs. 9, 16) that exact focusing depends on a subtle cancellation between the v -shift term proportional to f and the deflection term controlled by f' . The v -shift is related to the quantum phase shift and it was shown in refs. [8] that string effects modify the phase shift to take into account the finite size λ_s of the string. Unless a miraculous cancellation occurs, there will be a quantum-string uncertainty of $O(\lambda_s)$ on v_F , implying that an infinite energy density situation is avoided. This point too deserves, of course, further study.

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