CERN-TH.4958/88

SUPERSYMMETRIZATION OF SCALAR FIELD THEORIES

J. Gamboa and J. Zanelli

Centro de Estudios Científicos de Santiago
Casilla 16443, Santiago 9, Chile
and
Departamento de Física
Facultad de Ciencias, Universidad de Chile
Santiago, Chile

and

M. Ruiz-Altaba

CERN - Geneva

A B S T R A C T

We develop a formalism for constructing the vacuum functional and supersymmetrizing a scalar field theory with the help of its ground state representation. The field theory problem is first transformed into a quantum-mechanical one for which the ground state representation is well defined. The theory is then supersymmetrized by "taking the square root" of the Hamiltonian. Standard approximation techniques are used to construct the vacuum functional with which spontaneous supersymmetry breaking can be analyzed.

CERN-TH.4958/88

January 1988

1. - INTRODUCTION

A few years ago, Darboux [1] showed that an elliptic D -dimensional differential operator can always be written in the form $H = \sum_{i=1}^D Q_i^{(\nu)+} Q_i^{(\nu)}$, where the Q 's are first-order linear operators. The ν labels the eigenstates of H and there is one representation of this kind for each eigenvalue of H [2]. The Q 's are in general singular operators, except for the lowest eigenvalue of H ($\nu=0$). This is the so-called ground state representation (GSR) of H . It plays a special rôle in quantum mechanics [3] and supersymmetry [4].

Recently, Gozzi [4] has found another application for this representation: it provides a recipe for constructing supersymmetric extensions of purely bosonic quantum theories. This idea, originally proposed for one-dimensional systems [5], has been extended to higher dimensions [6], to generally covariant systems [7], and to particles in interaction with external fields [8].

Despite all these applications, it is still not known how to generalize this prescription to other field theories and use it to understand supersymmetry breaking. In this paper we examine the case of scalar field theories. The strategy is to put the field theory in a box and then transform the field theory problem into one in quantum mechanics, whose GSR is well defined. The supersymmetric theory follows as a direct application of the arguments in Refs. [4] and [6]. We show that the Riccati equation [9] relating the potential to the ground state functional can always be transformed into a Schrödinger equation for a two-dimensional harmonic oscillator in an external field. Consequently, the vacuum functional for a scalar field theory can always be calculated to any desired accuracy, using the standard quantum mechanical methods. This paper is organized as follows. In Section 2 we first review the GSR in quantum mechanics, then extend it to scalar field theories and finally carry through its supersymmetrization. We illustrate the procedure with two examples in Section 3. Section 4 contains our conclusions.

2. - GSR IN QFT AND ITS SUPERSYMMETRIZATION

A) GSR in Quantum Mechanics

Consider a bosonic quantum system described by the three-dimensional Hamiltonian

$$H = -\frac{1}{2} \nabla^2 + U(\vec{r}) \quad (2.1)$$

Since no fermions are present, the ground state wave function is nodeless [10] and can always be written as

$$\psi_0 = e^{-V} \quad (2.2)$$

with V a real function. Schrödinger's equation for the ground state is of the form

$$H \psi_0 = 0$$

if the potential $U(\vec{r})$ has been conveniently adjusted. Substituting (2.2) into (2.3), we find that the potential U is related to V through the Ricatti equation

$$U(\vec{r}) = \frac{1}{2} (\vec{\nabla} V)^2 - \frac{1}{2} \nabla^2 V \quad (2.4)$$

Substituting (2.4) into (2.1), one can easily check that

$$H = \frac{1}{2} \vec{Q}^\dagger \cdot \vec{Q} \quad (2.5)$$

with

$$\begin{aligned} \vec{Q} &= \vec{\nabla} + \vec{\nabla} V \\ \vec{Q}^\dagger &= -\vec{\nabla} + \vec{\nabla} V \end{aligned} \quad (2.6)$$

The form (2.5) is the GSR of the Hamiltonian (2.1).

The supersymmetric extension of (2.5) requires the introduction of fermionic variables ψ_i ($i=1,2,3$) satisfying the following Clifford algebra:

$$\begin{aligned} \{\psi_i, \psi_j\} &= \{\psi_i^\dagger, \psi_j^\dagger\} = 0 \\ \{\psi_i, \psi_j^\dagger\} &= \delta_{ij} + \Sigma_{ij} \end{aligned} \quad (2.7)$$

Here, $\Sigma_{ij} = -\Sigma_{ji}$ is a bosonic operator whose explicit form depends on the representation used for the ψ_i . It is responsible for the spin-orbit coupling in supersymmetric quantum mechanics [6] and can be chosen to be zero as long as no gauge fields are present [8]. We limit ourselves to this case in what follows and set $\Sigma_{ij} = 0$.

Given (2.7), the supercharges are

$$\begin{aligned} Q &= Q_i \psi_i \\ Q^\dagger &= Q_i^\dagger \psi_i^\dagger \end{aligned} \quad (2.8)$$

They satisfy the standard supersymmetric algebra

$$\begin{aligned}\{\phi, \phi\} &= \{\phi^+, \phi^+\} = 0 \\ \{\phi, \phi^+\} &= 2H_s \\ [\phi, H_s] &= [\phi^+, H_s] = 0\end{aligned}\quad (2.9)$$

where H_s is, by definition, the supersymmetric Hamiltonian. H_s can be computed explicitly, using (2.6)-(2.8):

$$H_s = \frac{1}{2} \vec{P}^2 + \frac{1}{2} (\vec{\partial} V)^2 - \nabla_i \nabla_j V [\psi_i, \psi_j^\dagger] \quad (2.10)$$

Note that (2.10) follows also from the supersymmetry-invariant Lagrangian

$$L = \frac{1}{2} \dot{\vec{X}}^2 - \frac{1}{2} (\vec{\partial} V)^2 + \bar{\psi}^\dagger (\partial_t - V'') \bar{\psi} \quad (2.11)$$

which arises from dimensional reduction of the Wess-Zumino model.

B) GSR in Field Theory

Consider a self-interacting field theory, two-dimensional for simplicity, whose Hamiltonian is:

$$H = \frac{1}{2} \int dx \left\{ \pi^2(x) + u^2(\varphi) \right\} \quad (2.12)$$

with

$$u^2(\varphi) = \varphi'^2 + \mu^2 \varphi^2 + \mathcal{H}_{int}(\varphi) \quad (2.13)$$

where $\mathcal{H}_{int}(\phi)$ is an interaction Hamiltonian density of the general form

$$\mathcal{H}_{int}(\varphi) = g(x_1, x_2, x_3, x_4) \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4) \quad (2.14)$$

We do not consider interactions at a single point to avoid ultra-violet divergences in which we are not presently interested.

The canonical variables ϕ and π satisfy the usual commutation rules

$$\begin{aligned}[\varphi(x), \varphi(y)] &= [\pi(x), \pi(y)] = 0 \\ [\varphi(x), \pi(y)] &= i \delta(x, y)\end{aligned}\quad (2.15)$$

In order to construct the GSR, let us regularize the theory (2.12) by putting it in a square box of length π . The canonical variables can then be Fourier-expanded:

$$\begin{aligned}\pi(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \pi_n e^{-inx} \\ \varphi(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \varphi_n e^{-inx}\end{aligned}\quad (2.16)$$

The coefficients π_n, φ_n satisfy the commutation relations

$$\begin{aligned}[\varphi_n, \varphi_m] &= [\pi_n, \pi_m] = 0 \\ [\varphi_n, \pi_m] &= i \delta_{n,-m}\end{aligned}\quad (2.17)$$

and the hermiticity condition

$$\begin{aligned}\varphi_n^\dagger &= \varphi_{-n} \\ \pi_n^\dagger &= \pi_{-n}\end{aligned}\quad (2.18)$$

In terms of the Fourier coefficients, the Hamiltonian (2.12) becomes

$$H = \frac{1}{2} \sum_{n=-\infty}^{+\infty} (\pi_n^\dagger \pi_n + U_n^2 (\varphi_n^\dagger, \varphi_n)) \quad (2.19)$$

with

$$U_n^2 = (n^2 + \mu^2) \varphi_n^\dagger \varphi_n + \mathcal{H}_n^{(int)} \quad (2.20)$$

We restrict ourselves to H_n of the form $H_n(\phi_n^\dagger, \phi_n)$. The Hamiltonian (2.19) describes an infinite set of particles, each living in a two-dimensional space with co-ordinates $\phi_n, \phi_n^\dagger$.

The next issue is the connection with the ground state. Fortunately, this problem was solved almost thirty years ago by Coester and Haag [11], who showed that the vacuum functional for the theory (2.12) is of the form (2.2), with $V = V(\phi)$. Furthermore, V has the remarkable property $V(\phi) = \sum_k \Lambda(\phi_k)$ if space is discretized in k cells. The field theory is thus isomorphic to quantum mechanics and (2.19) may be written as

$$H = \frac{1}{2} \sum_{n=-\infty}^{+\infty} Q_n^\dagger Q_n \quad (2.21)$$

with

$$\begin{aligned} Q_n &= \frac{1}{\sqrt{2}} \left(\pi_n - i \frac{\partial \Lambda}{\partial \varphi_n} \right) \\ Q_n^+ &= \frac{1}{\sqrt{2}} \left(\pi_n^+ + i \frac{\partial \Lambda}{\partial \varphi_n^+} \right) \end{aligned} \quad (2.22)$$

It is easy to check that an equation of the Riccati type relates V to U_n^2 :

$$\frac{\partial \Lambda}{\partial \varphi_n} \frac{\partial \Lambda}{\partial \varphi_n^+} - \frac{\partial^2 \Lambda}{\partial \varphi_n \partial \varphi_n^+} = (n^2 + \mu^2) \varphi_n^+ \varphi_n + \mathcal{H}_n(\varphi_n^+, \varphi_n) \quad (2.23)$$

The supersymmetrization of (2.21) is straightforward. First define the fermionic variables (α, β are spinorial indices)

$$\begin{aligned} \{\psi_{n\alpha}, \psi_{m\beta}\} &= \{\psi_{n\alpha}^+, \psi_{m\beta}^+\} = 0 \\ \{\psi_{n\alpha}, \psi_{m\beta}^+\} &= \delta_{\alpha\beta} \delta_{nm} \end{aligned} \quad (2.24)$$

The nilpotent supercharges are

$$\begin{aligned} Q_{n\alpha} &= Q_n \psi_{n\alpha} \\ Q_{n\alpha}^+ &= Q_n^+ \psi_{n\alpha}^+ \end{aligned} \quad (2.25)$$

The Fourier modes of the supersymmetric Hamiltonian density defined by

$$\{Q_{n\alpha}, Q_{m\beta}\} = 2 \delta_{nm} \delta_{\alpha\beta} \mathcal{H}_n^S \quad (2.26)$$

are, explicitly,

$$\mathcal{H}_n^S = \frac{1}{2} \pi_n^+ \pi_n + \frac{1}{2} \frac{\partial \Lambda}{\partial \varphi_n} \frac{\partial \Lambda}{\partial \varphi_n^+} - \frac{\partial^2 \Lambda}{\partial \varphi_n \partial \varphi_n^+} [\psi_{n\alpha}, \psi_{n\alpha}^+] \quad (2.27)$$

The superalgebra closes

$$\begin{aligned} \{Q_{n\alpha}, Q_{m\beta}\} &= \{Q_{n\alpha}^+, Q_{m\beta}^+\} = 0 \\ \{Q_{n\alpha}, Q_{m\beta}^+\} &= 2 \delta_{\alpha\beta} \delta_{nm} \mathcal{H}_n^S \\ [\mathcal{H}_n^S, Q_{m\alpha}] &= [\mathcal{H}_n^S, Q_{m\alpha}^+] = 0 \end{aligned} \quad (2.28)$$

and coincides with the superalgebra (2.9) mode by mode. The modes (2.27) arise from a Hamiltonian density

$$\mathcal{H}^S = \frac{1}{2} \pi^2(x) + \frac{1}{2} \frac{\delta V}{\delta \varphi} \frac{\delta V}{\delta \varphi} - \frac{\delta^2 V}{\delta \varphi(x) \delta \varphi(x')} [\psi_\alpha(x), \psi_\alpha^+(x')] \quad (2.29a)$$

which corresponds precisely to the Hamiltonian for the Wess-Zumino model.

Before finishing this section, let us briefly discuss the spin-orbit coupling in field theory^{*}). Consider the triplet $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} - U(\vec{\phi}) \quad (2.30)$$

with $U(\vec{\phi})$ a polynomial in $\vec{\phi} \cdot \vec{\phi}$. The Hamiltonian in the discretized version is

$$\begin{aligned} H &= \sum_{n=-\infty}^{+\infty} \sum_{i=1}^3 [\pi_{ni}^+ \pi_{ni} - U_n^z(\varphi_{ni}^+, \varphi_{ni})] \\ &= \sum_{n=-\infty}^{+\infty} \sum_{i=1}^3 Q_{ni}^+ Q_{ni} \end{aligned} \quad (2.31)$$

with the charges as in (2.22) with an extra index i .

In order to supersymmetrize the theory, introduce spinors ψ_{ni} realized as

$$\begin{aligned} \psi_{ni} &= \sigma_i \otimes (\tau_+)_n \\ \psi_{ni}^+ &= \sigma_i \otimes (\tau_-)_n \end{aligned} \quad (2.32)$$

where σ_i are Pauli matrices, $\otimes$ means tensor product, and the notation $(\tau_\pm)_n$ implies "one $\tau_\pm$ for each mode", with $\tau_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. The supercharges are

$$\begin{aligned} Q_n &= Q_{ni} \sigma_i \otimes (\tau_+)_n \\ Q_n^+ &= Q_{ni}^+ \sigma_i \otimes (\tau_-)_n \end{aligned} \quad (2.33)$$

which satisfy the superalgebra (2.28) with H_n^S given by

^{*}) Rigorously speaking, it is an "isospin-orbit" coupling, since the number of fields plays the approximate rôle of space-time dimensionality.

$$H_n^S = \begin{pmatrix} \vec{Q}_n^+ \cdot \vec{Q}_n + i(\vec{Q}_n^+ \times \vec{Q}_n) \cdot \vec{\sigma} & 0 \\ 0 & \vec{Q}_n^+ \cdot \vec{Q}_n - i(\vec{Q}_n^+ \times \vec{Q}_n) \cdot \vec{\sigma} \end{pmatrix} \quad (2.34)$$

Here, the term $(\vec{Q}_n^+ \times \vec{Q}_n) \cdot \vec{\sigma}$ is the field-theoretic analogue of the spin-orbit coupling discussed in Ref. [6].

3. - EXAMPLES

A) Two-Dimensional Klein-Gordon Field

Although somewhat trivial, this example illustrates very well the procedure developed in Section 2. The potential is of the form (2.13) with $H_{\text{int}} = 0$. Putting the theory in a box, the operators Q_n are given by (2.23) with

$$\frac{\partial \Lambda}{\partial \varphi_n} = \sqrt{n^2 + \mu^2} \varphi_n \quad (3.1)$$

The Hamiltonian can be written in either of the two forms

$$\begin{aligned} H_+ &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} Q_n^+ Q_n \\ H_- &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} Q_n Q_n^+ \end{aligned} \quad (3.2)$$

We define the solution to this ordering problem (more on this below) as

$$\begin{aligned} H &= \frac{1}{2} (H_+ + H_-) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} (Q_n^+ Q_n + Q_n Q_n^+) \\ &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} : Q_n^+ Q_n : \end{aligned} \quad (3.3)$$

The supersymmetrization of the system requires the introduction of fermions ψ_n satisfying the commutation relations (2.25). The anticommutator of the supercharges (2.26) yields the supersymmetric Hamiltonian

$$H_n^S = \frac{1}{2} \pi_n^+ \pi_n + \frac{1}{2} (n^2 + \mu^2) \varphi_n^+ \varphi_n \quad (3.4)$$

which coincides with the purely bosonic Hamiltonian. This is indeed the case for the free Wess-Zumino model [12].

The ordering chosen to define (3.3) agrees with the usual "normal" one. A simple calculation shows that Schrödinger's equations for (3.2) in the ground state are

$$H_{\pm} \psi_0 = \pm E_0 \psi_0$$

with $E_0 = \sum_{n=-\infty}^{+\infty} \sqrt{n^2 + \mu^2}$. Hence $H\psi_0 = 0$, so our ordering (3.3) yields a good GSR. A necessary and sufficient condition the zero-point energy to vanish is

$$Q_n \psi_0 = 0 \quad (3.5)$$

which can be easily integrated to yield

$$\psi_0 = \exp \left[-\frac{1}{2} \sum_{n=-\infty}^{+\infty} \sqrt{n^2 + \mu^2} \varphi_n^+ \varphi_n \right] \quad (3.6)$$

Taking its inverse Fourier transform, we find the usual vacuum functional [13]:

$$\psi_0[\varphi] = \exp \left[-\frac{1}{2} \int dx \varphi(x) \left(-\frac{\partial^2}{\partial x^2} + \mu^2 \right)^{1/2} \varphi(x) \right] \quad (3.7)$$

This functional is normalizable, hence supersymmetry is unbroken.

Note that in the limit $\mu \rightarrow 0$, (3.7) is just the ground state for the bosonic string [7]. Only in the limit $\mu = 0$ do the operators Q_n satisfy the hermiticity condition $Q_n^+ = Q_{-n}$, which restricts the sum in the exponent (3.6) to the positive values of n . Although the string Hamiltonian (in the orthonormal gauge) is given by (3.5), one might worry that all the other constraints ($L_n, n > 0$) do not arise in our limiting procedure. This happens because the free Klein-Gordon Hamiltonian corresponds to a theory whose Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial \varphi)^2 \quad (3.8)$$

lacks the gauge invariance of the full

$$\mathcal{L} = -\frac{1}{2} \sqrt{|g|} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \quad (3.9)$$

because the conformal factor ($g_{\alpha\beta} = \Omega \cdot \eta_{\alpha\beta}$) has been fixed to $\Omega = -1/\sqrt{|g|}$. The Hamiltonian of (3.9) vanishes (a feature common to all physical systems with general co-ordinate invariance) and the formalism must start from the constraints of the theory. The GSR representation for, and fermionization of, the bosonic string have already been discussed in Ref. [7].

B) Scalar Field Interacting with a Scalar Source

This second example is also soluble by standard methods, but in order to make the problem equivalent to case A, we need to find a formal solution to the Ricatti equation. The Hamiltonian is given by (2.12) with $H_{\text{int}} = 2g(x)\phi(x)$. In a box, the theory takes the form (2.19) with $H_n(\phi_n^+, \phi_n^-) = 2g_n\phi_n$, where g_n are the Fourier coefficients of $g(x)$. U is related to the logarithm of the vacuum functional through the Ricatti equation (2.23). Letting

$$\Lambda = -\log F \quad (3.10)$$

the Ricatti equation becomes

$$\frac{\partial^2 F}{\partial \varphi_n^+ \partial \varphi_n^-} - [(n^2 + \mu^2)\varphi_n^+ \varphi_n^- + 2g_n \varphi_n] F = 0 \quad (3.11)$$

Changing variables by

$$\begin{aligned} \xi &= \varphi_n + \varphi_n^+ \\ \eta &= \varphi_n - \varphi_n^+ \end{aligned} \quad (3.12)$$

the equation becomes

$$\frac{\partial^2 F}{\partial \xi^2} - \frac{\partial^2 F}{\partial \eta^2} - \left[\frac{1}{4}(n^2 + \mu^2)(\xi^2 - \eta^2) + g_n(\xi + \eta) \right] F = 0 \quad (3.13)$$

which can be Euclideanized by letting $\xi \rightarrow i\xi$. The resulting equation is then Schrödinger's in two dimensions for a harmonic oscillator in an external field, which can be solved exactly [14] by shifting the field modes. The vacuum functional is thus very similar to the free Klein-Gordon case, namely

$$\psi_0[\varphi] = \exp \left[-\frac{1}{2} \int dx (\varphi - \varphi_g) \left(-\frac{\partial^2}{\partial x^2} + \mu^2 \right)^{1/2} (\varphi - \varphi_g) \right] \quad (3.14)$$

where ϕ_g is a solution of

$$\left(-\frac{\partial^2}{\partial x^2} + \mu^2 \right) \varphi_g(x) = -g(x) \quad (3.15)$$

which can always be formally integrated. The supersymmetrization is now straightforward as in example A.

We thus conclude that both our very simple examples have well-behaved (normalizable) vacuum functionals and supersymmetry is unbroken.

4. - CONCLUSIONS

We have studied the ground state representation (GSR) and its supersymmetrization for scalar field theories without gauge fields. In two simple but illustrative cases we have constructed the vacuum functionals explicitly.

According to the supersymmetrization technique developed above, the field theory problem can be mapped into one in quantum mechanics for which the Ricatti equation (2.24) can always be written in the form

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} + \left[\lambda_n^2 (\xi^2 + \eta^2) + \mathcal{H}_n(\xi, \eta) \right] F = 0 \quad (4.1)$$

In other words, a two-dimensional interacting field theory is reduced to the solution of Schrödinger's equation for two simple harmonic oscillators coupled by $\mathcal{H}_n$. For a three-dimensional field theory, the problem is equivalent to two coupled two-dimensional oscillators, and so on.

Equation (4.1) can be solved by the approximate procedures usual in quantum mechanics, thus enabling the (approximate) construction of vacuum functionals for any field theory, with quartic interactions as in (2.14). For instance, the choice in (4.1) of

$$\mathcal{H}_n(\xi, \eta) = \xi^4 + \eta^4 \quad (4.2)$$

and the use of variational methods allows us to find an approximate vacuum functional of the form (3.7). Work is in progress to introduce gauge fields in our formalism and investigate supersymmetry breaking in realistic theories.

Another novelty of the method presented here is the possibility to generate, via Darboux transformations [15], integrable non-trivial field theories from scalar field theories with known exact solutions. These results will be presented elsewhere.

ACKNOWLEDGEMENTS

J.G. and J.Z. thank R. Hojman and M. Montenegro for useful discussions, and FONDECYT-Chile for financial support. M.R.-A. thanks the Centro de Estudios Científicos de Santiago for hospitality during part of this work.

REFERENCES

- [1] G. Darboux - C.R. Acad.Sci. (Paris) 94 (1982) 1456.
- [2] R. Hojman and S. Hojman - "Darboux transformations in classical mechanics", Universidad de Chile report (1987), unpublished.
- [3] E. Schrödinger - Proc.Roy.Irish Acad. A46 (1940) 9; A46 (1940) 183; A47 (1941) 53;
L. Infeld and T.E. Hull - Rev.Mod.Phys. 23 (1951) 21.
- [4] E. Gozzi - Phys.Lett. 129B (1983) 432.
- [5] E. Witten - Nucl.Phys. B1285 (1981) 513.
- [6] J. Gamboa and J. Zanelli - Phys.Lett. 165B (1985) 91.
- [7] J. Gamboa and J. Zanelli - "Ground state representations and the square root of the bosonic string", to appear in Class.Quant.Grav.
- [8] J. Gamboa and J. Zanelli - "Supersymmetric Quantum Mechanics of Fermions Minimally Coupled to Gauge Fields", to appear in J.Phys. A.
- [9] P. Kumar, M. Ruiz-Altaba and B.S. Thomas - Phys.Rev.Lett. 57 (1986) 2749.
- [10] A. Galindo and R. Tarrach - Ann.Phys. 173 (1987) 430.
- [11] F. Coester and R. Haag - Phys.Rev. 117 (1960) 1137.
- [12] See, e.g., P. Ramond - "Field Theory, a Modern Primer", Benjamin (1981), New York.
- [13] T. Barnes and G.I. Ghandour - Nucl.Phys. B166 (1979) 125;
C. Kimstedt - Nuovo Cimento 53 (1979) 133.
- [14] See, e.g., L.D. Landau and E.M. Lifshitz - "Quantum Mechanics", Mir (1958), Moscow.
- [15] W.M. Zheng - J.Math.Phys. 25 (1984) 88;
P. Roy and R. Raychaudhury - Phys.Rev. D33 (1986) 594.