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THE NAMBU-GOTO STRING:
ITS PHASE-SPACE PATH-INTEGRAL

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ABSTRACT

Using the Batalin-Fradkin-Vilkovisky formalism for constrained systems, the space-phase path-integral representation of the free bosonic string propagator is considered and compared with the Polyakov approach. A Gribov problem exists for this system with the consequence that the integration measure over modular space is left undetermined. We initially consider the case of the scalar particle with special emphasis on analogies with the string case. The question of Wick rotations in modular space is also discussed.

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I. INTRODUCTION

Polyakov's path-integral formulation [1] for the quantization of string theories [2] has led to much insight into their structure and properties [3]. Profound but not yet well understood relationships with apparently independent fields in pure mathematics are being discovered [4].

Its construction proceeds by analogy with the functional integration over the gauge fields of Yang-Mills theories. In this case, the well-known Faddeev-Popov trick [5] instructs us to integrate over a gauge slice defined by some gauge-fixing condition, with measure the determinant of the gauge variation of this condition. For physical quantities, results obtained this way are gauge independent.

In Polyakov's approach to bosonic string theories (similar structures appear for string theories with world-sheet supersymmetries [1]), the group of invariant transformations is the semi-direct product of world-sheet diffeomorphisms and local Weyl rescalings of the intrinsic world-sheet metric. The functional integral is taken over a well-defined gauge slice of the space of space-time embeddings of the world-sheet and world-sheet metrics, with the Faddeev-Popov determinant as the integration measure which ensures that physical quantities are independent of the specific gauge slice used [1,3,6,7].

For a given physical process this construction has to be repeated for each possible world-sheet topology. When appropriately summed, these contributions define a perturbative expansion for the quantity of interest.

For each topology, Polyakov's integral reduces to a finite dimensional integral over the Teichmüller parameters of the world-sheet restricted to their fundamental domain, and a functional integral over the conformal mode of the world-sheet metric [6,7].

For some critical number of degrees of freedom however, the conformal mode decouples and the quantum theory is invariant under the full group of world-sheet symmetries defined above. For example, in the case of the bosonic string in a flat space-time this leads to the critical value 26 for the space-time dimension [1], together with a Regge-like spectrum of states with a tachyon and massless "spin 1" or "spin 2" states on the leading trajectory [8].

These critical numbers find their origin in the conformal anomaly of the theory. Their relation to the world-sheet gravitational anomaly and to the Schwinger term in the Virasoro algebra (holomorphic anomaly) is well understood [9].

Introducing Faddeev-Popov ghosts to express the Faddeev-Popov determinant, one obtains a gauge-dependent effective action invariant under a BRST symmetry. This symmetry is a consequence of the original invariance of the system [3]. It is realized at the quantum level only when the same conditions which render the Polyakov path-integral independent of the conformal mode are met [10]. Existence of this BRST symmetry has led to important applications in recent developments of second-quantized string field theories [11].

Although it leads to well-defined quantities and (probably) to a physically consistent (unitary) theory of conformal classes on Riemann surfaces, Polyakov's approach to the path-integral for string theories remains very formal, leaving questions which warrant a better understanding unanswered.

Polyakov's construction for example, is carried out for space-time and world-sheet metrics of Euclidean signature. It is assumed that, as in quantum field theory, one can analytically continue the results "back" to a Minkowski signature by an inverse Wick rotation.

Also, measures for integrals over spaces of functions are implicitly defined by gaussian integrals over the corresponding tangent spaces (by analogy with integrals in finite-dimensional curved manifolds). Metrics introduced on these tangent spaces follow from the requirements of ultra-locality and invariance under world-sheet diffeomorphisms in an essentially unique way. Furthermore, they lead to (what seems to be) a unique and well-specified (though not explicit) measure for the integration over Teichmüller parameters [6,7].

A general procedure however exists for the quantization of systems with constraints, such as Yang-Mills and string theories. Dirac's original work on this problem was carried out within the framework of canonical quantization [12], which was then extended by Faddeev [13] and Popov [5] to a path-integral formulation.

Taking advantage of the full BRST invariance resulting from the constraints, Batalin, Fradkin and Vilkovisky (BFV) were able to obtain a complete generalization of these formulations for systems whose algebra of constraints has field-dependent structure functions and does not close off-shell [14,15]. Using the methods of canonical quantization, the BRST invariant quantum theory is then obtained. It is explicitly covariant under the symmetries generated by the constraints, as well as being equivalent to Dirac's quantization [15].

From the quantized BRST invariant theory, one can derive a phase-space functional integral representation for matrix elements of observable quantities. The integral over phase-space is taken with the Liouville measure and an effective action containing a gauge-fixing term. The path-integral is (formally) invariant under any BRST transformation [15]. The variation of the measure compensates that of the gauge-fixing term and, by explicit construction, the functional integral is invariant under the symmetries of the system [16]. For specific choices of the gauge-fixing function, one recovers [15] the phase-space path-integral given by Faddeev [13].

The BFV framework thus shows how the canonical, BRST and path-integral approaches for constrained systems are related and in some sense, unified. In the case of Yang-Mills theories, it is also the context in which the Faddeev-Popov trick finds its justification [13,15].

In this paper, we apply the BFV approach to the bosonic string (open and closed) propagating in a Minkowski space-time. More specifically, we consider the path-integral representation for the free string propagator between two fixed space-time configurations, and compare our results with those obtained for the same process within Polyakov's approach [17,18].

Since the BFV formulation provides a real time description of the system, comparing the Euclidean formulation, our results should tell us how a Wick rotation is to be performed in the case of string theory. A complete justification of the Wick rotation in the case of interacting strings however, cannot be hoped for with regards to the process under consideration here. Splitting and joining of strings may introduce new aspects to this problem. They might be studied by considering the three-string vertex within the BFV approach.

Since the measure of the phase-space path-integral is explicitly known, it should in turn determine how the integrals over world-sheet metrics and space-time embeddings must be carried out, particularly with reference to the measure for Teichmüller parameters.

We first examine these questions in the simple case of a scalar point-like particle. Some of the details presented here can be found in the literature [19-21]. They are nevertheless included for the sake of clarity and completeness. Other features of the system are emphasized to help understand analogies with the string case. This is done in Section 2.

We then shift our attention to the bosonic string and Section 3 reviews Dirac's approach, with the ensuing careful discussion of the issue of gauge fixing. Section 4 extends the discussion to the BFV framework, which leads to

the BRST quantization of the system [10, 22-24]. We find that a Gribov problem exists for the bosonic string.

The phase-space path-integral representation of the free string propagator is considered in Section 5 and explicitly evaluated both for open and closed strings. The results are compared with those obtained within Polyakov's approach [17,18]. Consequences of the Gribov problem are also discussed before presenting conclusions in Section 6. Some of the points under consideration here for the bosonic string can be found elsewhere [10,23-28]. Unfortunately they are not always treated without confusion [25-27] or even correctly [28].

Finally, we adopt notations given in Ref. [15], to which we also refer for a review of the BFV framework. Space-time and world-sheet coordinate indices are respectively labelled by $\mu, \nu, \dots = 0, 1, \dots, (D-1)$ and $\alpha, \beta, \dots = 0, 1$. We take $\eta^{\mu\nu} = (- + \dots +)$ as Minkowski metric, and world-sheet metrics of signature $(- +)$.

2. THE POINT PARTICLE

Let us consider the non-linear action

$$S[x^\mu] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\dot{x}^2(\tau)} = \int_{\tau_1}^{\tau_2} d\tau L(\dot{x}) \quad (2.1)$$

where τ parametrizes the particle world-line and $x^\mu(\tau)$ are D functions which define its space-time embedding. (As usual, dots above quantities stand for derivatives with respect to this time evolution parameter τ).

This action is analogous to the Nambu-Goto [29] action for the bosonic string. It is proportional to the total length of the world-line between some initial and final configuration, measured with the metric induced by the space-time Minkowski metric.

We are interested here in a process where the initial and final configurations are specified by the space-time positions x^μ_1, x^μ_2 of the particle at $\tau = \tau_1, \tau = \tau_2$ ($\tau_2 > \tau_1$) respectively.

The Lagrangian formulation of this system, subject to these boundary conditions, is well known [2]. We have the equations of motion:

$$\frac{d}{d\tau} \left(\frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) = 0, \quad x^\mu(\tau = \tau_i) = x^\mu_i \quad (i = 1, 2) \quad (2.2)$$

They are equivalent to the conservation of the total energy- and angular-momentum of the system, which is a consequence of the invariance of the action under space-time Poincaré transformations:

$$P_\mu(\tau) = \frac{m \dot{x}_\mu(\tau)}{\sqrt{-\dot{x}^2(\tau)}}, \quad M_{\mu\nu}(\tau) = P_\mu(\tau)x_\nu(\tau) - P_\nu(\tau)x_\mu(\tau) \quad (2.3)$$

With the space-time coordinates transforming as 1-dimensional scalar fields, the system is also invariant under world-line diffeomorphisms which leave the end points $\tau = \tau_i$ ($i=1,2$) fixed. For infinitesimal transformations we thus have

$$\tilde{\tau}(\tau) = \tau - \eta(\tau) \quad \text{or} \quad \delta\tau = \tilde{\tau}(\tau) - \tau = -\eta(\tau) \quad (2.4a)$$

$$\tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau) \quad \text{or} \quad \delta x^\mu(\tau) = \tilde{x}^\mu(\tau) - x^\mu(\tau) = \eta(\tau) \dot{x}^\mu(\tau) \quad (2.4b)$$

with $\eta(\tau_i) = 0$ ($i=1,2$).

To solve (2.2), a parametrization in τ has to be specified, i.e. a gauge has to be fixed. Any of the proper-time gauges provides a convenient choice

$$-\dot{x}^2(\tau) = m^2 \lambda_0^2 \quad (2.5)$$

Here λ_0 is a strictly positive constant (proper-time parametrization corresponds to $\lambda_0 = 1/m$).

Then the solution is

$$x^\mu(\tau) = x_1^\mu + \frac{\Delta x^\mu}{\Delta\tau} (\tau - \tau_1) \quad (2.6a)$$

with the constraint

$$\frac{(\Delta x)^2}{(\lambda_0 \Delta\tau)^2} + m^2 = 0 \quad \left(P_\mu(\tau) = \frac{\Delta x_\mu}{\lambda_0 \Delta\tau} \right) \quad (2.6b)$$

following from the gauge fixing condition (2.5). (We define $\Delta x^\mu = x_2^\mu - x_1^\mu$ and $\Delta\tau = \tau_2 - \tau_1$).

The corresponding classical action is:

$$S_c[x^\mu] = -m^2(\lambda_0 \Delta\tau) = \frac{1}{2} \left(\frac{(\Delta x)^2}{c} - m^2 c \right) \quad (2.7)$$

where $C = \lambda_0 \Delta\tau$. As a consequence of the invariance under world-line diffeomorphisms, any dependence on the gauge parameter λ_0 must always appear through this combination. This will become clear from what follows.

Let us now consider the Hamiltonian formulation of the system within Dirac's approach to constrained dynamics [12]. We have the degrees of freedom $x^\mu(\tau)$ and their conjugate momenta $p_\mu(\tau)$ with the Poisson brackets

$$\{x^\mu(\tau), p_\nu(\tau)\} = \delta^\mu_\nu \quad (2.8)$$

the primary constraint

$$\phi(\tau) \equiv \frac{1}{2} (p_\mu^2 + m^2) = 0 \quad (2.9)$$

and the canonical Hamiltonian

$$H_0(x^\mu, p_\mu) = \dot{x}^\mu p_\mu - L(\dot{x}) = 0 \quad (2.10)$$

Thus the full set of constraints is $\{\phi(\tau)\}$ and the total Hamiltonian is simply given by

$$H = \lambda(\tau) \phi(\tau) = \frac{1}{2} \lambda(\tau) (p_\mu^2 + m^2) \quad (2.11)$$

where $\lambda(\tau)$ is an arbitrary function. The corresponding equations of motion are

$$\dot{x}^\mu(\tau) = \{x^\mu(\tau), H\} = \lambda(\tau) p^\mu(\tau) \quad (2.12a)$$

$$\dot{p}^\mu(\tau) = \{p^\mu(\tau), H\} = 0 \quad (2.12b)$$

with $x^\mu(\tau_i) = x^\mu_{i_0}$ ($i=1,2$). They are solved by:

$$x^\mu(\tau) = x^\mu_{i_0} + \Delta x^\mu \frac{\int_{\tau_1}^{\tau} d\tau' \lambda(\tau')}{\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau)} \quad (2.13a)$$

$$p^\mu(\tau) = \frac{\Delta x^\mu}{\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau)} \quad (2.13b)$$

with the constraint (2.9)

$$\frac{(Ax)^2}{\left(\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau)\right)^2} + m^2 = 0 \quad (2.13c)$$

As (2.12) shows, only functions $\lambda(\tau)$ which do not vanish in the interval $\tau_1 \leq \tau \leq \tau_2$ can be considered [19]. Their sign is arbitrary; we take it positive, $\lambda(\tau) > 0$, so that the positive (negative) energy solutions propagate in the future (past). The physical meaning of this freedom in the sign of $\lambda(\tau)$ is clear [19,20].

Equations (2.9) and (2.10) follow from the first-order action:

$$S[x^\mu, p_\mu, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu - \frac{1}{2} \lambda (p^2 + m^2) \right] \quad (2.14)$$

subject to the boundary conditions $x^\mu(\tau_i) = x^\mu_i$ ($i=1, 2$). Here, $\lambda(\tau) > 0$ plays the role of a Lagrange multiplier for the constraint $\phi(\tau)$.

For the solution (2.13), (2.14) takes the value

$$S_c[x^\mu, p_\mu, \lambda] = \frac{1}{2} \left[\frac{(Ax)^2}{\left(\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau)\right)} - m^2 \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) \right] \quad (2.15)$$

Invariance of the non-linear action (2.1) under world-line diffeomorphisms (with $\tau = \tau_1$, $\tau = \tau_2$ fixed) translates into the invariance of this first-order action under transformations generated by the constraint ϕ . For an arbitrary infinitesimal function $\epsilon(\tau)$, consider the variations [19]:

$$\delta x^\mu(\tau) = \{x^\mu(\tau), \epsilon(\tau) \phi(\tau)\} = \epsilon(\tau) p^\mu(\tau) \quad (2.16a)$$

$$\delta p^\mu(\tau) = \{p^\mu(\tau), \epsilon(\tau) \phi(\tau)\} = 0 \quad (2.16b)$$

$$\delta \lambda(\tau) = \dot{\epsilon}(\tau) \quad (2.16c)$$

with $\epsilon(\tau = \tau_i) = 0$ ($i=1, 2$). Then

$$\delta S[x^\mu, p_\mu, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \frac{1}{2} \frac{d}{d\tau} \left[\epsilon (p^2 + m^2) \right] = 0 \quad (2.17)$$

Defining $\eta(\tau)$ by $\epsilon(\tau) = \lambda(\tau) \eta(\tau)$, (2.16) is recognised as a reparametrization in τ generated by $\eta(\tau)$, see (2.4).

In particular, the transformation of $\lambda(\tau)$ is that of a world-line einbein, as must be expected since $\lambda(\tau)$ is indeed such a quantity (see Appendix A). Thus for any finite transformation generated by the constraint, the multiplier $\lambda(\tau)$ changes as follows

$$\lambda(\tau) \longrightarrow \tilde{\lambda}(\tau) = \lambda(f(\tau)) \frac{df(\tau)}{d\tau} \quad (2.18)$$

for some function $f(\tau)$ such that $f(\tau_i) = \tau_i$ ($i=1,2$). We differ with Ref.[20] on this point.

This implies that $\left(\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) \right)$ is a geometrically invariant quantity for the local symmetries of the system. It also explains why $\lambda(\tau)$ solely appears through this combination for any classical solution, see (2.13) and (2.15).

Gauge fixing of the system, i.e. the choice of a specific world-line parametrization, is tantamount in this Hamiltonian formulation to a choice of the multiplier $\lambda(\tau)$. Its variation under the symmetries generated by the constraint determines how this space of functions is organized into orbits of the gauge group.

From (2.16c), it is clear that the necessary and sufficient condition for two Lagrange multipliers to belong to the same orbit is that they have equal values for the invariant $\left(\int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) \right)$. All orbits of the symmetry group are thus labelled by all positive constants λ_0 , and any element belonging to any of these orbits can be given as $\lambda(\tau) = \lambda_0 \frac{df(\tau)}{d\tau}$ for some function $f(\tau)$ with $f(\tau_i) = \tau_i$ ($i=1,2$).

A good gauge slice of the space of Lagrange multipliers is therefore defined by [19, 20]

$$\lambda(\tau) = \lambda_0 \quad (2.19)$$

with λ_0 a constant running over the real positive axis. This choice corresponds to the proper-time gauges defined in (2.5).

The geometrical invariant for the corresponding orbits takes the value $\lambda_0 \Delta\tau = c$, which is interpreted as the total proper-time of the particle between its initial and final configurations.

The quantity c is also recognised as the parameter which classifies different geometrical world-line structures that cannot be related by diffeomorphisms [17], in much the same way that Teichmüller and modular parameters label different conformal classes of Riemann surfaces of fixed topology. The constant c is thus the modular parameter of the world-line.

For any quantity invariant under world-line diffeomorphisms, any dependence on the Lagrange multiplier can only appear through a dependence on c .

To conclude our survey of the Hamiltonian formulation, let us consider its canonical quantization. Except for $\lambda(\tau)$, all degrees of freedom become operators with the commutation relations:

$$[x^\mu, p_\nu] = i\hbar \delta^\mu_\nu \quad (2.20)$$

The Schrödinger equation in the Hilbert space defined by this algebra is

$$H|\Psi, \tau\rangle = i\hbar \frac{\partial}{\partial \tau} |\Psi, \tau\rangle \quad (2.21a)$$

where

$$H = \frac{1}{2} \lambda (p^2 + m^2) \quad (2.21b)$$

Physical states are defined by the constraint

$$(p^2 + m^2)|\Psi, \tau\rangle = 0. \quad (2.22)$$

By (2.21), this also makes them independent of any world-line parametrization.

Using the wave-function representation of (2.20) where

$$x^\mu = x^\mu, \quad p_\mu = -i\hbar \frac{\partial}{\partial x^\mu} \quad (2.23a)$$

one obtains for physical states the Klein-Gordon equation

$$(-\hbar^2 \square + m^2)\Psi(x) = 0 \quad (2.23b)$$

The discussion presented above started with the non-linear action (2.1). A linear action can also be considered by introducing an intrinsic world-line einbein coupled to the scalar fields $x^\mu(\tau)$. However as explained in Appendix A, the Hamiltonian formulation of the system remains exactly the same, with the Lagrange multiplier $\lambda(\tau)$ now identified with the einbein. The only difference between these two actions is that (2.1) cannot be used to describe massless scalar particles.

Let us now turn to the BFM formulation [14,15] for this system. Its phase-space is extended as follows [15]. First, one introduces a degree of freedom $\pi(\tau)$ conjugate to $\lambda(\tau)$

$$\{ \lambda(\tau), \pi(\tau) \} = 1 \quad (2.24)$$

with the first-class constraint

$$\pi(\tau) = 0 \quad (2.25)$$

Associated to the two constraints $G_1(\tau) = \pi(\tau)$ and $G_2(\tau) = \phi(\tau)$, one introduces respectively two Grassmann variables $\eta^1(\tau)$ and $\eta^2(\tau)$ and their conjugate momenta $\mathcal{P}_1(\tau)$ and $\mathcal{P}_2(\tau)$ (also Grassmann variables) with the Poisson brackets [15]:

$$\{ \eta^a(\tau), \mathcal{P}_b(\tau) \} = -\delta^a_b, \quad a, b = 1, 2 \quad (2.26)$$

From the algebra of constraints

$$\{ G_a(\tau), G_b(\tau) \} = 0, \quad a, b = 1, 2 \quad (2.27a)$$

$$\{ H_0(\tau), G_a(\tau) \} = 0, \quad a, b = 1, 2 \quad (2.27b)$$

one obtains the BRST generator [15]

$$Q_B(\tau) = \eta^a(\tau) G_a(\tau) = \eta^1(\tau) \pi(\tau) + \frac{1}{2} \eta^2(\tau) (P^2(\tau) + m^2) \quad (2.28a)$$

and the BRST invariant Hamiltonian [15]:

$$H_B(\tau) = H_0(\tau) = 0. \quad (2.28b)$$

The ghost charge is defined by

$$Q_c(\tau) = i \eta^a(\tau) \mathcal{P}_a(\tau) = i \eta^1(\tau) \mathcal{P}_1(\tau) + i \eta^2(\tau) \mathcal{P}_2(\tau) \quad (2.29)$$

The BRST variation of any quantity X is obtained from $\delta_B X = \{ X, Q_B \}$, so that

$$\delta_B x^\mu = \eta^2 p^\mu, \quad \delta_B p^\mu = 0; \quad \delta_B \lambda = \eta^1, \quad \delta_B \pi = 0 \quad (2.30a)$$

$$\delta_B \eta^1 = 0, \quad \delta_B \mathcal{P}_1 = -\pi; \quad \delta_B \eta^2 = 0, \quad \delta_B \mathcal{P}_2 = -\frac{1}{2} (P^2 + m^2) \quad (2.30b)$$

By construction [15]:

$$\{Q_B, Q_B\} = 0 \quad (2.31)$$

In order to have a formulation which is explicitly invariant under the local symmetries of the system, here embodied in BRST transformations, we have to choose BRST invariant boundary conditions in the extended phase-space.

Different alternatives are possible [15], but keeping in mind the path-integral representation of the quantum system, we restrict our attention to one of them [15], namely

$$x^\mu(\tau_i) = x_i^\mu, \quad \pi(\tau_i) = 0, \quad i = 1, 2 \quad (2.32a)$$

$$\eta^2(\tau_i) = 0, \quad \mathcal{P}_1(\tau_i) = 0, \quad i = 1, 2 \quad (2.32b)$$

This choice is indeed invariant under (2.30).

Time evolution of the system is determined by an effective Hamiltonian [15]:

$$H_{\text{eff}} = H_B - \{\Psi, Q_B\} = -\{\Psi, Q_B\} \quad (2.33)$$

where Ψ is a function over the extended phase-space of ghost number (-1). Its role is to fix a gauge for the system. The appropriate choice cannot be underestimated as shall be discussed later.

The corresponding equations of motion, together with the boundary conditions (2.32), also follow from the first-order effective action [15]:

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu + \dot{\lambda} \pi + \dot{\eta}^1 \mathcal{P}_1 - \eta^2 \dot{\mathcal{P}}_2 - H_{\text{eff}} \right] \quad (2.34)$$

Let us consider the choice

$$\Psi(\tau) = \lambda(\tau) \mathcal{P}_2(\tau) \quad (2.35)$$

which leads to

$$H_{\text{eff}} = \frac{1}{2} \lambda (p_\perp^2 + m^2) - \mathcal{P}_2 \eta^1 \quad (2.36)$$

and

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu + \dot{\lambda} \pi + \dot{\eta}^1 \mathcal{P}_1 - \eta^2 \dot{\mathcal{P}}_2 + \mathcal{P}_2 \eta^1 - \frac{1}{2} \lambda (p^2 + m^2) \right] \quad (2.37)$$

Checking that these two quantities are indeed BRST invariant, provided $\eta^2(\tau)$ vanishes at the end points, is straightforward enough. From (2.36) we have the equations of motion

$$\dot{x}^\mu = \lambda p^\mu, \quad \dot{p}^\mu = 0; \quad \dot{\lambda} = 0, \quad \dot{\pi} = -\frac{1}{2} (p^2 + m^2) \quad (2.38a)$$

$$\dot{\eta}^1 = 0, \quad \dot{\mathcal{P}}_1 = -\mathcal{P}_2; \quad \dot{\eta}^2 = \eta^1, \quad \dot{\mathcal{P}}_2 = 0 \quad (2.38b)$$

With the boundary conditions (2.32), their solution is

$$\pi(\tau) = 0, \quad \eta^1(\tau) = \eta^2(\tau) = \mathcal{P}_1(\tau) = \mathcal{P}_2(\tau) = 0 \quad (2.39a)$$

$$\lambda(\tau) = \lambda_0, \quad x^\mu(\tau) = x_1^\mu + \frac{\Delta x^\mu}{\Delta \tau} (\tau - \tau_1), \quad p^\mu(\tau) = \frac{\Delta x^\mu}{\lambda_0 \Delta \tau} \quad (2.39b)$$

with the constraint

$$\frac{(\Delta x)^2}{(\lambda_0 \Delta \tau)^2} + m^2 = 0 \quad (2.39c)$$

It also gives the values

$$Q_B = 0, \quad Q_C = 0 \quad (2.39d)$$

$$S_{\text{eff}} = \frac{1}{2} \left[\frac{(\Delta x)^2}{\lambda_0 \Delta \tau} - m^2 \lambda_0 \Delta \tau \right] \quad (2.39e)$$

This is the solution that was previously obtained with $\lambda(\tau) = \lambda_0$. All additional phase-space variables are equal to zero. Here λ_0 is any arbitrary positive constant.

As was discussed in (2.19), the gauge condition $\dot{\lambda} = 0$ (with $\lambda(\tau) > 0$) resulting from the choice (2.35) defines a good gauge slice. It corresponds to the class of proper-time gauges and completely fixes the gauge freedom of the system.

Note that by solving for η^1 and $\mathcal{P}_2$ in (2.37), one recovers exactly the results obtained by the method in Ref. [30] applied to the action (2.14), with this gauge fixing condition $\hat{\lambda}=0$. Furthermore, η^2 and $(-i\mathcal{P}_1)$ correspond to the usual reparametrization ghosts $c(\tau)$ and $\bar{c}(\tau)$.

BRST quantization, i.e. canonical quantization of the BFM formulation, is defined by the algebra of commutators and anticommutators

$$[x^\mu, p_\nu]_- = i\hbar \delta^\mu_\nu, \quad [\lambda, \pi]_- = i\hbar, \quad [\eta^a, \mathcal{P}_b]_+ = -i\hbar \delta^a_b \quad (a, b=1, 2) \quad (2.40)$$

In the corresponding Hilbert space, physical states are defined by the conditions

$$Q_B |\Psi\rangle = 0, \quad Q_c |\Psi\rangle = 0 \quad (2.41)$$

In the proper-time gauges, time evolution of a state is determined by the quantum effective Hamiltonian (2.36).

BRST invariance of the system follows from the properties

$$[Q_B, Q_B]_+ = 0, \quad [H_{\text{eff}}, Q_B]_- = 0 \quad (2.42)$$

In the wave-function representation of (2.40) where

$$x^\mu = x^\mu, \quad p_\mu = -i\hbar \frac{\partial}{\partial x^\mu}; \quad \lambda = \lambda, \quad \pi = -i\hbar \frac{\partial}{\partial \lambda}; \quad \eta^a = \eta^a, \quad \mathcal{P}_a = -i\hbar \frac{\partial}{\partial \eta^a} \quad (a=1, 2) \quad (2.43)$$

the constraints (2.41) imply that a physical state $\Psi(x, \lambda, \eta^1, \eta^2; \tau)$ satisfies the Klein-Gordon equation and is independent of λ , η^1 and η^2 . Using the Schrödinger equation, it then follows that Ψ is also independent of the world-line parametrization.

We have thus completely recovered the results obtained previously with Dirac's formulation, at the classical as well as at the quantum level. The advantage however, of the BFM approach is that it provides an explicit phase-space path-integral representation for matrix elements of observables which is invariant under the gauge symmetries of the system [14,15].

In the present case, the probability amplitude for the particle to propagate from x^μ_1 to x^μ_2 is represented by:

$$Z(x^\mu_2, x^\mu_1) = \int [dx^\mu d p_\mu d\lambda d\pi d\eta^1 d\mathcal{P}_1 d\eta^2 d\mathcal{P}_2] e^{iS_{\text{eff}}} \quad (2.44)$$

with S_{eff} the effective action (2.34).

The functional integral is taken with the phase-space Liouville measure and the boundary conditions (2.32) (with $\lambda(\tau) > 0$). By construction, (2.44) is BRST invariant [14,15]. The quantized system being BRST invariant, any BRST transformation induces a change in the Liouville measure which is cancelled by the variation of the gauge fixing term $\{\Psi, Q_B\}$ in the effective action.

The integral (2.44) can explicitly be carried out by time slices. In the proper-time gauges defined by (2.35), it factorizes into a ghost and a bosonic contribution as:

$$Z(x_2^\mu, x_1^\mu) = \int [d\lambda d\pi] e^{i \int_{\tau_1}^{\tau_2} d\tau \dot{\lambda} \pi} Z_F Z_B(x_2^\mu, x_1^\mu; \lambda) \quad (2.45a)$$

with

$$Z_F = \int [d\eta^1 d\mathcal{Q}_1 d\eta^2 d\mathcal{Q}_2] e^{i \int_{\tau_1}^{\tau_2} d\tau (\dot{\eta}^1 \mathcal{Q}_1 - \dot{\eta}^2 \mathcal{Q}_2 + \mathcal{Q}_2 \dot{\eta}^1)} \quad (2.45b)$$

$$Z_B(x_2^\mu, x_1^\mu; \lambda) = \int [dx^\mu dP_\mu] e^{i \int_{\tau_1}^{\tau_2} d\tau (\dot{x}^\mu P_\mu - \frac{1}{2} \lambda (P^2 + m^2))} \quad (2.45c)$$

The integral over π and λ enforces the proper-time gauge condition $\dot{\lambda}=0$. It thus reduces to an integral over the gauge parameter $\lambda(\tau)=\lambda_0$ with $\lambda_0 > 0$. The integral Z_F over the ghosts η^a and their conjugate momenta $\mathcal{Q}_a$ in some sense specifies the measure for this integral [19].

Up to numerical factors, one obtains:

$$Z_F = \Delta\tau \quad (2.46a)$$

and

$$Z_B(x_2^\mu, x_1^\mu; \lambda) = c^{-D/2} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{c} - m^2 c \right]} \quad (2.46b)$$

where $c = \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) = \lambda_0 \Delta\tau$.

As expected, any dependence on $\lambda(\tau)$ appears through a dependence on c only. We have for the propagator (2.44)

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty dc \, c^{-D/2} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{c} - m^2 c \right]} \quad (2.47)$$

In order to make this quantity well-defined, we regulate it by inserting in the integral a factor $e^{-\epsilon c/2}$ (with $\epsilon > 0$ infinitesimal) [19]. Other regularizations are possible. Formally they are all equivalent and effectively give an infinitesimal negative imaginary part to m^2 or c . We then finally arrive at:

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty dc \, c^{-D/2} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{c} - (m^2 - i\epsilon)c \right]} \quad (2.48)$$

This is the proper-time representation of the propagator for a scalar particle in D dimensions. A Fourier analysis gives the usual field-theoretical Feynman propagator:

$$Z(x_2^\mu, x_1^\mu) = \int \frac{d^D p}{(2\pi)^D} e^{ip\Delta x} \frac{-i}{p^2 + m^2 - i\epsilon} \quad (2.49)$$

Several remarks are called for. Firstly, the negative imaginary part of m^2 required to make the propagator (2.47) well defined also gives it its correct causal structure. Secondly, the usual Wick rotation in configuration or momentum space corresponds to the following transformation on the modular parameter c :

$$c \longrightarrow c_E = ic \quad (2.50)$$

Indeed, (2.48) then becomes

$$Z_E(x_2^\mu, x_1^\mu) = \int_0^\infty dc_E \, c_E^{-D/2} e^{-\frac{1}{2} \left[\frac{(\Delta x)^2}{c_E} + m^2 c_E \right]} \quad (2.51)$$

which is the result obtained in the Euclidean formulation. And finally, as expected from the BRST invariance, (2.48) is invariant under world-line diffeomorphisms since it is given as an integral over the modular parameter c . Any dependence on the gauge fixing parameter λ_0 and on the end points τ_1 and τ_2 combines into the reparametrization invariant quantity $c = \lambda_0 \Delta\tau$.

Any measure for the integration over c would lead to such an invariant propagator. Through the integral over the ghosts η^a and $\mathcal{P}_a$ [19], the BFM phase-space path-integral fixes this measure uniquely and gives the correct propagator for a scalar particle.

One of the reasons for this result is that the function Ψ chosen in (2.35) defines a complete gauge fixing of the system. It selects one and only one element in each orbit of the gauge group, so that the path-integral reduces to an integral over the modular classes of the world-line.

Since it is often said [14,15,25,27,28] that the BFM path-integral is independent of the function Ψ (the so-called Fradkin-Vilkovisky theorem [14]), let us now consider the choice

$$\Psi = \frac{1}{\beta} \mathcal{G}_1 (\lambda - \lambda_0) + \mathcal{G}_2 \lambda \quad (2.52)$$

where λ_0 is a fixed positive constant and β a free parameter.

After redefining π and $\mathcal{G}_1$ by

$$\pi \longrightarrow \beta \pi, \quad \mathcal{G}_1 \longrightarrow \beta \mathcal{G}_1 \quad (2.53)$$

the limit $\beta \rightarrow 0$ is taken. One then obtains the effective action

$$S_{\text{eff}} = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu - \eta^2 \dot{\mathcal{G}}_2 + \mathcal{G}_2 \eta' - \pi (\lambda - \lambda_0) - \frac{1}{2} \lambda (p^2 + m^2) \right] \quad (2.54)$$

The analysis of the classical system defined by this action as well as its BRST invariance, its equivalence with the results obtained by the methods of Ref.[30] and its canonical quantization are left to the reader (see also Section 4). Only the path-integral obtained with (2.52) need be considered here.

Since the phase-space functional measure is invariant under (2.53) [15], the path-integral representation of the propagator is still given by

$$\mathcal{Z}(x_2^\mu, x_1^\mu; \lambda_0) = \int [dx^\mu d p_\mu d\lambda d\pi d\eta' d\mathcal{G}_1 d\eta^2 d\mathcal{G}_2] e^{i S_{\text{eff}}} \quad (2.55)$$

with the effective action (2.54). This factorizes as

$$\mathcal{Z}(x_2^\mu, x_1^\mu; \lambda_0) = \int [d\lambda d\pi d\eta' d\mathcal{G}_1 d\eta^2 d\mathcal{G}_2] e^{i \int_{\tau_1}^{\tau_2} d\tau [-\pi (\lambda - \lambda_0) - \eta^2 \dot{\mathcal{G}}_2 + \mathcal{G}_2 \eta']} \cdot \mathcal{Z}_B(x_2^\mu, x_1^\mu; \lambda) \quad (2.56)$$

The integral over π enforces the condition $\lambda(\tau) = \lambda_0$. The function (2.52) thus defines an overcomplete gauge fixing; it selects one element of only one orbit of the gauge group.

The time slice evaluation of the remaining integrals (Z_B is given in (2.46b)) leads to a well defined quantity (when the number of time steps in τ becomes infinite) only when the integrals over λ, π and $\eta, \mathcal{G}_a$ are combined. Up to numerical factors , one obtains

$$Z(x_2^\mu, x_1^\mu; \lambda_0) = \Delta\tau \, c^{-D/2} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{c} - m^2 c \right]} \quad (2.57)$$

where $c = \lambda_0 \Delta\tau$.

This result contradicts the usual interpretation [14,15] of the Fradkin-Vilkovisky theorem[14]. Using two different gauge fixing functions Ψ we have obtained different expressions for what should be the same quantity (if that interpretation were correct).

Actually, the correct meaning of the theorem is that the phase-space path-integral representation of an observable is independent of the gauge fixing function Ψ , provided the functions which one considers define gauge slices which are related to each other by gauge transformations. In other words, the BFM path-integral only depends on the equivalence classes of gauge fixing functions. Its BRST invariance means exactly this and nothing more. BRST transformations induce motions only along the orbits but not transversally to them. This does not imply that the path-integral is independent of the choice of Ψ .

For example, if rather than (2.52) we use

$$\Psi = \frac{1}{\beta} \mathcal{G}_1 (\lambda - \lambda_0 \dot{f}) + \mathcal{G}_2 \lambda \quad (2.58)$$

with $f(\tau)$ a fixed function such that $f(\tau_i) = \tau_i$ ($i=1,2$), an explicit calculation also gives (2.57). Indeed $\lambda(\tau) = \lambda_0$ and $\lambda(\tau) = \lambda_0 \dot{f}(\tau)$ are elements of the same orbit. Moreover, (2.57) is invariant under world-line diffeomorphisms since it only depends on $(\lambda_0 \Delta\tau)$ and $(\Delta\tau)$ which remain fixed by these transformations.

However, the correct choice of the equivalence class of functions Ψ leading to a complete but not overcomplete gauge fixing is a delicate problem which must be considered for each system separately. For the particle we saw that $\Psi = \mathcal{G}_2 \lambda$ leads to a correct gauge fixing of the system, but that the equivalence class defined by (2.58) determines an overcomplete gauge choice.

In order to obtain the correct propagator as a sum over all orbits, one might integrate (2.57) over λ_0 , as:

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty d\lambda_0 \, \mu(\lambda_0 \Delta\tau, \Delta\tau) \Delta\tau \, c^{-D/2} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{c} - m^2 c \right]} \quad (2.59)$$

where $\mu(\lambda_0 \Delta\tau, \Delta\tau)$ is a measure for the integration. (It is understood that as in (2.48), m^2 has an infinitesimal negative imaginary part to make (2.59) well defined).

A priori, this measure is arbitrary. If we insist on having an expression which is independent not only of world-line parametrizations but also of $\Delta\tau$ and which is a summation over modular classes, (2.59) should be an integral over $c = \lambda_0 \Delta\tau$ only. We then have:

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty dc \mu(c) c^{-D/2} e^{\frac{i}{2} \left[\frac{(\Delta x)^2}{c} - (m^2 - i\epsilon)c \right]} \quad (2.60)$$

with $\mu(c)$ an arbitrary function. For example, taking $\mu(c) = c^\alpha$, (2.60) can be expressed as

$$Z(x_2^\mu, x_1^\mu) = \int \frac{d^D p}{(2\pi)^D} e^{ip \Delta x} \frac{-i \Gamma(\alpha+1)}{(p^2 + m^2 - i\epsilon)^{\alpha+1}} \quad (2.61)$$

This freedom in the choice of $\mu(c)$ is related to the freedom in the choice of functional measures for the integration over world-line metrics and embeddings [31] in a formal approach to the scalar particle [17] similar to Polyakov's approach to string theories [1].

In the case of the particle, we know that the correct choice is $\mu(c) = 1$. However, nothing in the Polyakov formulation nor in the present one with the gauge fixing function (2.52) can lead uniquely to it (unless one accepts here the argument that the integral over λ_0 in (2.59) must be linear since the Liouville measure $[d\lambda]$ is linear).

Within the path-integral approach to the first quantization of the particle, only the discussion presented above using the BFV approach with the gauge fixing condition $\dot{\lambda} = 0$ shows that this is indeed the correct answer [19].

It is this result, namely that the measure for the integration over the modular parameter is uniquely fixed, which motivated the present work. Applying the same method to the bosonic string, it seems that similarly it should allow for a better understanding of the measure over modular space.

3. THE BOSONIC STRING: ITS GAUGE FIXING IN THE CANONICAL APPROACH

Let us consider the bosonic string with the non-linear Nambu-Goto action [19]:

$$S[x^\mu] = \frac{-1}{2\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[(\dot{x}x')^2 - \dot{x}^2 x'^2 \right]^{1/2} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \mathcal{L}(\dot{x}, x') \quad (3.1)$$

Here $\tau_1 \leq \tau \leq \tau_2$ and $0 \leq \sigma \leq \pi$ parametrize the string world-sheet and $x^\mu(\sigma, \tau)$ are D functions which define its space-time embedding. (As usual $\dot{x}^\mu \equiv \partial_\tau x^\mu$, $x'^\mu \equiv \partial_\sigma x^\mu$ and $(2\pi\alpha')$ is the inverse string tension).

This action is proportional to the total area spanned by the world-sheet for some process, measured with the metric induced by the space-time Minkowski metric.

We consider here the free propagation of the string between initial and final space-time configurations specified by two loops $\mathcal{L}_i^\mu$ ($i=1,2$) for the closed string or two curved arcs α_i^μ ($i=1,2$) for the open string. They are defined by functions $x_i^\mu(\sigma)$ at $\tau=\tau_i$ ($i=1,2$) for some arbitrary parametrization in σ . The choice of such a parametrization is necessary, but physical quantities should not depend on it. This parametrization should also be compatible with the boundary conditions imposed on the system.

The Lagrangian formulation of this system is well known [2,3]. We have the equations of motion

$$\partial_\alpha p_\mu^\alpha = 0 \quad (3.2a)$$

where

$$p_\mu^0 = \frac{-1}{2\pi\alpha'} \left[(\dot{x}x')^2 - \dot{x}^2 x'^2 \right]^{-1/2} \left[(\dot{x}x') x'_\mu - x'^2 \dot{x}_\mu \right] \quad (3.2b)$$

$$p_\mu^1 = \frac{-1}{2\pi\alpha'} \left[(\dot{x}x')^2 - \dot{x}^2 x'^2 \right]^{-1/2} \left[(\dot{x}x') \dot{x}_\mu - \dot{x}^2 x'_\mu \right] \quad (3.2c)$$

with the boundary conditions

$$i) \quad x^\mu(\sigma, \tau_i) = x_i^\mu(\sigma), \quad i=1,2 \quad (3.2d)$$

$$ii) \quad a) \text{ closed string: periodicity in } \sigma \rightarrow \sigma + \pi \quad (3.2e)$$

$$b) \text{ open string: } p_\mu^1(\sigma=0, \pi, \tau) = 0 \quad (3.2f)$$

Equations (3.2a) are equivalent to the conservation of the world-sheet energy- and angular- momentum currents. This conservation is a consequence of the invariance of the action under space-time Poincaré transformations.

With the space-time coordinates transforming as 2-dimensional scalar fields, the system is also invariant under world-sheet diffeomorphisms leaving the boundaries fixed. Denoting $\xi^0 = \tau$ and $\xi^1 = \sigma$, for infinitesimal transformations we have

$$\tilde{\xi}^\alpha(\xi) = \xi^\alpha - \eta^\alpha(\xi) \quad \text{or} \quad \delta \xi^\alpha = \tilde{\xi}^\alpha(\xi) - \xi^\alpha = -\eta^\alpha(\xi) \quad (3.3a)$$

$$\tilde{x}^\mu(\xi) = x^\mu(\xi) \quad \text{or} \quad \delta x^\mu(\xi) = \tilde{x}^\mu(\xi) - x^\mu(\xi) = \eta^\alpha \partial_\alpha x^\mu \quad (3.3b)$$

with

$$i) \quad \eta^0(\sigma, \tau_i) = 0, \quad i=1,2 \quad (3.3c)$$

$$ii) \text{ a) closed string: periodicity in } \sigma \rightarrow \sigma + \pi \quad (3.3d)$$

$$b) \text{ open string: } \eta^1(\sigma=0, \pi; \tau) = 0 \quad (3.3e)$$

To solve (3.2), a world-sheet parametrization has to be specified. A convenient choice is any of the proper-time gauges

$$\dot{x}x' = 0, \quad \dot{x}^2 + \lambda_0^2 x'^2 = 0 \quad (3.4)$$

with λ_0 a strictly positive constant (the conformal parametrization corresponds to $\lambda_0 = 1$).

The solutions are then the well known mode expansions [2,3]:

a) closed string:

$$x^\mu(\sigma, \tau) = q^\mu + (\alpha_0^\mu + \bar{\alpha}_0^\mu) \lambda_0 (\tau - \tau_1) + \frac{1}{2} i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{n} \left(\alpha_n^\mu e^{-2in\lambda_0(\tau-\tau_1) + 2in\sigma} + \bar{\alpha}_n^\mu e^{-2in\lambda_0(\tau-\tau_1) - 2in\sigma} \right) \quad (3.5a)$$

with $\alpha_0^\mu = \bar{\alpha}_0^\mu$

b) open string:

$$x^\mu(\sigma, \tau) = q^\mu + \alpha_0^\mu \lambda_0 (\tau - \tau_1) + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{n} \alpha_n^\mu e^{-in\lambda_0(\tau-\tau_1)} \cos n\sigma \quad (3.5b)$$

with the Virasoro constraints

a) closed string:
$$L_n \equiv \frac{1}{4\alpha'} \sum_m \alpha_m^\mu \alpha_{n-m\mu} = 0, \quad n, m \in \mathbb{Z}$$

$$\bar{L}_n \equiv \frac{1}{4\alpha'} \sum_m \bar{\alpha}_m^\mu \bar{\alpha}_{n-m\mu} = 0, \quad n, m \in \mathbb{Z} \quad (3.5c)$$

b) open string:
$$L_n \equiv \frac{1}{4\alpha'} \sum_m \alpha_m^\mu \alpha_{n-m\mu} = 0, \quad n, m \in \mathbb{Z} \quad (3.5d)$$

which follow from the gauge fixing conditions (3.4).

The integration constants which define these solutions are related to the initial and final configurations $x_i^\mu(\sigma)$ ($i=1,2$), whose parametrization must be compatible with the boundary conditions (3.2 e,f) in the proper-time gauges. We thus have:

a) closed string:
$$x_i^\mu(\sigma) = \sum_{n=-\infty}^{+\infty} x_{i,n}^\mu e^{-2in\sigma}, \quad i=1,2 \quad (3.6a)$$

with $x_{i,n}^\mu = x_{i,-n}^{\mu*}$

b) open string:
$$x_i^\mu(\sigma) = x_{i,0}^\mu + \sum_{n=1}^{\infty} x_{i,n}^\mu \cos n\sigma, \quad i=1,2 \quad (3.6b)$$

One can then solve for the integration constants $q_i^\mu, \alpha_n^\mu, \bar{\alpha}_n^\mu$ in terms of the coefficients $x_{i,n}^\mu$, which must satisfy the Virasoro gauge conditions. The Nambu-Goto action then reduces to:

a) closed string:
$$S_{cl}(x_i^\mu, c) = \frac{1}{4\pi\alpha'c} \left[(x_{2,0} - x_{1,0})^2 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{2\pi nc}{\sin 2\pi nc} (x_{2,n}^\mu e^{-i\pi nc} - x_{1,n}^\mu e^{i\pi nc}) \cdot (x_{2,-n\mu} e^{-i\pi nc} - x_{1,-n\mu} e^{i\pi nc}) \right] \quad (3.7a)$$

b) open string:
$$S_{op}(x_i^\mu, c) = \frac{1}{4\pi\alpha'c} \left[(x_{2,0} - x_{1,0})^2 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\pi nc}{\sin \pi nc} \left((x_{2,n}^2 + x_{1,n}^2) \cos \pi nc - 2x_{2,n}x_{1,n} \right) \right] \quad (3.7b)$$

where $c = \frac{\lambda_0 \Delta\tau}{\pi}$. As we will see, any dependence on the gauge parameter λ_0 must always appear through this combination. This is a consequence of the invariance under world-sheet diffeomorphisms.

However, when $(\lambda_0 \Delta\tau/\pi)$ is a rational number, there are non-zero values of n where one cannot solve for $\alpha_n^\mu, \bar{\alpha}_n^\mu$ in terms of $x_{i,n}^\mu$. In order to avoid this problem (which does not appear in the Euclidean formulation), c (or λ_0) has to be given an infinitesimal imaginary part. This is implicitly understood in (3.7). Later we find this requirement again, but for different reasons.

In the Hamiltonian formulation of the system [12,2], we have the degrees of freedom $x^\mu(\sigma, \tau)$ and their conjugate momenta $\pi_\mu(\sigma, \tau) = p_\mu^\circ(\sigma, \tau)$, with the Poisson brackets

$$\{x^\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)\} = \delta_\nu^\mu \delta(\sigma - \sigma') \quad (3.8)$$

They satisfy the primary constraints

$$\phi_+(\sigma, \tau) = 0, \quad \phi_-(\sigma, \tau) = 0 \quad (3.9a)$$

where

$$\phi_\pm(\sigma, \tau) = \frac{1}{2} \pi_\alpha' \phi_\pm^\mu(\sigma, \tau) \phi_{\pm\mu}(\sigma, \tau) \quad (3.9b)$$

$$\phi_\pm^\mu(\sigma, \tau) = \pi^\mu_\alpha(\sigma, \tau) \pm \frac{1}{2\pi_\alpha'} \partial_\sigma x^\mu(\sigma, \tau) \quad (3.9c)$$

The canonical Hamiltonian density is

$$\mathcal{H}_0(\sigma, \tau) = \dot{x}^\mu(\sigma, \tau) \pi_\mu(\sigma, \tau) - \mathcal{L}(\dot{x}, x') = 0 \quad (3.10)$$

From the Poisson brackets

$$\{x^\mu(\sigma, \tau), \phi_\pm(\sigma', \tau)\} = \pi_\alpha' \phi_\pm^\mu(\sigma', \tau) \delta(\sigma - \sigma') \quad (3.11a)$$

$$\{\pi^\mu_\alpha(\sigma, \tau), \phi_\pm(\sigma', \tau)\} = \pm \frac{1}{2} \phi_\pm^\mu(\sigma', \tau) \partial_\sigma \delta(\sigma - \sigma') \quad (3.11b)$$

$$\{\phi_\eta(\sigma, \tau), \phi_{\eta'}(\sigma', \tau)\} = \delta_{\eta, \eta'} \eta (\phi_\eta(\sigma, \tau) + \phi_{\eta'}(\sigma', \tau)) \partial_\sigma \delta(\sigma - \sigma'), \quad \eta, \eta' = +, - \quad (3.11c)$$

one obtains the full set of first-class constraints (ϕ_+, ϕ_-) and the total Hamiltonian

$$H = \int_0^\pi d\sigma [\lambda^+(\sigma, \tau) \phi_+(\sigma, \tau) + \lambda^-(\sigma, \tau) \phi_-(\sigma, \tau)] \quad (3.12)$$

with $\lambda^+(\sigma, \tau), \lambda^-(\sigma, \tau)$ arbitrary functions.

The corresponding equations of motion

$$\dot{x}^\mu(\sigma, \tau) = \pi_\alpha' [\lambda^+(\sigma, \tau) \phi_+^\mu(\sigma, \tau) + \lambda^-(\sigma, \tau) \phi_-^\mu(\sigma, \tau)] \quad (3.13a)$$

$$\dot{\pi}^\mu_\alpha(\sigma, \tau) = \frac{1}{2} \partial_\sigma [\lambda^+(\sigma, \tau) \phi_+^\mu(\sigma, \tau) - \lambda^-(\sigma, \tau) \phi_-^\mu(\sigma, \tau)] \quad (3.13b)$$

also follow from the first-order action

$$S[x^\mu, \pi_\mu, \lambda^+, \lambda^-] = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - \lambda^+ \phi_+ - \lambda^- \phi_-] \quad (3.14)$$

subject to the boundary conditions

$$i) \quad x^\mu(\sigma, \tau_i) = x^\mu_i(\sigma) \quad , \quad i=1,2 \quad (3.15a)$$

$$ii) \ a) \text{ closed string: periodicity in } \sigma \rightarrow \sigma + \pi \quad (3.15b)$$

$$b) \text{ open string: } (\lambda^+ \phi_+^\mu - \lambda^- \phi_-^\mu)(\sigma=0, \pi; \tau) = 0 \quad (3.15c)$$

(Solving for π^μ and λ^+, λ^- , (3.15c) indeed reduces to (3.2f)).

Thus λ^+ and λ^- are Lagrange multipliers for the constraints ϕ_+ and ϕ_- .

Actually, for reasons similar to those for the particle, only functions λ^+, λ^- such that $(\lambda^+ + \lambda^-)$ has a definite sign can be considered. This can be seen in different ways. The arbitrariness in this sign has the same physical interpretation as before.

We take it positive, $(\lambda^+ + \lambda^-) > 0$, so that the Nambu-Goto action is recovered from (3.14) by solving for π^μ and λ^+, λ^- . Invariance of the non-linear action (3.1) under world-sheet diffeomorphisms (leaving the boundaries fixed) translates into an invariance of (3.14) for transformations generated by the constraints ϕ_+, ϕ_- .

For arbitrary infinitesimal functions $\epsilon^+(\sigma, \tau), \epsilon^-(\sigma, \tau)$ with

$$i) \quad (\epsilon^+ + \epsilon^-)(\sigma, \tau_i) = 0 \quad , \quad i=1,2 \quad (3.16a)$$

$$ii) \ a) \text{ closed string: periodicity in } \sigma \rightarrow \sigma + \pi \quad (3.16b)$$

$$b) \text{ open string: } (\lambda^- \epsilon^+ - \lambda^+ \epsilon^-)(\sigma=0, \pi; \tau) = 0 \quad (3.16c)$$

(3.14) is invariant under the variations [15,24,26]

$$\delta x^\mu(\sigma, \tau) = \{x^\mu(\sigma, \tau), \phi_\epsilon(\tau)\} = \pi \alpha' [\epsilon^+(\sigma, \tau) \phi_+^\mu(\sigma, \tau) + \epsilon^-(\sigma, \tau) \phi_-^\mu(\sigma, \tau)] \quad (3.17a)$$

$$\delta \pi^\mu(\sigma, \tau) = \{\pi^\mu(\sigma, \tau), \phi_\epsilon(\tau)\} = \frac{1}{2} \partial_\sigma [\epsilon^+(\sigma, \tau) \phi_+^\mu(\sigma, \tau) - \epsilon^-(\sigma, \tau) \phi_-^\mu(\sigma, \tau)] \quad (3.17b)$$

$$\delta \lambda^+(\sigma, \tau) = \partial_\tau \epsilon^+(\sigma, \tau) - \lambda^+(\sigma, \tau) \partial_\sigma \epsilon^+(\sigma, \tau) + \partial_\sigma \lambda^+(\sigma, \tau) \epsilon^+(\sigma, \tau) \quad (3.17c)$$

$$\delta \lambda^-(\sigma, \tau) = \partial_\tau \epsilon^-(\sigma, \tau) + \lambda^-(\sigma, \tau) \partial_\sigma \epsilon^-(\sigma, \tau) - \partial_\sigma \lambda^-(\sigma, \tau) \epsilon^-(\sigma, \tau) \quad (3.17d)$$

where

$$\phi_\epsilon(\tau) = \int_0^\pi d\sigma \left[\epsilon^+(\sigma, \tau) \phi_+(\sigma, \tau) + \epsilon^-(\sigma, \tau) \phi_-(\sigma, \tau) \right] \quad (3.18)$$

Defining $\eta^\alpha(\sigma, \tau)$ by

$$\epsilon^+ = \lambda^+ \eta^0 + \eta^1, \quad \epsilon^- = \lambda^- \eta^0 - \eta^1 \quad (3.19)$$

(3.17) is recognized as a reparametrization in (σ, τ) generated by $\eta^\alpha(\sigma, \tau)$ (see (3.3)).

Let us concentrate on the quantity defined by:

$$g_{\alpha\beta} = e^\phi \frac{2}{\lambda^+ + \lambda^-} \begin{pmatrix} -\lambda^+ \lambda^- & \frac{1}{2}(\lambda^+ - \lambda^-) \\ \frac{1}{2}(\lambda^+ - \lambda^-) & 1 \end{pmatrix} \quad (3.20)$$

where $\phi(\sigma, \tau)$ is an arbitrary fixed function (with $\det g_{\alpha\beta} = e^{2\phi}$).

With the variations $\delta \lambda^\pm$ given above, we then have

$$\delta g_{\alpha\beta} = -(\eta^\gamma \partial_\gamma \phi + \partial_\gamma \eta^\gamma) g_{\alpha\beta} + \eta^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma} \partial_\beta \eta^\gamma + g_{\beta\gamma} \partial_\alpha \eta^\gamma \quad (3.21)$$

As must be expected since $g_{\alpha\beta}$ is indeed such a quantity (see Appendix B), (3.21) is the variation of a world-sheet metric tensor under the infinitesimal reparametrization $\tilde{\xi}^\alpha(\xi)$ given in (3.3) and a conformal rescaling, as:

$$g_{\alpha\beta}(\xi) \longrightarrow \tilde{g}_{\alpha\beta}(\tilde{\xi}) = e^{\rho(\tilde{\xi})} \frac{\partial \xi^\gamma}{\partial \tilde{\xi}^\alpha} \frac{\partial \xi^\delta}{\partial \tilde{\xi}^\beta} g_{\gamma\delta}(\xi) \quad (3.22a)$$

with

$$\rho = -(\eta^\gamma \partial_\gamma \phi + \partial_\gamma \eta^\gamma) \quad (3.22b)$$

The reason for this last factor is clear. Since the function ϕ is fixed, for each reparametrization the determinant of the metric has to be readjusted such that

$$\det \tilde{g}_{\alpha\beta}(\tilde{\xi}) = \det g_{\alpha\beta}(\xi) \quad (3.23)$$

From these remarks, it is easy to obtain the variations of the Lagrange multipliers λ^+, λ^- for finite transformations induced by the constraints ϕ_+, ϕ_- . For any diffeomorphism $\tilde{\xi}^\alpha(\xi)$ leaving the boundaries of the world-sheet fixed, $\tilde{\lambda}^+$ and $\tilde{\lambda}^-$ are obtained from the parametrization (3.20) and the transformation

$$\tilde{g}_{\alpha\beta}(\tilde{\xi}) = \left| \det \frac{\partial \xi(\tilde{\xi})}{\partial \tilde{\xi}} \right|^{-1} \left[\frac{\det g_{\alpha\beta}(\tilde{\xi})}{\det g_{\alpha\beta}(\xi(\tilde{\xi}))} \right]^{1/2} \frac{\partial \xi^\alpha(\tilde{\xi})}{\partial \tilde{\xi}^\alpha} \frac{\partial \xi^\beta(\tilde{\xi})}{\partial \tilde{\xi}^\beta} g_{\alpha\beta}(\xi(\tilde{\xi})) \quad (3.24)$$

These results also allow us to discuss the gauge fixing of the Nambu-Goto string in its Hamiltonian formulation. According to Teichmüller's theory, this problem is indeed well understood in the case of metrics for Riemann surfaces [4,6,7]. Since for the process which is studied here the world-sheet is a manifold only for the closed string, let us first consider this system.

The topology of the world-sheet is then a cylinder, i.e. a Riemann surface. It is well known [17] that the space of the metrics of Euclidean signature which can be defined on this surface is organized into conformal classes labelled by one strictly positive modular parameter. Two metrics fall into the same class if there is a world-sheet diffeomorphism and a Weyl rescaling which transforms one into the other (as in (3.22a)). For this group of transformations, a good gauge slice of the space metrics is thus obtained by selecting one element of each conformal class.

We will assume here that the same result essentially holds for world-sheet metrics of Minkowski signature, namely that conformal classes are still characterized by one strictly positive modular parameter. This assumption will be made more precise below.

Since any metric can be parametrized as in (3.20) (with $(\lambda^+ + \lambda^-) > 0$), the space of functions $\{\phi, \lambda^+, \lambda^-\}$ is also organized into conformal classes. These are the orbits of the group of diffeomorphisms and Weyl rescalings (as defined in (3.22a)). For infinitesimal transformations, we have the variations given in (3.17 c,d.) and

$$\delta\phi = \delta\phi + \eta^\alpha \partial_\alpha \phi + \partial_\alpha \eta^\alpha \quad (3.25)$$

Particularly, variations of the Lagrange multipliers only depend on world-sheet reparametrizations, and not on Weyl rescalings of the metric. Hence, any good gauge slice of the space of metrics also defines a good gauge slice of the space of Lagrange multipliers λ^+, λ^- .

Now as can be shown, the one-parameter family of metrics

$$g_{\alpha\beta} = \begin{pmatrix} -\lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad (3.26)$$

with λ_0 a strictly positive constant, specifies one and only one element in each conformal class.

Therefore, a complete gauge fixing of the Nambu-Goto string is defined by

$$\lambda^+(\sigma, \tau) = \lambda_0, \quad \lambda^-(\sigma, \tau) = \lambda_0 \quad (3.27)$$

where λ_0 is a constant running over the real positive axis. This choice corresponds to the proper-time gauges defined in (3.4).

From (3.20), (3.22) and (3.26), the form of an arbitrary element of the conformal class labelled by λ_0 can also be given. One finds:

$$\lambda^\pm(\sigma, \tau) = \lambda_0 \frac{(1 + \partial_\tau f^\pm(\sigma, \tau)) \pm \lambda_0^{-1} \partial_\tau f^\mp(\sigma, \tau)}{(1 + \partial_\sigma f^\pm(\sigma, \tau)) \pm \lambda_0 \partial_\sigma f^\mp(\sigma, \tau)} \quad (3.28)$$

with $f^\pm(\sigma, \tau_i) = 0$ ($i=1,2$) and periodicity in $(\sigma \rightarrow \sigma + \pi)$. For the open string considered below, the last condition is replaced by $f^\pm(\sigma=0, \pi; \tau) = 0$.

This discussion of the gauge fixing is based on the assumption stated above concerning the conformal structure of the space of world-sheet metrics of Minkowski signature. One can to some extent however, check our conclusions. The variations $\delta\lambda^+$, $\delta\lambda^-$ given in (3.17 c,d) determine how the space of Lagrange multipliers is organized into orbits of world-sheet diffeomorphisms.

In this space, let us consider more particularly a point specified by two functions $(\lambda^+(\tau), \lambda^-(\tau))$ independent of σ . For example, they could be given in the parametrization (3.28) with $f^\pm(\sigma, \tau)$ independent of σ . Proper-time gauges are such configurations.

From (3.17 c,d), any point $(\lambda^+(\tau) + \delta\lambda^+(\sigma, \tau), \lambda^-(\tau) + \delta\lambda^-(\sigma, \tau))$ infinitesimally close to the point $(\lambda^+(\tau), \lambda^-(\tau))$ on the same gauge orbit as it must necessarily be given by

$$\delta\lambda^\pm(\sigma, \tau) = \partial_\tau \epsilon^\pm(\sigma, \tau) \mp \lambda^\pm(\tau) \partial_\sigma \epsilon^\pm(\sigma, \tau) \quad (3.29)$$

for some functions $\epsilon^+(\sigma, \tau), \epsilon^-(\sigma, \tau)$ which satisfy the boundary conditions (3.16).

Then

$$\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \frac{1}{2} (\delta \lambda^+(\sigma, \tau) + \delta \lambda^-(\sigma, \tau)) = 0 \quad (3.30)$$

Hence, the quantity

$$\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \frac{1}{2} (\lambda^+(\tau) + \lambda^-(\tau)) \quad (3.31)$$

is a geometrical invariant [24] which characterizes the gauge orbit passing through $(\lambda^+(\tau), \lambda^-(\tau))$, i.e. it is a modular parameter.

If the assumption made above is correct, this should be the only modular parameter necessary to uniquely specify any gauge orbit. This in turn implies that given any variations $\delta \lambda^+(\sigma, \tau)$, $\delta \lambda^-(\sigma, \tau)$ at the point $(\lambda^+(\tau), \lambda^-(\tau))$ which satisfy (3.30) only, there should exist functions $\epsilon^+(\sigma, \tau)$, $\epsilon^-(\sigma, \tau)$ which solve (3.29).

However, when the invariant quantity $\frac{1}{\pi^2} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \frac{1}{2} (\lambda^+(\tau) + \lambda^-(\tau))$ takes rational values, this is not always possible. To solve this problem (which does not appear in the Euclidean formulation [24]), one has to give the quantity (3.31) an infinitesimal imaginary part.

Therefore, in order to justify the gauge fixing of the system discussed above, we have to assume that for the cylinder, conformal classes of world-sheet metrics of Minkowski signature are characterized by one modular parameter with an infinitesimal imaginary part and a strictly positive real part.

In the proper-time gauges (3.27), the geometrical invariant (3.31) reduces to $(\pi \lambda_0 \Delta \tau)$. We will choose the modular parameter as $c = \frac{\lambda_0 \Delta \tau}{\pi}$; this can be interpreted as the ratio of the total proper-time to the length of the world-sheet. Thus c (or λ_0) is assumed to have an infinitesimal imaginary part. This requirement was previously found in solving the boundary conditions for classical solutions.

Finally, note that the invariant quantity (3.31) is indeed a modular and not a Teichmüller parameter. The latter takes positive as well as negative values, transforming as $(c \rightarrow -c)$ under the modular group which acts by exchanging the boundaries of the cylinder.

In the process considered here, the restriction to modular space follows from the boundary conditions at $\tau = \tau_1$ and $\tau = \tau_2$, or also from the requirement $(\lambda^+ + \lambda^-) > 0$. Its physical meaning is that positive (negative) -energy states propagate forward (backward) in time.

Although the world-sheet of the open string is not a manifold, the corresponding gauge fixing easily follows from the results above. Indeed, such a world-sheet can be viewed as an orbifold constructed from a cylinder parametrized by $\tau_1 \leq \tau \leq \tau_2$ and $-\pi \leq \sigma \leq \pi$, with points identified under $\sigma \rightarrow -\sigma$. Metrics on the open string world-sheet are in a one-to-one correspondence with even metrics on the cylinder.

The family of metrics (3.26) is thus also a good gauge slice of the metrics for the open string. Therefore, in the space of Lagrange multipliers λ^+, λ^- , the proper-time gauges (3.27) also define a complete gauge fixing for the open Nambu-Goto string.

For any value of the proper-time gauge parameter λ_0 , one can solve the Hamiltonian equations of motion (3.13) subject to the boundary conditions (3.15). The solutions (3.5) are recovered, and the first-order action (3.14) reduces to the expressions (3.7).

The canonical quantization of the system in any of the proper-time gauges leads to exactly the same result as in the conformal gauge $\lambda_0 = 1$. These are well known [2,3].

Finally, note that a formulation of the system using a linear action [32] leads to the same Hamiltonian formulation as the one discussed here for the non-linear Nambu-Goto action (see Appendix B).

4. THE BFM APPROACH

In the BFM approach to the Nambu-Goto string, its phase-space is extended as follows [15].

First, one introduces conjugate momenta $\pi_-(\sigma, \tau), \pi_+(\sigma, \tau)$ to $\lambda^-(\sigma, \tau), \lambda^+(\sigma, \tau)$ respectively, with

$$\left\{ \lambda^\eta(\sigma, \tau), \pi_{\eta'}(\sigma', \tau) \right\} = \delta_{\eta'}^\eta \delta(\sigma - \sigma'), \quad \eta, \eta' = +, - \quad (4.1)$$

and the first-class constraints

$$\pi_-(\sigma, \tau) = 0, \quad \pi_+(\sigma, \tau) = 0 \quad (4.2)$$

In addition, Grassmann variables are associated to the constraints $G_1 = \pi_-$, $G_2 = \pi_+$, $G_3 = \phi_-$, $G_4 = \phi_+$. Respectively, these are $\eta^\alpha(\sigma, \tau)$ ($\alpha = 1, 2, 3, 4$) with their conjugate momenta $\mathcal{P}_\alpha(\sigma, \tau)$ ($\alpha = 1, 2, 3, 4$). We have the Poisson brackets:

$$\{ \eta^a(\sigma, \tau), \mathcal{P}_b(\sigma', \tau) \} = -\delta_b^a \delta(\sigma - \sigma') , \quad a, b = 1, 2, 3, 4 \quad (4.3)$$

From the algebra of constraints

$$\{ G_a(\sigma, \tau), G_b(\sigma', \tau) \} = \int_0^\pi d\sigma'' \, \mathcal{U}_{ab}^c(\sigma, \sigma'; \sigma'') G_c(\sigma'', \tau) \quad (4.4a)$$

$$\{ H_0(\tau), G_a(\sigma, \tau) \} = 0 \quad (4.4b)$$

with the (only) non-vanishing structure coefficients defined through (3.11c), one obtains the BRST generator [15]

$$Q_B = \int_0^\pi d\sigma \left[\eta^1 \pi_- + \eta^2 \pi_+ + \eta^3 \phi_- + \eta^4 \phi_+ - \eta^3 \partial_\sigma \eta^3 \mathcal{P}_3 + \eta^4 \partial_\sigma \eta^4 \mathcal{P}_4 \right] \quad (4.5)$$

and the BRST invariant Hamiltonian [15]

$$H_B(\tau) = H_0(\tau) = 0 \quad (4.6)$$

The ghost charge is defined by

$$Q_C = i \int_0^\pi d\sigma \, \eta^a \mathcal{P}_a \quad (4.7)$$

BRST variations of all degrees of freedom are

$$\delta_B x^\mu = \pi \alpha' (\eta^3 \phi_-^\mu + \eta^4 \phi_+^\mu) , \quad \delta_B \pi^\mu = \frac{1}{2} \partial_\sigma (-\eta^3 \phi_-^\mu + \eta^4 \phi_+^\mu) \quad (4.8a)$$

$$\delta_B \lambda^- = \eta^1 , \quad \delta_B \pi_- = 0 ; \quad \delta_B \lambda^+ = \eta^2 , \quad \delta_B \pi_+ = 0 \quad (4.8b)$$

$$\delta_B \eta^1 = 0 , \quad \delta_B \mathcal{P}_1 = -\pi_- ; \quad \delta_B \eta^2 = 0 , \quad \delta_B \mathcal{P}_2 = -\pi_+ \quad (4.8c)$$

$$\delta_B \eta^3 = \eta^3 \partial_\sigma \eta^3 , \quad \delta_B \mathcal{P}_3 = -\phi_- + 2 \partial_\sigma \eta^3 \mathcal{P}_3 + \eta^3 \partial_\sigma \mathcal{P}_3 \quad (4.8d)$$

$$\delta_B \eta^4 = -\eta^4 \partial_\sigma \eta^4 , \quad \delta_B \mathcal{P}_4 = -\phi_+ - 2 \partial_\sigma \eta^4 \mathcal{P}_4 - \eta^4 \partial_\sigma \mathcal{P}_4 \quad (4.8e)$$

and by construction [15]:

$$\{ Q_B, Q_B \} = 0 \quad (4.9)$$

Again, different choices of BRST invariant boundary conditions are possible [15]. Keeping in mind the path-integral representation, we take [15]:

$$x^\mu(\sigma, \tau_i) = x^\mu(\sigma), \quad \pi_\mp(\sigma, \tau_i) = 0, \quad i = 1, 2 \quad (4.10a)$$

$$\eta^3(\sigma, \tau_i) = 0 = \eta^4(\sigma, \tau_i), \quad \mathcal{P}_1(\sigma, \tau_i) = 0 = \mathcal{P}_2(\sigma, \tau_i), \quad i = 1, 2 \quad (4.10b)$$

Time evolution of the system is governed by an effective Hamiltonian, whose density is

$$\mathcal{H}_{eff}(\sigma, \tau) = \mathcal{H}_B(\sigma, \tau) - \{ \Psi(\sigma, \tau), Q_B(\tau) \} = - \{ \Psi(\sigma, \tau), Q_B(\tau) \} \quad (4.11)$$

Here $\Psi(\sigma, \tau)$ is a function over the extended phase-space, of ghost number (-1). It plays the role of a gauge fixing term.

The equations of motion which correspond to (4.11) together with the boundary conditions (4.10) follow from the first-order effective action

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[\dot{x}^\mu \pi_\mu + \dot{\lambda}^- \pi_- + \dot{\lambda}^+ \pi_+ + \dot{\eta}^1 \mathcal{P}_1 + \dot{\eta}^2 \mathcal{P}_2 - \dot{\eta}^3 \mathcal{P}_3 - \dot{\eta}^4 \mathcal{P}_4 - \mathcal{H}_{eff} \right] \quad (4.12)$$

As we know from the analysis for the point particle, the choice of Ψ is important. It should be such that one and only one element of each gauge orbit is selected. In turn, the integration measure for modular space would then be uniquely specified.

From the discussion in the previous Section, we know that such a gauge fixing is achieved by

$$\lambda^+(\sigma, \tau) = \lambda_0, \quad \lambda^-(\sigma, \tau) = \lambda_0 \quad (4.13)$$

with λ_0 a constant running over the real axis (with an infinitesimal imaginary part), or equivalently by the choice (3.28) (with $\lambda^\mu(\sigma, \tau)$ now fixed functions).

It is apparently impossible however, within the present formulation, to find a function Ψ leading to such a gauge fixing. The gauge fixing of (4.13) is specified by at least three equations. For example:

$$\partial_\tau(\lambda^+ + \lambda^-) = 0, \quad \partial_\sigma(\lambda^+ + \lambda^-) = 0, \quad \lambda^+ - \lambda^- = 0 \quad (4.14)$$

The formalism here, on the other hand, allows for only two gauge fixing conditions since we only have two Lagrange multipliers for this purpose, namely π_- and π_+ .

If non-local gauge fixing functions Ψ could be chosen, the problem would be overcome. For example,

$$\lambda^+ = \frac{1}{2\pi\Delta\tau} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma (\lambda^+ + \lambda^-) = \lambda^- \quad (4.15)$$

leads to (4.13) (with $(\lambda^+ + \lambda^-) > 0$).

Thus, there does not (seem to) exist a local function Ψ corresponding to a complete gauge fixing of the Nambu-Goto string, i.e. in which each gauge orbit is intersected once and only once by the gauge slice. This is a Gribov problem for the bosonic string. Such a problem was discovered first in the context of Yang-Mills theories [33], with the same interpretation [34].

This problem is not specific to the BFV formulation. It appears whatever the approach used to quantize these constrained systems, be it a path-integral [35] or a canonical one (as for example the method of Ref.[30] used in Ref.[10] for the bosonic string, where again only two gauge fixing conditions can be introduced).

Practically speaking, the consequence of the Gribov problem is that the integration measure over the space of orbits is not specified, even by the BFV formulation (which determines it in the absence of a Gribov problem). Some additional criterium is required to fix it uniquely. In the case of string theory, this may be achieved by requiring modular invariance for physical observables. With regards to the simple process under consideration however, this does not apply since the modular group is trivial.

As in (2.52) for the particle, let us choose a function Ψ which leads to an overcomplete gauge fixing by selecting one point of only one gauge orbit:

$$\Psi = \frac{1}{\beta} \mathcal{P}_1 (\lambda^- - \lambda_0) + \frac{1}{\beta} \mathcal{P}_2 (\lambda^+ - \lambda_0) + \mathcal{P}_3 \lambda^- + \mathcal{P}_4 \lambda^+ \quad (4.16)$$

Here, λ_0 is a fixed positive constant (with an infinitesimal imaginary part) and β a free parameter.

The corresponding effective action, effective Hamiltonian and equations of motion are easily obtained, together with their BRST invariance. To enforce the overcomplete gauge fixing $(\lambda^+ = \lambda_0 = \lambda^-)$, we rewrite the fields $\pi_\pm$, $\mathcal{P}_1$, $\mathcal{P}_2$ as $\beta\pi_\pm$, $\beta\mathcal{P}_1$, $\beta\mathcal{P}_2$ respectively, and then take the limit $\beta \rightarrow 0$.

With these redefinitions, the equations of motion become:

$$\dot{x}^\mu = \pi_\alpha' (\lambda^- \phi_-^\mu + \lambda^+ \phi_+^\mu) \quad , \quad \dot{\pi}^\mu = \frac{1}{2} \partial_\sigma (-\lambda^- \phi_-^\mu + \lambda^+ \phi_+^\mu) \quad (4.17a)$$

$$\lambda^- = \lambda_0 \quad , \quad \pi_- = -\phi_- + 2\partial_\sigma \eta^3 \mathcal{P}_3 + \eta^3 \partial_\sigma \mathcal{P}_3 \quad (4.17b)$$

$$\lambda^+ = \lambda_0 \quad , \quad \pi_+ = -\phi_+ - 2\partial_\sigma \eta^4 \mathcal{P}_4 - \eta^4 \partial_\sigma \mathcal{P}_4 \quad (4.17c)$$

$$\eta^1 = 0 \quad , \quad \mathcal{P}_1 + \mathcal{P}_3 = 0 \quad ; \quad \eta^2 = 0 \quad , \quad \mathcal{P}_2 + \mathcal{P}_4 = 0 \quad (4.17d)$$

$$\dot{\eta}^3 = \eta^1 - \lambda^- \partial_\sigma \eta^3 + \partial_\sigma \lambda^- \eta^3 \quad , \quad \dot{\mathcal{P}}_3 = -\lambda^- \partial_\sigma \mathcal{P}_3 - 2\partial_\sigma \lambda^- \mathcal{P}_3 \quad (4.17e)$$

$$\dot{\eta}^4 = \eta^2 + \lambda^+ \partial_\sigma \eta^4 - \partial_\sigma \lambda^+ \eta^4 \quad , \quad \dot{\mathcal{P}}_4 = \lambda^+ \partial_\sigma \mathcal{P}_4 + 2\partial_\sigma \lambda^+ \mathcal{P}_4 \quad (4.17f)$$

They also follow from the effective action

$$\begin{aligned} S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \big[& \dot{x}^\mu \pi_\mu - \eta^3 \dot{\mathcal{P}}_3 - \eta^4 \dot{\mathcal{P}}_4 - (\lambda^- - \lambda_0) \pi_- - (\lambda^+ - \lambda_0) \pi_+ \\ & - \lambda^- \phi_- - \lambda^+ \phi_+ + \mathcal{P}_1 \eta^1 + \mathcal{P}_2 \eta^2 + \mathcal{P}_3 \eta^3 + \mathcal{P}_4 \eta^4 \\ & + 2\lambda^- \partial_\sigma \eta^3 \mathcal{P}_3 + \lambda^- \eta^3 \partial_\sigma \mathcal{P}_3 - 2\lambda^+ \partial_\sigma \eta^4 \mathcal{P}_4 - \lambda^+ \eta^4 \partial_\sigma \mathcal{P}_4 \big] \end{aligned} \quad (4.18)$$

Some of these equations are now constraints. Solving (4.17 b,c,d) leaves us with the dynamical degrees of freedom $x^\mu, \pi_\mu, \eta^3, \mathcal{P}_1, \eta^4, \mathcal{P}_2$ and the action

$$\begin{aligned} S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \big[& \dot{x}^\mu \pi_\mu + \eta^3 \dot{\mathcal{P}}_1 + \eta^4 \dot{\mathcal{P}}_2 - \lambda_0 (\phi_- + \phi_+) \\ & - 2\lambda_0 \partial_\sigma \eta^3 \mathcal{P}_1 - \lambda_0 \eta^3 \partial_\sigma \mathcal{P}_1 + 2\lambda_0 \partial_\sigma \eta^4 \mathcal{P}_2 + \lambda_0 \eta^4 \partial_\sigma \mathcal{P}_2 \big] \end{aligned} \quad (4.19)$$

With the boundary conditions

$$i) \quad x^\mu(\sigma, \tau_i) = x_i^\mu(\sigma) \quad , \quad i=1,2 \quad (4.20a)$$

$$(\phi_- + 2\partial_\sigma \eta^3 \mathcal{P}_1 + \eta^3 \partial_\sigma \mathcal{P}_1)(\sigma, \tau_i) = 0 \quad , \quad i=1,2 \quad (4.20b)$$

$$(\phi_+ - 2\partial_\sigma \eta^4 \mathcal{P}_2 - \eta^4 \partial_\sigma \mathcal{P}_2)(\sigma, \tau_i) = 0 \quad , \quad i=1,2 \quad (4.20c)$$

$$\eta^3(\sigma, \tau_i) = 0 = \eta^4(\sigma, \tau_i) \quad , \quad \mathcal{P}_1(\sigma, \tau_i) = 0 = \mathcal{P}_2(\sigma, \tau_i) \quad , \quad i=1,2 \quad (4.20d)$$

$$ii) a) \text{ closed string: periodicity in } \sigma \rightarrow \sigma + \pi \quad (4.20e)$$

$$b) \text{ open string: } \partial_\sigma x^\mu(\sigma=0, \pi; \tau) = 0 \quad (4.20f)$$

$$(\eta^3 - \eta^4)(\sigma=0, \pi; \tau) = 0 = (\mathcal{P}_1 - \mathcal{P}_2)(\sigma=0, \pi; \tau) \quad (4.20g)$$

and the constraint equations (4.17 b,c,d), this system is equivalent to the original one.

All time evolution equations follow from (4.19), as well as from the effective Hamiltonian density

$$\mathcal{H}_{eff} = \lambda_0(\phi_- + \phi_+) + 2\lambda_0 \partial_\sigma \eta^3 \mathcal{P}_1 + \lambda_0 \eta^3 \partial_\sigma \mathcal{P}_1 - 2\lambda_0 \partial_\sigma \eta^4 \mathcal{P}_2 - \lambda_0 \eta^4 \partial_\sigma \mathcal{P}_2 \quad (4.21)$$

provided one now uses the Poisson brackets (since $\mathcal{P}_3 = -\mathcal{P}_1$, $\mathcal{P}_4 = -\mathcal{P}_2$):

$$\{ \eta^3(\sigma, \tau), \mathcal{P}_1(\sigma', \tau) \} = \delta(\sigma - \sigma') = \{ \eta^4(\sigma, \tau), \mathcal{P}_2(\sigma', \tau) \} \quad (4.22)$$

(4.20 b,c) replace the boundary conditions imposed on π_- , π_+ in (4.10), as follows from (4.17 b,c). It may seem strange at first that both x^μ , η^3 , η^4 and their conjugate momenta π_μ , $\mathcal{P}_1$, $\mathcal{P}_2$ have boundary conditions at both end points $\tau = \tau_i$ ($i=1,2$). This is nevertheless consistent with the equations and the correct physical solutions are obtained.

With the constraints (4.17 b,c,d), the BRST variations (4.8) (which have the same expressions after the aforementioned field redefinitions) reduce for x^μ , π_μ , η^3 , $\mathcal{P}_1$, η^4 , $\mathcal{P}_2$ to:

$$\delta_B x^\mu = \pi \alpha' (\eta^3 \phi_-^\mu + \eta^4 \phi_+^\mu) \quad , \quad \delta_B \pi^\mu = \frac{1}{2} \partial_\sigma (-\eta^3 \phi_-^\mu + \eta^4 \phi_+^\mu) \quad (4.23a)$$

$$\delta_B \eta^3 = \eta^3 \partial_\sigma \eta^3 \quad , \quad \delta_B \mathcal{P}_1 = \phi_- + 2 \partial_\sigma \eta^3 \mathcal{P}_1 + \eta^3 \partial_\sigma \mathcal{P}_1 \quad (4.23b)$$

$$\delta_B \eta^4 = -\eta^4 \partial_\sigma \eta^4 \quad , \quad \delta_B \mathcal{P}_2 = \phi_+ - 2 \partial_\sigma \eta^4 \mathcal{P}_2 - \eta^4 \partial_\sigma \mathcal{P}_2 \quad (4.23c)$$

For the remaining degrees of freedom, they are consistent with the constraints.

With the boundary conditions (4.20), these transformations leave the effective action (4.19) invariant. They are generated by the BRST charge

$$Q_B = \int_0^\pi d\sigma \left[\eta^3 \phi_- + \eta^4 \phi_+ + \eta^3 \partial_\sigma \eta^3 \mathcal{P}_1 - \eta^4 \partial_\sigma \eta^4 \mathcal{P}_2 \right] \quad (4.24)$$

with (4.22). Through the field redefinitions mentioned above, this is also the expression obtained from (4.5).

The choice (4.16) thus leads to the proper-time gauge specified by λ_0 . More generally, one could consider

$$\Psi = \frac{1}{\beta} \mathcal{P}_1 (\lambda^- - \lambda_0^-) + \frac{1}{\beta} \mathcal{P}_2 (\lambda^+ - \lambda_0^+) + \mathcal{P}_3 \lambda^- + \mathcal{P}_4 \lambda^+ \quad (4.25)$$

where λ_0^-, λ_0^+ are functions given as in (3.28) for some fixed λ_0 and $\frac{1}{2}(\sigma, \tau)$ ($\sigma=0,1$). The corresponding analysis leads to essentially the same equations. From BRST invariance, it should lead to exactly the same results for physical solutions, since they can only depend on the particular orbit chosen, specified by λ_0 , and not on a particular point on that orbit. In practice, the proper-time choice is more convenient; all equations can be solved.

With the boundary conditions (4.20) and the redefinitions

$$\eta^3 = c^-, \quad \mathcal{P}_1 = \frac{i}{2\pi} b_{--}; \quad \eta^4 = c^+, \quad \mathcal{P}_2 = \frac{i}{2\pi} b_{++} \quad (4.26)$$

the action then becomes

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[\dot{x}^\mu \pi_\mu - \lambda_0 (\phi_- + \phi_+) + \frac{i}{2\pi} b_{--} (\partial_\tau + \lambda_0 \partial_\sigma) c^- + \frac{i}{2\pi} b_{++} (\partial_\tau - \lambda_0 \partial_\sigma) c^+ \right] \quad (4.27)$$

This is essentially recognised as the starting point of the original BRST quantization of the (open) bosonic string [10] (where the conformal gauge was taken). The Grassmann fields $\eta^3, \eta^4, \mathcal{P}_1, \mathcal{P}_2$ are seen to correspond to the usual Faddeev-Popov reparametrization ghosts $c^\pm$ and $b_{\pm\pm}$ [3].

More generally, with the choice (4.25) and the corresponding field redefinitions, by solving for $\pi^\mu, \eta^1, \eta^2, \mathcal{P}_3, \mathcal{P}_4$ the effective action (4.12) reduces to the gauge fixed action obtained in the Lagrangian approach, be it that of Ref.[30] used in Ref.[10], or Polyakov's path-integral approach [1]. The conditions $\lambda^+ = \lambda_0^+, \lambda^- = \lambda_0^-$ then, specify the gauge-fixed intrinsic metric through (3.20).

These remarks clearly show that the usual results of BRST quantization [10, 23] obtained in the conformal gauge $\lambda_0=1$, are essentially recovered. The only difference is that in all mode expansions the τ -dependence is replaced here by a dependence on $\lambda_0(\tau-\tau_1)$, as shown for example in (3.5) for the bosonic degrees of freedom.

Solving the classical equations of motion with the boundary conditions (4.20), the solutions (3.5a,b) as well as the constraints (3.5c,d) are recovered, with all other fields in the extended phase-space vanishing. The BRST and ghost charges are then equal to zero, and the effective action (4.19) reduces to the expressions given in (3.7).

BRST quantization in any proper-time gauge proceeds straightforwardly, giving the algebra of commutators and anticommutators for the mode expansions as in the conformal gauge $\lambda_0 = 1$ [10, 23].

Details are not discussed here. We only recall that BRST invariance (i.e. nilpotency of the BRST charge) of the quantized theory is possible if and only if $D = 26$ and the first excitation level is massless (i.e. the Regge intercept is equal to 1) [10]. This is equivalent [3] to the vanishing of the central charge of the Virasoro algebra. This implies that the full group of world-sheet diffeomorphisms continuously connected to the identity is then an exact symmetry of the quantum theory. Moreover, physical states are characterized by $Q_B = 0$ and $Q_C = -1/2$. These are well known results [10].

5. THE BFM PHASE-SPACE PATH-INTEGRAL

From the discussion in the previous Section we know that, due to a Gribov problem for the bosonic string, the BFM phase-space path-integral will not determine the integration measure for the modular parameter.

With the overcomplete gauge fixing following from (4.16) (or (4.25)) we can nevertheless consider the path-integral restricted to the orbit specified by the fixed value λ_0 and arbitrarily integrate it over all orbits with a measure to be determined by other requirements (as was discussed for the particle).

In order that this path-integral be invariant for the world-sheet diffeomorphisms, the quantized system must be BRST invariant [15]. In the following, we thus assume that $D = 26$, so that the path-integral for a fixed orbit will only depend on $(\lambda_0 \Delta \tau)$ and not on any specific point on that orbit. Again, with the choice (4.16), its explicit evaluation is the easiest.

With the usual field redefinitions as $\beta \rightarrow 0$, which leave the Liouville measure invariant [15], the path-integral representation of the string propagator becomes:

$$Z(x_2^*(\sigma), x_1^*(\sigma), \lambda_0) = \int [d\mu] e^{iS_{\text{eff}}} \quad (5.1)$$

Here, $[d\mu]$ is the Liouville measure over the extended phase-space, and S_{eff} is given in (4.18). The functional integrals are taken with the boundary conditions

$$i) \text{ same as in (4.10)} \quad (5.2a)$$

$$ii) \text{ same as in (4.20 e, f, g)} \quad (5.2b)$$

The integrals over $\pi_{\mp}$ enforce the overcomplete gauge fixing conditions $\lambda^{\pm} = \lambda_0$, $\lambda^- = \lambda_0$ for fixed λ_0 .

With the boundary conditions (5.2b) for proper-time gauges, the usual mode expansions in σ are possible for all degrees of freedom in the extended phase-space. The functional integrals simply reduce to path-integrals for harmonic oscillators. These can be carried out using standard methods [36]. The details of these rather straightforward calculations are not presented here.

Consider first the case of the closed string. Proceeding as just indicated, the propagator (5.1) essentially factorizes as:

$$Z_{\text{cl}}(x_2^{\mu}(\sigma), x_1^{\mu}(\sigma), \lambda_0) = Z_L^+(0, \lambda_0) Z_F^+(0, \lambda_0) Z_L^-(0, \lambda_0) Z_F^-(0, \lambda_0) Z_B(0, \lambda_0) \cdot \prod_{n=1}^{\infty} [Z_L^+(n, \lambda_0) Z_F^+(n, \lambda_0) Z_L^-(n, \lambda_0) Z_F^-(n, \lambda_0) Z_B(n, \lambda_0)] e^{i S_{\text{cl}}(x_2^{\mu}, \lambda_0 \Delta\tau)} \quad (5.3)$$

Here, $S_{\text{cl}}(x_2^{\mu}, \lambda_0 \Delta\tau)$ is the classical action of the closed string given in (3.7a).

$Z_L^{\pm}(n, \lambda_0)$ are the integrals over the n th mode of the Lagrange multipliers $\lambda^{\pm}$ and $\pi_{\pm}$, $Z_F^{\pm}(n, \lambda_0) (\eta^{\mu\nu}, Z_F^{\pm}(n, \lambda_0))$ are the integrals over the n th mode of BFV ghosts η^1, η^3 and $\mathcal{P}_1, \mathcal{P}_3 (\eta^{\mu\nu}, \eta^2, \eta^4, \mathcal{P}_2, \mathcal{P}_4)$ and $Z_B(n, \lambda_0)$ are the integrals over the n th mode of the bosonic fields α^{μ} and π_{μ} .

All these integrals can explicitly be computed by time slices. For the zero modes, we have (up to numerical factors)

$$Z_B(0, \lambda_0) = (2\pi\alpha' \lambda_0 \Delta\tau)^{-D/2} \quad (5.4a)$$

$$Z_L^+(0, \lambda_0) Z_F^+(0, \lambda_0) = \Delta\tau = Z_L^-(0, \lambda_0) Z_F^-(0, \lambda_0) \quad (5.4b)$$

As in the case of the particle, in order to obtain a well defined quantity when the number N of time steps in τ becomes infinite, the integrals $Z_L^{\pm}$ and $Z_F^{\pm}$ must be combined. This holds for the zero-modes as well as the nonzero-modes.

For the non-zero modes, following [36], one obtains for the bosonic integral:

$$Z_B(n, \lambda_0) = \left(2\pi \alpha' \lambda_0 \Delta\tau \cdot \frac{\sin 2n\lambda_0 \Delta\tau}{2n\lambda_0 \Delta\tau} \right)^{-D} \quad (5.5)$$

For the other integrals one explicitly obtains:

$$Z_L^\pm(n, \lambda_0) Z_F^\pm(n, \lambda_0) = (\Delta\tau)^2 J(\lambda_0, \Delta\tau) \lim_{N \rightarrow \infty} \det(-i B_\pm) \quad (5.6)$$

where $J(\lambda_0, \Delta\tau)$ is a Jacobian factor such that for $n = 0$ this expression reduces to $(\Delta\tau)^2$ [36].

The matrices $B_\pm$ are defined as

$$B_\pm = \mathbb{1} \otimes M \mp 2in\lambda_0 \sigma_2 \otimes \mathbb{1} \quad (5.7a)$$

where the first factors in the tensor products are 2×2 matrices (this structure appears because of the even and odd modes in σ ; σ_2 is the usual Pauli matrix) and the second factors are $N \times N$ matrices (N is the number of time steps) with

$$M_{nm} = \begin{cases} 0 & \text{if } n=m \\ 0 & \text{if } n+m \text{ is even} \\ \frac{4}{\Delta\tau} \frac{nm}{(n^2 - m^2)} & \text{if } n+m \text{ is odd} \end{cases} \quad (5.7b)$$

$n, m = 1, \dots, N$

(this matrix is the discretized version of ∂_τ)

Combining the contributions of the right - and left - moving modes and using the identity

$$\lim_{N \rightarrow \infty} (-M^2)_{nm} = \delta_{n,m} \left(\frac{n\pi}{\Delta\tau} \right)^2 \quad (5.8)$$

one obtains

$$Z_L^+(n, \lambda_0) Z_F^+(n, \lambda_0) Z_L^-(n, \lambda_0) Z_F^-(n, \lambda_0) = (\Delta\tau)^4 \left(\frac{\sin 2n\lambda_0 \Delta\tau}{2n\lambda_0 \Delta\tau} \right)^2 \quad (5.9)$$

Therefore

$$Z_{cl}(x_2^*(\sigma), x_1^*(\sigma), \lambda_0) = (\Delta\tau)^2 (2\pi \alpha' \lambda_0 \Delta\tau)^{-D/2} \cdot \prod_{n=1}^{\infty} \left[(\Delta\tau)^4 (2\pi \alpha' \lambda_0 \Delta\tau)^{-D} \left(\frac{\sin 2n\lambda_0 \Delta\tau}{2n\lambda_0 \Delta\tau} \right)^{-D+2} \right] \cdot e^{i S_{cl}(x_i^*, \lambda_0 \Delta\tau)} \quad (5.10)$$

Note the well known fact [3] that the contribution from the BFM ghosts cancels that of unphysical states for each non-zero mode. Also, as a consequence of BRST invariance (for $D=26$), (5.10) depends on τ_1 , τ_2 , and λ_0 .

only through quantities invariant under world-sheet diffeomorphisms, namely $\Delta\tau$ and the modular parameter $(\lambda_0 \Delta\tau = \pi c)$.

Since the infinite product diverges, (5.10) is not well defined. As we are interested only in its dependence on λ_0 , $\Delta\tau$ and $x_i^\mu(\sigma)$ ($i=1,2$), we use ζ -function regularization to regulate the infinite product [17]. Up to (possibly infinite) numerical factors, we then have:

$$\prod_{n=1}^{\infty} \alpha = \alpha^{-1/2}, \quad \prod_{n=1}^{\infty} e^{\alpha n} = e^{-\alpha/12} \quad (5.11)$$

and

$$Z_{cl}(x_2^\mu(\sigma), x_1^\mu(\sigma), \lambda_0) = (\lambda_0 \Delta\tau)^{-D/2+1} e^{\frac{1}{6}\hat{v}(D-2)\lambda_0 \Delta\tau} \cdot \prod_{n=1}^{\infty} (1 - e^{-4\hat{v}n\lambda_0 \Delta\tau})^{-D+2} \cdot e^{iS_{cl}(x_i^\mu, \lambda_0 \Delta\tau)} \quad (5.12)$$

The second factor actually represents the regulated contribution of the zero-point energy fluctuations and corresponds to the presence of a tachyon [2,3,17] in the spectrum of the theory. See (5.14) below.

We assumed throughout that λ_0 has an infinitesimal imaginary part, for reasons already discussed above. However in order to make the infinite product in (5.12) well defined, this imaginary part must be negative. This will furthermore give the propagator its correct causal structure.

Note that all the dependence on $(\Delta\tau)$ (as opposed to $\lambda_0 \Delta\tau$), which followed from the integration over the ghosts, has cancelled out. Thus their net effect lies with the fact that the power of the second and third factors is $(D-2)$ rather than D , as would be the case if unphysical states were also propagating.

The full closed string propagator is obtained by integrating (5.12) over all gauge orbits (specified by λ_0) and all parametrizations Σ_λ of the initial and final string configurations ℓ_i^μ ($i=1,2$).

The corresponding measures should be invariant under world-sheet diffeomorphisms. For the integration over λ_0 it should be given as a function which depends only on $(\Delta\tau)$ and $(\lambda_0 \Delta\tau)$, such as $\nu_{cl}(\lambda_0, \Delta\tau) = \lambda_0^{-1} \mu_{cl}(\lambda_0 \Delta\tau, \Delta\tau)$.

We then have:

$$Z_{cl}(\ell_2^\mu, \ell_1^\mu) = \int [d\Sigma_1 d\Sigma_2] \int_0^\infty dc \mu_{cl}(c, \Delta\tau) c^{-13} e^{4i\pi c} \cdot \prod_{n=1}^{\infty} (1 - e^{-4i\pi nc})^{-24} e^{iS_{cl}(x_i^\mu, c)} \quad (5.13)$$

where $c = \lambda_0 \Delta\tau / \pi$ is the modular parameter of the cylinder (and $D=26$).

If we insist on having an expression independent of any reference to the world-sheet parametrization, i.e. to $\Delta\tau$, the measure $\mu_d(c, \Delta\tau)$ should only depend on the modular parameter. We assume this to be the case.

Restricting (5.13) to point-states [17], we have:

$$Z_d(x_2^{\mu}, x_1^{\mu}) = \int_0^{\infty} dc \mu_d(c) c^{-13} e^{4i\pi c} \prod_{n=1}^{\infty} (1 - e^{-4i\pi n c})^{-24} e^{\frac{i}{4\pi\alpha'c} (\Delta x)^2} \quad (5.14a)$$

$$= \sum_{N=0}^{\infty} d_0(N) \int_0^{\infty} dc \mu_d(c) c^{-13} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{(2\pi\alpha'c)} - (2\pi\alpha'c) \frac{4}{24} (N-1) \right]} \quad (5.14b)$$

where

$$\sum_{N=0}^{\infty} d_0(N) q^{N-1} = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24} \quad (5.15)$$

In view of (2.60), (5.14) is identified with the total propagator of an infinite set of free particles of masses $\alpha' m^2 = 4(N-1)$ ($N=0, 1, \dots$) (which is indeed the mass spectrum of the closed bosonic string) and degeneracies given by (5.15), which are those of the open string [17]. This shows that an infinitesimal negative imaginary part for the modular parameter c indeed gives the propagator its correct causal structure.

For the particle, the correct measure is known to be $\mu_d(c) = 1$ supporting the same choice in (5.13) for the closed bosonic string propagator. For world-sheets of other topologies, the modular group should restrict the form of this measure.

The calculation of the open string propagator is similar. Taking advantage of the boundary conditions in σ , one can extend its range to $-\pi \leq \sigma \leq \pi$ and regard the would-be right- and left- moving fields as the values which a same field takes in the intervals $0 \leq \sigma \leq \pi$ and $-\pi \leq \sigma \leq 0$ respectively (the other choice is of course possible). In other words, one uses the orbifold character of the world-sheet alluded to in Section 3.

The system then essentially reduces to a closed bosonic string (now with periodicity in $\sigma \rightarrow \sigma + 2\pi$) with half the number of fields. These remarks facilitate the calculation of (5.1) in this case. One obtains (up to numerical factors):

$$Z_{op}(x_2^\mu(\sigma), x_1^\mu(\sigma), \lambda_0) = (\Delta\tau) (\pi\alpha') \lambda_0 \Delta\tau)^{-D/2} \cdot \prod_{n=1}^{\infty} \left[(\Delta\tau)^2 (\pi\alpha') \lambda_0 \Delta\tau \right]^{-D/2} \left(\frac{\sin n\lambda_0 \Delta\tau}{n\lambda_0 \Delta\tau} \right)^{-D/2+1} e^{iS_{op}(x_i^\mu, \lambda_0 \Delta\tau)} \quad (5.16)$$

with $S_{op}(x_i^\mu, \lambda_0 \Delta\tau)$ the classical action of the open string given in (3.7b).

Through ζ -function regularization of the infinite product, this reduces to (up to an infinite numerical factor including dependence on α'):

$$Z_{op}(x_2^\mu(\sigma), x_1^\mu(\sigma), \lambda_0) = (\lambda_0 \Delta\tau)^{-D/2+1/2} e^{i\frac{(D-2)}{24}\lambda_0 \Delta\tau} \cdot \prod_{n=1}^{\infty} (1 - e^{-2in\lambda_0 \Delta\tau})^{-\frac{(D-2)}{2}} e^{iS_{op}(x_i^\mu, \lambda_0 \Delta\tau)} \quad (5.17)$$

The same remarks as for the closed string apply here. The second factor follows from the zero-point energy fluctuations, and corresponds to the presence of a tachyon in the spectrum of the theory. (See (5.19) below). The infinitesimal imaginary part of λ_0 must be negative to make the infinite product converge. It also gives the propagator its correct causal structure. Here again, the net effect of the integration over the BGV ghosts is to cancel out the contributions of the unphysical states. Finally, (5.17) only depends on $(\Delta\tau)$ and $(\lambda_0 \Delta\tau)$, as must follow from BRST invariance.

Integrating over all parametrizations Σ_i of the initial and final string configurations α_i^μ ($i=1,2$) and over all gauge orbits with an invariant measure $\nu_{op}(\lambda_0, \Delta\tau) = \left(\frac{\Delta\tau}{\lambda_0}\right)^{D/2} \mu_{op}(\lambda_0 \Delta\tau, \Delta\tau)$, the full open string propagator is:

$$Z_{op}(\alpha_2^\mu, \alpha_1^\mu) = \int [d\Sigma_1 d\Sigma_2] \int_0^\infty dc \mu_{op}(c, \Delta\tau) c^{-13} e^{i\pi c} \cdot \prod_{n=1}^{\infty} (1 - e^{-2i\pi n c})^{-12} e^{iS_{op}(x_i^\mu, c)} \quad (5.18)$$

where $c = \lambda_0 \Delta\tau / \pi$ (and $D=26$).

For the same reasons as before, we consider measures $\mu_{op}(c)$ which only depend on the modular parameter. Restricting (5.18) to point-states [17], we have:

$$Z_{op}(x_2^\mu, x_1^\mu) = \int_0^\infty dc \mu_{op}(c) c^{-13} e^{i\pi c} \prod_{n=1}^{\infty} (1 - e^{-2i\pi n c})^{-12} e^{\frac{i}{4\pi\alpha'c} (\Delta x)^2} \quad (5.19a)$$

$$= \sum_{N=0}^{\infty} d'_0(N) \int_0^\infty dc \mu_{op}(c) c^{-13} e^{\frac{1}{2}i \left[\frac{(\Delta x)^2}{(2\pi\alpha'c)} - (2\pi\alpha'c) \frac{1}{\alpha'} (N-1) \right]} \quad (5.19b)$$

where

$$\sum_{N=0}^{\infty} d'_0(N) q^{N-1} = q^{-1} \prod_{n=1}^{\infty} (1 - q^{2n})^{-12}, \quad d'_0(2N+1) = 0 \quad (5.20)$$

(5.19) is the total propagator of an infinite set of free particles of masses

$\alpha' m^2 = N-1$ ($N=0,1,\dots$) (which is indeed the mass spectrum of the open bosonic string) and degeneracies $d'_0(N)$. Again this supports the choice $\mu_{op}(c)=1$ as the correct measure for the open bosonic string propagator.

It is now possible to compare our results (5.13) and (5.18) with those obtained within the Polyakov approach [17,18]. The latter are obtained from ours by the transformation

$$C \longrightarrow C_E = iC \quad (5.21)$$

where C_E stands for the modular parameter in the Euclidean formulation. Thus, in the present representation of the theory, (5.21) specifies how a Wick rotation is to be implemented in modular space. Actually, this may have been expected since c measures the total proper-time of the string. We also expect this result to be true for any world-sheet topology, even in the presence of interactions.

Note that this is also consistent with our requirement of an infinitesimal negative imaginary part for the modular parameter for consistency of the formulation in Minkowski space-time.

It may appear that Polyakov's approach determines the measure $\mu(c)$ uniquely. This however, is not the case. In its usual presentation [6,7], measures are implicitly defined through gaussian integrals over the tangent spaces of field configurations, with some invariant metrics introduced on these spaces. It seems that the freedom in the choice of those measures was not fully taken into account. Indeed, since such a construction is carried out for each conformal class separately, it cannot uniquely specify the integration measure over the modular parameters.

This aspect of Polyakov's approach was considered in Ref.[31] in the case of the particle, where the result (2.60) was obtained. We expect the same to happen for string theory within that approach. The requirement of modular invariance however is expected to uniquely solve this problem in most cases. But it does not do so for the process considered here.

Nevertheless, given the arguments above and a calculation in the light-cone gauge [17,18], we know that the correct choices must be $\mu_{cl}(c)=1=\mu_{op}(c)$. The Polyakov approach implicitly makes this choice [1,6,7].

A more detailed comparison of the two formulations is possible [25]. For the closed string, with ζ -function regularization, the complete contribution of the bosonic modes is (see (5.4a) and (5.5)):

$$Z_B^{cl}(\lambda_0 \Delta\tau) = \left[(\lambda_0 \Delta\tau) e^{-\frac{1}{3}i\lambda_0 \Delta\tau} \prod_{n=1}^{\infty} (1 - e^{-4in\lambda_0 \Delta\tau})^2 \right]^{-D/2} \quad (5.22)$$

Combining that of the Lagrange multipliers and BFM ghosts, we obtain:

$$Z_g^d(\lambda_0 \Delta\tau) = \left[(\lambda_0 \Delta\tau) e^{-\frac{1}{3} i \lambda_0 \Delta\tau} \prod_{n=1}^{\infty} (1 - e^{-4i n \lambda_0 \Delta\tau})^2 \right] \quad (5.23)$$

The full propagator (5.13) is then expressed as

$$Z_{cl}(\ell_2^\mu, \ell_1^\mu) = \int [d\Sigma_1, d\Sigma_2] \int_0^\infty \frac{dc}{c} \mu_{cl}(c) Z_g^d(c) Z_B^d(c) e^{iS_d(x_i^\mu, c)} \quad (5.24)$$

with $\pi c = \lambda_0 \Delta\tau$.

Performing the Wick rotation (5.21), (5.24) leads to the expression given in Eq.(4.1) of Ref.[17] (for $\mu_{cl}(c) = 1$) for the Polyakov path-integral for the same process.

Z_B^d and Z_g^d correspond to the bosonic and Faddeev-Popov determinants respectively. These can formally be rewritten as functional integrals over x^μ and Faddeev-Popov ghosts with a measure defined precisely through this relation. This measure is not the Liouville measure on those functional spaces.

Moreover, dc/c (i.e. $\mu_{cl}(c) = 1$) is the modular measure obtained in Ref.[17] from the Teichmüller deformations and conformal Killing vectors of the cylinder. It can easily be shown that this is also the Weil-Petersson measure of the cylinder which the work of Ref.[7] instructs us to use. But, as already mentioned, the arbitrary factor $\mu_{cl}(c)$ (which in the BFM approach is a consequence of a Gribov problem) corresponds in the Polyakov approach to a freedom in the definition of formal measures in the spaces of world-sheet metrics and space-time embeddings. Implicitly, the correct choice $\mu_{cl} = 1$ is always made.

A similar analysis for the open string leads to

$$Z_{op}(a_2^\mu, a_1^\mu) = \int [d\Sigma_1, d\Sigma_2] \int_0^\infty \frac{dc}{\sqrt{c}} \mu_{op}(c) Z_g^{op}(c) Z_B^{op}(c) e^{iS_{op}(x_i^\mu, c)} \quad (5.25)$$

where $\pi c = \lambda_0 \Delta\tau$, and

$$Z_B^{op}(\lambda_0 \Delta\tau) = [p(\lambda_0 \Delta\tau)]^{-D/2}, \quad Z_g^{op}(\lambda_0 \Delta\tau) = (\lambda_0 \Delta\tau)^{-1/2} p(\lambda_0 \Delta\tau) \quad (5.26a)$$

with

$$p(\lambda_0 \Delta\tau) = (\lambda_0 \Delta\tau) e^{-\frac{1}{12} i \lambda_0 \Delta\tau} \prod_{n=1}^{\infty} (1 - e^{-2i n \lambda_0 \Delta\tau}) \quad (5.26b)$$

Once again, performing the Wick rotation (5.21), (5.25) leads exactly to the expression Eq. (2.2) of Ref.[18] for the Polyakov integral (with $\mu_{op}(c)=1$). Z_8^{op} and Z_9^{op} correspond to the bosonic and Faddeev-Popov determinants. The measure $dc/\sqrt{c}$ (i.e. $\mu_{op}(c)=1$) is obtained in Ref. [18] from the Teichmüller deformations of the world-sheet. The discussion for the closed string likewise follows from these remarks.

6. CONCLUSION

In this paper we have applied the BFM formulation [14,15] of constrained systems to the Nambu-Goto action for the bosonic string. Since this approach allows for a real time description and leads to a phase-space path-integral with a specific measure invariant for the symmetries of the system [14,15], it may in principle bring new insight into questions related to the usual Euclidean Polyakov approach for the quantization of this system, such as the measure over Teichmüller and modular parameters and the problem of the Wick rotation of the theory.

These questions were first considered in the case of the free propagation of the scalar particle. Although this is a well known system [19-21], a detailed discussion was given in order to emphasize the analogies with the string case, particularly those concerning the question of gauge fixing and integration measure for modular space [19]. We also explicitly showed that the usual interpretation of the Fradkin-Vilkovisky theorem [14] is not correct.

We then considered the free propagator of closed and open bosonic strings. After a detailed discussion of gauge fixing, we found that this system possesses a Gribov problem [35]. As a consequence the measure for modular space is not determined from the formalism, and another requirement is necessary. In general, modular invariance should suffice to fix it uniquely but not in the case at hand.

We indicated that this arbitrariness in the modular measure corresponds in the Polyakov approach to a (not fully appreciated) freedom in the formal definitions [1,6,7] of measures over configuration space. This was first realized in the case of the particle in Ref.[31]. With regards to the string it appears that the usual choice implicitly made is indeed the correct one.

We found how a Wick rotation is to be performed in modular space, and how in this real time description a well defined propagator and its correct causal structure are obtained by having an infinitesimal negative imaginary part for the modular parameters. This last requirement is also necessary for a correct gauge fixing of the system.

Although these results are obtained here only for the free string, we expect them to hold also in the presence of interactions.

Finally, a detailed comparison of our expressions for the propagator with those obtained [17,18] within the Polyakov approach (in $D=26$) established the full quantum equivalence (at least for this process) of the two formalisms [25].

In this respect, let us note that our results follow from the Nambu-Goto action [29] as well as from the linear action [32] used in Polyakov's approach [1]. Clearly, this is because the conformal mode of the world-sheet metric does not couple classically, so that it does not appear in the BFV formulation of the theory.

In other words, at the quantum level, the BFV approach explicitly preserves the Weyl invariance of the linear action, whereas the Polyakov approach explicitly preserves its invariance for world-sheet diffeomorphisms.

For this reason we consider the present work really as the phase-space path-integral quantization of the Nambu-Goto string and not of the so-called Polyakov string.

In order to obtain from the BFV formalism the Polyakov path-integral for $D \leq 26$ [1,22], one needs an action in which the conformal mode couples classically. Presumably, this is possible with the action considered in Ref.[37,23], and constitutes a possible extension of our work.

Another interesting possibility entails the repetition of the same analysis [16] for the spinning string [38,32] (and the corresponding spinning particle [39,20,31]), having especially in mind problems related to supermoduli [40] of super-Riemann surfaces. Although a super-Gribov is also to be expected for these parameters, the BFV phase-space path-integral may bring new insight into the problem of the integration of the supermoduli and the superghosts [41]. It should also indicate which transformation in supermoduli space corresponds to a Wick rotation in space-time.

Finally, concerning the latter point, for the bosonic as well as for the spinning string, one could consider within the same BFV approach, the 3-string (and generally N-string) vertex (already studied for bosonic strings within the Euclidean Polyakov formulation in Ref.[42]). Using the free string propagator, any 1-loop string amplitude could be reconstructed and expressed as an integral over the usual number of (super)modular parameters associated to the lengths and twists of the corresponding (super)world-sheets, with a measure to be determined from (super)modular invariance.

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Appendix A

Let us consider the linear action for the point-particle:

$$S[x^\mu, e] = \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau \left[e^{-1} \dot{x}^2 - e m^2 \right] \quad (\text{A.1})$$

where $e(\tau)$ is an intrinsic world-line einbein coupled to $x^\mu(\tau)$.

The Lagrangian approach is straightforward. It is left to the reader [2].

In the Hamiltonian formulation we have the degrees of freedom $x^\mu(\tau)$, $e(\tau)$ and their conjugate momenta $p_\mu = e^{-1} \dot{x}_\mu$ and $\pi_e = 0$, with the Poisson brackets

$$\{x^\mu(\tau), p_\nu(\tau)\} = \delta^\mu_\nu, \quad \{e(\tau), \pi_e(\tau)\} = 1 \quad (\text{A.2})$$

The canonical Hamiltonian is

$$H_0 = \dot{x}^\mu p_\mu + \dot{e} \pi_e - \frac{1}{2} \left[e^{-1} \dot{x}^2 - e m^2 \right] = \frac{1}{2} e (p^2 + m^2) \quad (\text{A.3})$$

and we have the primary constraint

$$\pi_e = 0 \quad (\text{A.4})$$

Applying Dirac's general analysis [12] to this system, the complete set of constraints is found to be only first-class, with

$$\pi_e(\tau) = 0, \quad \phi(\tau) \equiv \frac{1}{2} (p^2(\tau) + m^2) = 0 \quad (\text{A.5})$$

and the total Hamiltonian is

$$H = \frac{1}{2} e(\tau) (p^2(\tau) + m^2) + v(\tau) \pi_e(\tau) \quad (\text{A.6})$$

where $v(\tau)$ is an arbitrary function.

The corresponding equations of motion

$$\dot{x}^\mu = \{x^\mu, H\} = e p^\mu, \quad \dot{p}^\mu = \{p^\mu, H\} = 0 \quad (\text{A.7a})$$

$$\dot{e} = \{e, H\} = v, \quad \dot{\pi}_e = \{\pi_e, H\} = -\frac{1}{2}(p^2 + m^2) \quad (\text{A.7b})$$

and the constraints (A.5) follow from the first-order action:

$$S[x^\mu, p_\mu, e, \pi_e] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu + \dot{e} \pi_e - \frac{1}{2} e (p^2 + m^2) - v \pi_e \right] \quad (\text{A.8})$$

Solving for $v(\tau)$ this is equivalent to

$$S[x^\mu, p_\mu, e] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu - \frac{1}{2} e (p^2 + m^2) \right] \quad (\text{A.9a})$$

with

$$\pi_e = 0 \quad (\text{A.9b})$$

(A.9a) is the first-order action (2.14) obtained from the non-linear action (2.1), where the Lagrange multiplier λ is identified with the einbein e .

Solving for $p^\mu = e^{-1} \dot{x}^\mu$ (A.9a) reduces to the linear action (A.1), which for $m \neq 0$ reduces to (2.1) by solving the constraint $\phi(\tau) = 0$ (with $e(\tau) > 0$).

Appendix B

Let us consider the linear action for the bosonic string [32]:

$$S[x^\mu, g_{\alpha\beta}] = \frac{-1}{4\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{-\det g_{\alpha\beta}} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu \quad (\text{B.1})$$

where $g_{\alpha\beta}(\sigma, \tau)$ is an intrinsic world-sheet metric coupled to $x^\mu(\sigma, \tau)$.

In order to simplify the Hamiltonian formulation, let us trade the three metric degrees of freedom $g_{\alpha\beta}$ for three other functions defined by

$$\phi = \ln(-\det g_{\alpha\beta})^{1/2}, \quad g^\pm = \frac{\sqrt{-\det g_{\alpha\beta}} \pm g_{01}}{g_{11}} \quad (\text{B.2a})$$

or

$$g_{\alpha\beta} = e^\phi \frac{2}{g^+ + g^-} \begin{pmatrix} -g^+ g^- & \frac{1}{2}(g^+ - g^-) \\ \frac{1}{2}(g^+ - g^-) & 1 \end{pmatrix} \quad (\text{B.2b})$$

We then have for (B.1):

$$S[x^\mu, g^+, g^-] = \frac{-1}{4\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \frac{2}{g^+ + g^-} \left(-\dot{x}^2 + (g^+ - g^-) \dot{x} x' + g^+ g^- x'' \right) \quad (B.3)$$

Because of the Weyl invariance of (B.1), the conformal mode ϕ does not couple. The Lagrangian formulation for (B.3) is straightforward and is not considered here.

In the Hamiltonian formulation, we have the degrees of freedom x^μ, g^+, g^-, ϕ and their conjugate momenta $\pi_\mu, \pi_+, \pi_-, \pi_\phi$ with the corresponding Poisson brackets.

The canonical Hamiltonian density is

$$\mathcal{H}_0 = \frac{1}{2} \pi_{\alpha'} g^+ \left(\pi^\mu + \frac{1}{2\pi\alpha'} \partial_\sigma x^\mu \right)^2 + \frac{1}{2} \pi_{\alpha'} g^- \left(\pi^\mu - \frac{1}{2\pi\alpha'} \partial_\sigma x^\mu \right)^2 \quad (B.4)$$

and we have the primary constraints

$$\pi_+ = 0, \quad \pi_- = 0, \quad \pi_\phi = 0 \quad (B.5)$$

Applying Dirac's general procedure [12], the complete set of constraints is found to be only first-class with

$$\pi_\phi = 0, \quad \pi_+ = 0, \quad \phi_+ \equiv \frac{1}{2} \pi_{\alpha'} \left(\pi^\mu + \frac{1}{2\pi\alpha'} \partial_\sigma x^\mu \right)^2 = 0 \quad (B.6a)$$

$$\pi_- = 0, \quad \phi_- \equiv \frac{1}{2} \pi_{\alpha'} \left(\pi^\mu - \frac{1}{2\pi\alpha'} \partial_\sigma x^\mu \right)^2 = 0 \quad (B.6b)$$

and the total Hamiltonian is

$$H = \int_0^\pi d\sigma \left[g^+ \phi_+ + g^- \phi_- + v^+ \pi_+ + v^- \pi_- + v^\phi \pi_\phi \right] \quad (B.7)$$

where v^+, v^- and v^ϕ are arbitrary functions.

The corresponding equations of motion and the constraints (B.6) follow from the first-order action

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[\dot{x}^\mu \pi_\mu + \dot{g}^+ \pi_+ + \dot{g}^- \pi_- + \dot{\phi} \pi_\phi - g^+ \phi_+ - g^- \phi_- - v^+ \pi_+ - v^- \pi_- - v^\phi \pi_\phi \right] \quad (B.8)$$

Solving for v^+, v^- and v^ϕ , this is equivalent to

$$S[x^\mu, \pi_\mu, g^+, g^-] = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma [\dot{x}^\mu \pi_\mu - g^+ \phi_+ - g^- \phi_-] \quad (\text{B.9a})$$

with

$$\pi_- = 0, \quad \pi_+ = 0, \quad \pi_\phi = 0 \quad (\text{B.9b})$$

(B.9a) is the first-order action (3.14) obtained from the non-linear Nambu-Goto action (3.1), where the Lagrange multipliers λ^+, λ^- are identified with the metric components g^+, g^- .

Solving for π_μ , (B.9a) reduces to the linear action (B.1), which reduces to (3.1) by solving the constraints $\phi_+ = 0, \phi_- = 0$ (with $(g^+ + g^-) > 0$).

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