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ABOUT THE LOWER BOUND ON THE FIXED ANGLE SCATTERING AMPLITUDE

AT HIGH ENERGY

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A B S T R A C T

Assuming that the phase is small we prove that the fixed angle scattering amplitude in potential scattering cannot decrease faster than $\exp -Ck\theta \log k$, where θ is the scattering angle and k the centre-of-mass momentum. This result agrees with the explicit calculation of T.T. Wu for the Gaussian potential ($\exp -C\theta k/\log k$) and with a previous bound derived from the Mandelstam representation with a finite number of subtractions. The applicability of this result outside the framework of potential scattering is discussed.

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Many years ago, Cerulus and I derived a fixed-angle lower bound on the scattering amplitude from Mandelstam representation [1]:

$$|F(k, \theta)| > \exp[-C(\theta)k \log k] \quad (1)$$

where k is the centre-of-mass momentum and θ the centre-of-mass scattering angle. The basic ingredients of the proof were:

- (i) the analyticity of the scattering amplitude in the cut $\cos\theta$ plane;
- (ii) the fact that, because of positivity and of dispersion relations, the forward scattering amplitude cannot decrease faster than k^{-4} (or s^{-2}) [2];
- (iii) an overall polynomial boundedness in s and $\cos\theta$ in the $\cos\theta$ complex plane.

Unfortunately, the Mandelstam representation was never proved, not even in perturbation, and explicit models possessing essentially the Mandelstam analyticity do not have the overall polynomial boundedness. For instance, the Veneziano amplitude [3] violates the lower bound (1). This is because Regge trajectories, being linear, give a contribution s^t , i.e., exceeding any power of s for $t \rightarrow +\infty$. Recently, Gross and Mende [4] have pointed out that the string scattering amplitude in the tree approximation violates (1).

Nevertheless, one may wonder if the lower bound (1) cannot be obtained from a completely different approach, with no recourse to analyticity considerations. Indeed, many years ago also, T.T. Wu has investigated the behaviour of the scattering amplitude at fixed angle in potential scattering [5]. What he has done is to treat explicitly a number of examples and he found no violation of the lower bound (1). In cases where the Born approximation decreases faster than $\exp-K \log K$ for large q where K is the momentum transfer $|\vec{k}_i - \vec{k}_f|$ he finds that this approximation is not valid at fixed angle and, in particular, for the Gaussian potential, the Born approximation is $\exp-\text{const} \times K^2$, i.e., $\exp-\text{const} \times k^2 \sin^2 \theta / 2$ while he finds for the fixed-angle amplitude

$$\exp[-\text{const} \times k \sqrt{\log k}] \quad (2)$$

Here we want to generalize these considerations to arbitrary radial potentials at the cost of an assumption which is probably justified at high energy, namely that the phase of the scattering amplitude is bounded in a way which will be later specified.

We define the scattering amplitude as

$$F = \sum (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta) \quad , \quad \delta_l \text{ real} \quad (3)$$

and

$$f = \frac{F}{k} . \quad (4)$$

Hence the differential cross-section is

$$\frac{d\sigma}{d\Omega} = |f|^2 . \quad (5)$$

We know that, according to Hunziker for instance [6], the difference between the scattering amplitude and the Born approximation decreases like an inverse power of k

$$|f(\vec{k}_i, \vec{k}_f) - f_{\text{Born}}(\vec{k}_i, \vec{k}_f)| < \frac{C}{k^{5/2}} \quad (6)$$

with

$$f_{\text{Born}} = -\frac{1}{4\pi} \int d^3r \exp(-i \vec{K} \cdot \vec{r}) V(r) . \quad (7)$$

The assumptions of Hunziker on the potential are rather strong but can be considerably weakened, following for instance the methods of [7], if one wants to restrict oneself to a comparison with the lowest Born approximation like in Eq. (6).

We write the unitarity condition as

$$\text{Im} F(\theta_{12}) = \frac{1}{4\pi} \int F^*(\theta_{13}) F(\theta_{23}) d\Omega_3 . \quad (8)$$

At this point, we introduce explicitly our assumption on the phase of the amplitude.

- If the potential is attractive, we assume

$$|\text{Arg} F| < \frac{\pi}{2} \quad (9)$$

and deduce from (8)

$$0 < \text{Arg} F < \frac{\pi}{2} \quad (10)$$

- If the potential is repulsive we assume

$$\frac{\pi}{2} < \text{Arg} F < \frac{3\pi}{2} \quad (11)$$

and get from (8)

$$\frac{\pi}{2} < \text{Arg } F < \pi. \quad (12)$$

In both cases we get from (8):

$$\text{Im } F(\theta_{12}) > \frac{1}{4\pi} \int d\Omega_3 | \text{Re } F(\theta_{13}) | | \text{Re } F(\theta_{23}) | \quad (13)$$

$$\text{Im } F(\theta_{12}) > \frac{1}{4\pi} \int d\Omega_3 \text{Im } F(\theta_{13}) \text{Im } F(\theta_{23}) \quad (14)$$

If we define

$$R(\theta) = \inf_{0 \leq \theta' \leq \theta} | \text{Re } F(\theta') |$$

$$I(\theta) = \inf_{0 \leq \theta' \leq \theta} \text{Im } F(\theta') \quad (15)$$

we can exploit the fact that all contributions add up in (13) and (14) and get:

$$\text{Im } F(\theta + \Delta) \geq \frac{1}{4\pi} S(\Delta, \theta) I(2\Delta) I(\theta) \quad (16)$$

where $S(\Delta, \theta)$ represents the area of intersection of two spherical discs on the unit sphere, with centres whose distance is an angle $\Delta + \theta$, and spherical radii 2Δ and θ . S is larger than the area of the spherical disc with spherical radius $\Delta/2$ for $\theta > \Delta/2$ and we get:

$$\text{Im } F(\theta + \Delta) > \frac{1 - \cos \frac{\Delta}{2}}{2} I(2\Delta) I(\theta) \quad (17)$$

for $\theta > \Delta/2$.

Hence, we have also:

$$\inf_{\frac{\Delta}{2} < \theta' \leq \theta} (\text{Im } F(\theta' + \Delta)) > \frac{1 - \cos \frac{\Delta}{2}}{2} I(2\Delta) I(\theta). \quad (18)$$

Assume now that Δ is chosen sufficiently small, so that

$$\frac{1 - \cos \frac{\Delta}{2}}{2} I(0) < 1. \quad (19)$$

then

$$\inf_{0 < \theta < 2\Delta} \text{Im } F(\theta) > \frac{1 - \cos \frac{\Delta}{2}}{2} I(2\Delta) I(0), \quad (20)$$

and therefore, provided condition (19) is fulfilled:

$$I(\theta + \Delta) > \frac{1 - \cos \frac{\Delta}{2}}{2} I(2\Delta) I(\theta) \quad (21)$$

In high energy potential scattering, condition (19) is fulfilled for Δ less than some fixed value. Indeed, because of the validity of the Born approximation in the forward direction

$$\begin{aligned} I(0) &\simeq k^2 \int d\Omega |f_{\text{Born}}|^2 \\ &\simeq \pi \int_{-\infty}^0 dt |f_{\text{Born}}(t)|^2 \end{aligned} \quad (22)$$

where

$$t = 2k^2(\cos \theta - 1). \quad (23)$$

We also get from Eq. (13) by integrating over a disk of spherical radius $\Delta/2$

$$I(3\Delta) > \frac{1 - \cos \frac{\Delta}{2}}{2} (R(2\Delta))^2. \quad (24)$$

By repeated use of (21) combined with (24), we get

$$I(\theta) > \left(\frac{1 - \cos \frac{\Delta}{2}}{2} \right)^{\frac{\theta}{\Delta}} \left[\frac{1 - \cos \frac{\Delta}{6}}{2} (R(\frac{2\Delta}{3}))^2 \right]^{\frac{\theta}{\Delta} + 1} \quad (25)$$

Here, strictly speaking, θ/Δ should be an integer. Take now $\Delta = \lambda/k$, i.e., $t \approx \lambda^2$. Then

$$R(\frac{2\Delta}{3}) = k \inf_{|t| \leq 4\lambda^2/9} (f_B) - O\left(\frac{1}{k^{3/2}}\right) > Ck$$

and hence, in the small angle approximation for Δ

$$I(\theta) > \left(\frac{\lambda^2}{16k^2} \right)^{\frac{\theta k}{2}} \left(\frac{\lambda^2}{36} C^2 \right)^{\frac{\theta k}{2} + 1}$$

Therefore

$$I(\theta) > \exp \left[-C' \theta k \log \frac{k}{k_0} \right], \quad (26)$$

which, mysteriously, coincides with the lower bound (1).

One may notice that the expression obtained by T.T. Wu for the Gaussian potential contains $\sqrt{\log k}$ in the exponent while (26) contains $\log k$. In fact, for the

Gaussian potential we can also get as lower a bound a similar expression. In that case $f_B \sim \exp-C''|t|$ and if we take $\Delta = \mu(\sqrt{\log k}/k)$, we get

$$R\left(\frac{2\Delta}{3}\right) > k \left(k\right)^{-\frac{4\mu^2}{9}} - O\left(\frac{1}{k^{3/2}}\right)$$

Therefore it is possible to take μ small enough so that $R > C''/k$ and then, with $\lambda = \mu\sqrt{\log k}$, one gets indeed

$$I(\theta) > \exp(-\text{const } \theta k \sqrt{\log k}) \quad (27)$$

Similarly, for a potential with a Fourier transform behaving like

$$\exp - K^3, \text{ for large } K \quad (t = K^2),$$

one would get

$$I(\theta) > \exp\left[-\text{const } \theta k (\log k)^{3/4}\right],$$

etc. The result (26) is not very surprising in the case of a superposition of Yukawa potentials for which the Mandelstam representation holds with a finite number of subtractions [8]. In addition, in this case, the Born approximation is anyway analytic in a cut plane and cannot decrease faster than $\exp-|K|$.

Finally, I would like to discuss the applicability of these considerations outside the domain of potential scattering.

First, we must keep the assumption that the phase remains, for instance, in $|\text{Arg} F| < \pi/2$. Such an assumption is sometimes obviously violated like in pp scattering beyond the "dip" at a t value between 1 and 2 $(\text{GeV})^2$, depending on the energy. However, this may not be so serious if smaller angles dominate in the unitarity integral. The other assumption is that the contributions from the inelastic channels to the unitarity do not interfere destructively with the elastic contribution. In other words, we assume that the "overlap function", as it is called by Van Hove [8], is positive in the physical region. There is also a slight technical complication in that (21) no longer strictly holds, because $I(0)$ is only controlled by the Froissart bound [10], [11], $I(0) < k^2(\log k)^2$. However, by taking $\Delta \leq \lambda/k$, one still has

$$I(\theta + \Delta) > C \frac{1 - \cos \frac{\Delta}{2}}{2} I(2\Delta) \frac{I(\theta)}{(\log k)^2},$$

and this is enough to get the lower bound (26).

In conclusion, the bound (1) seems to be hard to kill, except if there is a conspiracy of various contributions to unitarity which does not seem to be very likely. In Ref. [4] it is remarked that the same thing might happen for the string scattering amplitude after summation over all possible graphs.

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REFERENCES

- [1] F. Cerulus and A. Martin - Phys.Lett. 8 (1964) 80.
See also:
T. Kinoshita - Phys.Rev.Lett. 12 (1964) 256;
A. Martin - Nuovo Cimento 37 (1965) 671.
- [2] Y.S. Jin and A. Martin - Phys.Rev. 135B (1964) 1369.
- [3] G. Veneziano - Nuovo Cimento 57 (1968) 190.
- [4] D.J. Gross and P.F. Mende - Preprint Princeton University, PUPT-1062 (1987).
- [5] T.T. Wu - Phys.Rev. 143 (1966) 1110.
- [6] W. Hunziker - Helv.Phys.Acta 36 (1963) 838.
- [7] A. Martin - Course given at the University of Washington, Notes by D.A. Krueger and W.M. Dante, Preprint, unpublished (1964).
- [8] J.D. Bessis - J.Math.Phys. 6 (1965) 637 and 820.
- [9] L. Van Hove - Nuovo Cimento 28 (1963) 789.
- [10] M. Froissart - Phys.Rev. 123 (1961) 1053.
- [11] A. Martin - Nuovo Cimento 42 (1966) 930.