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NOWHERE-VANISHING SPINORS AND TRIALITY ROTATIONS IN 8-MANIFOLDS

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ABSTRACT

Using methods from differential geometry and index theorems, we give simple derivations of the criteria for the existence of nowhere-vanishing spinor fields on compact Riemannian manifolds. These had been derived previously using obstruction theory. The possibility of the non-existence of nowhere-vanishing spinor fields arises only in two, four or eight dimensions, and of these eight is the dimension that presents the most interesting features. This is the same as the dimension of the transverse space for the superstring, and we make some observations that follow from this fact.

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1. Introduction

It is well known that the condition for a compact, oriented n -dimensional manifold M to admit a nowhere-vanishing vector field is that the Euler number, χ , of M must be zero. It is natural to enquire what the analogous conditions on M must be if it is to admit nowhere-vanishing fields in other representations of the tangent space group. In the case of spinor fields, this question arose recently [1,2], in the context of superstring theories in the light-cone gauge. One can convert back and forth between the Green-Schwarz formalism and the Neveu-Schwarz-Ramond formalism provided that there exists a nowhere-vanishing Majorana-Weyl spinor field on the eight-dimensional transverse space [1]. Essentially, one is exploiting the triality properties of the $SO(8)$ tangent space group of the transverse space: one can convert between a vector V^a in the representation 8_v and a spinor ψ in the representation 8_s by means of a distinguished nowhere-vanishing spinor field ξ in the eight-dimensional conjugate spinor representation 8_c . Normalizing ξ to have unit length (which we can always do as long as it is nowhere-vanishing), the relation between V^a and ψ is given by

$$\psi = V^a \Gamma_a \xi, \quad V^a = \bar{\xi} \Gamma^a \psi. \quad (1.1)$$

The question about the existence of nowhere-vanishing spinor fields was originally posed in the context of string theory, where the manifold M is eight-dimensional. However, it is interesting to address the question in the more general framework of an n -dimensional manifold M . In practice, in string theory, M would typically be a non-compact eight-manifold (for example $R^2 \times K_6$, where K_6 is a 6-dimensional Calabi-Yau space), although if one were considering compactifications to two dimensions (such as the “worldsheets for worldsheets” of ref. [3]), M could be compact. In fact, just as for nowhere-vanishing vectors, there is only any possibility of an obstruction to the existence of nowhere-vanishing spinor fields if M is compact.

The conditions under which M admits nowhere-vanishing spinors were established in ref. [4], by relating the problem to the topological question of reducing the structure group of the spinor bundle from $Spin(8)$ to $Spin(7)$. This purely topological problem had in fact been considered earlier in [5]. However, in both the papers the proofs relied heavily on the use of the mathematics of obstruction theory and, from the point of view of a physicist, could not be said to be particularly transparent. The purpose of this paper is to present a simple proof based on differential geometric methods, which in fact is a rather straightforward extension of Chern’s intrinsic derivation of the Gauss-Bonnet theorem [6]. Accordingly, we begin by giving a brief review in section 2 of Chern’s proof, together with a simple example that illustrates the key ideas. In section 3, we turn to the case of spinor fields by generalizing Chern’s argument for vector fields. In order to have any possibility of an obstruction to the existence of a nowhere-vanishing spinor field, it is necessary, as we shall explain later, that the real dimension of the spinor should be less than or equal to the dimension, n , of M . Since the dimension of spinors grows rapidly with n , this means that only in certain low dimensions can there be any obstruction. Specifically, the only dimensions one need consider are 1, 2, 4 and 8. Of these, $n = 8$ is really the only case that presents any features of interest. It is amusing that this is precisely the dimension that arose originally in string theory. In section 4, we give an alternative derivation of the

criterion for the existence of nowhere-vanishing spinor fields in eight dimensions, using an index theorem argument inspired by connections with string theory. Finally, in section 5 we discuss some issues related to compactifications of superstring theories.

2. The Gauss-Bonnet theorem

The Euler number of an n -dimensional manifold M may be defined in the following way. If one considers a vector field V on M , then in general it cannot be non-zero everywhere on M , but rather it can be deformed so that there are just a finite number of isolated points x_i at which the vector vanishes. At each such point one can define the *index* a_i of the corresponding singularity, as the integer that characterizes the order of the zero of V at x_i . The Euler number χ is then given by

$$\chi = \sum_i a_i. \quad (2.1)$$

Although the values of the indices of the individual singularities can be changed by deforming the vector field, the sum given by (2.1) is a topological invariant. As a special case, (2.1) implies that M admits a nowhere-vanishing vector field if and only if χ is zero.

The Gauss-Bonnet theorem states that for an even-dimensional, compact and closed Riemannian manifold M , χ may be expressed as an integral over M of a certain polynomial in the curvature,

$$\chi = \int_M \Omega, \quad (2.2)$$

where

$$\Omega = \frac{1}{p! (4\pi)^p} \epsilon^{i_1 i_2 \dots i_n} \Theta_{i_1 i_2} \wedge \dots \wedge \Theta_{i_{n-1} i_n}, \quad (2.3)$$

$\Theta_{ij} = \frac{1}{2} R_{ijkl} e^k \wedge e^l$ is the curvature 2-form and $n = 2p$. Since χ is a topological invariant, the curvature may be calculated using any metric on M .

It is easy to see that Ω defined by (2.3) is closed, $d\Omega = 0$, and hence locally Ω may be expressed as the exterior derivative of an $(n-1)$ -form A . Clearly, however, if χ is non-zero A cannot be globally defined on M , since otherwise one would arrive at a contradiction by applying Stokes' theorem to obtain $\int_M \Omega = \int_M dA = \int_{\partial M} A = 0$, where the last step follows from the fact that M is compact and hence $\partial M = \emptyset$. In other words, Ω is closed, but not exact. The basic idea of Chern's proof of the Gauss-Bonnet theorem is to work with the $(2n-1)$ -dimensional bundle space E of unit tangent vectors to M , and show that Ω lifts to a closed n -form $\pi^*(\Omega)$ on E that is exact, i.e.

$$\pi^*(\Omega) = d\Phi, \quad (2.4)$$

where Φ is a globally defined $(n-1)$ -form on E and π is the projection map of the bundle.

Since the fibers F of the bundle E are tangent vectors of unit length ($V^i V^i = 1$), it follows that $F = S^{n-1}$, the $(n-1)$ -sphere. As with any fiber bundle, a cross-section

$X : M \rightarrow E$ is an injective map, and hence maps M into a diffeomorphic copy (i.e. a submanifold) in E . A global cross-section of E is nothing but a nowhere-vanishing vector field on M . Now in general, M will not admit a nowhere-vanishing vector field, and so a cross-section X will contain singularities at the zeroes of the vector field. However, we can excise a small region (an n -ball) around each point in M where the vector field vanishes, giving a manifold M' with boundary. Thus a cross-section $X(M')$ will now be a diffeomorphic copy of M' . The boundary $\partial X(M')$ consists of a disjoint sum of $(n-1)$ -spheres, corresponding to the boundaries of the n -balls around the singular points. Thus we have

$$\int_{M'} \Omega = \int_{X(M')} \pi^*(\Omega) = \int_{X(M')} d\Phi = \int_{\partial X(M')} \Phi. \quad (2.5)$$

The point now is that the integral of Φ can easily be evaluated over each of the S^{n-1} boundaries whose sum is equal to ∂X , and for each S^{n-1} component of the boundary one finds that $\int_{S^{n-1}} \Phi$ is just equal to a_i , the index of the singularity within the corresponding excised region. Thus by taking the limit as the radii of the n -balls tend to zero, we see from (2.1) and (2.2) that the Gauss-Bonnet theorem is established.

This, then, is the strategy of Chern's proof. It is one of the early examples of a technique that has now become standard in the study of characteristic classes. It is also a technique that has found application in the physics literature, in the study of anomalies [6]. To complete the details of the derivation, it remains to derive the explicit expression for the globally defined $(n-1)$ -form Φ of equation (2.4). We shall present here the steps of Chern's derivation [6], so that in the next section we can invoke the same ideas in our proof for spinor fields.

Let e^i be an orthonormal frame on M . Since we have a non-singular metric on M , we can just as well work with 1-forms as with vectors in order to derive the Gauss-Bonnet theorem. Consider, then, a 1-form u , with frame components u_i :

$$u = u_i e^i. \quad (2.6)$$

We shall take this to be of unit norm;

$$u^i u_i = 1. \quad (2.7)$$

Define θ_i to be the covariant exterior derivative of u_i :

$$\theta_i = Du_i \equiv du_i + \omega_{ij} u_j, \quad (2.8)$$

where ω_{ij} is the connection 1-form defined by $de^i = -\omega^i_j \wedge e^j$, i.e. $De^i = 0$. Note that the tangent-frame indices $i, j, \dots$ may be raised and lowered freely, using the Kronecker delta. θ_i has the property that $du = \theta_i \wedge e^i$. It is easy to check that θ_i satisfies

$$u_i \theta_i = 0, \quad D\theta_i \equiv d\theta_i + \omega_{ij} \wedge \theta_j = \Theta_{ij} u_j, \quad (2.9)$$

where $\Theta_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ is the curvature 2-form on M . Note that u_i and θ_i are *covariant*, i.e. that they transform as vectors under $SO(n)$ rotations of the tangent frame e^i .

We now construct a sequence of $(n-1)$ -forms $\Phi_{(r)}$, and a sequence of n -forms $\Psi_{(r)}$, defined by

$$\Phi_{(r)} = \epsilon_{ij \dots k \ell m \dots n p} u_i \theta_j \wedge \dots \wedge \theta_k \wedge \Theta_{\ell m} \wedge \dots \wedge \Theta_{np}, \quad (2.10)$$

$$\Psi_{(r)} = \epsilon_{ijk \dots \ell m n \dots p q} \Theta_{ij} \wedge \theta_k \wedge \dots \wedge \theta_\ell \wedge \Theta_{mn} \wedge \dots \wedge \Theta_{pq}, \quad (2.11)$$

for $r = 0, 1, \dots, p-1$, where $n = 2p$ is the (even) dimension of the manifold M . In each expression the “...” stands for a number of intervening θ 's or Θ 's. The number of Θ 's in (2.10) is r , while in (2.11) the total number of Θ 's is $r+1$. The remaining indices on ϵ in each case are soaked up by the appropriate counterbalancing number of θ 's. By using the Bianchi identity $D\Theta_{ij} = 0$, and after some algebra, one can show that [6]

$$d\Phi_{(r)} = \Psi_{(r-1)} - \frac{(n-2r-1)}{2(r+1)} \Psi_{(r)}. \quad (2.12)$$

(This is analogous to the “descent equations” of reference [7].) By solving iteratively, one obtains expressions for $\Psi_{(r)}$ as a sum of terms $\sum_{s=0}^r c_{(r)s} d\Phi_{(s)}$ for certain constant coefficients $c_{(r)s}$. In particular,

$$\Psi_{(p-1)} = \sum_{m=0}^{p-1} \frac{2^{p-m} p!}{m! (2p-2m-1)!!} d\Phi_{(m)}. \quad (2.13)$$

Now from (2.3) and (2.11), we see that $\Psi_{(p-1)}$ is a certain multiple of Ω , so that in fact the globally defined $(n-1)$ -form Φ of equation (2.4) is given by

$$\Phi = (2\pi)^{-p} \sum_{m=0}^{p-1} \frac{2^{-m}}{m! (2p-2m-1)!!} \Phi_{(m)}. \quad (2.14)$$

The boundary $\partial X(M')$ is the sum of copies of the fiber F , which is an $(n-1)$ -sphere, corresponding to the $(n-1)$ -sphere bounding the excised region around each singularity of the 1-form field in the base manifold M . If a given singularity is *simple* (i.e. the index a_i is equal to 1), then the corresponding component of the boundary ∂X will just be a single copy of the S^{n-1} fiber. In other words, as one moves around the points on the sphere surrounding the chosen singularity, each unit tangent vector is visited exactly once. If the singularity has index a_i , the corresponding component of the boundary will be a_i copies of S^{n-1} . This may be seen by deforming the 1-form field in the neighborhood of the singularity, so as to separate out the multiple singularity into simple ones, which would each then be surrounded by a copy of S^{n-1} . Thus it just remains to evaluate the integral of Φ , given by (2.14), over the S^{n-1} fiber F . Now the components u_i of the 1-form u may be thought of as coordinates on the fiber (recall that there are really only $(n-1)$ u_i 's because of (2.7)), and one may put the following metric on the bundle space E :

$$ds^2 = e^i e^i + \theta_i \theta_i \quad (2.15)$$

(this is analogous to the metric one uses in Kaluza-Klein theories). Thus when we evaluate the integral $\int_F \Phi$, with Φ given by (2.14), only the term $m = 0$ will contribute, since only for this term do all the legs of the $(n - 1)$ -form lie in the fiber directions (see equation (2.10)). In fact, for the same reason, we may replace each θ_j by du_j within the integrand, so that

$$\int_F \Phi_{(0)} = \int_F \epsilon_{i_1 \dots i_k} u_i du_{j_1} \dots du_{j_k}. \quad (2.16)$$

The integrand is clearly just the volume form on the $(n - 1)$ -sphere, and after evaluating (2.16) and using (2.1) and (2.5), we obtain the Gauss-Bonnet formula (2.2).

A simple example illustrates some of the essential features of the procedure. Consider the case where M is the 2-sphere, with its standard metric $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$. We choose as an orthonormal basis of 1-forms $e^1 = d\theta$, $e^2 = \sin \theta d\phi$. We write the components u_i of the 1-form u as

$$u_1 = r \cos \psi, \quad u_2 = r \sin \psi. \quad (2.17)$$

After a simple calculation, we find that the metric (2.15) becomes

$$ds^2 = dr^2 + r^2 (d\psi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2. \quad (2.18)$$

Note that we have not yet imposed the unit-norm condition (2.7) on u_i , so (2.18) is a four-dimensional metric on the tangent bundle of S^2 . Restricting to the bundle of *unit* 1-forms, i.e. $r = 1$, we get the 3-metric

$$ds^2 = (d\psi + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2 \quad (2.19)$$

as the metric on the bundle E . This is in fact like the standard metric on S^3 , written in Euler angle coordinates, except that there is an antipodal identification since ψ has period 2π (see (2.17)) rather than 4π . Thus E in this case is the manifold $\mathbf{RP}^3$.

Calculating (2.10) and (2.14) in this case, we find

$$\Phi = \frac{1}{2\pi} (d\psi + \cos \theta d\phi). \quad (2.20)$$

This is indeed globally defined on $\mathbf{RP}^3$. Now the 1-form field u , with components given by (2.17), has singularities at the north and south poles of S^2 , where the basis 1-form e^2 becomes degenerate. Thus we excise a 2-ball (a disk) around each pole. Now as one moves around the boundary of the disk (i.e. change ϕ continuously by 2π , while keeping θ fixed and close to zero or π), then for a fixed value of ψ one realizes each possible unit 1-form u . This is equivalent to changing ψ continuously by 2π at fixed ϕ , so indeed we see that the boundary of each disk is diffeomorphic to a copy of the S^1 fiber. We now integrate (2.20) over the S^1 fiber sitting over each of the poles. Since the period of ψ (i.e. the fiber volume) is 2π , we therefore obtain the result that for S^2 ,

$$\chi = 2. \quad (2.21)$$

Finally, it is interesting to note that global structure of the tangent bundle of S^2 , which admits the metric (2.18), is precisely the $\mathbf{Z}_2$ factoring of the Taub-BOLT manifold discussed

by Page in [8], for which he found a Ricci-flat metric. This may be seen by comparing the “bolt” and the asymptotic behaviour of (2.18) and the Page metric.

3. Nowhere-vanishing spinor fields

We saw in the previous section that the potential obstruction to the existence of a nowhere-vanishing vector field could be viewed as an obstruction to the existence of a nowhere-vanishing cross-section of the bundle of unit-length tangent vectors or 1-forms. The fiber F in that case was the $(n-1)$ -sphere. In the case of spinor fields, we must now look at the bundle of unit-norm spinors, for which the fiber F will be the $(D-1)$ -sphere, where D is the real dimension of the fundamental spinor representation of the tangent space group $SO(n)$ of the manifold M . The general question of the existence of cross-sections of bundles has been extensively studied in obstruction theory, and here we shall just summarize some relevant facts. A more detailed discussion may be found in reference [4].

In general, if the lowest non-vanishing homotopy group of the fiber F is $\pi_p(F)$, then the first obstruction that could be encountered will lie in the cohomology group $H^{p+1}(M; \pi_p(F))$; this is known as the *primary* obstruction. Its vanishing is necessary for the existence of a global cross-section of the bundle*. If it does vanish, there are then secondary obstructions that might arise, which would lie in higher cohomology groups of the manifold M . However we shall see that for our problem, only the primary obstruction need be considered, and hence its vanishing is the necessary and sufficient condition for a global cross-section to exist. This is because we are considering the particularly simple case for which the fiber F is the $(D-1)$ -sphere S^{D-1} , and so the first $(D-2)$ homotopy groups vanish. Furthermore, $\pi_{D-1}(S^{D-1})$ is the group of integers $\mathbf{Z}$, and hence the primary obstruction lies in $H^D(M; \mathbf{Z})$. This means in particular that if the real dimension D of the spinor representation exceeds the dimension n of the manifold M , then there can be no obstruction to the existence of nowhere-vanishing spinor fields. Furthermore, secondary obstructions would only need to be considered if D were strictly less than n ; this in fact never occurs. Because the dimension of spinors grows rapidly with n , it is clear that the possibility even of the primary obstruction only arises in certain low dimensions. For $n \geq 9$, one can easily establish that $D > n$, so we need only examine the cases of n less than or equal to 8. Table 1 lists the real dimension of the spinor representations of $SO(n)$ for $n \leq 9$.

* One should really use cohomology groups with coefficients in the *bundle of local groups* associated with the homotopy groups evaluated at each point in M . However, we do not need to concern ourselves with this subtlety since the bundle of spinors in which we are interested is orientable.

Spinors in n dimensions		
n	D	Maj./Weyl
1	1	M
2	2	M or W
3	4	–
4	4	W
5	8	–
6	8	M or W
7	8	M
8	8	M and W
9	16	M

Table 1

Referring to Table 1, we see that the dimensions that require investigation are $n = 1, 2, 4$ and 8 , where in each case the real dimension of the spinor representation is equal to the dimension n of the manifold M . We shall deal first with the case $n = 8$, proceeding by analogy with Chern's proof of the Gauss-Bonnet theorem described in section 2. The difference is that rather than considering the bundle of 1-form fields with components u_i satisfying (2.7), we now consider the bundle of real spinor fields u , satisfying $\bar{u}u = 1$, or in terms of components,

$$u^\alpha u_\alpha = 1, \quad (3.1)$$

where $\alpha = 1 \cdots 8$ is an eight-dimensional spinor index of $SO(8)$, say the 8_c . Instead of θ_i defined by (2.8), we now define θ_α by

$$\theta_\alpha = Du_\alpha \equiv du_\alpha + \frac{1}{4}\omega_{ij}(\Gamma^{ij})_{\alpha\beta}u^\beta, \quad (3.2)$$

where Γ^i are the 16×16 Dirac matrices of eight dimensions and $(\Gamma^{ij})_{\alpha\beta}$ denotes the components of the chiral projection of Γ^{ij} onto the 8_c representation. Spinor indices are understood to be raised and lowered with the charge conjugation matrix C ; $(\Gamma^{ij})_{\alpha\beta}$ is antisymmetric in α and β .

The rest of the derivation now proceeds precisely in parallel with section 2, with the 8_v vector indices $i, j, \dots$ in (2.7)-(2.11), (2.15) and (2.16) replaced by 8_c spinor indices $\alpha, \beta, \dots$. The curvature 2-form Θ is also now taken to be in the spinor representation $\Theta_{\alpha\beta}$:

$$\Theta_{\alpha\beta} \equiv \frac{1}{4}\Theta_{ij}(\Gamma^{ij})_{\alpha\beta}. \quad (3.3)$$

Thus defining a spinor analogue $\tilde{\chi}$ of the Euler number χ , we have the result that in eight dimensions

$$\tilde{\chi} = \int_M \tilde{\Omega}, \quad (3.4)$$

where $\tilde{\Omega}$, the analogue of Ω in (2.2), is given by

$$\tilde{\Omega} = \frac{1}{4!(4\pi)^4} \epsilon^{\alpha_1 \dots \alpha_8} \Theta_{\alpha_1 \alpha_2} \dots \Theta_{\alpha_7 \alpha_8}. \quad (3.5)$$

Substituting from (3.3), and working out the Γ -matrix algebra, we find

$$\tilde{\Omega} = \frac{1}{(4\pi)^4} (\text{tr } \Theta^4 - \frac{1}{4} (\text{tr } \Theta^2)^2) + \frac{1}{2} \Omega. \quad (3.6)$$

But $\text{tr } \Theta^4$ and $(\text{tr } \Theta^2)^2$ can be re-expressed in terms of the first and second Pontrjagin forms P_1 and P_2 on M , to give

$$\tilde{\Omega} = \frac{1}{16} (P_1^2 - 4P_2 + 8\Omega). \quad (3.7)$$

Thus from (3.4), the analogue for spinors of the Euler number is given by

$$\tilde{\chi} = \frac{1}{16} (p_1^2 - 4p_2 + 8\chi), \quad (3.8)$$

where p_1^2 and p_2 are the Pontrjagin numbers $\int_M P_1^2$ and $\int_M P_2$ respectively. The analogue of (2.1) becomes

$$\tilde{\chi} = \sum_i \tilde{a}_i, \quad (3.9)$$

where $\tilde{a}_i$ is the index of the i 'th zero of the spinor field. In particular, we see that M admits a nowhere-vanishing Majorana-Weyl spinor field if and only if $\tilde{\chi}$ given by (3.8) is zero. Thus we have derived, by differential geometric methods, the same condition as was obtained in [4] using obstruction theory. However, here we also have the stronger result (3.9), that relates $\tilde{\chi}$ to the number of zeroes of the spinor field.

The same analysis can be repeated for a spinor in the 8_s rather than the 8_c representation of the $SO(8)$ tangent space group. Because the opposite chiral projection of Γ^{ij} is being taken, one would now find that the sign of the χ term in (3.8) is reversed. If one asked both for an 8_c and an 8_s nowhere-vanishing spinor field, one would therefore find that the Euler number χ of M must vanish. This is not surprising, since given such non-vanishing spinor fields ξ_c and ξ_s , one can form the non-vanishing vector field $\bar{\xi}_c \Gamma^i \xi_s$. (That this vector field should be non-vanishing is a special property following from the Γ -matrix algebra of eight dimensions. In higher even dimensions, where non-vanishing right- and left-handed spinor fields always exist, the analogous vector field can have zeroes, and thus there is no conflict with the non-existence of nowhere-vanishing vector fields in manifolds with non-zero Euler number.)

The vector and spinor representations also have matching dimensions for $n = 1, 2$ and 4 , and so the possibility of an obstruction to the existence of nowhere-vanishing spinor fields exists for these dimensions too. For $n = 4$, the group $Spin(4)$ is isomorphic to $SU(2) \times SU(2)$, and hence bundles over 4-manifolds are classified by the second Chern class for the left and right $SU(2)$, which correspond to left-handed and right-handed spinors

respectively. The curvatures of the bundles are $\Theta_{ij}^\mp \equiv \frac{1}{2}(\Theta_{ij} \mp \check{\Theta}_{ij})$, i.e. the anti-self-dual and self-dual parts of the Riemannian curvature, where $\check{\Theta}_{ij} = \frac{1}{2}\epsilon_{ijkl}\Theta_{kl}$. The criterion for the existence of non-vanishing left-handed or right-handed spinors is therefore that Ω^- or Ω^+ , defined by

$$\Omega^\mp = \pm \frac{1}{32\pi^2} \epsilon^{ijkl} \Theta_{ij}^\mp \wedge \Theta_{kl}^\mp, \quad (3.10)$$

should integrate to zero. In terms of the Pontrjagin and Euler numbers of M , this becomes the condition that χ^- or χ^+ , given by

$$\chi^\mp = \frac{1}{4}(p_1 \mp 2\chi), \quad (3.11)$$

should be zero. The only non-trivial example, where only one of χ^- or χ^+ is zero, is provided by the $K3$ manifold.

In the case $n = 2$, the two-dimensional analogue of (3.5) now leads to the condition that $\tilde{\chi}$, given by

$$\tilde{\chi} = \mp \frac{1}{2}\chi, \quad (3.12)$$

must vanish if M is to admit a nowhere-vanishing spinor field. In other words, the condition is the same as that for vectors. Finally, the case $n = 1$ is trivial since the tangent space group $SO(1)$ is trivial, so global cross-sections always exist both for vectors and spinors.

Since (3.8) and (3.12) are related to the numbers of zeroes of a spinor field in 8 and 2 dimensions, it follows that these expressions must be integers, at least if the compact manifold M is a spin manifold. In the case of (3.12) this is nothing new, since all compact 2-manifolds have even Euler number. (It follows, for example, from the fact that all 2-manifolds are Kähler.) The result that (3.8) must be an integer for compact eight-dimensional spin manifolds is less obvious. As we shall see in the next section, though, it can also be deduced from an index theorem.

4. Nowhere-vanishing spinors and twisted spin bundles

As we noted in the Introduction, the existence of a nowhere-vanishing Majorana-Weyl spinor field ξ in an eight-dimensional manifold, transforming, say, under the 8_c of $SO(8)$, enables one to map back and forth between the 8_s representation of Majorana-Weyl spinors of the opposite handedness and the 8_v vector representation. In terms of a notation that we will use extensively in this section, we can restate this by saying that if ξ is a global cross-section of $\Delta^\pm$ (the bundle of 8_s or 8_c right-handed or left-handed spinor fields), then it provides a natural invertible mapping from $\Delta^\mp$ to Λ^1 , the bundle of 1-forms (which are equivalent to vectors since the manifold M is endowed with a non-singular Riemannian metric). The mapping is implemented by the (off-diagonal) 8×8 blocks of the Dirac matrices of eight dimensions, $(\Gamma_i)^\alpha_\beta$, as in (1.1).

We shall now present an alternative proof, based on the Atiyah-Singer index theorem [9], that the existence of the nowhere-vanishing spinor ξ in eight dimensions implies that $\tilde{\chi}$ defined by (3.8) must vanish. To do this, we need to consider the index for the so-called

twisted spin complex, i.e. we compute the index for a Dirac operator acting on the tensor product of the spin bundle and some vector bundle V . Instead, therefore of the usual Dirac operator that interchanges the Δ^+ and Δ^- spin bundles, we now have an operator P that interchanges $\Delta^+ \otimes V$ and $\Delta^- \otimes V$. An example would be where V is a principal bundle with structure group G , and P is the gauged Dirac operator in the background of a Yang-Mills field minimally coupled to the spinors. For our present purposes, we shall consider the cases where V is $\Delta^\pm$ and where V is Λ^1 . The covariant derivative D_i in the Dirac operator $\mathcal{D} = \Gamma^i D_i$ is then the usual one acting on bispinors or spinor-vectors respectively. We denote these Dirac operators by $\mathcal{D}_\pm$ and $\mathcal{D}_1$ respectively.

Suppose now that a nowhere-vanishing spinor ξ in Δ^- or Δ^+ exists. We thus have the invertible mapping described above between $\Delta^\pm$ and Λ^1 . Now although ξ will not in general be covariantly constant with respect to the usual covariant derivative on spinors, we may define a new derivative, D'_i , by

$$D'_i \psi = D_i \psi + \frac{1}{8} (\bar{\xi} \Gamma^{ab} D_i \xi) \Gamma_{ab} \psi, \quad (4.1)$$

where ψ is an arbitrary spinor, with respect to which ξ is covariantly constant. D'_i is a covariant derivative constructed from a connection with torsion, $D'_i = \partial_i + \frac{1}{4} \omega'_i{}^{ab} \Gamma_{ab}$, where $\omega'_i{}^{ab} = \omega_i{}^{ab} + \frac{1}{2} \bar{\xi} \Gamma^{ab} D_i \xi$. One can easily verify, by making a Fierz transformation, that when ψ is taken to be ξ , $D'_i \xi$ is indeed zero. Thus we may define new Dirac operators $\mathcal{D}'_\pm$ and $\mathcal{D}'_1$ for the spin complex twisted by $\Delta^\pm$ and Λ^1 respectively. The invertible map described above, implemented by the spinor ξ which satisfies $D'_i \xi = 0$, therefore implies that the spectra of $\mathcal{D}'_\pm$ and $\mathcal{D}'_1$ are identical. In particular, this means that the zero-eigenvalue modes are in one-to-one correspondence, and so the index $I'_\pm$ of $\mathcal{D}'_\pm$ is the same as the index I'_1 of $\mathcal{D}'_1$. However, the index is a topological invariant, independent of (smooth) deformations of the connection. By supposition, ξ is regular everywhere in M , so we can interpolate smoothly between $\mathcal{D}$ and $\mathcal{D}'$ by putting a factor λ in front of the torsion term in (4.1), which can be varied smoothly between 0 and 1. Thus we have the result that if the nowhere-vanishing spinor ξ in $\Delta^\pm$ exists on M , then

$$I_1 - I_\pm = 0. \quad (4.2)$$

There is a direct way to calculate the indices $I_\pm$ and I_1 for the operators $\mathcal{D}_\pm$ and $\mathcal{D}_1$. The Atiyah-Singer index theorem for a twisted spin complex $\Delta^\pm \otimes V$ states that the corresponding index I_V is given by [9]

$$I_V = \int_M \hat{A}(M) \wedge \text{ch}(F, V), \quad (4.3)$$

where $\hat{A}(M)$ is the A-roof genus of the usual spin complex and $\text{ch}(F, V)$ is the Chern character of the vector bundle V . It is defined by

$$\text{ch}(F, V) = \text{tr} \exp\left(\frac{i}{2\pi} F\right), \quad (4.4)$$

where F is the curvature of the vector bundle V , and the trace is that appropriate to the bundle. (See also [10] for a physicist's discussion of these issues.) Now when V is Λ^1 , F is just Θ in the usual vector representation of the tangent space group. In a manifold M of (even) dimension $n = 2p$, Θ_{ij} may be skew-diagonalized so that locally

$$\frac{1}{2\pi}\Theta_{ij} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 & x_p \\ & & & & -x_p & 0 \end{pmatrix}. \quad (4.5)$$

Thus a simple calculation gives

$$\text{ch}(\Theta, \Lambda^1) = 2 \sum_{a=1}^p \cosh x_a. \quad (4.6)$$

When $V = \Delta^\pm$, the Chern character is $\text{ch}(\Theta, \Delta^\pm)$, where Θ is in the $\Delta^\pm$ spinor representation of the tangent space group (i.e. $\frac{1}{4}\Gamma^{ij}\Theta_{ij}$ projected onto the positive or negative eigenspaces of the chirality operator). Thus, from (4.5), $\frac{1}{2\pi}\Theta$ is the chiral projection of $\frac{1}{2}(x_1\Gamma_{12} + x_2\Gamma_{34} + \cdots + x_p\Gamma_{n-1,n})$. This leads to a nice factorization in (4.4), yielding straightforwardly the result

$$\text{ch}(\Theta, \Delta^\pm) = 2^{p-1} \left(\prod_{a=1}^p \cosh\left(\frac{x_a}{2}\right) \pm \prod_{a=1}^p \sinh\left(\frac{x_a}{2}\right) \right). \quad (4.7)$$

The A-roof genus $\hat{A}(M)$ is given by

$$\hat{A}(M) = \prod_{a=1}^p \frac{x_a/2}{\sinh(x_a/2)} \quad (4.8)$$

(see, for example, [10]). These formulae may all be re-expressed in terms of the Pontrjagin forms P_q and the Euler form Ω , using the relations

$$P_1 = \sum_a x_a^2; \quad P_2 = \sum_{a < b} x_a^2 x_b^2; \quad P_3 = \sum_{a < b < c} x_a^2 x_b^2 x_c^2; \quad \dots, \quad (4.9)$$

$$\Omega = \prod_a x_a, \quad (4.10)$$

which follow from (4.5) and the standard definitions of the Pontrjagin and Euler classes [10]. We note in passing that in eight dimensions the top form of $\hat{A}(M) \wedge \text{ch}(\Theta, \Delta^\pm)$ is the

same as the top form of $\frac{1}{2}(L(M) \pm \Omega)$, where $L(M)$ is the Hirzebruch L -polynomial of the signature complex, so from (4.3) we find that

$$I_{\pm} = \frac{1}{2}(\tau \pm \chi), \quad (4.11)$$

where τ is the Hirzebruch signature and χ is the Euler number.

We now proceed to calculate the quantity

$$I_1 - I_{\pm} = \int_M \hat{A}(M) \wedge \left(\text{ch}(\Theta, \Lambda^1) - \text{ch}(\Theta, \Delta^{\pm}) \right) \quad (4.12)$$

for the case of M an 8-manifold. From (4.6) and (4.7), we find

$$\text{ch}(\Theta, \Lambda^1) - \text{ch}(\Theta, \Delta^{\pm}) = \frac{1}{16} \left(\sum_a x_a^4 - 2 \sum_{a < b} x_a^2 x_b^2 \mp 8 \prod_a x_a \right). \quad (4.13)$$

Note in particular that the 0-form and 4-form components of $\text{ch}(\Theta, \Lambda^1)$ and $\text{ch}(\Theta, \Delta^{\pm})$ are equal, and have cancelled in (4.13). From (4.9) and (4.10), we find after some elementary algebra that

$$\text{ch}(\Theta, \Lambda^1) - \text{ch}(\Theta, \Delta^{\pm}) = \frac{1}{16} (P_1^2 - 4P_2 \mp 8\Omega), \quad (4.14)$$

In other words,

$$I_1 - I_{\pm} = \frac{1}{16} (p_1^2 - 4p_2 \mp 8\chi). \quad (4.15)$$

Thus comparing with (4.2), which was derived under the hypothesis that a nowhere-vanishing spinor exists on M , we see that we have reproduced the criterion of section 3 that the existence of such a spinor implies the vanishing of the right hand side of (4.15).

As an interesting parenthetical remark we note that the expression (4.11) for $I_{\pm}$, derived using the index theorem for the twisted spin complex, can be understood also by more elementary means. When the vector bundle V is taken to be Δ^+ , the corresponding Dirac operator $\mathcal{D}_+$ interchanges the bundles $\Delta^+ \otimes \Delta^+$ and $\Delta^- \otimes \Delta^+$. But these are, respectively, the same as $\Lambda^0 \oplus \Lambda^2 \oplus \Lambda_+^4$ and $\Lambda^1 \oplus \Lambda^3$, where Λ^q denotes the bundle of q -forms and the “+” subscript on Λ_+^4 denotes the bundle of *self-dual* 4-forms. These equivalences may be seen simply from the multiplication of $8_s \otimes 8_s$ and $8_c \otimes 8_s$ in $SO(8)$. Thus the index I_+ may be re-expressed as

$$I_+ = b_0 + b_2 + b_4^+ - b_1 - b_3, \quad (4.16)$$

where b_q is the q 'th Betti number and the “+” superscript on b_4^+ denotes the number of *self-dual* harmonic 4-forms. Repeating for the case where $V = \Delta^-$, the index I_- for this case can be re-expressed as

$$I_- = b_1 + b_3 - b_0 - b_2 - b_4^-. \quad (4.17)$$

Adding (4.16) and (4.17), we see that $I_+ + I_- = b_4^+ - b_4^-$, which is nothing but τ , the Hirzebruch signature (see, for example, [10]). Subtracting (4.17) from (4.16), and using the relation $b_q = b_{8-q}$ which follows from Hodge duality, we find that $I_+ - I_- = \chi$, the Euler number of M . Thus solving again for $I_{\pm}$, we recover (4.11).

We noted earlier in this section that (4.13), which is the Chern character for the virtual bundle $(\Lambda^1 - \Delta^{\pm})$ in the 8-manifold M , contains no 0-form or 4-form. Thus in evaluating (4.12), only the 0-form component of $\hat{A}(M)$ gives any contribution. This means that the expression (4.15) for $I_1 - I_{\pm}$ is insensitive to the fact that it is the *spin* complex $\Delta^{\pm}$ that is being twisted with the virtual bundle. We would obtain the same result, for example, if M were a complex manifold and we computed the index of the Dolbeault complex, again twisted with the same virtual bundle, since this in this case $\text{index}(\bar{\partial}_V) = \int_M \text{td}(T_c(M)) \wedge \text{ch}(\Theta, V)$, where $\text{td}(T_c(M)) = 1 + \frac{1}{2}c_1 + \dots$ is the Todd class of the complex tangent bundle of M (see, for example, [10]). In the case of the twisted signature complex, the index would be 16 times (4.15), since in this case the Chern character has to be evaluated with Θ replaced by 2Θ in (4.4) [10].

5. Conclusions

In this paper we have seen that the only dimensions for which there might be an obstruction to the existence of nowhere-vanishing spinor fields are when $n = 2, 4$ or 8 , and we have derived the topological criteria that must be satisfied if such a spinor field is to exist. The reason that these particular dimensions are singled out is that only for these non-trivial cases is the real dimension of the spinor representation of $SO(n)$ less than or equal to the dimension n of the manifold. Of course, in the case of vectors, the vector representation of $SO(n)$ is always equal to n , which is why the possibility of an obstruction to the existence of a nowhere-vanishing vector field can arise in any (even) dimension. In the Introduction, we raised the question of the conditions under which a compact manifold M admits a nowhere-vanishing field which transforms under some general representation of the tangent space group. As discussed in section 2, and more extensively in [4], an obstruction can only arise if the real dimension of the representation of $SO(n)$ is less than or equal to n . This can in fact occur only for the vector representation with arbitrary n , and for the spinor representations when $n = 1, 2, 4$ or 8 . (There is, of course, the case of antisymmetric 2-index tensors in three dimensions, but these are trivially equivalent to vectors, by Hodge duality.) Thus only for vectors, and spinors in the cases we have described, is there any possibility of the non-existence of nowhere-vanishing fields.

As was observed in [4], it is easy to construct examples of 8-manifolds for which the topological number $\tilde{\chi}$ of (3.8) is non-zero, and which therefore do not admit nowhere-vanishing Majorana-Weyl spinor fields. For example if $M = S^8$, which has $p_1^2 = p_2 = 0$ and $\chi = 2$, we have $\tilde{\chi} = 1$, which implies that any Majorana-Weyl spinor field on the eight-sphere must have at least one zero. Another class of examples described in [4] is provided by any M that is the product of a 2-manifold and a 6-manifold, each having non-zero Euler number.

If M is an 8-manifold that admits a metric with positive Ricci scalar (such as S^8 with its standard metric) then by Lichnerowicz's theorem [11] it follows that there can be no

zero modes of the Dirac operator on M , and so in particular the index of the Dirac operator must be zero. Since this is given by $\int_M \hat{A}(M)$, and the 8-form in (4.8) is $\frac{1}{5760}(7P_1^2 - 4P_2)$ [10], it follows that in such a manifold

$$7p_1^2 - 4p_2 = 0. \quad (5.1)$$

(We are, of course, assuming here that M is a spin manifold.) Now in any 8-manifold, the Hirzebruch signature τ is given by [10] $\tau = \frac{1}{45}(7p_2 - p_1^2)$, which together with (5.1) implies that $p_2 = 7\tau$ and $p_1^2 = 4\tau$. Thus if M admits a metric with positive Ricci scalar, then from (3.8) we have

$$\tilde{\chi} = \frac{1}{2}(\chi - 3\tau). \quad (5.2)$$

In [12], it is shown that any 8-dimensional quaternionic Kähler manifold that admits a metric with positive scalar curvature satisfies $\chi = 3\tau$, and so such manifolds provide non-trivial examples which admit nowhere-vanishing spinor fields. These examples include the quaternionic projective plane, $P_2(\mathbf{H}) = Sp(3)/(Sp(2) \cdot Sp(1))$, which has $p_1^2 = 4$, $p_2 = 7$ and $\chi = 3$, and the coset spaces $SU(4)/S(U(2) \times U(2))$ and $G_2/SO(4)$ [12].

Another class of manifolds for which (3.8) vanishes is provided by complex manifolds of vanishing first Chern class [4]. This follows from the fact that in a complex manifold the first and second Pontrjagin classes are related to the Chern classes as follows:

$$P_1 = -2c_2 + c_1^2; \quad P_2 = 2c_4 + c_2^2 - 2c_1c_3, \quad (5.3)$$

and furthermore for complex manifolds of real dimension eight,

$$\chi = c_4. \quad (5.4)$$

Thus substituting into (3.8) we find

$$\tilde{\chi} = \frac{1}{16}c_1(c_1^3 - 4c_1c_2 + 8c_3), \quad (5.5)$$

whence the result. Of course here, as in all cases where $\tilde{\chi}$ is zero by virtue of a non-trivial cancellation between $-\chi$ and $p_1^2 - 4p_2$, nowhere-vanishing spinor fields exist only for one handedness of spinor. With the conventions we have adopted, they are contained in the left-handed spin bundle, Δ^- , and transform as an 8_c under $SO(8)$. The class $\tilde{\chi}$ for spinor fields of the opposite handedness will be given by reversing the sign of χ in (3.8). Thus if $\tilde{\chi} = 0$ for the 8_c spinor, it follows that for the 8_s spinor we have $\tilde{\chi} = -\chi$. This means that right-handed spinor fields would have the same number of zeroes as for vector fields. This is what one would expect, since given a nowhere-vanishing left-handed spinor ξ , (1.1) gives an invertible mapping between vectors and right-handed spinors. It is also true, of course, that for every example of a manifold where $-\chi = p_1^2 - 4p_2$ there is another which has $\chi = p_1^2 - 4p_2$, obtained by reversing the orientation of the original manifold. Thus the entire discussion can be repeated for these cases, but with the left and right spin bundles interchanged.

The fact that nowhere-vanishing Majorana-Weyl spinor fields exist in complex manifolds with $c_1 = 0$ is of interest in considering compactifications of the ten-dimensional superstring. The original motivation for studying the problem of nowhere-vanishing spinors was that such a field is required in order to transform between the Green-Schwarz and Neveu-Schwarz-Ramond formulations of the superstring in the light-cone-gauge [1,2,4]. A wide class of compactifications of the superstring have been investigated in which the transverse eight-dimensional space is taken to be a Kähler space, whose metric is such that the Ricci form satisfies

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} S, \quad (5.6)$$

where S is some globally defined scalar constructed from the Riemann tensor and its covariant derivatives [2,13]. Now $\frac{1}{2\pi} R_{a\bar{b}}$ is a representative of the first Chern class of a Kähler manifold, so (5.6) implies that $c_1 = 0$. Thus not only is the transformation between the Green-Schwarz and the Neveu-Schwarz-Ramond formalism possible when the transverse space is $R^2 \times$ (6-dimensional Calabi-Yau) (this is the case of physical interest for compactification to four spacetime dimensions; being non-compact, this 8-manifold always admits nowhere-vanishing spinors), but it is also possible when M is a compact Kähler 8-manifold, whose metric satisfies (5.6). This case would be of interest in the “worldsheets for worldsheets” investigated in [3].

There is another interesting connection between the results of [4] and this paper and superstring theory, related to an observation first made in [14]. The effective action for the superstring contains a term Y quartic in the Riemann tensor, which in the light-cone gauge may be written as [15]

$$Y = \int d^8 \psi_1 d^8 \psi_2 \exp(\bar{\psi}_1 \Gamma^{ij} \psi_1 \bar{\psi}_2 \Gamma^{kl} \psi_2 R_{ijkl}), \quad (5.7)$$

where ψ_1 and ψ_2 are two 8_c anticommuting spinors and (5.7) is evaluated using the rules of Berezin integration. Y may also be written as [15]

$$Y = t^{i_1 \dots i_8} t^{j_1 \dots j_8} R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8}, \quad (5.8)$$

where the tensor $t^{i_1 \dots i_8}$ is defined in [16]. A convenient way to define it is as that tensor, antisymmetric in each of the first, second, third and fourth pairs of indices, such that

$$t^{i_1 \dots i_8} M_{i_1 i_2} \dots M_{i_7 i_8} = 24 \text{tr } M^4 - 6(\text{tr } M^2)^2 + \frac{1}{2} \epsilon^{i_1 \dots i_8} M_{i_1 i_2} \dots M_{i_7 i_8}, \quad (5.9)$$

where M_{ij} is an arbitrary antisymmetric tensor. From this definition it is clear that (5.8) has the form

$$Y = Y_0 + Y_1 + Y_2, \quad (5.10)$$

where the terms on the right denote those terms in (5.8) involving 0, 1 and 2 ϵ tensors respectively. Now from (3.6) and (5.9), we see that

$$\tilde{\Omega} = \frac{1}{4! (4\pi)^4} t^{i_1 \dots i_8} \Theta_{i_1 i_2} \dots \Theta_{i_7 i_8}. \quad (5.11)$$

Thus the scalar “integrand” in (3.4), obtained by taking the Hodge dual of $\tilde{\Omega}$, is

$$\begin{aligned} *\tilde{\Omega} &= \frac{1}{4!(8\pi)^4} t^{i_1 \dots i_8} \epsilon^{j_1 \dots j_8} R_{i_1 i_2 j_1 j_2} \dots R_{i_7 i_8 j_7 j_8} \\ &= \frac{1}{4!(8\pi)^4} (Y_2 + \frac{1}{2} Y_1). \end{aligned} \tag{5.12}$$

This implies that on 8-manifolds where a nowhere-vanishing spinor field exists, the integral of (5.10) over M may be replaced by the integral of $L = (Y_0 - Y_2)$. (This fact had already been noted in Kähler manifolds of vanishing first Chern class in [14].) In fact L , which, unlike Y , admits a generalization to arbitrary dimension, can be shown to be the correct term in the covariant ten-dimensional effective action [14, 17] that reproduces the four loop metric beta function derived in [18].

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