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STRING FIELD THEORY AS GENERAL RELATIVITY OF LOOPS

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A B S T R A C T

Invariance under general loop diffeomorphisms $\delta X^\mu(\sigma) = \epsilon^\mu[X, \sigma]$ is used as a guide to construct a field theory of interacting closed strings. Local loop reparametrizations $\delta X^\mu(\sigma) = \epsilon[X, \sigma] \partial_\sigma X^\mu(\sigma)$ act on the tangent loop space, much as the local Lorentz rotations of general relativity. In terms of Fourier modes, the theory is obtained by gauging the semi-direct sum of the Kac-Moody extended Poincaré algebra with the Virasoro algebra. In the supersymmetric case, we discuss the relevant super-Kac-Moody $\mathfrak{e}$ (super)-Virasoro algebra. The nilpotency of the corresponding BRST operator yields the superloop field equations.

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There are by now several geometric approaches to string field theory¹⁾. The one we advocate here is a natural extension of general relativity to "general loop relativity", the geometrical principle being invariance under general loop diffeomorphisms

$$\delta X^\mu(\sigma) = \varepsilon^\mu[X, \sigma] \quad (1)$$

$X^\mu(\sigma)$ describes a loop in a D-dimensional spacetime, and $\varepsilon^\mu[X, \sigma]$ is an arbitrary functional of $X^\mu(\sigma)$.

Since we will work most of the time in terms of Fourier modes, we rewrite (1) as

$$\delta x_m^\mu = \varepsilon_m^\mu(x) \quad (2)$$

where x_m^μ is the Fourier coefficient of $X^\mu(\sigma)$:

$$X^\mu(\sigma) = \sum_{m=-\infty}^{\infty} x_m^\mu e^{im\sigma} \quad (3)$$

ε_m^μ is now an arbitrary function of the coordinates x_m^μ (in Fourier space any loop functional becomes a function of the infinite set of coordinates x_m^μ).

Our purpose is to find the analogue of general relativity, in the infinite-dimensional loop space $\{x_m^\mu\}$.

This will be done by generalizing to loop space the geometrical method of refs. 2), 3) for the construction of (super)gravity theories. The method can be summarized as follows. The starting point is the choice of a Lie algebra $\mathcal{G}$. The vielbein one-form μ^A (A runs on the adjoint of $\mathcal{G}$) on the group manifold $G = \exp \mathcal{G}$ is taken to be the fundamental field of the theory. For $\mathcal{G} = \text{Poincaré Lie algebra}$, μ^A is identified with the ordinary vielbein for A corresponding to the translation generators, and with the spin connection for A corresponding to the rotation generators. A dynamical theory of the "potential" μ^A requires the notion of "soft" group manifold, i.e. a deformation of $G = \exp \mathcal{G}$ whose potential

does not need to satisfy the Cartan-Maurer equations of G . The extent of deformation is measured by the curvature $R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C$ with $C^A_{BC} = G$ -structure constants. A lagrangian D-form $\mathcal{L}$ can now be constructed in terms of R^A and μ^A following some general building principles³⁾. The field equations for μ^A are obtained by considering the variations $\delta\mathcal{L}/\delta\mu^A = 0$ on the whole group manifold. The theory described by $\mathcal{L}$ is automatically invariant under diffeomorphisms on the soft group manifold, since all operations used in $\mathcal{L}$, i.e. wedge products and exterior derivatives, are diffeomorphic invariant. These diffeomorphisms encompass the whole invariance structure of the theory, and are interpreted in spacetime as general coordinate and gauge transformations. For the case of pure gravity, the horizontality of the curvature R^A (i.e. the vanishing of the Lorentz components^{*)} $R^A_{ab\,cd}$) implies that the theory lives on the coset space $\{x^\mu\}$: Poincaré/Lorentz, and ensures gauge invariance under local Lorentz rotations. We refer to this construction as Poincaré/Lorentz "gauging".

We wish to apply a similar program in loop space, and obtain a theory invariant under the diffeomorphisms (2).

Also, this theory should be gauge invariant under the local loop reparametrizations:

$$\delta X^\mu(\sigma) = \varepsilon [\chi, \sigma] \partial_\sigma X^\mu(\sigma) \quad (4)$$

since these do not physically change the loop $X^\mu(\sigma)$. The invariance (4) is to be considered on the same conceptual footing as local Lorentz invariance in general relativity.

How do we implement these geometrical requirements in the choice of G , the algebra to be gauged by our candidate field theory of loops?

G must contain the ordinary Poincaré Lie algebra, so as to recover gravity in the constant loop limit (i.e. discarding all Fourier modes with $n \neq 0$). G must also contain the Virasoro algebra generating the reparametrizations (4).

*) the intrinsic components of R^A are defined by $R^A \equiv R^A_{BC} \mu^B \wedge \mu^C$.

A natural choice is :

$$\mathbb{G} = \text{Kac-Moody extension of Poincaré} \ltimes \text{Virasoro} \\ (\text{generators: } P_a^n, M_{ab}^n) \quad (\text{generators: } L^n) \quad (5)$$

where $\ltimes$ indicates the semidirect sum. The generators of $\mathbb{G}$ satisfy the commutation relations :

$$[P_a^m, P_b^m] = 0 \quad (6a)$$

$$[M_{ab}^m, M_{cd}^m] = \eta_{ad} M_{bc}^{m+m} + \eta_{bc} M_{ad}^{m+m} - \eta_{ac} M_{bd}^{m+m} - \eta_{bd} M_{ac}^{m+m} \quad (6b)$$

$$[M_{ab}^m, P_c^m] = \eta_{bc} P_a^{m+m} - \eta_{ac} P_b^{m+m} \quad (6c)$$

$$[L^m, L^m] = (m-m) L^{m+m} \quad (6d)$$

$$[L^m, P_a^m] = m P_a^{m+m} \quad (6e)$$

$$[L^m, M_{ab}^m] = m M_{ab}^{m+m} \quad (6f)$$

For a comprehensive review on Kac-Moody and Virasoro algebras, see ref. 4). Since we are discussing the classical theory, we do not include central extensions in (6). Notice that the algebra of the 0-modes can be retrieved as a contraction of (6), by multiplying each n-mode by a scale factor $(\alpha')^n$. Then $\alpha' \rightarrow 0$ yields the ordinary Poincaré algebra, and the Virasoro algebra disappears (L^0 decouples). It is suggestive to identify α' with the inverse of the string tension, since $\alpha' \rightarrow 0$ yields the constant loop limit.

Let us justify further the choice (6) for $\mathbb{G}$:

i) in Fourier space, the reparametrizations (4) act on the loop coordinates as

$$\delta_\pi x_m^a = (\pi-m) x_{m-\pi}^a \quad (7)$$

where δ_κ is the variation due to the action of the Virasoro generator $L^\kappa = i e^{i\sigma} \partial_\sigma$. In loop space, L^κ can therefore be represented by

$$L^\kappa = \sum_{m=-\infty}^{\infty} (\kappa-m) x_{m-\kappa}^a \frac{\partial}{\partial x_m^a} \quad (8)$$

and indeed satisfies the commutation relations (6e) with the Kac-Moody "momenta" P_a^m (represented by $\frac{\partial}{\partial x_m^a}$).

ii) the algebra (6) allows a geometric interpretation of loop space as the coset manifold

$$\text{loop space} = \frac{G}{H} = \frac{\text{Kac-Moody Poincaré} \ltimes \text{Virasoro}}{\text{Kac-Moody Lorentz} \ltimes \text{Virasoro}} \quad (9)$$

Horizontality of the curvatures in the H-directions ensures gauge invariance of the theory under the subgroup $H = \text{Kac-Moody Lorentz} \ltimes \text{Virasoro}$.

The gauging of the algebra proposed in (6) is a straightforward generalization of the Poincaré gauging for pure gravity. The curvature definitions are (we consistently omit wedge symbols) :

$$\begin{aligned} R_m^a &= dV_m^a - \omega_s^{ab} V_{m-s}^b + (m-s) A_s V_{m-s}^a \equiv DV_m^a \\ R_m^{ab} &= d\omega_m^{ab} - \omega_s^{ac} \omega_{m-s}^{cb} + (m-s) A_s \omega_{m-s}^{ab} \\ F_m &= dA_m + \frac{2s-m}{2} A_s A_{m-s} \end{aligned} \quad (10)$$

where V_n^a , ω_n^{ab} and A_n are the projections along the P_a^n , M_{ab}^n and L^n directions of the soft group manifold vielbein. Eqs. (10), and the Jacobi identities for the structure constants of (6), imply the Bianchi identities:

$$\begin{aligned} dR_m^a + R_s^{ab} V_{m-s}^b - \omega_s^{ab} R_{m-s}^b - (m-s) F_s V_{m-s}^a + (m-s) A_s R_{m-s}^a &\equiv \\ DR_m^a + R_s^{ab} V_{m-s}^b - (m-s) F_s V_{m-s}^a &= 0 \end{aligned}$$

$$\begin{aligned}
 dR_m^{ab} + R_s^{ac} \omega_{m-s}^{cb} - \omega_s^{ac} R_{m-s}^{cb} - (m-s) F_s \omega_{m-s}^{ab} + (m-s) A_s R_{m-s}^{ab} &\equiv \\
 DR_m^{ab} - (m-s) F_s \omega_{m-s}^{ab} &= 0 \\
 dF_m - (2s-m) F_s A_{m-s} &= 0.
 \end{aligned}
 \tag{11}$$

One could now attempt to construct an action for the soft algebra (10), according to the building principles of ref. 3). However, when supersymmetry enters the stage, the situation simplifies considerably. The Bianchi identities for the supersymmetric extension of (10) are satisfied only on shell, and therefore can be exploited to arrive at the field equations of the theory⁵⁾. This generalizes the well-known fact that in supergravity theories without auxiliary fields the supersymmetry algebra closes only modulo the equations of motion.

What is the right supersymmetric extension of (10) ? Classically, it could be any extension of the Poincaré algebra (in any spacetime dimension D) to a Kac-Moody superalgebra, or Kac-Moody free differential superalgebra^{*)} $\oplus$ (super)Virasoro algebra. For example, if one wants to retrieve matter coupled D = 10, N = 1 supergravity in the constant loop limit, the relevant infinite-dimensional free differential algebra is the Kac-Moody extension of the free differential algebra discussed in ref. 7). Note that at the classical level nothing prevents the gauging of the Kac-Moody extension ($\oplus$ Virasoro) of the D = 4, N = 1 superPoincaré algebra, whose curvatures are :

$$\begin{aligned}
 R_m^a &= dV_m^a - \omega_s^{ab} V_{m-s}^b - \frac{i}{2} \bar{\psi}_s \Gamma^a \psi_{m-s} + (m-s) A_s V_{m-s}^a \equiv DV_m^a - \frac{i}{2} \bar{\psi}_s \Gamma^a \psi_{m-s} \\
 R_m^{ab} &= d\omega_m^{ab} - \omega_s^{ac} \omega_{m-s}^{cb} + (m-s) A_s \omega_{m-s}^{ab} \\
 \rho_m &= d\psi_m - \frac{1}{4} \omega_s^{ab} \Gamma_{ab} \psi_{m-s} + (m-s) A_s \psi_{m-s} \equiv D\psi_m
 \end{aligned}
 \tag{12}$$

*) free differential algebras provide an elegant algebraic framework for supergravity theories with antisymmetric tensors, see refs. 6), 3).

The corresponding Bianchi identities yield the superloop field equations⁵⁾, generalizing the supergravity field equations of the ordinary superPoincaré theory.

The quantization of loop relativity, however, may presumably restrict both the dimension of spacetime (i.e. the range of the index a in P_m^a) and the choice of gauge matter one can append to the supergravity multiplet. Indeed, following the prescription of refs. 8), we consider the BRST operator for the "soft" algebra (10):

$$Q = g^A \tilde{T}_A - \frac{1}{2} C_{BC}^A g^B g^C h_A + R_{BC}^A g^B g^C h_A \quad (13)$$

where $\tilde{T}_A$ are the tangent vectors $\tilde{P}_a^n, \tilde{M}_{ab}^n, \tilde{L}^n$ dual to the soft one-forms $V_n^a, \omega_n^{ab}, A_n$, and g^A, h_B are the corresponding ghosts and antighosts. (C_{BC}^A are the structure constants of the algebra (6).) The tangent vectors $\tilde{T}_A$ close on the soft algebra :

$$[\tilde{P}_a^n, \tilde{P}_b^m] = -2 R_{r ab}^c \tilde{P}_c^r - 2 R_{r ab}^{cd} \tilde{M}_{cd}^r - 2 F_{r ab} \tilde{L}^r \quad (14)$$

all other commutators as in (6) because of the horizontality of R^A in the Kac-Moody Lorentz and Virasoro directions. The quantum BRST operator has to be normal ordered in the ghosts and antighosts (positive moded g and negative moded h are creation operators), and moreover one should (perhaps) include central extensions k, c in the Kac-Moody and in the Virasoro algebras. Then $Q^2 = 0$ yields not only the Bianchi identities (and the field equations as well in the supersymmetric case^{*)}), but constrains some group quantities like $\dim G$ and c_G , the quadratic Casimir of G . Consider for example a generic G -Kac-Moody $\oplus$ Virasoro algebra, with the central extensions k and c . Then the corresponding quantum BRST operator is nilpotent only if⁹⁾

$$k = -c_G, \quad c = 26 + 2 \dim G \quad (15)$$

*) Then one could consider the cubic action $\int \mathcal{D}x_n^a Q^3$ (see also Bars and Yankielowicz in ref. 1)), whose variational equations certainly admit the $Q^2 = 0$ solutions (and possibly more).

For the $N = 1$ supersymmetric case one finds¹⁰⁾

$$k = 0, \quad c = 10 + 2 \dim G$$

It is clear that it becomes interesting to find the analogous constraints for free differential Kac-Moody (super)algebras coupled to (super)Virasoro. This analysis is in progress and will be reported in a separate publication.

On a more speculative level, one can also wonder whether loop relativity can in turn be extended to p-membrane relativity. In principle we can consider the mappings

$$\frac{G'}{H} \rightarrow G$$

instead of the $S^1 \rightarrow G$ mappings giving rise to Kac-Moody algebras. Then the generators would be labelled by a set of numbers characterizing the harmonics on G'/H , and the structure constants would contain the corresponding Clebsch-Gordon coefficients. These algebras do exist, in implicit form, in Kaluza-Klein compactified theories¹¹⁾. For closed 2-membranes, for example, the relevant generators would be labelled by the numbers corresponding to the harmonics on S^2 (spherical harmonics).

Finally, a few words to situate our work in relation to the existing literature. The approach to string field theory that we think closer to ours is the one proposed by Bars and Yankielowicz in ref. 1). One of the main differences lies in the fact that they consider as algebra to be gauged only the Virasoro algebra. This, we feel, does not provide a fully geometrical picture as the one we discuss in the present paper. Indeed general loop diffeomorphisms are completely absent in their approach, and the coset space on which their theory is built is $\text{Virasoro}/L^0$, which is not parametrized by the loop space coordinates x_n^a .

In summary: we have proposed a conceptually simple extension of general relativity to a closed string field theory. The extended geometrical principles are implemented by gauging (super)Kac-Moody & Virasoro algebras, which generalize the (super)algebras gauged by the corresponding (super)gravity limits. We obtain a (super)string field theory, guaranteed by construction to reduce, in the constant loop limit, to the (matter coupled) (super)gravity of our choice. This choice will be restricted in the quantized theory⁵⁾.

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