EUROPEAN ORGANISATION FOR NUCLEAR RESEARCH
CERN - SPS DIVISION

CERN SPS/88-2 (AMS)
LHC Note 66

CERN LIBRARIES, GENEVA

CM-P00061416

Normal Forms for Hamiltonian Maps
and Non Linear Effects in a LHC Model

A. Bazzani
INFN-Sezione di Padova

P. Mazzanti
INFN-Sezione di Bologna

G. Servizi
Dipartimento di Fisica della Università di Bologna

G. Turchetti
Dipartimento di Fisica della Università di Bologna
INFN-Sezione di Bologna

Document received in CERN
PRIVATE COMMUNICATION
not to be quoted or copied without author's permission

Prévessin - 22nd January, 1988

NORMAL FORMS FOR HAMILTONIAN MAPS AND NON LINEAR EFFECTS IN A LHC MODEL

A. Bazzani
INFN-Sezione di Padova

P. Mazzanti
INFN-Sezione di Bologna

G. Servizi
Dipartimento di Fisica della Università di Bologna
INFN-Sezione di Bologna

G. Turchetti
Dipartimento di Fisica della Università di Bologna
INFN-Sezione di Bologna

Abstract

We describe the motion of a particle in the lattice of a hadronic accelerator using the formalism of symplectic maps ^[3,4].

In this context we revisit the Courant-Snyder's theory ^[5] and we stress that the reduction in normal form of a symplectic map ^[6,7,8], in the non resonant case, is just the natural generalisation of the Courant-Snyder's theory to include the non linear effects.

We analyse briefly the first order results for the tune shift and the smear of the quadratic map and we show that it is possible to recover a continuum limit from a discrete description of the lattice.

A study of an one-dimensional model for LHC is made. We have computed the tune shift and the smear in various situations using the normal forms method in the program ^[11]. We compare the normal form results at various perturbative orders with tracking results obtained using a thin lens model for the same lattice.

1. Single particle dynamics

We shall analyse an one-dimensional model for the transverse motion of a particle in a hadronic accelerator. In the standard description of the betatron motion along the x axis, the following differential equation holds [7] :

$$\frac{d^2x}{ds^2} + \omega^2(s)x = \sum_{n \geq 2} \frac{K_n(s)}{n!} x^n \quad (1-1)$$

where s is the arc length of the designed orbit and x the horizontal distance from this orbit.

The quantities K_n are related with the analytic expansion in x of the magnetic field B_y :

$$B_y(s) = -B_o(s) + \sum_{n \geq 1} B_n(s)x^n \quad (1-2)$$

according to the definition:

$$K_n(s) = \frac{eB_n(s)}{p_0 c} \quad n \geq 1 \quad (1-3)$$

with p_0 the designed moment of the machine. In the equation $K_n(s)$ is the contribute of the $2(n+1)$ -pole component of the field B_y . Conversely the quantity $\omega^2(s)$ defined by:

$$\omega^2(s) = \frac{1}{\rho^2(s)} - K_1(s)$$

where $\rho(s)$ is the local radius of curvature, determines the betatron oscillations on the x axis due to the quadrupole component $K_1(s)$.

We note that, when we neglect the edge-effects in the lattice structure of a circular accelerator, $\omega(s)$ and $K_n(s)$ are discontinuous functions: namely step functions. As a consequence their Fourier components in the basis $\exp(im \frac{2\pi}{L} s)$, where L is the length

of the superperiod of the machine, decrease as $\frac{1}{m}$, so that the perturbation theory could not be applied to the non linear differential equation (1-1) from a rigorous point of view.

Let us consider, for the moment, the linear equation:

$$\frac{d^2x}{ds^2} + \omega^2(s)x = 0 \quad (1-5)$$

According to the Courant-Snyder's theory we write the solution in the form:

$$x(s) = A\sqrt{\beta(s)} \sin(\psi(s) + \delta) \quad (1-6)$$

with

$$\psi(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)} \quad \text{and} \quad \beta(L) = \beta(0)$$

In order to evaluate the functions $\beta(s)$ and $\psi(s)$ in each element of the lattice, it is convenient to rewrite the solution (1-6) using the transfer matrix $\mathbf{M}(s)$ defined according to:

$$\begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix} = \mathbf{M}(s) \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} \quad (1-7)$$

$\mathbf{M}(s)$ is a symplectic matrix because the equation (1-5) can be derived from the Hamiltonian:

$$H(x, \dot{x}, s) = \frac{\dot{x}^2}{2} + \frac{\omega^2(s)}{2}x^2$$

If we substitute in equation (1-7) the solution (1-6), the transfer matrix $\mathbf{M}(s)$ takes the form:

$$\mathbf{M}(s) = \mathbf{U}(s)\mathbf{R}(\psi(s))\mathbf{U}^{-1}(0) \quad (1-8)$$

where:

$$\mathbf{U}(s) = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ \gamma(s) & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{with} \quad \gamma(s) = \frac{\dot{\beta}(s)}{2\sqrt{\beta(s)}} \quad (1-9)$$

and

$$\mathbf{R}(\psi(s)) = \begin{pmatrix} \cos \psi(s) & \sin \psi(s) \\ -\sin \psi(s) & \cos \psi(s) \end{pmatrix}$$

is a rotation of angle $\psi(s)$.

As a consequence of the periodicity of $\beta(s)$ we have:

$$\mathbf{U}(0) = \mathbf{U}(L)$$

and for $s = L$ equation (1-8) becomes:

$$\mathbf{M}(L) \equiv \mathbf{T} = \mathbf{U}(0)\mathbf{R}(\psi(L))\mathbf{U}^{-1}(0) \quad (1-10)$$

We call $\mathbf{T}$ the transfer matrix on a superperiod of the machine and we stress the following property:

$$\mathbf{M}(nL) = \mathbf{T}^n \quad (1-11)$$

due to the periodicity of the lattice. Equation (1-10) tell us that $\mathbf{M}(L)$ is a rotation up to a similarity transformation, so the transfer matrix over n superperiods reads:

$$\mathbf{T}^n = \mathbf{U}(0)\mathbf{R}(n\psi(L))\mathbf{U}^{-1}(0) \quad (1-12)$$

The angle $\psi(L)$ is a characteristic of the accelerator and the quantity:

$$\nu = \frac{\psi(L)}{2\pi} \quad (1-13)$$

is called the tune over a superperiod. Now instead of solving equation (1-5), we can calculate $\mathbf{T}$ as the composition:

$$\mathbf{T} = \mathbf{M}_N \mathbf{M}_{N-1} \cdots \mathbf{M}_1 \quad (1-14)$$

where $\mathbf{M}_j$ is the transfer matrix of the j -th magnetic element of the lattice and N is the number of elements in the superperiod. Each matrix $\mathbf{M}_j$ appears in the composition (1-14) in the same order as the corresponding element in the lattice. By equation (1-10) it follows that:

$$2 \cos(2\pi\nu) = \text{Tr}(\mathbf{T}) = t_{1,1} + t_{2,2}$$

and

$$\beta(0) = \frac{t_{1,2}}{\sin(2\pi\nu)} \quad \gamma(0) = \frac{\cos(2\pi\nu) - t_{1,1}}{\sqrt{t_{1,2} \sin(2\pi\nu)}}$$

In order to establish a complete analogy between the Courant-Snyder's theory based on the differential equation (1-5) and the transfer map approach based on equation (1-10), we must be able to change our reference section ($s = 0$) on the machine. Let us denote

with s_k the coordinate of the end point of the $(k-1)$ -th element of the lattice as shown in the above figure.

Then we have:

$$\begin{pmatrix} x(s_k + L) \\ \dot{x}(s_k + L) \end{pmatrix} = M_{k-1} \cdots M_1 M_N \cdots M_k \begin{pmatrix} x(s_k) \\ \dot{x}(s_k) \end{pmatrix} \quad (1-15)$$

and using the equations (1-8), (1-10) and (1-14), we can write:

$$M_{k-1} \cdots M_1 M_N \cdots M_k = (M_{k-1} \cdots M_1) T(M_{k-1} \cdots M_1)^{-1} = U(s_k) R(2\pi\nu) U^{-1}(s_k) \quad (1-16)$$

In fact we can prove that the symplectic matrix:

$$M_{k-1} \cdots M_1 U(0)$$

has a unic decomposition of the form:

$$M_{k-1} \cdots M_1 U(0) = \begin{pmatrix} \sqrt{\beta_k} & 0 \\ \gamma_k & \frac{1}{\sqrt{\beta_k}} \end{pmatrix} R(\phi)$$

so that the equation (1-16) holds with:

$$U(s_k) = \begin{pmatrix} \sqrt{\beta_k} & 0 \\ \gamma_k & \frac{1}{\sqrt{\beta_k}} \end{pmatrix} \quad (1-17)$$

We observe that the similarity transformation (1-10) which reduce a symplectic matrix to a rotation is itself defined up to a rotation. This arbitrariness is fixed by requiring that $U(s)$ has a special structure such as $u_{1,2} = 0$.

The non linear effects on the particle motion in an accelerator will be treated by generalising the transfer map approach developed in the linear case. We first consider the transfer map for a single magnetic element of the lattice; in our approximation we assume that the magnetic field B_y is independent of s in each element. The equations of motion read:

$$\begin{cases} \dot{x} = \omega p \\ \dot{p} = -\omega x + \lambda x^2 \end{cases} \quad (1-18)$$

with $\lambda = \omega^{-\frac{3}{2}} \frac{K_2}{2}$. To simplify the presentation we have taken into account in (1-18) only the sextupole effects; it will be clear that the presence of other multipoles does not introduce any conceptual difficulties. It is convenient to use a complex variable z defined by:

$$z = x + ip \quad (1-19)$$

We can rewrite the equations (1-18) in the integral form:

$$z(s) = e^{-i\omega s} z(0) + \frac{i}{4} \lambda \int_0^s e^{-i\omega(s-t)} [z(t) + z^*(t)]^2 dt \quad (1-20)$$

Denoting with l_0 the length of our element, the first order Taylor expansion in $\eta = \omega l_0$ reads:

$$z(l_0) = e^{-i\omega l_0} [z(0) + \frac{i}{4} \lambda l_0 (z(0) + z^*(0))^2] + o(\eta^2) \quad (1-21)$$

The error behaves as $\frac{\lambda}{4} \frac{\eta^2}{2\omega} [Re(z(s^*))]^2$ with $s^* \in [0, l_0]$.

The expression (1-21) is called the kick approximation and provides an exactly symplectic map. Introducing the data of a LHC dipole, the typical range of the error is $10^{-2} - 10^{-3}$.

One can improve the result by computing the transfer map according to:

$$z(l_0) = \mathbf{M} \left(\frac{l_0}{n} \right) \circ \mathbf{M} \left(\frac{l_0}{n} \right) \circ \dots \circ \mathbf{M} \left(\frac{l_0}{n} \right) z(0) \quad n - \text{times} \quad (1-22)$$

In this case it is easy to see that the total error decreases essentially as $\frac{1}{n}$.

Equation (1-20) can also be solved by iterations; after $N - 1$ steps, a polynomial map of order N in the initial conditions is obtained:

$$z(l_0) = e^{-i\omega l_0} z + \sum_{m=2}^N P_m(z, z^*; l_0) \quad (1-23)$$

The coefficients of P_m are at least of order $o(\eta^{2m-1})$ so that if we stop after $N - 1$ steps, we make an error of the order $\eta^{2N-1}|z|^N$.

This method can be very convenient with respect to the previous one. However the map (1-23) is symplectic only up to terms of order z^N ; this means that the jacobian of this map is a polynomial of the form $1 + o(z^N)$. As a consequence the map is contracting or expanding where the jacobian is smaller or larger than 1. We call a map like (1-23) a truncation of order N of a symplectic map and we denote it by the symbol $[]_N$.

The transfer map $\mathbf{T}$ on the superperiod is obtained by composing the transfer maps of the single elements. We require that $\mathbf{T}$ is a truncation of order N of a symplectic map. This is possible if the transfer map of each element is either a symplectic map or a symplectic truncation of order not less than N .

2. Normal forms

As in the linear case we look for a change of variables:

$$z = \Phi(\xi, \xi^*) = \sum_{n \geq 1} \Phi_n(\xi, \xi^*) \quad (2-1)$$

where $\Phi_n(\xi, \xi^*)$ are homogeneous polynomials of order n , such that:

$$\mathbf{T} = \Phi \circ \xi e^{i\Omega(\xi\xi^*)} \circ \Phi^{-1} \quad (2-2)$$

i. e. in the new variables the map is simply a rotation with a non constant angle:

$$\xi' = e^{i\Omega(\xi\xi^*)} \xi \quad (2-3)$$

The map (2-3) is called the non resonant normal form of the map $\mathbf{T}$.

It is easy to see that if we substitute Φ with $\Phi \circ \xi e^{i\Xi(\xi\xi^*)}$ in the equation (2-2), the normal form is unaffected. So Φ is univocally determined requiring a particular structure.

If we change continuously the reference section along the machine, the equation (2-2) is replaced by:

$$\mathbf{T}(s) = \Phi(s) \circ \xi e^{i\Omega(\xi\xi^*, s)} \circ \Phi^{-1}(0) \quad (2-4)$$

where $\Phi(L) = \Phi(0) = \Phi$ and $\Omega(\xi\xi^*, L) = \Omega(\xi\xi^*)$. While the computation of (2-4) involves additional technical difficulties, it is easy to compute the whole transfer map of the superperiod on a different section. Exactly as in the linear case, denoting with $\mathbf{T}_k$ the transfer map of the section s_k , we have:

$$\mathbf{T}_k = (\mathbf{M}_{k-1} \cdots \mathbf{M}_1) \mathbf{T} (\mathbf{M}_{k-1} \cdots \mathbf{M}_1)^{-1} \quad (2-5)$$

The normal form transformation then reads:

$$\mathbf{T}_k = \Phi_k \circ \xi e^{i\Omega(\xi\xi^*)} \circ \Phi_k^{-1}$$

where Φ_k is related to Φ according to:

$$\Phi_k = \mathbf{M}_{k-1} \circ \cdots \circ \mathbf{M}_1 \circ \Phi \circ \xi e^{i\Xi(\xi\xi^*)}$$

The rotation $\xi e^{i\Xi(\xi\xi^*)}$ is chosen so that Φ_k has the same structure as Φ . We see

also that the quantity $\frac{\Omega(\xi\xi^*)}{2\pi}$ is a characteristic of the machine and it is the natural

generalisation of the linear tune: in particular $\nu = \frac{\Omega(0)}{2\pi}$.

If the linear tune is non resonant the Birkhoff theorems [6] assure the possibility of calculating the normal form at all perturbative orders and provide the recursive algorithms to compute it. Conversely if ν is resonant (i. e. $\nu = \frac{j}{m}$) the perturbative solution of equation (2-2) is defined only up to order $n < m-1$. In this case a generalized normal form can be introduced. Equation (2-2) is replaced by:

$$\mathbf{T} = \Phi \circ \mathbf{N} \circ \Phi^{-1}$$

where $N(\xi, \xi^*)$ is no longer a rotation but a symplectic map such that:

$$N(e^{2\pi i\nu}\xi, e^{-2\pi i\nu}\xi^*) = e^{2\pi i\nu}N(\xi, \xi^*) \quad (2-6)$$

N is called the resonant normal form with respect to the rotation $e^{2\pi i\nu}$ of the map T ; its analytic expansion reads:

$$N = \xi e^{2\pi i\nu} + N_3 \xi^2 \xi^* + \dots + N_{m-1} \xi (\xi \xi^*)^{\lfloor \frac{m-2}{2} \rfloor} + R_{m-1} \xi^{*m-1} + \dots$$

The relevant informations in the normal form can be extracted by using the interpolating Hamiltonian [8] $H(Z, Z^*)$ such that if $Z = \psi(\xi, \xi^*; t)$ is the corresponding phase flow with initial condition ξ then $e^{i\nu t}Z$ interpolates the normal form according to:

$$N^n(\xi, \xi^*) = e^{i\nu n} \psi(\xi, \xi^*; n)$$

3. The quadratic map

The quadratic map introduced by Henon [10] is the most simple example of symplectic map which presents all the features of a general map. We can obtain it starting from the most general symplectic quadratic map:

$$w' = e^{-i2\pi\nu} \left[w + iAe^{-i\phi} (e^{2i\phi} w^2 + 2ww^* + e^{-2i\phi} w^{*2}) \right] \quad (3-1)$$

After a rotation and a scaling $w = \frac{z}{4A} e^{-i\phi}$ it reduces to:

$$z' = e^{-i2\pi\nu} \left[z + \frac{i}{4} (z + z^*)^2 \right] \quad (3-2)$$

or in real coordinates:

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = R(2\pi\nu) \begin{pmatrix} x \\ p + x^2 \end{pmatrix}$$

In figure 1 we can see a phase plot of the quadratic map with $\nu = .21$. The islands structure is due to the presence of a five order resonance near the elliptic fixed point.

This means that the tune of the orbits $(x_{(j)}, y_{(j)})$ with initial value $(x_{(0)}, y_{(0)}) = (x, 0)$, defined by:

$$\nu(x) = \frac{1}{2\pi} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \arccos \frac{y_{(j)}y_{(j-1)} + x_{(j)}x_{(j-1)}}{|(x_{(j)}, y_{(j)})| |(x_{(j-1)}, y_{(j-1)})|}$$

where $|(x_{(j)}, y_{(j)})| = \sqrt{x_{(j)}^2 + y_{(j)}^2}$

assumes the resonant value $\nu = \frac{1}{5}$ for some x . Indeed if we look at the figure 2, which shows the tune shift of the orbits (i. e. in our case $\nu(x) - .21$), we note a plateau at the value -0.1 whose width is equal to the islands width along the x axis; this is the so called phase-locked effect.

Another quantity which is interesting in the analysis of the non linear effects, is the smear σ defined as the deviation of the orbits from a circle:

$$\sigma(x) = \frac{\langle (|(y_{(j)}, x_{(j)})| - \langle |(y_{(j)}, x_{(j)})| \rangle)^2 \rangle^{\frac{1}{2}}}{\langle |(y_{(j)}, x_{(j)})| \rangle}$$

where the symbol $\langle \cdot \rangle$ denotes the mean value along the orbit. We expect that the smear increases with x as the non linear terms become larger until x reaches the islands zone (see figure 3). In this zone the smear presents a sudden jump and a valley. This effects is due to the presence of the hyperbolic points with their separatrices and of the elliptic points in the center of the islands.

In figure 4 we plot the dynamic aperture of the quadratic map as a function of the linear tune ν . To explain all the features of this function is a very hard task and until now it is an open problem. We have only local results: for example for $x \ll 1$ we can explain the curve by means of the position of the hyperbolic fixed point of the quadratic map while the zero value at $\nu = \frac{1}{3}$ is due to the properties of the three order resonance.

The first order results for the normal form of the quadratic map (3-2) are given in the appendix. Here we recall that using the normal form the tune shift is defined by:

$$\delta\nu = \frac{\Omega(\rho)}{2\pi} - \nu$$

where

$$\rho = |\xi| = |\Phi^{-1}(z, z^*)| = |z| + \dots$$

and the smear σ is given by:

$$\sigma = \frac{\langle (r - \langle r \rangle)^2 \rangle^{\frac{1}{2}}}{\langle r \rangle}$$

where $r(\theta) = |\Phi(\rho e^{i\theta}, \rho e^{-i\theta})|$ and $\langle r \rangle = \frac{1}{2\pi} \int_0^{2\pi} r(\theta) d\theta$.

The first order results for the tune shift and the smear explicitly read:

$$\delta\nu = -\frac{1}{32\pi} \left[3 \cot \frac{\nu}{2} + \cot \frac{3\nu}{2} \right] \rho^2 \quad (3-3)$$

$$\sigma = \frac{1}{16} \left[2 \cot^2 \frac{\nu}{2} + 2 \cot^2 \frac{3\nu}{2} + 20 \right]^{\frac{1}{2}} \rho \quad (3-4)$$

The quadratic map can be used as a model for a simple cell, which is schematically represented by a linear part plus a sextupole in the kick approximation. The transfer map for such a cell reads:

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \mathbf{M} \begin{pmatrix} 1 & L_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p + \frac{K_2}{2} L_s x^2 \end{pmatrix} \quad (3-5)$$

where L_s is the length of the sextupole and $\mathbf{M}$ is a symplectic matrix describing a linear element. Introducing the normal coordinates for the linear part of the mapping (3-5) according to:

$$\begin{cases} x = \sqrt{\beta} X \\ p = \gamma X + \frac{1}{\sqrt{\beta}} P \end{cases} \quad (3-6)$$

we recover the standard form of the quadratic map (3-2) performing the scaling:

$$z = \frac{K_2}{2} L_s \beta^{\frac{3}{2}} (X + iP) \quad (3-7)$$

Using the formulas (3-3) and (3-4) the tune-shift and the smear now read:

$$\delta\nu = -\frac{1}{32\pi} \left[3 \cot \frac{\nu}{2} + \cot \frac{3\nu}{2} \right] \beta^3 \left(\frac{K_2}{2} \right)^2 L_s^2 \epsilon \quad (3-8)$$

$$\sigma = \frac{1}{16} \left[2 \cot^2 \frac{\nu}{2} + 2 \cot^2 \frac{3\nu}{2} + 20 \right]^{\frac{1}{2}} \frac{K_2}{2} L_s \beta^{\frac{3}{2}} \sqrt{\epsilon} \quad (3-9)$$

where $\epsilon = \sqrt{X^2 + P^2}$ is the emittance.

We can also write an analytical expression for the dynamic aperture of our simple cell:

$$A = \frac{H(2\pi\nu)}{\frac{K_2}{2} L_s \beta} \quad (3-10)$$

where $H(\alpha)$ is the dynamical aperture of the standard quadratic map (see figure 4).

A simple result we can obtain immediately from this model is the continuum limit. Letting L be the equivalent length of the linear part of the cell:

$$2\pi\nu = \frac{L}{\beta}$$

then the map (3-5) reads:

$$\begin{pmatrix} X' \\ P' \end{pmatrix} = \mathbf{R} \left(\frac{L}{\beta} \right) \begin{pmatrix} X \\ P + \frac{K_2}{2} \frac{L_s}{L} \beta^{\frac{3}{2}} X^2 L \end{pmatrix} \quad (3-11)$$

In the limit $L \rightarrow 0$ with $\frac{L}{L_s}$ fixed, the orbits of the map (3-11) become the solutions of the Hamiltonian system:

$$H = \frac{1}{2\beta} (X^2 + P^2) - \frac{K_2}{3!} \beta^{\frac{3}{2}} \frac{L_s}{L} X^3 \quad (3-12)$$

We can observe that in the continuum limit the tune of the cell goes to zero as the length of the cell itself. The quantity which has a finite limit is the hamiltonian frequency $\omega = \frac{2\pi\nu}{L} \rightarrow \frac{1}{\beta}$. So we can calculate the frequency shift due to the non linearity:

$$\delta\omega = \lim_{L \rightarrow 0} 2\pi \frac{\delta\nu}{L} = -\frac{5}{12} \beta^4 \left(\frac{K_2}{2} \right)^2 \left(\frac{L_s}{L} \right)^2 \epsilon \quad (3-13)$$

This is precisely the result given by the canonical perturbation theory on the Hamiltonian (3-12). Then the tune shift over a superperiod will be given by:

$$\delta\nu = -\frac{5}{12} \frac{\beta^3}{(2\pi)^2 \nu} \left(\frac{K_2}{2} \right)^2 \left(\frac{L_s l}{L} \right)^2 \epsilon \quad (3-14)$$

where l is the length of the superperiod.

A similar result can be obtained for the smear with:

$$\sigma = \frac{1}{6} \sqrt{\frac{5}{6}} \beta^{\frac{5}{2}} \frac{K_2}{2} \frac{L_s}{L} \epsilon \quad (3-15)$$

For the dynamic aperture, observing that $\lim_{\alpha \rightarrow 0} \frac{H(\alpha)}{\alpha} = 1$ we recover the continuum limit:

$$A = \frac{1}{\frac{K_2}{2} \beta^2} \frac{L}{L_s}$$

as it is easy to check by computing the coordinates of the hyperbolic critical point of the Hamiltonian (3-12).

4. Analysis of LHC model

We have considered an one dimensional model for the LHC superperiod lattice with $27 + \frac{1}{2}$ FODO plus a linear element. The tune of a superperiod is fixed at $\nu = 7.785$ and the initial value of β is 169.5 m.

The physical aperture is 2 cm. The FODO cell used has the above schematic representation.

The data for each magnetic element are given in the table 1.

We have analysed three different situations for the superperiod lattice:

- 1) only systematic sextupoles
- 2) systematic and chromatic sextupole
- 3) all the multipoles up to decapoles.

For each case we have computed: the tracking with the map $\mathbf{T}$, the tracking with the truncated map $[\mathbf{T}]_N$ for $N = 5, 10, 15$ and the normal form at the same perturbative orders $N = 5, 10, 15$.

The transfer map for each magnetic element has been calculated using the kick approximation with 10 kicks.

The quantitative results for the three different situations are summarized in the figures 5, 6, 7, 8, 9, 10 and the tables 2, 3, 4, 5. The tracking with 1000 iterations gives a dynamic aperture respectively of 4 cm, 5.6 cm, 3.6 cm.

Let us consider the cases 1) and 2).

In the tables 2, 3 we compare the tune shift and the smear obtained with the tracking and the normal form of order 10. The error of the normal form dynamic, defined as the variance of the sequence of numbers:

$$|\Phi^{-1}(z_n, z_n^*)| \quad z_n = \mathbf{T}^n z \quad (4-1)$$

is also reported. Indeed we recall that $|\Phi^{-1}(z, z^*)|$ is a first integral of motion for the orbit as it is easy to see from equation (2-2).

The figures 5, 6, 7, 8 show the plot of the tune shift and the smear as functions of the distance x from the origin calculated by means of the normal forms at various perturbative orders. For both situations a very good agreement between the normal form and the tracking is observed within the physical aperture. In the case 2) the non linear effects are considerably smaller because of the presence of the chromatic sextupoles corrections.

For the last situation which includes octupoles and decapoles, we report in the tables 4, 5 the smear and the tune shift for all the computed perturbative orders. For $x \geq 2$ cm we observe a minimum in the error when we increase the perturbative order, due to the asymptotic nature of the normal form series.

The figures 9, 10 show the variation with x of the tune shift and the smear. Because of the presence of higher multipoles, we stress that in this case the first perturbative orders are not significant.

Within the physical aperture the agreement with the tracking results is excellent; moreover for the tune shift a significant agreement at perturbative orders $N = 7, 9$, which present the smallest error, is also observed until the dynamic aperture is reached.

In the figures 11, 12, 13 we compare the orbits of the map $\mathbf{T}$, its truncations $[\mathbf{T}]_N$ at various perturbative orders and the normal form $\Phi e^{i\Omega} \Phi^{-1}$ at order 15 in the analytic expansions.

The truncation effect is evident: as we move away from the origin the orbits become thick and lose their stability so that no real dynamic information can be extracted from this map. Of course when we increase the truncation order, this effect appears farther from the origin if the number of points of the orbit is kept fixed. For a large number of iterations $\sim 10^8 - 10^9$ the region where the map behaves as a symplectic one becomes narrower.

The normal form conversely allows to recover a symplectic map from $[\mathbf{T}]_N$ as shown by the first plot of figure 11 and its break up is due only to the presence of resonances closed to the linear tune. Such resonances lead to orbits which are no longer diffeomorphic to a circle but exhibit loops whose number is equal to the order of the resonance.

In the region where such resonances effect are not present, the normal form results hold for a very large number of iterations typically $n \sim 10^9$, as we derive from some numerical evidences [15].

Anyway, to have rigorous relations among the perturbative order of the normal form, the region where we can apply the normal form dynamic and the error from the real motion is up to now an open problem.

The last plot of figure 13 refers to an equivalent map of the form:

$$\begin{pmatrix} X' \\ P' \end{pmatrix} = \mathbf{R}(2\pi\nu) \begin{pmatrix} X \\ P + \lambda_2 X^2 + \lambda_3 X^3 + \lambda_4 X^4 \end{pmatrix} \quad (4-2)$$

This is obtained after a suitable linear transformation on the map $[\mathbf{T}]_N$ which make the neglected coefficients as small as possible. This map is symplectic and its simple form allows to compute the dynamic aperture which is a reasonable approximation of the real one, if the neglected terms are really small.

In the present case the phase plots of $\mathbf{T}$ and of the equivalent map show a good agreement.

The physical meaning is that the full machine can in some cases be well described not by continuous mean machine but better by a periodic machine with a cell formed by a linear element plus a non linear one with sextupoles, octupoles and decapoles.

APPENDIX

We shall solve explicitly the second perturbative order of the equation (2-2) for the quadratic map:

$$z' = e^{-i\alpha} \left[z + \frac{i}{4}(z + z^*)^2 \right] \quad (1)$$

where α is a non resonant angle.

This will be a good exercise to show some important properties of the normal forms and to have an idea of the computational technique.

Let us represent the transformation $z = \Phi(\xi, \xi^*)$ which brings (1) into normal form by means of a complex generating function:

$$F(\xi, z^*) = \xi z^* + \sum_{n \geq 2} F_n(\xi, z^*) \quad (2)$$

so that Φ is implicitly given by the equations:

$$\begin{cases} z = \xi + \sum_{n \geq 2} \frac{\partial F_n}{\partial z^*}(\xi, z^*) \\ z^* = \xi^* - \sum_{n \geq 2} \frac{\partial F_n}{\partial \xi}(\xi, z^*) \end{cases} \quad (3)$$

Yet the transformation Φ is a real symplectic transformation in complex variables; this implies some constraints for the generating function F which assure that the equations (3) and their complex conjugate represent the same map.

If we explicit this constraints at the first perturbative order we obtain the following equation for F_3 :

$$F_3(\xi, z^*) + F_3^*(z^*, \xi) = 0 \quad (4)$$

Let us substitute the equations (3) in the functional equation:

$$\mathbf{M} \circ \Phi(\xi, \xi^*) = \Phi \circ \xi e^{-i\Omega}(\xi \xi^*) \quad (5)$$

and retain only the second order in ξ, z^* :

$$\frac{\partial F_3}{\partial z^*}(e^{-i\alpha}\xi, e^{i\alpha}z^*) - e^{-i\alpha}\frac{\partial F_3}{\partial z^*}(\xi, z^*) = \frac{i}{4}(\xi + z^*)^2 \quad (6)$$

Then it is not difficult to compute:

$$\frac{\partial F_3}{\partial z^*}(\xi, z^*) = \frac{i}{4}\frac{1}{e^{-i\alpha}-1}\xi^2 + \frac{i}{2}\frac{1}{e^{i\alpha}-1}\xi z^* + \frac{i}{4}\frac{1}{e^{3i\alpha}-1}z^{*2} \quad (7)$$

which is consistent with the constraint (4) as one can verify very easily. Then the second order of the transformation Φ reads:

$$\Phi(\xi, \xi^*) = \frac{i}{4}\frac{1}{e^{-i\alpha}-1}\xi^2 + \frac{i}{2}\frac{1}{e^{i\alpha}-1}\xi\xi^* + \frac{i}{4}\frac{1}{e^{3i\alpha}-1}\xi^{*2} \quad (8)$$

Now if we iterate the quadratic map retaining only the second order terms we have the map:

$$[\mathbf{M}^n]_2 = e^{-in\alpha} \left\{ z + \frac{i}{4} \left[\frac{e^{-in\alpha}-1}{e^{-i\alpha}-1} z^2 + \frac{e^{in\alpha}-1}{e^{i\alpha}-1} 2zz^* + \frac{e^{3in\alpha}-1}{e^{3i\alpha}-1} z^{*2} \right] \right\}$$

which is no longer a symplectic map but only a truncation of a symplectic map. Then we cannot obtain any dynamical information on the true map using directly such a map; anyway if we perform the normal form reduction we recover the same transformation Φ as in the equation (5). Indeed the following equation holds:

$$\mathbf{M}^n = \Phi \circ \xi e^{in\Omega}(\xi\xi^*) \circ \Phi^{-1} \quad (9)$$

This means that not only the normal form reduction allows to recover a symplectic map from a truncation but also the normal form error in determining the position of an orbit is independent from the number of iterations we are considering. What depends on the

number of iterations is the motion on the orbit itself i. e. the error is in the phase $e^{-in\Omega}$ but this is irrelevant if we are concerning with the stability of the motion.

Higher orders calculation involves more technical difficulties ^[15,16] due to the necessity of composing two polynomial maps. This problem can be solved developing algorithms which compute numerically the coefficients of the composition of two polynomials at every order; in this way we don't need the help of an algebraic manipulator.

Acknowledgements

This work was done at the SPS division of CERN and we wish express our gratitude for the opportunity to work there. We would like to thank W. Scandale for many helpful discussions and suggestions.

REFERENCES

- 1) V. I. Arnold "Méthodes mathématiques de la mécanique classique"- Ed. MIR (1976).
- 2) G. D. Birkhoff "Dynamical systems" Ed. Am. Math. Soc. (1927).
- 3) A. Dragt "Nonlinear dynamics and the beam-beam interaction" Ed. J. C. Herrera Am. Inst. of Phys. (1979).
- 4) A. J. Dragt, J. M. Finn, J. Math. Phys. 17, 12 (1976) - 17, 2215 (1976).
- 5) E. Courant, H. Snyder, Ann. Phys. (N. Y.) 3, 1 (1958).
- 6) G. D. Birkhoff, Acta Mathematica 43, 1 (1927).
- 7) G. Turchetti "Perturbative methods for hamiltonian maps" Lectures given at the Medellin Summer School 1986. Ed. A. W. Saenz World Scientific in press.
- 8) A. Bazzani "Teoria perturbativa per mappe hamiltoniane" Tesi della Università di Padova (1987).
- 9) E. Courant, R. Ruth, W. Weng, SLAC Pub 3415 (1984).
- 10) M. Henon, C. Heiles Astron. J. , 69, 73 (1964).
- 11) G. Servizi, G. Turchetti, Computer Physics Comm. 32, 201 (1984).
- 12) "The large hadron collider in the Lep tunnel", CERN 87-05 (1987).
- 13) L. M. Healy LEP 585 (1987).
- 14) J. Roels, M. Henon Bull. Astr. Soc. , 2, 267 (1967).
- 15) G. Servizi, G. Turchetti, Nuovo Cimento 95B, 121 (1986).
- 16) A. Bazzani "Normal forms for symplectic maps in $\mathbf{R}^{2n}$ " - submitted to Celestial Mechanics (1987).

TABLE 1

Typical LHC periods

Cell length=99.9592 m
Phase advance= $\frac{\pi}{2}$
Number of quadrupoles=2
Effective quadrupoles length=3.08 m
Number of dipoles=8
Effective dipoles length=9.54 m
Number of sextupoles=2
Effective sextupoles length=.3 m
Maximun beta value=169.5 m
Minimun beta value=29.5 m
Bending radius=2668.5 m

LHC magnets

Maximun dipole field=10 T
 ω^2 focusing quadrupole= $9.3818 \cdot 10^{-3} \text{m}^{-2}$
 ω^2 defocusing quadrupole= $-9.3803 \cdot 10^{-3} \text{m}^{-2}$
systematic sextupole component of dipole= $-2.4 \cdot 10^{-3} \text{m}^{-3}$
systematic octupole component of dipole= $-.01 \text{m}^{-4}$
systematic sextupole component of dipole= 44.8m^{-5}
focusing sextupole $k_2 = .099 \text{m}^{-3}$
defocusing sextupole $k_2 = .049 \text{m}^{-3}$

Table 2

<i>Only systematic sextupoles</i>	<i>Tune shift</i>	<i>Smear</i>	<i>Normal form error</i>	<i>X</i>
<i>tracking</i>	- 0.46626E-03	0.22020E-02		
<i>normal form order 10</i>	- 0.46429E-03	0.22020E-02	0.21622E-11	x = .4 cm
<i>tracking</i>	- 0.18406E-02	0.53019E-02		
<i>normal form order 10</i>	- 0.18390E-02	0.53043E-02	0.81397E-08	x = .8 cm
<i>tracking</i>	- 0.40738E-02	0.95627E-02		
<i>normal form order 10</i>	- 0.40754E-02	0.95567E-02	0.11135E-05	x = 1.2 cm
<i>tracking</i>	- 0.70960E-02	0.15296E-01		
<i>normal form order 10</i>	- 0.70913E-02	0.15136E-01	0.41966E-04	x = 1.6 cm
<i>tracking</i>	- 0.10787E-01	0.27039E-01		
<i>normal form order 10</i>	- 0.10704E-01	0.23599E-01	0.88827E-03	x = 2.0 cm
<i>tracking</i>	- 0.16630E-01	0.69570E-01		
<i>normal form order 10</i>	- 0.14392E-01	0.38708E-01	0.27191E-01	x = 2.4 cm
<i>tracking</i>	- 0.22736E-01	0.44847E-01		
<i>normal form order 10</i>	- 0.17028E-01	0.57148E-01	0.41193E-01	x = 2.8 cm
<i>tracking</i>	- 0.29097E-01	0.44907E-01		
<i>normal form order 10</i>	- 0.19726E-01	0.85730E-01	0.82324E-01	x = 3.2 cm

Table 3

<i>All sextupoles</i>	<i>Tune shift</i>	<i>Smear</i>	<i>Normal form error</i>	<i>X</i>
<i>tracking</i>	- 0.15976E-03	0.19916E-02		
<i>normal form of order 10</i>	- 0.15967E-03	0.19908E-02	0.20777E-12	x = .4 cm
<i>tracking</i>	- 0.64027E-03	0.40602E-02		
<i>normal form of order 10</i>	- 0.63669E-03	0.40618E-02	0.24726E-10	x = .8 cm
<i>tracking</i>	- 0.14309E-02	0.62806E-02		
<i>normal form of order 10</i>	- 0.14260E-02	0.62776E-02	0.29438E-08	x = 1.2 cm
<i>tracking</i>	- 0.25274E-02	0.86866E-02		
<i>normal form of order 10</i>	- 0.25198E-02	0.86796E-02	0.91931E-07	x = 1.6 cm
<i>tracking</i>	- 0.39111E-02	0.11321E-01		
<i>normal form of order 10</i>	- 0.39073E-02	0.11301E-01	0.13930E-05	x = 2.0 cm
<i>tracking</i>	- 0.55856E-02	0.14239E-01		
<i>normal form of order 10</i>	- 0.55738E-02	0.14206E-01	0.13522E-04	x = 2.4 cm
<i>tracking</i>	- 0.75170E-02	0.17771E-01		
<i>normal form of order 10</i>	- 0.74954E-02	0.17582E-01	0.98612E-04	x = 2.8 cm
<i>tracking</i>	- 0.96682E-02	0.23046E-01		
<i>normal form of order 10</i>	- 0.96286E-02	0.21884E-01	0.60627E-03	x = 3.2 cm
<i>tracking</i>	- 0.12138E-01	0.36446E-01		
<i>normal form of order 10</i>	- 0.11888E-01	0.27957E-01	0.36001E-02	x = 3.6 cm
<i>tracking</i>	- 0.15160E-01	0.91576E-01		
<i>normal form of order 10</i>	- 0.14111E-01	0.36638E-01	0.44957E-01	x = 4.0 cm
<i>tracking</i>	- 0.19761E-01	0.48284E-01		
<i>normal form of order 10</i>	- 0.16021E-01	0.47255E-01	0.47327E-01	x = 4.4 cm
<i>tracking</i>	- 0.22960E-01	0.42999E-01		
<i>normal form of order 10</i>	- 0.17272E-01	0.56206E-01	0.64411E-01	x = 4.8 cm
<i>tracking</i>	- 0.26578E-01	0.42694E-01		
<i>normal form of order 10</i>	- 0.17831E-01	0.60798E-01	0.94408E-01	x = 5.2 cm

Table 4 - Full LHC

	<i>Tune shift</i>	<i>Smear</i>	<i>Normal form error</i>	<i>Perturbative order</i>
<i>Tracking results</i>	0.47019E-05	0.20068E-02		
<i>Normal form results</i> <i>x = .4 cm</i>	0.00000E+00	0.19764E-02	0.18565E-04	2
	-0.36647E-05	0.20294E-02	0.71565E-06	3
	-0.36646E-05	0.20080E-02	0.13347E-06	4
	0.47930E-05	0.20072E-02	0.20173E-07	5
	0.47931E-05	0.20073E-02	0.55268E-08	6
	0.47209E-05	0.20073E-02	0.33654E-09	7
	0.47209E-05	0.20073E-02	0.75643E-10	8
	0.47208E-05	0.20073E-02	0.10265E-10	9
	0.47208E-05	0.20073E-02	0.12840E-11	10
	0.47208E-05	0.20073E-02	0.30786E-12	11
	0.47208E-05	0.20073E-02	0.20704E-12	12
	0.47208E-05	0.20073E-02	0.20573E-12	13
	0.47208E-05	0.20073E-02	0.20564E-12	14
	0.47208E-05	0.20073E-02	0.20565E-12	15
<i>Tracking results</i>	0.11365E-03	0.41824E-02		
<i>Normal form results</i> <i>x = .8 cm</i>	0.00000E+00	0.39510E-02	0.14588E-03	2
	-0.14593E-04	0.43559E-02	0.11686E-04	3
	-0.14590E-04	0.41986E-02	0.39145E-05	4
	0.11951E-03	0.41766E-02	0.13448E-05	5
	0.11953E-03	0.41776E-02	0.67554E-06	6
	0.11496E-03	0.41810E-02	0.91798E-07	7
	0.11496E-03	0.41808E-02	0.36977E-07	8
	0.11495E-03	0.41807E-02	0.98874E-08	9
	0.11495E-03	0.41807E-02	0.25533E-08	10
	0.11495E-03	0.41807E-02	0.93661E-09	11
	0.11495E-03	0.41807E-02	0.18226E-09	12
	0.11495E-03	0.41807E-02	0.87596E-10	13
	0.11495E-03	0.41807E-02	0.25458E-10	14
	0.11495E-03	0.41807E-02	0.15429E-10	15
<i>Tracking results</i>	0.58247E-03	0.66433E-02		
<i>Normal form results</i> <i>x = 1.2 cm</i>	0.00000E+00	0.59239E-02	0.48254E-03	2
	-0.32605E-04	0.72025E-02	0.60034E-04	3
	-0.32586E-04	0.67359E-02	0.25935E-04	4
	0.63728E-03	0.65849E-02	0.15869E-04	5
	0.63780E-03	0.65922E-02	0.10811E-04	6
	0.58655E-03	0.66442E-02	0.25034E-05	7
	0.58649E-03	0.66412E-02	0.13178E-05	8
	0.58634E-03	0.66347E-02	0.51247E-06	9
	0.58633E-03	0.66355E-02	0.21980E-06	10
	0.58628E-03	0.66366E-02	0.12141E-06	11
	0.58628E-03	0.66365E-02	0.34433E-07	12
	0.58633E-03	0.66363E-02	0.21119E-07	13
	0.58633E-03	0.66364E-02	0.12236E-07	14
	0.58632E-03	0.66364E-02	0.93017E-08	15

Table 5 - Full LHC.

	<i>Tune shift</i>	<i>Smear</i>	<i>Normal form error</i>	<i>Perturbative order</i>
<i>Tracking results</i>	0.17445E-02	0.94891E-02		
<i>Normal form results $x = 1.6$ cm</i>	0.00000E+00	0.78951E-02	0.11317E-02	2
	-0.57421E-04	0.10694E-01	0.18970E-03	3
	-0.57339E-04	0.97517E-02	0.92169E-04	4
	0.20246E-02	0.91890E-02	0.90482E-04	5
	0.20315E-02	0.92197E-02	0.75380E-04	6
	0.17465E-02	0.95643E-02	0.25996E-04	7
	0.17452E-02	0.95424E-02	0.15686E-04	8
	0.17446E-02	0.94663E-02	0.79065E-05	9
	0.17444E-02	0.94734E-02	0.52265E-05	10
	0.17432E-02	0.94975E-02	0.37059E-05	11
	0.17432E-02	0.94963E-02	0.13209E-05	12
	0.17447E-02	0.94889E-02	0.91480E-06	13
	0.17448E-02	0.94894E-02	0.97490E-06	14
	0.17443E-02	0.94918E-02	0.83023E-06	15
<i>Tracking results</i>	0.38544E-02	0.13012E-01		
<i>Normal form results $x = 2.0$ cm</i>	0.00000E+00	0.98647E-02	0.22314E-02	2
	-0.88654E-04	0.14873E-01	0.46076E-03	3
	-0.88403E-04	0.13341E-01	0.23491E-03	4
	0.49006E-02	0.11842E-01	0.34679E-03	5
	0.49517E-02	0.11956E-01	0.33557E-03	6
	0.38636E-02	0.13398E-01	0.15477E-03	7
	0.38509E-02	0.13288E-01	0.10115E-03	8
	0.38567E-02	0.12798E-01	0.63666E-04	9
	0.38530E-02	0.12831E-01	0.60762E-04	10
	0.38385E-02	0.13071E-01	0.50065E-04	11
	0.38385E-02	0.13065E-01	0.20691E-04	12
	0.38610E-02	0.12951E-01	0.15853E-04	13
	0.38661E-02	0.12962E-01	0.28551E-04	14
	0.38525E-02	0.13019E-01	0.26066E-04	15
<i>Tracking results</i>	0.69162E-02	0.17858E-01		
<i>Normal form results $x = 2.4$ cm</i>	0.00000E+00	0.11833E-01	0.40631E-02	2
	-0.12582E-03	0.19730E-01	0.12986E-02	3
	-0.12521E-03	0.17565E-01	0.10081E-02	4
	0.10032E-01	0.14330E-01	0.13122E-02	5
	0.10296E-01	0.14744E-01	0.14292E-02	6
	0.69883E-02	0.19209E-01	0.10681E-02	7
	0.69108E-02	0.18762E-01	0.86086E-03	8
	0.69851E-02	0.16646E-01	0.84793E-03	9
	0.69536E-02	0.16759E-01	0.90386E-03	10
	0.68377E-02	0.18223E-01	0.88700E-03	11
	0.68390E-02	0.18193E-01	0.71998E-03	12
	0.70492E-02	0.17219E-01	0.71687E-03	13
	0.71447E-02	0.17396E-01	0.94385E-03	14
	0.69537E-02	0.18117E-01	0.91245E-03	15

Figure 1

Figure 2

Figure 3

DYNAMIC APERTURE – QUADRATIC MAP

Figure 4

Figure 5

Figure 6

Figure 7

SMEAR – LHC ALL SEXTUPOLES

Figure 8

Figure 9

Figure 10

PHASE PLOT - FULL LHC

Figure 11

PHASE PLOT — FULL LHC

Figure 12

PHASE PLOT – FULL LHC

Figure 13