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SUPERSTRING AND GRADED GRASSMANNIANS

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ABSTRACT

We introduce the infinite dimensional graded Grassmannian manifolds in terms of free field operators, and study their properties. We show the embedding of the graded $\text{Diff } S^1/S^1$ manifold in the graded Grassmannians and discuss some physical applications.

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At present, there are two attractive views of string theory, both based on holomorphic geometry. The first is the formulation of quantum string theory as integrable analytic geometry on the universal moduli space of Riemann surfaces [1]. The other is based on the concept of loop space and formulated as holomorphic vector bundle over the manifold $\text{Diff } S^1/S^1$ [2]. In both cases, there exist one-to-one embedding of the base manifold into the infinite dimensional Grassmannians [3-11]. There are various advantages of working with the Grassmannians mainly that most computations become algebraic as well as having the promise of providing a non-perturbative treatment for moduli spaces of all Riemann surfaces, including the infinite genus ones.

Recently, we have formulated closed string theory as Hermitian geometry on Grassmannians [11] and it is natural to generalize this approach to the closed superstring and the heterotic string. This is not a trivial exercise since the mathematical tools that were available in the bosonic case [3,4] are not fully developed in the supersymmetric one. Indeed as we shall see, the known cases [12,13] turn out to be not physically relevant.

In this letter we shall start to build the foundation of the generalized approach to the superstring by introducing the graded Grassmannians (denoted by SGr) in terms of Bogoliubov transformations on the first quantized Hilbert space of a Fermi-Bose system. This can also be equivalently written in terms of the supersymmetric ghost system. We study the properties of these manifolds and construct the Hamiltonian and the currents, and comment on the possible supersymmetric KP hierarchy. We give the embedding of the graded manifold $\text{super Diff } S^1/S^1$ into SGr and present a simple way of computing the curvature of the superholomorphic vector bundle based on this manifold.

The simplest way to obtain the graded Grassmannian manifolds SGr is to make use of free field operators. Following Ueno and Yamada [13], let A be the Clifford algebra over C with generators ψ_m^a, ψ_n^{a*} where a is a graded index ($a = 0, 1$). The indices m and n are taken to be integers or half-integers depending on whether they are in the Ramond sector (R) or in the Neveu-Schwarz (NS) sector.

We impose the following quantization conditions

$$\begin{aligned} [\psi_m^a, \psi_n^b] &= 0 \\ [\psi_m^{a*}, \psi_n^{b*}] &= 0 \\ [\psi_m^{a*}, \psi_n^b] &= \delta^{ab} \delta_{m+n,0} \end{aligned} \quad (1)$$

where the graded commutator is defined by

$$[\psi_m^a, \psi_n^b] = \psi_m^a \psi_n^b - (-1)^{ab} \psi_n^b \psi_m^a$$

Then, it is easy to see that the space spanned by one and the normal-ordered products $:\psi_m^a \psi_n^{*b}:$ has the Lie superalgebra structure which is the one-dimensional central extension of $GL(\infty|\infty)$, denoted by $\widetilde{GL}(\infty|\infty)$. Defining the vacuum by

$$\begin{aligned}\psi_n^a |0\rangle &= 0 & n > -\lambda^a \\ \psi_n^{a*} |0\rangle &= 0 & n > -\lambda^{a*}\end{aligned}\quad (2)$$

where λ^a and λ^{a*} are the weight factors of ψ_n^a and ψ_n^{a*} respectively. We can then form a subgroup of $\widetilde{GL}(\infty|\infty)$ defined by

$$G = \{g \in A \mid \exists g^{-1}; gVg^{-1} = V, gV^*g^{-1} = V^*\} \quad (3)$$

where V and V^* are the spaces

$$\begin{aligned}V &= \oplus \mathbb{C} \psi_n^a \\ V^* &= \oplus \mathbb{C} \psi_n^{a*}\end{aligned}\quad (4)$$

An element of g is represented by

$$g = \exp\left(-\sum_{m,n>0} \psi_{-m}^a T_{mn}^{ab} \psi_n^{b*}\right) \quad (5)$$

and it is easy to show that G indeed forms a group. We then identify the infinite dimensional Grassmann manifold SGr with $G|vac\rangle/GL(1)$.

Under the action of G the fields ψ_n^a and ψ_n^{a*} transform as

$$\begin{aligned}\psi_n'^a &= g^{-1} \psi_n^a g \\ \psi_n'^{a*} &= g^{-1} \psi_n^{a*} g\end{aligned}\quad (6)$$

and the new vacuum $|g\rangle = g|0\rangle$ will be annihilated by the states

$$\begin{aligned}\psi_n'^a |g\rangle &= 0 & n > -\lambda^a \\ \psi_n'^{a*} |g\rangle &= 0 & n > -\lambda^{a*}\end{aligned}\quad (7)$$

as follows from Eqs. (2) and (6). Thus the charges

$$\begin{aligned}Q_n^a &= \psi_n^a - \sum_{m>0} \psi_{-m}^b T_{mn}^{ba} & n > -\lambda^a \\ Q_n^{a*} &= \psi_n^{a*} + \sum_{m>0} T_{nm}^{ab} \psi_{-m}^{b*} & n > -\lambda^{a*}\end{aligned}\quad (8)$$

annihilate the new vacuum $|g\rangle$ as can be seen from Eqs. (5), (6) and (7). In terms of the complex variable z on the circle, we write

$$\begin{aligned}\psi^a(z) &= \sum_n \psi_n^a z^{-n-\lambda^a} \\ \psi^{a*}(z) &= \sum_n \psi_n^{a*} z^{-n-\lambda^{a*}} \\ T^{ab}(z, \omega) &= \sum_{m, n \geq 0} T_{mn}^{ab} z^{-m+\lambda^a-1} \omega^{-n+\lambda^{b*}-1}\end{aligned}\quad (9)$$

In this notation, the group elements g are expressed in terms of the contour integral

$$g = \exp\left(-\oint dz \oint d\omega \psi^a(z) T^{ab}(z, \omega) \psi^{b*}(\omega)\right) \quad (10)$$

while the conserved charges are

$$\begin{aligned}Q_n^a &= \oint dz \psi^b(z) W_n^{ba}(z) \\ Q_n^{a*} &= \oint dz W_n^{ab*}(z) \psi^{b*}(z)\end{aligned}\quad (11)$$

In Eq. (11) the functions W_n^{ab} and W_n^{ab*} are the Plucker co-ordinates of SGr and given by

$$\begin{aligned}W_n^{ab} &= z^{n+\lambda^a-1} \delta^{ab} - \sum_{m \geq 0} T_{mn}^{ab} z^{-m+\lambda^b-1} \\ W_n^{ab*} &= z^{n+\lambda^{a*}-1} \delta^{ab} + \sum_{m \geq 0} T_{nm}^{ab} z^{-m+\lambda^{b*}-1}\end{aligned}\quad (12)$$

However, since our main interest is in the superstring, we must find the correspondence with the superconformal treatment [14]. We introduce the superghost system

$$\begin{aligned}B(z, \theta) &= \beta(z) + \theta b(z) \\ C(z, \theta) &= c(z) + \theta \gamma(z)\end{aligned}\quad (13)$$

with superconformal weights $\hat{j}$ and $\frac{1}{2}-\hat{j}$ respectively. The mode expansions of the component fields are

$$\begin{aligned}b(z) &= \sum_{n \in \mathbb{Z}} b_n z^{-n-j} & c(z) &= \sum_{n \in \mathbb{Z}} c_n z^{-n-(1-j)} \\ \beta(z) &= \sum_{\alpha} \beta_{\alpha} z^{-\alpha-(1-\frac{1}{2})} & \gamma(z) &= \sum_{\alpha} \gamma_{\alpha} z^{-\alpha-(\frac{1}{2}-j)}\end{aligned}\quad (14)$$

where $\alpha \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and $\alpha \in \mathbb{Z}$ in the R sector, and $j = \hat{j} + \frac{1}{2}$. The quantization conditions are

$$\begin{aligned} \{b_n, c_m\} &= \delta_{n+m,0} \\ [\gamma_n, \beta_m] &= \delta_{n+m,0} \end{aligned} \quad (15)$$

with all other (anti) commutators vanishing. The vacuum is defined by

$$\begin{aligned} b_n |0\rangle &= 0 & n > -j & ; & c_n |0\rangle &= 0 & n \geq j \\ \beta_\alpha |0\rangle &= 0 & \alpha > \frac{1}{2} - j & ; & \gamma_\alpha |0\rangle &= 0 & \alpha \geq j - \frac{1}{2} \end{aligned} \quad (16)$$

For simplicity, we shall not deal with the equivalent charged vacuum which is present due to the bosons β and γ [14].

Let H be the Hilbert space of all complex square integrable superfields on S^1 , the superspace extension of the circle. In analogy with the bosonic case, we can polarize H by the action of B and C fields defined over this superspace. The graded extensions of the general linear group $GL(H)$ and the restricted unitary group $U_{\text{res}}(H)$ can be defined [4]. The subgroup $G(H)$ of $U_{\text{res}}(H)$ is now represented by

$$g = \exp \left(- \oint dz_1 d\theta_1 \oint dz_2 d\theta_2 B(z_1, \theta_1) T(z_1, \theta_1; z_2, \theta_2) C(z_2, \theta_2) \right) \quad (17)$$

where $T(z_1, \theta_1; z_2, \theta_2)$ is a smooth function on $S^1 \times S^1$. The two expressions for g in Eqs. (10) and (17) coincide if we identify

$$\begin{aligned} \gamma^a(z) &= \begin{bmatrix} \beta(z) \\ b(z) \end{bmatrix} & \gamma^{a*}(z) &= \begin{bmatrix} \gamma(z) \\ c(z) \end{bmatrix} \end{aligned} \quad (18)$$

$$\begin{aligned} T(z_1, \theta_1; z_2, \theta_2) &= T^{10}(z_1, z_2) + \theta_1 T^{00}(z_1, z_2) \\ &+ T^{11}(z_1, z_2) \theta_2 + \theta_1 \theta_2 T^{01}(z_1, z_2) \end{aligned}$$

with the weight assignments

$$\begin{aligned} \lambda^0 &= j - \frac{1}{2} & \lambda^{0*} &= \frac{3}{2} - j \\ \lambda^1 &= j & \lambda^{1*} &= 1 - j \end{aligned}$$

The conserved charges can now be expressed in terms of superspace integrals of the form

$$\begin{aligned} &\oint B(z, \varphi) w_n(z, \varphi, \theta) dz d\varphi \\ &\oint C(z, \varphi) w_n(z, \varphi, \theta) dz d\varphi \end{aligned}$$

where the w_n are super-Plucker co-ordinates.

A little thought reveals that the SGr we have defined is larger than what one would need since we know that the superconformal transformations in the (z, θ) space

are of a special type. What one would need is a restriction to reduce the SGr into a submanifold. We shall come back to this point later.

Since one of the main advantages of the Grassmannians is their intimate link with the KP hierarchy and the moduli space of Riemann surfaces, we must seek a similar relation in the graded case. To do this, we introduce the variables

$$X^{ab}(z) = \sum_{n \geq 0} X_n^{ab} z^{n+\lambda^a+\lambda^{b*}-1} \quad (19)$$

and define the Hamiltonian

$$H(X^{ab}) = \oint X^{ab}(z) j^{ab}(z) dz \quad (20)$$

where $j^{ab}(z) = :\psi^a(z)\psi^{b*}(z):$ are the currents. These currents form a non-trivial infinite dimensional graded algebra:

$$\begin{aligned} [j_n^{ab}, j_m^{cd}] = & (-1)^{ac+bd} \left((-1)^{bc} \delta^{bc} j_{n+m}^{ad} - (-1)^{ad} \delta^{ad} j_{n+m}^{cb} \right) \\ & + (-1)^{(a+1)(b+1)} (n+\lambda^a+\lambda^{b*}-1) \delta^{ad} \delta^{bc} j_{n+m,0} \end{aligned} \quad (21)$$

where

$$j_n^{ab}(z) = \sum_n j_n^{ab} z^{-n+\lambda^a+\lambda^{b*}-1}$$

The Hamiltonian defined in Eq. (20) annihilates the vacuum

$$H(X^{ab})|0\rangle = 0$$

and the action on the fields $\psi^a(z)$ and $\psi^{a*}(z)$ is:

$$\begin{aligned} e^{H(X)} \psi^a e^{-H(X)} &= e^{X^{ab}(z)} \psi^b(z) \\ e^{H(X)} \psi^{a*} e^{-H(X)} &= e^{-X^{ab}(z)} \psi^{b*}(z) \end{aligned} \quad (22)$$

which is clearly non-diagonal. This makes finding the vertex operators for ψ^a and ψ^{b*} a difficult task. The super-tau function is given by

$$\tau(X^{ab}, g) = \langle 0 | e^{H(X^{ab})} g | 0 \rangle \quad (23)$$

where the vacuum $\langle 0 |$ must be defined in such a way as to absorb any possible vacuum charge. The appearance of the variables X^{ab} suggests that the KP hierarchy equations [3]

$$\begin{aligned} L(x, \partial) w(z, x) &= z w(z, x) \\ \frac{\partial w}{\partial x_n} &= (L^n)_+ w \end{aligned}$$

where $L(x, \partial)$ is a pseudo-differential operator, $\partial = \partial/\partial x_1$, and $(L^n)_+$ is the differential part of L^n , must be modified by replacing L by a matrix L^{ab} and W by W^a . However, as we said before, we are more interested in obtaining the hierarchy connected with superconformal theories. From the Lagrangian of the superghost system, one easily finds that the supercurrent is

$$\dot{j}(z, \theta) = B(z, \theta) C(z, \theta) \quad (24)$$

With this we associate the Hamiltonian

$$H^{s.c.}(x_n, \lambda_n) = \oint dz d\theta X(z, \theta) j(z, \theta) \quad (25)$$

where

$$X(z, \theta) = \sum_{n \geq 0} x_n z^n + \theta \sum_{n \geq 0} \lambda_n z^{n-\frac{1}{2}} \quad (26)$$

The Hamiltonian in Eq. (20) reduces to that in Eq. (25) if we set

$$\begin{aligned} X_n^{00} &= X_n'' = x_n \\ X_n^{10} &= 0 \\ X_n^{01} &= \lambda_n \end{aligned} \quad (27)$$

The (x_n, λ_n) development of the B and C fields is now diagonal and given by

$$\begin{aligned} e^{H(x_n, \lambda_n)} B(z, \theta) e^{-H(x_n, \lambda_n)} &= e^{X(z, \theta)} B(z, \theta) \\ e^{H(x_n, \lambda_n)} C(z, \theta) e^{-H(x_n, \lambda_n)} &= e^{-X(z, \theta)} C(z, \theta) \end{aligned} \quad (28)$$

Thus, it remains to be seen what kind of SKP hierarchy is generated with the identifications in Eq. (27). However, we now know that the only deformation variables in the differential equations must be the x_n and λ_n , and not the full set x_n^{ab} . Such a kind of SKP hierarchy has been recently given by Manin and Radul [12], and one would have been tempted to take it as the hierarchy we are seeking. However, by studying the algebra of the currents in the Hamiltonian flow in that example, one discovers that it can be obtained from Eq. (20), after setting

$$\begin{aligned} X_n^{00} &= X_n'' = x_n & n \geq 1 \\ X_n^{10} &= X_n^{01} = \lambda_n & n \geq 1 \end{aligned} \quad (29)$$

The generators of the Manin-Radul algebra are $\partial/\partial x$, $\Theta_{2\ell}$, Θ and $\Theta_{2\ell-1}$ and related to the current j_n^{ab} by:

$$\begin{aligned}
 \frac{\partial}{\partial x} &= J_2^{00} + J_2^{11} \\
 \Theta &= J_1^{10} - J_1^{01} \\
 \Theta_{2\ell} &= J_{2\ell}^{00} + J_{2\ell}^{11} \\
 \Theta_{2\ell-1} &= J_{2\ell-1}^{10} + J_{2\ell-1}^{01}
 \end{aligned} \tag{30}$$

On the other hand, the currents appearing in Eq. (24) are given by different, inequivalent combinations of j_n^{ab} . Thus, unfortunately it is not related to the SKP hierarchy of Manin and Radul.

As a side remark, we mention that some information about the function $T(z_1, \theta_1; z_2, \theta_2)$ (or equivalently the matrices T_{mn}^{ab}) that comes from embedding the supermoduli space into the Grassmannian SGr (such embedding would certainly exist because of the generality of the SGr we defined). To do this we calculate the two-superpoint function for the superghosts

$$G(z_1, \theta_1; z_2, \theta_2) = \langle 0 | B(z_1, \theta_1) C(z_2, \theta_2) g | 0 \rangle \tag{31}$$

which can be calculated using Eqs. (16) and (17) to obtain:

$$G(z_1, \theta_1; z_2, \theta_2) = \frac{\theta_1 - \theta_2}{z_{12}} - T(z_2, \theta_2; z_1, \theta_1) \tag{32}$$

where $z_{12} = z_1 - z_2 - \theta_1 \theta_2$. On the other hand, the two-superpoint function for the superghost on the supertorus was recently calculated [15]

$$\begin{aligned}
 G(z_1, \theta_1; z_2, \theta_2 | \tau, \alpha) \\
 = -4D \left[\frac{(\text{Im } z_{12})^2}{2 \text{Im } \tau'} - \frac{1}{4\pi} \ln \left| \frac{\theta_1(z_{12} | \tau')}{\eta(\tau')} \right|^2 \right]
 \end{aligned} \tag{33}$$

where $\tau' = \tau + (\theta_1 + \theta_2)\alpha$. Comparing Eqs. (32) and (33) yields the function T corresponding to a supertorus.

We now address the question of the embedding of the graded manifold super Diff S^1/S^1 into SGr(H). In the bosonic case this is achieved via the smooth composite map [3]

$$\text{Diff } S^1 \rightarrow \mathcal{U}_{\text{res}}(H) \rightarrow \text{Gr}_*(H) = \frac{\mathcal{U}_{\text{res}}(H)}{\mathcal{U}(H_+) \times \mathcal{U}(H_-)} \tag{34}$$

Explicitly the map is

$$f \cdot h(z) = h(g(z)) \left| \frac{dg}{dz} \right|^j \quad (35)$$

where $f: S^1 \rightarrow S^1$, $h: S^1 \rightarrow \mathbb{C}$ and g is the inverse of f . The generalization is

$$f \cdot h(z, \theta) = h(g(z, \theta), \chi(z, \theta)) \left| \frac{\partial v}{\partial z} \right|^{\hat{j}} \quad (36)$$

where $f: S^1 \rightarrow S^1$, $h: S^1 \rightarrow \mathbb{C}$ and $v = g + \theta x$ is the inverse of f , while $\hat{j}$ is the superconformal weight of h . Here we shall not attempt to fully characterize the embedded manifold in terms of $SGr(H)$. The infinitesimal action of super Diff S^1 on h is then [14]:

$$\delta h = \left(\delta z \frac{\partial}{\partial z} + \delta \theta \frac{\partial}{\partial \theta} + \hat{j} \frac{\partial \epsilon}{\partial z} \right) h \quad (37)$$

where

$$\begin{aligned} \delta z &= \epsilon_0 + \theta \epsilon_1 \\ \delta \theta &= \epsilon_1 + \frac{1}{2} \theta \frac{d\epsilon_0}{dz} \\ \epsilon &= \delta z + \theta \delta \theta \end{aligned} \quad (38)$$

To find the Fourier representation of the structure constants of super Diff S^1 , we expand

$$\begin{aligned} h(z, \theta) &= h^0(z) + 2\theta h^1(z) \\ h^0(z) &= \sum_n h_n z^{-n} \quad ; \quad h^1(z) = \sum_n h_n z^{-n} \\ \epsilon^0(z) &= \sum_n \epsilon_n z^n \quad ; \quad \epsilon^1(z) = \sum_n \epsilon_n z^n \end{aligned} \quad (39)$$

Then we can write

$$\delta h_A = \epsilon^B f_{BA}^C(\hat{j}) h_C \quad (40)$$

where A, B, C are graded indices of the form (m, α) and

$$\begin{aligned} f_{mn}^{C(\hat{j})} &= [(\hat{j} - \frac{1}{2})m - n] \delta_{m+n}^P \delta_P^C \\ f_{m\beta}^{C(\hat{j})} &= [(\hat{j} - 1)m - \beta] \delta_{m+\beta}^\gamma \delta_\gamma^C \\ f_{\alpha n}^{C(\hat{j})} &= [(\hat{j} - \frac{1}{2})\alpha - \frac{1}{2}n] \delta_{\alpha+n}^\gamma \delta_\gamma^C \\ f_{\alpha\beta}^{C(\hat{j})} &= [2] \delta_{\alpha+\beta}^P \delta_P^C \end{aligned} \quad (41)$$

Note that for $\hat{j} = 3/2$, $f_{AB}^C (3/2)$ are the structure constants for super Diff S^1 in the adjoint representation and have the appropriate symmetry.

The generators for super Diff S^1 can thus be represented by $(L_A)_B^C = f_{AB}^{C(j)}$ when it acts on fields with weight $\hat{j}$. With this at hand, we can calculate the curvature of the line bundle whose base space is super Diff S^1/S^1 and whose fibres belong to the Fock space of a superfield with spin $\hat{j}$, as the pull-back of the bundle whose base space is the image of super Diff S^1/S^1 in SGr, with the same fibres. Since the latter is a superKähler manifold, we can write for the curvature

$$\omega(L_A, L_B) = \text{Str} (L_A L_B - (-1)^{AB} L_B L_A) \quad (42)$$

where the supertrace, Str, must be performed. The above assignment is only valid for Grassmann manifolds and is one of the advantages of working with this formalism since such calculations are purely algebraic. Using the representation of L_A , Eq. (42) can be rewritten as

$$\omega(L_A, L_B) = \sum_{\substack{C \leq 0 \\ D > 0}} (-1)^D [f_{AC}^{D(j)} f_{BD}^{C(j)} - (-1)^{AB} f_{BC}^{D(j)} f_{AD}^{C(j)}] \quad (43)$$

Substituting Eq. (41) into Eq. (43), and taking care that the indices α can either be integers (in the R sector) or $\frac{1}{2}$ integers (in the NS sector), we obtain after a straightforward calculation

$$\omega(L_A, L_B)^R = \begin{bmatrix} -(\hat{j} - \frac{1}{4}) m^3 \delta_{m+n,0} & 0 \\ 0 & -4(\hat{j} - \frac{1}{4}) \alpha^2 \delta_{\alpha+\beta,0} \end{bmatrix} \quad (44)$$

$$\omega(L_A, L_B)^{N.S} = \begin{bmatrix} [-(\hat{j} - \frac{1}{4}) m^3 + \frac{m}{4}] \delta_{m+n,0} & 0 \\ 0 & [-4(\hat{j} - \frac{1}{4}) \alpha^2 + \frac{1}{4}] \delta_{\alpha+\beta,0} \end{bmatrix} \quad (45)$$

The simplicity of the above computation should be contrasted with the complicated one in Ref. [16] where the same answer is obtained for the case $\hat{j} = 3/2$.

If we have instead a scalar superfield $X^\mu(z, \theta)$ on the Fock space of the vector bundle on super Diff S^1/S^1 , then the curvature of the line bundle can be calculated using the representation of the Virasoro algebra for the superstring field X^μ [10,11]

$$\omega(L_A, L_B) = [S(L_A), S(L_B)] - S([L_A, L_B]) \quad (46)$$

Computing the above quantity will give the central extension of the Virasoro algebra which is [14]

$$\omega(L_A, L_B) = \begin{bmatrix} [\frac{d}{2}(m^2-m) - 2am] \delta_{m+n,0}, & 0 \\ 0, & [\frac{d}{2}(\alpha^2 - \frac{1}{4}) - 2a] \delta_{\alpha+\beta,0} \end{bmatrix} \quad (47)$$

where the terms proportional to $2a$ arise by redefining the L_0 generator by [10]

$$L'_0 = L_0 + \frac{a}{2}$$

Obviously, by taking a product bundle of the superghost system with $\hat{j} = 3/2$ and the X^μ superfield with $d = 10$, then the curvature of the product bundle vanishes if $a = -5/8$ in the R sector and $a = 1/4$ in the NS sector.

We conclude by saying that many problems stay to be investigated, such as the generalizing of the Krichever maps from the supermoduli space to the graded Grassmannians and the true nature of the SKP hierarchy. This and related issues are under investigation.

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