ALL HODGE NUMBERS OF ALL
COMPLETE INTERSECTION CALABI-YAU MANIFOLDS

P.S. Green

Department of Mathematics
University of Maryland, College Park, MD 20742

T. Hübsch*

Department of Physics
University of Texas, Austin, TX 78712

and

C.A. Lütken

CERN - Geneva

ABSTRACT

We compute explicitly all the Hodge numbers for all Calabi-Yau manifolds realized as complete intersections of hypersurfaces in products of complex projective spaces. This determines the essential part of the matter superfield spectrum for the heterotic superstring compactified on any of these manifolds. We use various techniques presented in the recent literature to obtain 264 distinct Hodge diamonds, we exemplify these techniques, give all necessary details of our computation and append a suitable computer program.

* Supported by Robert A. Welch Foundation and NSF Grants : PHY 8503890 and PHY 8605978.

1 INTRODUCTION

The most extensively studied possibly realistic superstring-inspired model[1,2] involves compactification of the $E_8 \times E_8$ heterotic superstring[3,4,5,6] on a complex 3-dimensional manifold of $SU(3)$ holonomy constructed by G.Tian and S.-T.Yau[7]. In general, compact Kähler manifolds of complex dimension 3 with $SU(3)$ holonomy, so-called Calabi-Yau manifolds, are solutions to the compactification problem of the heterotic superstring[3,6] which yield an effective action in the 3+1-dimensional Minkowski space-time with unbroken $N = 1$ supergravity[4,5,6,8].

As a generalization of the few known examples[4,7,9] a large family of Calabi-Yau manifolds was established in Ref. [10,11] and soon studied in some detail[12,13,14,15]. All these manifolds are constructed as complete intersections of certain hypersurfaces in products of projective spaces and we shall adopt the acronym CICY[13,14] for them.

Upon compactification on *any* Calabi-Yau manifold, the spectrum of massless modes is derived[4,10] in terms of certain harmonic forms on the ‘internal’ Calabi-Yau manifold. The essential part of the massless spectrum is represented by :

$$\Phi_i^{27}(x) \cdot u^i, \quad u^i \in H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx H^{2,1}(\mathcal{M}), \quad (1.1)$$

$$\bar{\Phi}_{\bar{i}}^{27*}(x) \cdot v^{\bar{i}}, \quad v^{\bar{i}} \in H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^*) = H^{1,1}(\mathcal{M}), \quad (1.2)$$

here $H^p(\mathcal{M}, \mathcal{V})$ denotes the space of harmonic antiholomorphic p -forms valued in the bundle $\mathcal{V}$ over $\mathcal{M}$ and **27** and **27*** denote the corresponding representations of E_6 . Using the isomorphism indicated on the far right¹¹ we see that the spectrum of independent massless fields (corresponding to the number of harmonic u s and v s) is represented by :

$$b_{2,1} \cdot \mathbf{27} \oplus b_{1,1} \cdot \mathbf{27}^*. \quad (1.3)$$

Since each superfield transforming as **27** of E_6 can accommodate a generation of fermions of the Standard model[16], together with 12 exotic fermions and their 27 superpartners, there are $|b_{1,1} - b_{2,1}| = \frac{1}{2} |\chi_E|$ generations in (1.3) and $\min(b_{1,1}, b_{2,1})$ of **27** $\oplus$ **27*** mirror pairs among which the Higgs doublet(s) of the Glashow–Weinberg–Salam model are to be found. While it is straightforward, though tedious, to compute the Euler characteristic (χ_E) for any CICY, obtaining $b_{1,1}$ and/or $b_{2,1}$ separately is a much more delicate problem[11,12].

¹¹The equivalence in Eq. (1.1) is induced by the existence of a covariantly constant harmonic (3,0)-form and is valid for all Calabi-Yau manifolds.

Our main motivation here is to show that all the Hodge numbers can be (and now in fact are) determined for all CICYs. Since phenomenologically interesting Calabi-Yau manifolds ought to be multiply connected and it is believed that all CICYs are simply connected, after constructing an explicit CICY, one has to pass to a quotient, dividing out the action of some suitably chosen automorphisms of the CICY. However, the spectrum (1.3) is now determined for all CICYs, and thus serves as a starting point in this program.

It has been argued[13,19] that there are very few simple constructions predicting precisely three generations, and that they are either ill-defined or diffeomorphic to the first example in Ref. [7]. There is however no information on the number of more involved constructions such as in Ref. [20] and moreover, four-generation models seem to be relatively abundant. These questions clearly relate to an as yet non-existent complete classification of Calabi-Yau manifolds. While we have not attacked this problem in full generality, we feel that our exhibition of 264 distinct Hodge diamonds for the CICYs is a modest but possibly relevant step[21] in such a direction. That the classification of Calabi-Yau manifolds is not an issue of purely mathematical interest should be clear because the geometrical properties of candidate solutions to the string theory must play a role in determining its ground state.

Also, it has recently been claimed[22] that certain so-called *exactly soluble* 1+1-dimensional conformal field theories correspond to compactifications on Calabi-Yau manifolds, as exemplified by the quintic in $\mathbf{P}^4$. Knowing all the Hodge numbers for all the CICYs, one has more information to look for corresponding exactly soluble conformal field theories.

Complementing the iterative application of the Lefschetz Hyperplane Theorem (LHT hereafter)[17], and of classic results in the theory of complex surfaces[18], a technique for computing $b_{p,q}$ was recently introduced[12] which relies on the theory of exact and spectral sequences (TESS, for short). In addition to enumerating all possible Hodge diamonds for CICYs, we illustrate these methods (especially TESS) as well as an application of a technique introduced in Ref. [14], termed there ‘splitting’.

The paper is organized as follows : in section 2, we present our notation and review those computations which are independent of spectral sequences. We review exact sequences and sheaves in section 3 and discuss our application of these ideas and spectral sequences through TESS in section 4. The program of our computation is presented in section 5 together with the discussion of a possible obstruction to our strategy. We give our results in section 6 together with several observations and concluding remarks. Appendix A presents the detailed analysis of a generic case featuring the obstruction

to our general strategy of computation of $b_{2,1}$ and $b_{1,1}$, together with our resolution of the problem which applies to all such cases and therefore enables us to complete our computation. Finally, in appendix B we give, for the convenience of the reader, the interactive version of the computer code (in standard PASCAL) which we have used.

2 NOTATION AND PRELIMINARY COMPUTATIONS

Recall that all CICYs are defined[10,11] to be Calabi-Yau manifolds embedded as complete intersections of hypersurfaces X^a in products of complex projective spaces $\mathcal{W} = \mathbf{P}_1^{n_1} \times \cdots \times \mathbf{P}_m^{n_m}$. Each hypersurface X^a is defined as the zero locus of a suitably chosen homogeneous holomorphic polynomial :

$$X^a : \xi^a(z_r^\mu) = 0, \quad a = 1, \dots, K, \quad (2.1)$$

which is homogeneous of degree q_a^r with respect to the homogeneous coordinates of $\mathbf{P}_r^{n_r}$. The homogeneous polynomial ξ^a may also be thought of as a holomorphic section of the line bundle $\otimes_r (h_r)^{q_a^r}$, where h_r denotes the hyperplane bundle over $\mathbf{P}_r^{n_r}$, i.e. the holomorphic line bundle whose sections are homogeneous linear functions of the homogeneous coordinates of $\mathbf{P}_r^{n_r}$. We shall interpret the ξ^a in this manner without further comment throughout the remainder of the paper.

For any choice of q_a^r , the coefficients α can be chosen so that the space of solutions to the K simultaneous equations $\{\xi^a = 0, a = 1, \dots, K\}$ is indeed a manifold $(X^1 \cap \dots \cap X^K = \mathcal{M} \subset \mathcal{W})$ [11]. Moreover, for $\mathcal{M}$ to be a Calabi-Yau manifold, we must ensure that the first Chern class of $X^1 \cap \dots \cap X^K$ vanishes. This leads to the condition :

$$\sum_{a=1}^K q_a^r = n_r + 1, \quad r = 1, \dots, m. \quad (2.2)$$

It is further required that the matrix $[q_a^r]$ cannot be brought to a block-diagonal form with more than one block. This eliminates the cases where $\mathcal{M}$ is obviously decomposable as a product and, as we shall see, actually ensures that $\mathcal{M}$ is indecomposable.

The $m \times (K+1)$ array $[n_r \| q_a^r]$ is called a *configuration matrix*; an equivalence class of configuration matrices under permutations of all rows, and all columns except the first one, is called a *configuration*. A configuration is represented by a diagram by drawing an open circle with n_r legs for every $\mathbf{P}_r^{n_r}$ a factor in $\mathcal{W}$, a dot for every constraint and then connecting q_a^r legs of the r^{th} circle to the a^{th} dot. Indecomposable manifolds are represented by connected diagrams. When discussing configurations, we shall have the corresponding diagram in mind as well. It was proven[11] that each connected configuration satisfying Eq. (2.2) represents a *partial deformation class* of CICYs.

One-dot-decomposable (1DD) diagrams have a dot by deletion of which the diagram becomes disconnected. Connected diagrams which decompose by deletion of a leg are called one-leg-decomposable (1LD); the set of all 1DD diagrams includes all the 1LD ones. The disjoint component of a decomposing leg which contains the dot (circle) incident to the decomposing leg is called its dot-complement (circle-complement, respectively). Decomposing legs are indexed by $\ell_d - m_d - c_d$, where ℓ_d , m_d and c_d are the number of legs, dots and circles in the dot-complement.

The Euler character for any CICY can be computed by evaluating the expression[10] (for a proof see Ref. [14]) :

$$\begin{aligned} \chi_E[\mathcal{M}] &= \left[\left(\sum_{r,s,t=1}^m c_3^{rst} x_r x_s x_t \right) \cdot \prod_{b=1}^K \left(\sum_{u=1}^m q_b^u x_u \right) \right]_{\text{coefficient of } \prod_{r=1}^m (x_r)^{n_r}}, \quad (2.3) \\ c_3^{rst} &= \frac{1}{3} \left((n_r + 1) \delta^{rst} - \sum_{a=1}^K q_a^r q_a^s q_a^t \right), \\ \delta^{rst} &= \begin{cases} 1 & \text{for } r = s = t, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is somewhat cumbersome to evaluate by hand but is well adapted for machine computation, so χ_E may be easily computed for every configuration.

In order to explain our first algorithm for computing $b_{1,1}$ we begin by stating the

Lefschetz Hyperplane Theorem (LHT) : *Let $\mathcal{L}$ be a positive holomorphic line bundle over an $n + 1$ -dimensional complex manifold $\mathcal{W}$, and λ be a holomorphic section of $\mathcal{L}$. Then :*

$$\pi_i(\mathcal{W}, \lambda^{-1}(0)) = 0, \quad \text{for } i \leq n.$$

for a proof of which we refer to Ref. [17]. As a consequence of LHT, the inclusion of $\lambda^{-1}(0)$ in $\mathcal{W}$ induces

$$b_i(\lambda^{-1}(0)) = b_i(\mathcal{W}), \quad \text{for } i \leq n, \quad \text{and } b_n(\lambda^{-1}(0)) \geq b_n(\mathcal{W}).$$

The detailed definition of positivity need not concern us here; the reader is referred to Ref. [18], p.148. The basic facts which we use are summarized in the following

Proposition 1

i) *The hyperplane bundle is positive.*

- ii) If $\mathcal{L}$ is a positive line bundle over X and Y is a holomorphic submanifold of X , then $\mathcal{L}|_Y$ is positive over Y .
- iii) The tensor product of positive line bundles is positive.
- iv) If $\mathcal{L}$ is positive over X and $\mathcal{L}'$ is positive over X' , then $\mathcal{L} \otimes \mathcal{L}'$ is positive over $X \times X'$.

For the applications we have in view, λ will be ξ^a considered as a section over $\Pi_{q_r^a > 0} \mathbf{P}_r^{n_r}$ or an appropriate submanifold thereof. We call a constraint positive if the corresponding line bundle is. The configurations for which $b_{1,1}$ can be computed by iterative application of LHT and Proposition 1 are called *favorable* and are characterized by the condition that the indices a can be ordered so that for $a' < a$, either $q_r^a q_r^{a'} = 0$ for all r , or $q_r^a \leq q_r^{a'} q_r^a$ for all r . We shall shortly discuss the application of LHT example by example.

It was shown that $b_{2,1}$ and $b_{1,1}$ cannot be computed by TESS for certain CICYs, all of which belong to a class which have a (1DD) diagram[12]. The class of 1DD diagrams where the two subdiagrams of the complement of the decomposing dot describe two surfaces, S^1 and S^2 , and the decomposing dot represents a constraint defining a hypersurface in $S^1 \times S^2$, were studied separately[15] relying on the existing classification of complex surfaces[18]. It turns out that $\chi_E[S^i]$ may only be 3, ..., 10 and 12 and, using LHT and Eq. (2.3) respectively, two closed formulae for the Hodge numbers of such hypersurfaces were obtained in Ref. [15] :

$$b_2[\mathcal{M}] = \chi_E[S^1] + \chi_E[S^2] - 4, \quad 3 \leq \chi_E[S^i] \leq 10, \quad (2.4)$$

$$\chi_E[\mathcal{M}] = -2(12 - \chi_E[S^1])(12 - \chi_E[S^2]), \quad 3 \leq \chi_E[S^i] \leq 12. \quad (2.5)$$

This covers almost all cases, except when any one or both surfaces have $\chi_E[S^i] = 12$, in which case $\chi_E[\mathcal{M}] = 0$ and $b_2[\mathcal{M}] = 19^{12}$. Since $b_2 = b_{1,1}$ and $\chi_E = 2(b_{1,1} - b_{2,1})$ for any Calabi-Yau manifold, $b_{2,1}$ is determined as well.

The set of CICYs with a favorable configurations and those with a 1DD configuration overlap, but only partially. We now give examples for this :

$$\mathcal{M}_1 \in M_1 := \left[\begin{array}{c|cc} 3 & 0 & 1 & 3 \\ \hline 3 & 2 & 1 & 1 \end{array} \right]_{-120}^3 \cong \text{Diagram}$$

¹²This was computed by Chad Schoen using his results in Ref. [23]; see also Ref. [15]

is favorable, but not 1DD. Here and hereafter the superscript on the configuration denotes $b_2 = b_{1,1}$ and the subscript is χ_E . As $\mathcal{M}_1$ is favorable, we can use LHT iteratively and one starts by applying the first constraint (the one that is non-trivial with respect to the least number of factors in $\mathcal{W}$). Since it is trivial with respect to the first $\mathbf{P}^3$ it is non-positive over the entire $\mathcal{W}$, but it defines a quadric hypersurface of complex dimension 2 in the second $\mathbf{P}^3$ only (where it is positive), $\Delta_1 := [3\|2]$, with $\chi_E = 4$. LHT applies and tells that $b_i(\Delta_1) = b_i(\mathbf{P}^3)$ for $i = 0, 1$ and $b_2(\Delta_1) \geq b_2(\mathbf{P}^3)$. This is enough to deduce that $\chi_E(\Delta_1) = 2 + b_2(\Delta_1)$ and so $b_2(\Delta_1) = 2$. Next one considers¹³

$$\left[\begin{array}{c|cc} 3 & 3 & 1 \\ \Delta_1 & 1 & 1 \end{array} \right]_{-120}^3 \cong \text{Diagram},$$

for which LHT applies straightforwardly as all the constraints are non-trivial (and positive, since the degrees of homogeneity are positive) over both factor-spaces and finally, by application of the standard Künneth formula :

$$b_i(\mathcal{A} \times \mathcal{B}) = \sum_{j=0}^i b_j(\mathcal{A}) \cdot b_{i-j}(\mathcal{B}),$$

it follows that $b_2(\mathcal{M}_1) = b_2(\mathbf{P}^3) + b_2(\Delta_1) = 3$.

Consider for our second example :

$$\mathcal{M}_2 \in M_2 := \left[\begin{array}{c|cccc} 4 & 2 & 2 & 0 & 0 & 1 \\ 4 & 0 & 0 & 2 & 2 & 1 \end{array} \right]_{-32}^{12} \cong \text{Diagram},$$

which is both favorable and 1DD. Since $\mathbf{P}^4$ is of complex dimension 4, the e.g. first quadratic constraint (which is positive) will define a hypersurface, $\Delta_2 := [4\|2]$, of complex dimension 3 having, by LHT, $b_2(\Delta_2) = b_2(\mathbf{P}^4) = 1$. The second constraint (again positive) defines a hypersurface $\Delta_3 := [\Delta_2\|2]$ of complex dimension 2, so this time LHT guarantees only that $b_2(\Delta_3) \geq b_2(\Delta_2)$. To compute $b_2(\Delta_3)$, we proceed as in the first example and since $\chi_E(\Delta_3) = \chi_E[4\|22] = 8$, $b_2(\Delta_3) = 6$. The same is of course true of the other two constraints defining the same kind of surface in the other $\mathbf{P}^4$. The last constraint is positive over both surfaces and LHT applies to tell that

$$b_i(\mathcal{M}_2) = 2b_i(\Delta_3) \quad \text{for } i \leq 2,$$

¹³We hope to create no confusion by the slight abuse of the notation here : the first column of the configuration matrix represents the embedding space $\mathbf{P}^3 \times \Delta_1$.

which is what we need to conclude that $b_2(\mathcal{M}_2) = 12$.

The example :

$$\mathcal{M}_3 \in M_3 := \left[\begin{array}{c|cc} 2 & 0 & 3 \\ 2 & 2 & 1 \\ 1 & 2 & 0 \end{array} \right]_{-36}^9 \cong \begin{array}{c} \circ \\ \parallel \\ \bullet \end{array} - \begin{array}{c} \circ \\ \parallel \\ \bullet \end{array}$$

is not favorable but is 1DD. The second constraint acts in the product of the first surface, $\mathbf{P}_1^2$, and the surface defined by the first constraint in $\mathbf{P}_2^2 \times \mathbf{P}^1$ where it is positive. The latter surface, $\Delta_4 := \left[\begin{array}{c|c} 2 & 2 \\ 1 & 2 \end{array} \right]$, has $\chi_E = 10$, and by LHT $b_0 = 1$ and $b_1 = 0$, whereby $b_2[\mathcal{M}_2] = \chi_E - 2 = 8$. The second constraint however is not *manifestly* positive over Δ_4 as it is not positive over one of the factors (the $\mathbf{P}^1$) that went into the definition of Δ_4 . It was proven (Theorem 5 of Ref. [15]) that the line bundle corresponding to second constraint indeed is positive over Δ_4 even though not as a consequence of Proposition 1. This implies the applicability of LHT and by Eq. (2.4) we have :

$$b_2(\mathcal{M}_3) = b_2(\mathbf{P}^2) + b_2(\Delta_4) = 9.$$

Note however that if one assumed the applicability of LHT when analyzing the constraints in the reversed order, a false result $b_2 = 3$ would follow.

The set of configurations which are neither favorable nor 1DD comprises however the vast majority of CICYs and in the next section we shall describe the technique of exact and spectral sequences (TESS) that we use to compute $b_{2,2} = b_{1,1} = b_2$ and $b_{2,1} = b_{1,2}$ for all the remaining CICYs.

3 EXACT SEQUENCES AND SHEAVES

We shall first review some basic facts of homological algebra and sheaf theory which are essential to understand the techniques we used in our computations.

3.1 DEFINITIONS AND ELEMENTARY RESULTS

If A and B are linear spaces and $f : A \rightarrow B$ is a linear map, the *image* of f , denoted by $\text{Im}(f)$ is the subspace of B , consisting of all elements of the form $f(a)$ for some $a \in A$. The *kernel* of f , written $\ker(f)$ is the subspace of A consisting of those elements of A which are mapped to 0 by f . The map f is called *monic* or *injective* if $\ker(f) = 0$, and is *epic* or *surjective* if $\text{Im}(f) = B$. The sequence of linear spaces and maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* at B if $\text{Im}(f) = \ker(g)$. Similarly, a longer sequence is exact if the kernel of each map is the image of the preceding one.

The exactness of the ‘short exact sequence’ $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is equivalent to the injectivity of f and surjectivity of g . Such a sequence is also equivalent to an identification of A with a subspace of B , and C with the corresponding quotient :

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \text{ is exact} \quad \Leftrightarrow \quad A \approx f(A) \subseteq B, \quad C \approx \{B/f(A)\} \quad (3.1)$$

and consequently, $\dim C = \dim B - \dim A$.

All of the foregoing definitions also apply to maps and sequences of vector bundles and in particular to holomorphic vector bundles and holomorphic linear maps. If $f : A \rightarrow B$ is holomorphic and has constant rank, then $\ker(f)$ and $\text{Im}(f)$ are holomorphic subbundles of A and B respectively.

If $\mathcal{W}$ is a complex manifold and $\mathcal{M}$ is a submanifold, the normal bundle, $\mathcal{N}$, to $\mathcal{M}$ in $\mathcal{W}$ is *defined* to be the quotient bundle $\{\mathcal{T}_{\mathcal{W}}|_{\mathcal{M}}/\mathcal{T}_{\mathcal{M}}\}$, where $\mathcal{T}_{\mathcal{M}}$ and $\mathcal{T}_{\mathcal{W}}$ denote the tangent bundles of $\mathcal{M}$ and $\mathcal{W}$ respectively. Let $\mathcal{E}$ be a holomorphic bundle over $\mathcal{W}$ and ξ a holomorphic section of $\mathcal{E}$ such that $\mathcal{M}$ is the zero locus of ξ , i.e., $\xi^{-1}(0) = \mathcal{M} \hookrightarrow \mathcal{W}$. Then we have the map $D\xi : \mathcal{T}_{\mathcal{W}}|_{\mathcal{M}} \rightarrow \mathcal{E}|_{\mathcal{M}}$, which is defined independently of any connection¹⁴ and is a holomorphic linear mapping of bundles over $\mathcal{M}$. It is evident that $\ker(D\xi)$ is $\mathcal{T}_{\mathcal{M}}$. If, in addition, $D\xi$ is epic we have the short exact sequence :

$$0 \rightarrow \mathcal{T}_{\mathcal{M}} \xrightarrow{\iota} \mathcal{T}_{\mathcal{W}}|_{\mathcal{M}} \xrightarrow{D\xi} \mathcal{E}|_{\mathcal{M}} \rightarrow 0 \quad (3.2)$$

which allows us to identify $\mathcal{E}|_{\mathcal{M}}$ with $\mathcal{N}$. This is precisely the situation one obtains in the case of a non-singular complete intersection as in section 2. In the notation of that section, we take :

$$\mathcal{E} := \bigoplus_{a=1}^K \mathcal{E}^a, \quad \mathcal{E}^a := \bigotimes_{r=1}^m (h_r)^{q_r^a}, \quad (3.3)$$

and ξ to be the section of $\mathcal{E}$ whose a^{th} component is the constraint ξ^a in Eq. (2.1).

Our computational techniques also require some results from the theory of sheaves. A sheaf A over a manifold $\mathcal{W}$ assigns to each open subset U of $\mathcal{W}$, a set (actually a vector space in all the examples we shall consider) A_U and to each inclusion of open sets $U \subseteq V$, a map (for us a homomorphism) $r_U^V : A_V \rightarrow A_U$. These maps can generally

¹⁴This independence is easily seen since locally, $D\xi = \partial\xi + \Gamma\xi$ for some connection Γ . However, $\Gamma\xi$ vanishes wherever ξ does, so $D\xi = \partial\xi$ everywhere on $\mathcal{M}$.

be interpreted as restrictions and must satisfy $r_U^V \circ r_V^W = r_U^W$ for $U \subseteq V \subseteq W$ ¹⁵. The examples we will need are, in order of increasing generality:

- (A) Let $\mathcal{W}$ be a complex manifold and $U \subseteq V$ two open neighborhoods as above. $\mathcal{O}_{\mathcal{W}}(U)$ is defined to be the ring of holomorphic functions on U . The map r_U^V is the restriction map.
- (B) Let $\mathcal{E}$ be a holomorphic vector bundle over the complex manifold $\mathcal{W}$. We define $\mathcal{E}(U)$ to be the space of holomorphic sections of $\mathcal{E}|_U$. The map r_U^V is the restriction map.
- (C) Let $\mathcal{M}$ be a submanifold of the complex manifold $\mathcal{W}$ and let $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{M}$. We define $\mathcal{E}_{\mathcal{M}}$ to be the sheaf over $\mathcal{W}$ which assigns to each open $U \subseteq \mathcal{W}$ the space of holomorphic sections of $\mathcal{E}|_{(U \cap \mathcal{M})}$. Once more r_U^V is the restriction map.

If A and B are sheaves over $\mathcal{W}$, a map of sheaves, $f : A \rightarrow B$ assigns to each open U a map $f_U : A_U \rightarrow B_U$ which satisfy the obvious compatibility condition

$$f_U \circ r_U^{(A)V} = r_U^{(B)V} \circ f_V.$$

The definition of exactness for a sequence of linear maps of linear sheaves over $\mathcal{W}$ is not precisely what one would assume; a sequence of maps of sheaves is called exact if it is exact over sufficiently small open subsets. In particular, an exact sequence of holomorphic vector bundles induces an exact sequence of the corresponding sheaves, but the reverse is not true in general. See Refs. [18,24] for further details.

There exists a cohomology theory of sheaves which assigns to a manifold $\mathcal{W}$ and a linear sheaf A over $\mathcal{W}$ cohomology groups $H^q(\mathcal{W}, A)$, $q \geq 0$. What we need to know about sheaf cohomology is summarized in the following

Theorem 1

- i) If $\mathcal{E}$ is a holomorphic vector bundle over $\mathcal{W}$ the sheaf cohomology groups $H^q(\mathcal{W}, \mathcal{E})$ may be identified with the $\bar{\partial}$ -cohomology of $\mathcal{W}$ with coefficients in $\mathcal{E}$ ¹⁶.

¹⁵We have actually defined a presheaf rather than a sheaf, but since every presheaf determines a sheaf, and since this abuse of notation is well embedded in the mathematical vernacular, the distinction is unimportant for our purposes.

¹⁶These cohomology groups can also be identified with the space of harmonic $\mathcal{E}$ -valued $(0, q)$ -forms which has a natural (additive) Abelian group structure and for which the rank equals the dimension.

- ii) With $\mathcal{W}$, $\mathcal{M}$ and $\mathcal{E}$ as in example (C) above, $H^q(\mathcal{W}, \mathcal{E}_{\mathcal{M}}) = H^q(\mathcal{M}, \mathcal{E})$.
- iii) If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence of sheaves over $\mathcal{W}$, then the long cohomology sequence :

$$0 \rightarrow \dots \xrightarrow{\delta_{q-1}} H^q(\mathcal{W}, A) \xrightarrow{f_q} H^q(\mathcal{W}, B) \xrightarrow{g_q} H^q(\mathcal{W}, C) \xrightarrow{\delta_q} \dots \rightarrow 0$$

is exact. The maps f_q and g_q are induced from f and g .

Proofs of these statements are found in Refs. [18,24].

3.2 APPLICATION TO THE CICYs

Recall now that we wish to determine $H^{2,1}(\mathcal{M}) \approx H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ and possibly also $H^{1,1}(\mathcal{M}) \approx H^{2,2}(\mathcal{M}) \approx H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$. By statement iii) of Theorem 1, it follows that the short exact sequence (3.2) implies the existence of the long exact cohomology sequence :

$$0 \rightarrow \dots \xrightarrow{\delta_{q-1}} H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \xrightarrow{i_q} H^q(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \xrightarrow{j_q} H^q(\mathcal{M}, \mathcal{E}) \xrightarrow{\delta_q} \dots \rightarrow 0. \quad (3.4)$$

Using that $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx H^{2,0}(\mathcal{M}) = \emptyset = H^{2,3}(\mathcal{M}) \approx H^3(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ for any Calabi-Yau manifold¹⁷, we notice that :

$$\text{Im}(i_1) = \emptyset \quad \Leftrightarrow \quad H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx \{H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}})/j_0[H^0(\mathcal{M}, \mathcal{E})]\}, \quad (3.5)$$

but, to the best of our knowledge, there is *no a priori reason* for $\text{Im}(i_1) = \emptyset$.

To proceed, we need to determine $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ and $H^*(\mathcal{M}, \mathcal{E})$, and since the embedding $\mathcal{M} \hookrightarrow \mathcal{W}$ is defined using a section of $\mathcal{E}$, it appears to be most natural to derive the relevant cohomology groups on $\mathcal{M}$ from certain cohomology groups on $\mathcal{W}$. For this, we need to discuss the restriction maps :

$$\begin{aligned} \varrho_{\mathcal{T}}^* : \mathcal{T}_{\mathcal{W}} &\rightarrow \mathcal{T}_{\mathcal{W}}|_{\mathcal{M}}, & H^*(\mathcal{W}, \mathcal{T}_{\mathcal{W}}) &\rightarrow H^*(\mathcal{M}, \mathcal{T}_{\mathcal{W}}), \\ \varrho_{\mathcal{E}}^* : \mathcal{E} &\rightarrow \mathcal{E}|_{\mathcal{M}}, & H^*(\mathcal{W}, \mathcal{E}) &\rightarrow H^*(\mathcal{M}, \mathcal{E}), \end{aligned}$$

and we therefore focus first on the restriction $\varrho : \mathcal{O}_{\mathcal{W}} \rightarrow \mathcal{O}_{\mathcal{M}}$, which fits into the exact sequence :

$$\mathcal{E}^* \xrightarrow{\xi} \mathcal{O}_{\mathcal{W}} \xrightarrow{\varrho} \mathcal{O}_{\mathcal{M}} \rightarrow 0. \quad (3.6)$$

The restriction ϱ is epic because any holomorphic function on a sufficiently small open subset of $\mathcal{M}$ extends to some open subset of $\mathcal{W}$. The exactness at $\mathcal{O}_{\mathcal{W}}$ follows from the fact that both the kernel of ϱ and the image of ξ consists locally of all linear combinations of the constraint functions with holomorphic coefficients. Note that we are using the same symbol for a vector bundle and for the corresponding sheaf over $\mathcal{W}$.

¹⁷This can be proved e.g., relying on a theorem by Bochner[10] or, in sufficient generality, the results presented in the next section of the present paper (see Theorem 1 of Ref. [12]).

3.3 EMBEDDING BY A SINGLE CONSTRAINT

Let us now temporarily restrict to the case when $\mathcal{M}$ is embedded in $\mathcal{W}$ by a single constraint, so that the $\mathcal{E}$ bundle defined in Eq. (3.3) is of rank 1, i.e., $K = 1$. In this case, ξ is clearly monic on the bundle level except along $\mathcal{M}$, and is monic on the sheaf level because no open subset of $\mathcal{W}$ is contained in $\mathcal{M}$. Thus the exact sequence (3.6) becomes :

$$0 \rightarrow \mathcal{E}^* \xrightarrow{\xi} \mathcal{O}_{\mathcal{W}} \xrightarrow{\varrho} \mathcal{O}_{\mathcal{M}} \rightarrow 0. \quad (3.7)$$

Since the exactness of a sequence of sheaves over a manifold is not changed by tensoring with the sheaf of sections of any holomorphic vector bundle over the same manifold, by tensoring (3.7) with $\mathcal{E}$ and separately with $\mathcal{T}_{\mathcal{W}}$, we get the two short exact sequences :

$$0 \rightarrow \mathcal{O}_{\mathcal{W}} \rightarrow \mathcal{E} \rightarrow \mathcal{E}|_{\mathcal{M}} \rightarrow 0, \quad (3.8)$$

$$0 \rightarrow \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}} \rightarrow \mathcal{T}_{\mathcal{W}} \rightarrow \mathcal{T}_{\mathcal{W}}|_{\mathcal{M}} \rightarrow 0, \quad (3.9)$$

where the sheaves restricted to $\mathcal{M}$ are interpreted as sheaves over $\mathcal{W}$ in the sense of our example (C) above, and we have used that $\mathcal{E}^* \otimes \mathcal{E} = \mathcal{O}_{\mathcal{W}}$ since $\text{rank } \mathcal{E} = 1$.

For the manifolds in the scope of this paper, $\mathcal{W}$ is an product of m $\mathbf{P}_r^{n_r}$'s and therefore :

$$H^q(\mathcal{W}) = \bigoplus_{\sum \gamma_s = q} \bigotimes_{r=1}^m H^{\gamma_r}(\mathbf{P}_r^{n_r}) = \bigotimes_{r=1}^m H^0(\mathbf{P}_r^{n_r}) \approx \mathbf{C} \quad (3.10)$$

since $H^q(\mathbf{P}^n)$ vanishes except for $H^0(\mathbf{P}^n) \approx \mathbf{C}$. As a matter of fact, we have that $H^q(\mathcal{W}, \mathcal{E})$ vanishes except for :

$$\text{rank } H^0(\mathcal{W}, \mathcal{E}) = \text{rank } \bigotimes_{r=1}^m H^0(\mathbf{P}_r^{n_r}, (h_r)^{(n_r+1)}) = \prod_{r=1}^m \binom{2n_r+1}{n_r+1},$$

so that the only non-trivial part of the long exact cohomology sequence corresponding to (3.8) is :

$$0 \rightarrow H^0(\mathcal{W}) \rightarrow H^0(\mathcal{W}, \mathcal{E}) \rightarrow H^0(\mathcal{M}, \mathcal{E}) \rightarrow 0.$$

Thus $H^0(\mathcal{M}, \mathcal{E}) = \{H^0(\mathcal{W}, \mathcal{E})/H^0(\mathcal{W})\}$, so that $\text{rank } H^0(\mathcal{M}, \mathcal{E}) = \prod_{r=1}^m \binom{2n_r+1}{n_r+1} - 1$, which is precisely what one would obtain by methods of Ref. [25] which rely on polynomial deformations (PDM hereafter).

To assess $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ from the long exact cohomology sequence corresponding to

the short exact sequence of sheaves (3.9) one uses the Bott formula[28] :

$$\text{rank } H^q(\mathbf{P}^n, \Lambda^p T(\mathbf{P}^n) \otimes h^k) = \begin{cases} \binom{k+n+1+p}{p} \binom{k+n}{n-p} & q = 0 \quad \text{and } k \geq -p, \\ 1 & q = n - p \quad \text{and } k = -(n+1), \\ \binom{-k-n-2}{p} \binom{-k-p-1}{n-p} & q = n \quad \text{and } k \leq -(n+p+2), \\ 0 & \text{otherwise,} \end{cases} \quad (3.11)$$

and the decomposition[12] :

$$H^q(\mathcal{W}, \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}}) = \bigoplus_{r=1}^m \bigoplus_{\sum \gamma_t = q} \left[H^{\gamma_r}(\mathbf{P}_r^{n_r}, T(\mathbf{P}_r^{n_r}) \otimes (h_r)^{-(n_r+1)}) \otimes \bigotimes_{\substack{s=1 \\ s \neq r}}^m H^{\gamma_s}(\mathbf{P}_s^{n_s}, (h_s)^{-(n_s+1)}) \right],$$

since for $K = 1$, $\mathcal{E}^* = \bigotimes_{s=1}^m (h_s)^{-(n_s+1)}$ and $\mathcal{T}_{\mathcal{W}} = \bigoplus_{r=1}^m T(\mathbf{P}_r^{n_r})$. The cohomology groups in the first line of this decomposition will be non-vanishing only for $\gamma_r = n_r - 1$, while those in the second line are non-vanishing only for $\gamma_s = n_s$, so that $H^q(\mathcal{W}, \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}})$ vanishes except for $q = 3$, since $\sum_r n_r = \dim \mathcal{M} + K = 4$, and we have that

$$\text{rank } H^{K+2}(\mathcal{W}, \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}}) = \sum_{r=1}^m [(1) \cdot \prod_{\substack{s=1 \\ s \neq r}}^m (1)] = m.$$

In close analogy,

$$\begin{aligned} \text{rank } H^0(\mathcal{W}, \mathcal{T}_{\mathcal{W}}) &= \text{rank} \bigoplus_{r=1}^m \left[H^0(\mathbf{P}_r^{n_r}, T(\mathbf{P}_r^{n_r})) \otimes \bigotimes_{\substack{s=1 \\ s \neq r}}^m H^0(\mathbf{P}_s^{n_s}) \right] \\ &= \sum_{r=1}^m [(n_r(n_r+2)) \cdot \prod_{\substack{s=1 \\ s \neq r}}^m (1)] = \sum_{r=1}^m (n_r(n_r+2)). \end{aligned}$$

This exhausts the non-vanishing groups on $\mathcal{W}$ appearing in the long exact cohomology sequence corresponding to the short exact sequence of sheaves (3.9) which therefore breaks apart into two pieces :

$$0 \rightarrow H^0(\mathcal{W}, \mathcal{T}_{\mathcal{W}}) \rightarrow H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow 0 \quad \Rightarrow \quad H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \approx H^0(\mathcal{W}, \mathcal{T}_{\mathcal{W}}),$$

$$0 \rightarrow H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow H^3(\mathcal{W}, \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}}) \rightarrow 0 \quad \Rightarrow \quad H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \approx H^3(\mathcal{W}, \mathcal{E}^* \otimes \mathcal{T}_{\mathcal{W}}).$$

This yields $\text{rank } H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) = \sum_{r=1}^m n_r(n_r+2)$, matching precisely the result one would obtain by PDM[25], and $\text{rank } H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) = m$.

Now that all $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ and $H^*(\mathcal{M}, \mathcal{E})$ have been determined, we see that – for the case of $\mathcal{M} \hookrightarrow \mathcal{W}$ defined by a single constraint – the long exact cohomology sequence (3.4) breaks apart into :

$$0 \rightarrow H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow H^0(\mathcal{M}, \mathcal{E}) \rightarrow H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \rightarrow 0,$$

from which we conclude that the r.h.s. of the equivalence (3.5) is true and furthermore that $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ is correctly parametrized by PDM, so that

$$b_{2,1} = \text{rank } H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = \left[\prod_{r=1}^m \binom{2n_r+1}{n_r} - 1 \right] - \left[\sum_{r=1}^m n_r(n_r + 2) \right],$$

and

$$0 \rightarrow H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \rightarrow H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow 0 \quad \Rightarrow \quad H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \approx H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}),$$

so that $b_{1,1} = b_{2,2} = \text{rank } H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) = m$.

Thus, this relatively simple analysis[26] yields the exact values of both $b_{2,1}$ and $b_{1,1}$, independently of Eq. (2.3). Note however that there are only five Calabi-Yau manifolds embedded in some product of complex projective spaces by a single constraint :

$$[4\|5]_{-200}^1, \quad \left[\begin{array}{c|c} 2 & 3 \\ \hline 2 & 3 \end{array} \right]_{-162}^2, \quad \left[\begin{array}{c|c} 3 & 4 \\ \hline 1 & 2 \end{array} \right]_{-168}^2, \quad \left[\begin{array}{c|c} 2 & 3 \\ \hline 1 & 2 \\ \hline 1 & 2 \end{array} \right]_{-144}^3, \quad \left[\begin{array}{c|c} 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 2 \end{array} \right]_{-128}^4,$$

and we clearly need more general methods.

4 THE GENERAL CASE OF CICYS

We now turn to discuss the general $K > 1$ case, when $\mathcal{M} = \bigcap_{a=1}^K X^a$ and for which we again need to study the restriction map $\varrho : \mathcal{O}_{\mathcal{W}} \rightarrow \mathcal{O}_{\mathcal{M}}$. We need to continue the sequence (3.6) further to the left while preserving its exactness. Such an extension is called a *resolution* of the restriction $\varrho : \mathcal{O}_{\mathcal{W}} \rightarrow \mathcal{O}_{\mathcal{M}}$ and is accomplished by :

$$0 \rightarrow \wedge^K \mathcal{E}^* \rightarrow \dots \rightarrow \wedge^2 \mathcal{E}^* \rightarrow \mathcal{E}^* \rightarrow \mathcal{O}_{\mathcal{W}} \xrightarrow{\varrho} \mathcal{O}_{\mathcal{M}} \rightarrow 0 \quad (4.1)$$

where each map except the last one is induced by the section ξ of the bundle $\mathcal{E}$, acting on one component of $\wedge^K \mathcal{E}^*$ only. The sequence is exact even as a sequence of bundles over $\mathcal{W}$, except along $\mathcal{M}$. As before, it is exact as a sequence of sheaves over $\mathcal{W}$ because no open subset of $\mathcal{W}$ is contained in $\mathcal{M}$. Note that for $K = 1$, the sequence

(4.1) collapses to the short exact sequence of sheaves (3.7), as it should. Just as before, we may tensor the exact sequence of sheaves over $\mathcal{W}$ with the sheaf of sections of any holomorphic vectrobundle $\mathcal{V}$ over $\mathcal{W}$ to obtain the exact sequence of sheaves :

$$0 \rightarrow \wedge^K \mathcal{E}^* \otimes \mathcal{V} \rightarrow \dots \rightarrow \wedge^2 \mathcal{E}^* \otimes \mathcal{V} \rightarrow \mathcal{E}^* \otimes \mathcal{V} \rightarrow \mathcal{V} \xrightarrow{g} \mathcal{V}|_{\mathcal{M}} \rightarrow 0, \quad (4.2)$$

where again all the maps, except the last one, are induced by the section ξ of the bundle $\mathcal{E}$. We shall write the same symbol to denote a vector bundle and the sheaf of germs of holomorphic functions valued in that bundle. In particular, we shall use two exact sequences of sheaves with $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{E}$ replacing $\mathcal{V}$ in (4.2) to determine $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ and $H^*(\mathcal{M}, \mathcal{E})$ respectively.

4.1 SPECTRAL SEQUENCES

Since there are more than three terms for the general case $K > 1$ in this exact sequence, there is no corresponding long cohomology sequence. Therefore one cannot use in general the simple argument that allows us to derive (3.5) from the short exact sequence (3.2); instead, one has the *spectral sequence*¹⁸ :

$$\{E_i^{j,k}(\mathcal{V}), d_i\}, \quad i > 0, \quad d_i : E_i^{j,k}(\mathcal{V}) \rightarrow E_i^{j-i+1, k-i}(\mathcal{V})$$

of bi-graded groups where $d_i \circ d_i = 0$,

$$E_1^{j,k}(\mathcal{V}) := H^j(\mathcal{W}, \wedge^k \mathcal{E}^* \otimes \mathcal{V}),$$

and

$$E_i^{j,k}(\mathcal{V}) := \left\{ \frac{\ker(d_i : E_i^{j,k}(\mathcal{V}) \rightarrow E_i^{j-i+1, k-i}(\mathcal{V}))}{\text{Im}(d_i : E_i^{j+i-1, k+i}(\mathcal{V}) \rightarrow E_i^{j,k}(\mathcal{V}))} \right\}.$$

Since the number of constraints, K , is finite, there exists a *limit* to this sequence :

$$E_i^{j,k}(\mathcal{V}) = E_{i+1}^{j,k}(\mathcal{V}) =: E_{\infty}^{j,k}(\mathcal{V}), \quad \text{for } i \geq \hat{i} = \hat{i}(j, k), \quad \hat{i} \leq K.$$

Finally, to make connection of $E_{\infty}^{**}(\mathcal{V})$ and $H^*(\mathcal{M}, \mathcal{V})$, we note that there exists a *filtration* of $H^*(\mathcal{M}, \mathcal{V})$:

$$\varrho[H^q(\mathcal{W}, \mathcal{V})] =: \mathcal{F}_0 H^q(\mathcal{M}, \mathcal{V}) \subseteq \dots \subseteq \mathcal{F}_K H^q(\mathcal{M}, \mathcal{V}) := H^q(\mathcal{M}, \mathcal{V}) \quad (4.3)$$

such that

$$E_{\infty}^{q,0}(\mathcal{V}) = \mathcal{F}_0 H^q(\mathcal{M}, \mathcal{V}) \quad \text{and} \quad E_{\infty}^{q+k,k}(\mathcal{V}) = \mathcal{F}_k H^q(\mathcal{M}, \mathcal{V}) / \mathcal{F}_{k-1} H^q(\mathcal{M}, \mathcal{V}). \quad (4.4)$$

¹⁸For a general reference on spectral sequences see Ref. [27].

It is said that $\{E_i(\mathcal{V}), d_i\}$ converges to $H^q(\mathcal{M}, \mathcal{V})$. Even though this does not provide a *natural* isomorphism between $H^q(\mathcal{M}, \mathcal{V})$ and $\bigoplus_{k=0}^K E_\infty^{q+k,k}(\mathcal{V})$, it does follow that

$$\text{rank } H^q(\mathcal{M}, \mathcal{V}) = \sum_{k=0}^K \text{rank } E_\infty^{q+k,k}(\mathcal{V}), \quad (4.5)$$

which is all we use. On a $j \times k$ chart of $E_\infty^{j,k}(\mathcal{V})$, the groups in the sum in (4.5) lie on a diagonal and we refer to (4.5) as ‘collecting terms on the q -diagonal’.

Thus we first compute the ranks of the groups in $E_1(\mathcal{T}_\mathcal{W})$ and $E_1(\mathcal{E})$, many of which actually vanish. This information, together with the general properties of spectral sequences listed above, will usually be sufficient to determine E_∞ and therefore the ranks of the cohomology groups on $\mathcal{M}$. Once $\text{rank } H^q(\mathcal{M}, \mathcal{T}_\mathcal{W})$ and $\text{rank } H^q(\mathcal{M}, \mathcal{E})$ are obtained, we use the long exact sequence (3.4) to derive $\text{rank } H^0(\mathcal{M}, \mathcal{T}_\mathcal{M}) = b_{2,1}$ and $\text{rank } H^2(\mathcal{M}, \mathcal{T}_\mathcal{M}) = b_{2,2} = b_{1,1}$.

5 THE COMPUTATION OF $b_{2,1}$ AND $b_{1,1}$

First of all, we note that the proof of Theorem 4 of Ref. [12] allows a stronger version of the statement :

Theorem 2 *For a diagram representing an n -fold with no decomposing $(n-1)$ -leg, $E_1^{q+k,k}(\mathcal{V})$ vanishes for $q \geq n-1$.*

Proof: Exactly the same as for Theorem 4 in Ref. [12]. □

Consequently, the Corollary 2 of Ref. [12] may be rephrased :

Corollary 1 *If a diagram representing a Calabi-Yau 3-fold has no decomposing 1-legs $H^1(\mathcal{M}, \mathcal{T}_\mathcal{W})$ vanishes and no decomposing 2-legs implies $H^2(\mathcal{M}, \mathcal{E}) = \emptyset$, so that the sequences*

$$0 \rightarrow H^0(\mathcal{M}, \mathcal{T}_\mathcal{W}) \rightarrow H^0(\mathcal{M}, \mathcal{E}) \rightarrow H^1(\mathcal{M}, \mathcal{T}_\mathcal{M}) \rightarrow 0$$

$$0 \rightarrow H^1(\mathcal{M}, \mathcal{E}) \rightarrow H^2(\mathcal{M}, \mathcal{T}_\mathcal{M}) \rightarrow H^2(\mathcal{M}, \mathcal{T}_\mathcal{W}) \rightarrow 0$$

are exact.

Next we note that the same proof implies

Corollary 2 *Every diagram representing an n -fold with a decomposing $(n-1)$ -leg is equivalent to another one with no decomposing $(n-1)$ -leg which is obtained by replacing the circle-complement of the decomposing $(n-1)$ -leg with a single circle and the decomposing leg with two legs. Diagrammatically :*

This includes the three ‘identities between one-folds’ of Ref. [14].

It is obvious that in analysing configurations representing Calabi-Yau three-folds, it suffices to analyze only configurations with no decomposing 2-legs, so that there may be only three types of non-vanishing $E_1^{j,k}(\mathcal{T}_\mathcal{W})$ groups :

- (a) $E_1^{0,0}(\mathcal{T}_\mathcal{W}) = \bigoplus_{r=1}^m H^0(\mathbf{P}_r^{n_r}, T(\mathbf{P}_r^{n_r}))$;
- (b) $E_1^{K+2,K}(\mathcal{T}_\mathcal{W}) \approx \mathbf{C}^m$;
- (c) $E_1^{q+\kappa,\kappa}(\mathcal{T}_\mathcal{W}) \ni H^0(\mathbf{P}_r^{n_r}, T(\mathbf{P}_r^{n_r}) \otimes h_r^{-1}) \approx \mathbf{C}^{n_r+1}$,
 $\forall \{A, r : \sum_{a \in A} q_a^r = 1, \kappa = |A| + 1\}$.

Here $|A|$ denotes the cardinality of A , the set of indices labeling a subset of constraints which act only in a $(q + \kappa)$ -dimensional factor of the product $\mathcal{W}$. We note here that the possibly non-vanishing groups in (c) are not detected by PDM, and for $q = 0$ this leads to an incompleteness in the parametrization of $H^0(\mathcal{M}, \mathcal{T}_\mathcal{W})$ and thereby a redundancy in the parametrization of $H^1(\mathcal{M}, \mathcal{T}_\mathcal{M})$.

By analogous analysis of $E_1^{j,k}(\mathcal{E})$, one obtains[12] :

$$\begin{aligned}
 E_1^{j,k}(\mathcal{E}) &= \bigoplus_{a=1}^K \bigotimes_{r=1}^m H^0(\mathbf{P}_r^{n_r}, (h_r)^{q_a^r}), & j = k = 0, \\
 &= \bigoplus_{a=1}^k \left[\bigotimes_{r=1}^m H^0(\mathbf{P}_r^{n_r}) \right] \oplus \bigoplus_{a=1}^K \bigoplus_{\substack{b=1 \\ b \neq a}}^K \left[\bigotimes_{r=1}^m H^0(\mathbf{P}_r^{n_r}, (h_r)^{(q_a^r - q_b^r)}) \right], & j = 0, k = 1, \\
 &= \bigoplus_{b=1}^K \bigoplus_{\sum \gamma_r = j} \bigoplus_{|A|=k} \left[\bigotimes_{r=1}^m H^{\gamma_r}(\mathbf{P}_r^{n_r}, (h_r)^{(q_b^r - \sum_{a \in A} q_a^r)}) \right], & 1 < k < K.
 \end{aligned} \tag{5.1}$$

The possibly non-vanishing groups in the third line of this expression are not accounted for by PDM and for $j = k$ this implies incompleteness in the parametrization of $H^0(\mathcal{M}, \mathcal{E})$ and thereby in the parametrization of $H^0(\mathcal{M}, \mathcal{T}_\mathcal{M})$ as well.

Based on these formulae, the PASCAL program in appendix B computes the ranks of $E_1^{j,k}(\mathcal{T}_W)$ and $E_1^{j,k}(\mathcal{E})$.

Observe now that the $E_1^{j,k}(\mathcal{E})$ groups with $k > j$ would converge to $H^{-|k-j|}(\mathcal{M}, \mathcal{V})$, which must have rank 0, since the $\bar{\partial}$ -cohomology vanishes in negative dimensions. Therefore, there must exist one or more effective differentials d_i to ‘cancel out’ these groups when passing from $E_1(\mathcal{V})$ to $E_\infty(\mathcal{V})$. Since all the d_i ’s map from a q -diagonal into the $(q+1)$ -diagonal, the $E_1^{j,k}(\mathcal{V})$ with $k > j$ ‘cancel out’ elements from the $(j-k+1)$ -diagonal, leading to :

$$\sum_{k=0}^K \text{rank } E_1^{k,k}(\mathcal{E}) + \sum_{k=1}^K \sum_{j=1}^{k-1} (-1)^{k+j} \text{rank } E_1^{j,k}(\mathcal{E}) = \text{rank } H^0(\mathcal{M}, \mathcal{E})$$

if no element of any $E_1^{j,j}(\mathcal{E})$ is mapped into $E_1^{j',j'-1}(\mathcal{E})$ and cancelled out thereby. This would allow us to compute the rank of $H^0(\mathcal{M}, \mathcal{E})$ via Eq. (4.5) even if not all of the summands were known separately.

However, if there exists a pair of non-vanishing groups $\{E_1^{j,j}(\mathcal{E}), E_1^{j',j'-1}(\mathcal{E}); j \geq j'\}$, not knowing the specific differential d_i that ‘connects’ them prevents us from determining whether some of the elements of these groups are cancelled out or not. In such a case only the difference $\text{rank } H^0(\mathcal{M}, \mathcal{E}) - \text{rank } H^1(\mathcal{M}, \mathcal{E})$ is calculable and this seems to obstruct our computation of $b_{2,1}$ and $b_{1,1}$ since there is not enough information supplied into the long exact sequence (3.4). One of the CICYs for which this obstruction does not vanish is discussed in full detail in appendix A. Suffice it here to note that in all the nine cases found, these obstructions turn out to be innocuous as each configuration that features the obstruction can be shown to have same Hodge numbers as an alternative configuration for which the obstruction vanishes.

We have analyzed the 7868 configurations which were shown in Ref. [14] to be sufficient to represent all possible CICYs. Note that not one of these configurations has a decomposing 2-leg so that Theorem 2 applies.

1. First we used Eq. (2.4) to compute $b_2 = b_{1,1}$ for all 1DD configurations for which the complement of the decomposing dot represents a product of two complex surfaces. Note that this includes all 1LD configurations so that we have left out only non-1LD configurations.
2. We have next computed the ranks of $E_1^{j,k}(\mathcal{T}_W)$ and $E_1^{j,k}(\mathcal{E})$ using a program similar to that in appendix B for all the configurations not considered in 1. In all these cases we find, in agreement with Theorem 2 of Ref. [12], that the only non-vanishing groups in the tangent bundle spectral sequence are $E_1^{0,0}(\mathcal{T}_W)$ and $E_1^{K+2,K}(\mathcal{T}_W)$.

These are isomorphic to $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ and $H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$ respectively and thus, $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) = \emptyset$. Since we have left out only non-1LD configurations in 1., the sequences

$$0 \rightarrow H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow H^0(\mathcal{M}, \mathcal{E}) \rightarrow H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \rightarrow 0 \quad (5.2)$$

$$0 \rightarrow H^1(\mathcal{M}, \mathcal{E}) \rightarrow H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \rightarrow H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \rightarrow 0 \quad (5.3)$$

are now exact by Corollary 1. All we here need are the ranks of $H^0(\mathcal{M}, \mathcal{E})$ and $H^1(\mathcal{M}, \mathcal{E})$ to complete our computation.

3. In cases where there is no pair of non-vanishing groups $\{E_1^{j,j}(\mathcal{E}), E_1^{j',j'-1}(\mathcal{E}); j \geq j'\}$, there is no obstruction to determine $E_{\infty}(\mathcal{E})$ *even if no d_i 's are known explicitly*. Except for $E_1^{j,k}(\mathcal{E})$ with $j \leq k$, the effect of which has been discussed in the beginning of the section, $E_1^{**}(\mathcal{E}) = E_{\infty}^{**}(\mathcal{E})$. This provides rank $H^*(\mathcal{M}, \mathcal{E})$ which we used to determine $b_{2,1}$ and $b_{1,1}$ from the short exact sequences (5.2) and (5.3).

4. Finally, in the nine cases when there does exist an obstructing pair of non-vanishing groups $\{E_1^{j,j}(\mathcal{E}), E_1^{j',j'-1}(\mathcal{E}); j \geq j'\}$, one cannot obtain rank $H^0(\mathcal{M}, \mathcal{E})$ and rank $H^1(\mathcal{M}, \mathcal{E})$ but only the difference of the two. However, every such configuration represents non-singular 3-folds which also belong to a certain 'augmented' configuration, for which there is no obstruction. Thus we compute $b_{2,1}$ and $b_{1,1}$ using this augmented configuration and identify them with those of the manifolds which belong to the initial configuration. This identification is true since the augmentation we perform is a special case of 'splitting'[14] for which we prove in appendix A that the initial and the final configuration represent overlapping deformation classes of Calabi-Yau manifolds.

6 RESULTS

It would be desirable to present our results along with representative configurations, but it is clear that tabulating 264 configuration matrices, some of which are fairly big, simply requires too much space. Also, as Calabi-Yau manifolds may not be completely classified up to deformation by the Hodge numbers such a list would be arbitrary and possibly incomplete. For this reason we merely list the independent Hodge numbers and supply the details of our computation (see also the appendices).

The independent Hodge numbers $b_{2,1}$ and $b_{1,1}$ we obtained are displayed in Table 1. In view of the identities $b_{p,q} = b_{3-p,3-q}$ and $b_{p,q} = b_{q,p}$ valid for all Kähler manifolds as well as $b_{p,q} = b_{3-p,q} = b_{p,3-q}$ and

$$\begin{aligned} b_{p,q} &= 1 && \text{if both } p, q \equiv 0 \pmod{3}, \\ b_{p,q} &= 0 && \text{if exactly one } p, q \equiv 0 \pmod{3}, \end{aligned}$$

which are valid for all Calabi-Yau manifolds¹⁹, the information in the table determines

$b_{1,1}$	$b_{2,1}$
1	65, 73, 89, 101
2	46, 47, 50, 55, 56, 58, 59, 62, 64, 66, 68, 72, 76, 77, 83, 86
3	27, 31, 33, 35–41, 43–57, 59–61, 63, 66, 69, 72, 75
4	26, 28–56, 58–62, 64, 67, 68
5	25–53, 55–57, 59, 61
6	24–51, 54, 55
7	23–39, 41–43, 47–49
8	22–38, 40–44
9	21–33, 35–37, 39
10	20–32, 34, 35
11	19–23, 25–29, 31
12	18–22, 24, 27, 28
13	17, 19, 21, 23, 25
14	16, 18, 22, 23
15	21
16	20
19	19

Table 1: The independent Hodge numbers of all CICYs

all possible Hodge diamonds for CICYs.

As a by-product of our computation, we have explicitly computed that

$$\text{rank Im}(i_1 : H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \rightarrow H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}})) = 0$$

for all 7868 CICYs of Ref. [14], so that the sequence (5.2) is indeed exact for all of them. This is seen as follows :

1. For configurations with no decomposing 2-legs and no decomposing 1-legs Corollary 1 implies $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) = \emptyset$. Since $\text{Im}(i_1) \subseteq H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$, it follows that $\text{Im}(i_1) = \emptyset$.
2. For configurations with decomposing 1-legs, however, $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \neq \emptyset$ as proved in Ref. [12] where examples of this phenomenon were given as well. For example,

¹⁹At this point it is crucial that the compact complex three dimensional manifolds we consider have $SU(3)$ holonomy, rather than a subgroup of $SU(3)$.

which is the formula usually *assumed to be universally valid* in the physics literature[25].

However, in many cases the relation (6.2) is simply false[12]. In addition to the several examples in Ref. [12], our appendix A presents another such example CICY. In order to follow the analysis of Ref. [1] for a particular CICY with $E_\infty^{j,j}(\mathcal{E}) \neq \emptyset$ or $E_\infty^{j,j}(\mathcal{T}_W) \neq \emptyset$ for some $j \neq 0$, one needs some explicit parametrization of these cohomology groups since they contribute to $H^0(\mathcal{M}, \mathcal{E})$ and $H^0(\mathcal{M}, \mathcal{T}_W)$ respectively, and thereby to $H^1(\mathcal{M}, \mathcal{T}_M)$.

We note in passing that provided an explicit parametrization of $H^q(\mathbf{P}^n, \wedge^p T(\mathbf{P}^n) \otimes h^k)$ is obtained for $p = 0, 1$ and arbitrary q and k , not only $H^1(\mathcal{M}, \mathcal{T}_M)$ but $H^2(\mathcal{M}, \mathcal{T}_M)$ as well, and thereby the **27**'s as well as the **27***'s in Eqs. (1.1) and (1.2), would be parametrized explicitly generalizing the results of Ref. [25]. Moreover, by straightforward application of the spectral sequence $\{E_i^{j,k}, d_i\}$ (following from the exact sequence of sheaves (4.1)) one can prove that the (3, 0)- and (0, 3)-forms are correctly parametrized by Candelas in Ref. [25].

It is perhaps worth noting that the redundancy of PDM is always masked by its incompleteness i.e., the rank of the subgroup of $H^0(\mathcal{M}, \mathcal{T}_W)$ not detected by PDM is always less than or equal to that of the undetected subgroup of $H^0(\mathcal{M}, \mathcal{E})$. Therefore, mere counting of dimensions of the spaces provided by PDM is not conclusive and the computations begun in Ref. [12] and completed here are both necessary and sufficient for the general case.

APPENDICES

A THE OBSTRUCTION EXEMPLIFIED

We present here an example configuration for which there exists a pair of non-vanishing groups $\{E_1^{j,j}(\mathcal{E}), E_1^{j',j'-1}(\mathcal{E}); j \geq j'\}$ obstructing our straightforward computation of $H^*(\mathcal{M}, \mathcal{E})$ and thereby $b_{2,1}$ and $b_{1,1}$. We shall also present the augmentation of the configuration by which the deformation class of Calabi-Yau manifolds contained in the initial configuration and the augmented one are identical while the obstruction vanishes in the augmented configuration.

The simplest configuration for which the obstruction exists is

$$\mathcal{M}_4 \in M_4 := \left[\begin{array}{c|cccc} 4 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{array} \right]_{-72}^5 \cong \text{Diagram}$$

The tangent bundle spectral sequence is simpler as the only non-vanishing groups are $E_1^{0,0}(T_{\mathcal{W}})$ which is of rank $24+8+8=40$, corresponding to the linear reparametrizations of the $\mathbf{P}^4 \times \mathbf{P}_1^2 \times \mathbf{P}_2^2$ ($PGL(5; \mathbf{C}) \times PGL(3; \mathbf{C}) \times PGL(3; \mathbf{C})$) and $E_1^{7,5}(T_{\mathcal{W}})$ which is of rank 3, corresponding to the pullbacks of the three Kähler forms of the three projective spaces.

Using the first and the second line of Eq. (5.1) one obtains $E_1^{0,0}(\mathcal{E}) \approx \mathbf{C}^{83}$ and $E_1^{0,1}(\mathcal{E}) \approx \mathbf{C}^7$. $E_1^{0,0}(\mathcal{E})$ corresponds to a space spanned by monomials of degrees of homogeneity as specified by the configuration matrix of $\mathcal{M}_4$: the set of monomials which are cubic in $\mathbf{P}^4$, bi-linear in $\mathbf{P}^4 \times \mathbf{P}_1^2$, etc. $E_1^{0,1}(\mathcal{E})$ is represented by sections of $\mathcal{E}^* \otimes \mathcal{E}$, i.e., by ‘rescaling’ of the constraint polynomials: firstly, there are five ‘overall’ rescalings of the five homogeneous polynomial constraints which generates rank five of $E_1^{0,1}(\mathcal{E})$. The remaining two are seen by rescaling the fifth constraint by a complex multiple of the fourth one and vice versa. The map $d_1 : E_1^{0,1}(\mathcal{E}) \rightarrow E_1^{0,0}(\mathcal{E})$ must be onto, since $E_\infty^{0,1}(\mathcal{E})$ must vanish. Consequently, $\text{rank}[E_\infty^{0,0}(\mathcal{E}) = E_2^{0,0}(\mathcal{E})] = 83 - 7 = 76$. This is completely in agreement with Ref. [25].

The third line in Eq. (5.1) yields the following non-vanishing groups: $E_1^{4,4}(\mathcal{E})$, of rank 5, and $E_1^{4,3}(\mathcal{E})$, of rank 2. These are found as follows:

1. Taking $b = 1$ and $A = \{2, 3, 4, 5\}$ in Eq. (5.1), $E_1^{4,4}(\mathcal{E})$ is non-vanishing and given by the group

$$H^0(\mathbf{P}^4, h^{3-(1+1+0+0)}) \otimes H^2(\mathbf{P}_1^2, h_1^{0-(1+0+1+1)}) \otimes H^2(\mathbf{P}_2^2, h_2^{0-(0+1+1+1)}),$$

the rank of which can be found using Eq. (3.11) to be $\binom{4+1}{1} \cdot (1) \cdot (1) = 5$.

2. The group $E_1^{4,3}(\mathcal{E})$ has two terms in its decomposition as in Eq. (5.1), both of which have $A = \{1, 2, 3\}$, and $b = 4$ vs. $b = 5$ for the two terms respectively. In both cases the terms are

$$H^0(\mathbf{P}^4, h^{0-(3+1+1)}) \otimes H^0(\mathbf{P}_1^2, h_1^{1-(0+1+0)}) \otimes H^0(\mathbf{P}_2^2, h_2^{1-(0+0+1)}),$$

and have rank $(1) \cdot (1) \cdot (1) = 1$ so that $\text{rank } E_1^{4,3}(\mathcal{E}) = 2$.

The groups $E_1^{j,k}(\mathcal{E})$, $j \geq k > 0$ are not parametrized by the formalism set up in Ref. [25]; obtaining an explicit parametrization would enable us to follow the analysis of the Tian-Yau manifold[1,7].

The obstruction to the completion of our computation arises since we cannot determine the rank of $\text{Im}(d_1 : E_1^{4,4}(\mathcal{E}) \rightarrow E_1^{4,3}(\mathcal{E}))$. Thus we cannot decide if $\text{rank } H^1(\mathcal{M}, \mathcal{E})$ is 0, 1 or 2 and there is a complementary uncertainty in $\text{rank } H^0(\mathcal{M}, \mathcal{E}) = \text{rank } E_2^{0,0}(\mathcal{E}) + \text{rank } E_2^{4,4}(\mathcal{E})$. Via the sequences (5.2) and (5.3), the best we can state so far is that :

$$(b_{2,1}, b_{1,1}) \in \{(41, 5), (40, 4), (39, 3)\} \quad (\text{A.1})$$

all three of which agree with $\chi_E = -72$ as obtained by Eq. (2.3).

It turns out that all nine configurations for which this obstruction exists are of the form :

$$\left[\begin{array}{c|ccc} X & \mathbf{Q} & \hat{1} & \hat{1} & \hat{0} & \hat{0} \\ \hline 2 & \hat{0} & 1 & 0 & 1 & 1 \\ 2 & \hat{0} & 0 & 1 & 1 & 1 \end{array} \right] \cong \text{Diagram} \quad , \quad (\text{A.2})$$

where $\hat{0}$ is a column or a row 0-vector, $\hat{1}$ is a column vector, the only non-vanishing component of which is a single '1', and $\mathbf{Q}$ is a matrix. In particular, $[X \parallel \mathbf{Q}]$ is $[4 \parallel 3]$ in $\mathcal{M}_4$. We will show that the obstruction can be removed by altering the configuration changing the 'diamond' shaped subdiagram on the far left of (A.2) in such a way that the deformation class of Calabi-Yau manifolds represented has the same Hodge numbers.

Consider therefore the (partial) deformation class of complex surfaces Υ represented by the configuration :

$$\Upsilon \in Y := \left[\begin{array}{c|cc} 2 & 1 & 1 \\ \hline 2 & 1 & 1 \end{array} \right]_6^4 \cong \text{Diagram} \quad , \quad b_{p,q}(\Upsilon) = \begin{array}{ccccc} & & 1 & & \\ & 0 & 0 & & \\ 0 & 4 & 0 & & \\ & 0 & 0 & & \\ & & 1 & & \end{array} ,$$

and also the following configuration :

$$\Upsilon' \in Y' := \left[\begin{array}{c|ccc} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 2 & 0 & 1 & 1 \end{array} \right]_6^4 \cong \text{Diagram}$$

Replacing Y by Y' in a configuration is a special case of ‘splitting’^{#10} introduced in Ref. [14]. We now wish to prove

Proposition 2 *The Hodge numbers satisfy $b_{p,q}(\mathcal{M}) = b_{p,q}(\mathcal{M}')$ for $\mathcal{M}$ as in display (A.2) and*

$$\mathcal{M}' \in M' := \left[\begin{array}{c|cccccc} X & \mathbf{Q} & \hat{1} & \hat{1} & \hat{0} & \hat{0} & \hat{0} \\ 2 & \hat{0} & 1 & 0 & 1 & 0 & 1 \\ 1 & \hat{0} & 0 & 0 & 1 & 1 & 0 \\ \hline 2 & \hat{0} & 0 & 1 & 0 & 1 & 1 \end{array} \right] \cong \text{Diagram}$$

Proof :

We will show that a *generic* surface Υ' belonging to Y' has the property that its projection on the $\mathbf{P}^2 \times \mathbf{P}^2$ factor in its embedding space $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^2$ maps by a holomorphic equivalence onto a non-singular surface belonging to Y in such a way that the bundles corresponding to all remaining constraints correspond under the equivalence. It will follow that if M' is the result of replacing a copy of Y in the diagram of M with a copy of Y' , then there exist smooth varieties which belong to both M and M' .

The two constraints of Y' involving CP^1 can be written as

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = 0, \quad (z_0, z_1) \in \mathbf{P}^1,$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ are linear combinations of the homogeneous coordinates of the i^{th} $\mathbf{P}^2$. Points of $\mathbf{P}^2 \times \mathbf{P}^2$ which are in the image of the projection of Υ' must satisfy

$$\det \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix} = (\mathcal{A}_1 \mathcal{B}_2 - \mathcal{A}_2 \mathcal{B}_1) = 0, \quad (\text{A.3})$$

yielding an equation which is bi-linear in $\mathbf{P}^2 \times \mathbf{P}^2$. It is straightforward to show that this determinant equation, together with the remaining constraint, describes a smooth manifold onto which the projection maps Υ' by a holomorphic equivalence if and only

^{#10}In a sense, the first constraint has been split into two constraints and the embedding space has simultaneously been enlarged by a new $\mathbf{P}^1$ factor.

if the matrix in Eq. (A.3) is of rank 1, not 0, at every point of Υ' . For it to be of rank 0, all four elements would have to vanish but these four equations, $\mathcal{A}_i = \mathcal{B}_i = 0$, together with the third constraint in Y' , are 5 equations in the 4-dimensional $\mathbf{P}^2 \times \mathbf{P}^2$ and, generically, have no common solution. $\square$

Note now that splitting Y into Y' , there will be one more Kähler form, the pullback of which will appear in $E_1^{K+2,K}(\mathcal{T}_{\mathcal{W}})$ and thereby in $H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}})$. By Corollary 2 of Ref. [12], the sequences (5.2) and (5.3) are still exact, so that there is nothing to cancel out this obvious contribution to $\text{rank } H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}})$ and thus the third possibility is ruled out from (A.1).

In complete analogy, one next splits the third constraint of Y' :

$$Y := \left[2 \parallel \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]_6^4 \rightarrow Y'' := \left[2 \parallel \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{array} \right]_6^4 \cong \text{Diagram}$$

and the straightforward modification of Proposition 2 ensures that there are Calabi-Yau manifolds represented both by M' and M'' , where M'' is the result of replacing a copy of Y' in the diagram of M' with a copy of Y'' . By the same argument as before, $b_{1,1}$ increases by one through splitting and only the $b_{2,1} = 41$, $b_{1,1} = 5$, possibility from (A.1) remains.

Indeed, repeating the computation of $E_1(\mathcal{T}_{\mathcal{W}})$ and $E_1(\mathcal{E})$ for M_4 , M'_4 and M''_4 , one obtains the values of the ranks displayed in Table 2 and the obstruction is seen to

ranks	$E_1^{0,0}(\mathcal{T}_{\mathcal{W}})$	$E_1^{K+2,K}(\mathcal{T}_{\mathcal{W}})$	$E_1^{0,0}(\mathcal{E})$	$E_1^{0,1}(\mathcal{E})$	$E_1^{\Delta,\Delta-1}(\mathcal{E})$	$E_1^{\Delta,\Delta}(\mathcal{E})$
$M_4 :$	40	3	83	7	2	5
$M'_4 :$	43	4	86	7	1	5
$M''_4 :$	46	5	89	7	0	5

In this table $\Delta = \text{rank } \mathcal{W} - \text{rank } \mathbf{P}^2 \times \mathbf{P}^2$.

Table 2: Ranks of non-vanishing groups in $E_1(\mathcal{T}_{\mathcal{W}})$ and $E_1(\mathcal{E})$

vanish for M''_4 . By construction, M_4 and M''_4 represent deformation classes of Calabi-Yau manifolds which can be connected by a chain of overlapping deformation classes, so that $\mathcal{M}_4$ has $b_{2,1} = 41$ and $b_{1,1} = 5$ regardless of which of the configurations M_4 , M'_4 or M''_4 we represent it by.

The same procedure has been followed for all the nine cases we found, completing our computation of $b_{p,q}$ for all 7868 CICYs.

B THE PROGRAM

Together with Eqs. (2.5), the formulae in (a–c) of section 5 and Eq. (5.1) are sufficient for our computation but may become quite cumbersome for application to large configuration matrices. To facilitate the reproduction of our computation for any particular CICY, we present here the interactive version of the PASCAL program we have used.

To save some space, we assume the necessary binomial coefficients to be loaded in the array `ch`, compress the display of the code and omit the following procedures :

`clear`, which resets the arrays `n`, `q`, `w`, `ts`, `tc`, `ns` and `nc` to zero;

`display`, which displays or prints out $\text{rank } E_1^{**}(\mathcal{T}_W)$, $\text{rank } H^*(\mathcal{M}, \mathcal{T}_W)$, $\text{rank } E_1^{**}(\mathcal{E})$ and $\widetilde{\dim} H^*(\mathcal{M}, \mathcal{T}_W)$, stored in arrays `ts`, `tc`, `ns` and `nc` respectively. Special care must be taken with $\widetilde{\dim} H^*(\mathcal{M}, \mathcal{T}_W)$, which is computed ignoring the obstruction. Since the obstruction is easily seen in a chart of $\text{rank } E_1^{**}(\mathcal{E})$ (see section 5 and appendix A for details) this should create no problem.

PROGRAM COHOMOLOGY (input,output);

`const` `nx = 7; mx = 12; hx = 9; wx = 126;`

`var` `H,M,K : integer; n : array[1..mx] of integer; nc : array[0..3] of integer;`
`ns : array[0..(hx+3),0..hx] of integer; tc : array[0..3] of integer;`
`ts : array[0..(hx+3),0..hx] of integer; ch : array[1..hx,0..hx] of integer;`
`q : array[1..mx,1..hx] of integer; w : array[1..mx,1..wx] of integer;`

PROCEDURE `dimensions`; `var r,t : integer;`

`begin repeat writeln(' '); write('How many factors in W ?'); readln(M);`

`until((M<=mx) and (M>=0));`

`if (M<>0) then`

`begin repeat writeln('Enter the ',M:1,' dimensions : ');`

`if (M>1) then for r := 1 to (M-1) do read(n[r]); readln(n[M]);`

`t := 0; for r := 1 to M do t := t+n[r]; H := t-3;`

`until(t<=mx) and (t>3);`

`begin ts[(H+2),H] := M;`

`begin ts[0,0] := 0;`

`for r := 1 to M do ts[0,0] := ts[0,0] + ((n[r]+1)**2-1);`

`end; end; end; end;`

PROCEDURE `configuration`; `var a,r,z : integer;`

`begin writeln('Enter ',H:1,' degrees of homogeneity : '); for r := 1 to M do`

```

begin write(' ',n[r]:1,' || ');
  if (H>1) then for a := 1 to (H-1) do read(q[r,a]); readln(q[r,H]); end;
begin ns[0,0] := 0; for a := 1 to H do
  begin z := 1; for r := 1 to M do z := z*ch[(n[r]+q[r,a]),q[r,a]];
    ns[0,0] := ns[0,0] + z; end; end; end;

PROCEDURE donorm; var e,i,r,s,z,c : integer; t : array[1..mx] of integer;
begin for i := 1 to ch[H,K] do for e := 1 to H do
  begin for r := 1 to M do t[r] := w[r,i]+q[r,e]; s := 0; z := 1; c := 0;
    for r := 1 to M do if ((t[r] >= 0) or (t[r]=-(n[r]+1))) then
      begin if (t[r]>0) then z := z*ch[(n[r]+t[r]),t[r]];
        if (t[r]=-(n[r]+1)) then c := c+n[r]; s := s+1; end;
      if (s=M) then nc[c,K] := nc[c,K]+z; end; end; end;

PROCEDURE dotang; var i,r,s,l,z,c : integer;
begin for i := 1 to ch[H,K] do
  begin s := 0; l := 0; z := 0; c := 0; for r := 1 to M do if (l<2) then
    if ((w[r,i]=0) or (w[r,i]=-1) or (w[r,i]=-(n[r]+1))) then
      begin if (w[r,i]=-1) then
        begin l := l+1; z := n[r]+1; end;
        if (w[r,i]=-(n[r]+1)) then c := c+n[r]; s := s+1; end;
      if (l=1) then if (s=M) then ts[c,K] := ts[c,K]+z; end; end; end;

PROCEDURE wedging; var a,b,c,r : integer; d : array[1..mx,1..wx] of integer;
begin for a := 1 to wx do for r := 1 to M do d[r,a] := w[r,a]; c := 0;
  for b := (K+1) to H do
    begin for a := 1 to ch[(b-1),K] do
      begin c := c+1; for r := 1 to M do w[r,c] := d[r,a]-q[r,b]; end; end;
    K := K+1; end;

PROCEDURE dowedges; var f,g : integer;
begin K := 1; for g := 1 to M do for f := 1 to H do w[g,f] := -q[g,f];
  donorm; dotang; repeat wedging; donorm; dotang;
  until(K>=H); end;

PROCEDURE docohs; var j,q : integer;
begin for q := 0 to 3 do begin nc[q] := 0; tc[q] := 0; end;
  for q := 0 to 3 do for j := 0 to H do
    begin nc[q] := nc[q]+nc[(j+q),j]; tc[q] := tc[q]+ts[(j+q),j];end;

```

```

    for q := 1 to H do for j := 1 to q do
      nc[0] := nc[0]+((-1)**j)*nc[(q-j),q]; end;
begin clear;
  repeat dimensions; if (M<>0) then
    begin configuration; dowedges; docohs; display; clear; end;
  until(M<1); end.

```

REFERENCES

- [1] B.R.GREENE, K.H.KIRKLIN, P.J.MIRON and G.G.ROSS: *Nucl. Phys.* **B278** (1986)667.
- [2] S.KALARA and R.N.MOHAPATRA: *University of Minnesota preprint* UMN-TH-590-86(1986).
- [3] D.GROSS, J.HARVEY, E.MARTINEC and R.ROHM: *Nucl. Phys.* **B265** (1985)253.
- [4] P.CANDELAS, G.HOROWITZ, A.STROMINGER and E.WITTEN: *Nucl. Phys.* **B258** (1987)46.
- [5] J.S.GATES JR. and H.NISHINO: *Phys. Lett.* **173B** (1986)52, *ibid.* **189B** (1987)45, *Nucl. Phys.* **B291** (1987)205;
S.J.GATES JR. and S.VASHAKIDZE: *Nucl. Phys.* **B291** (1987)172.
- [6] M.B.GREEN, J.H.SCHWARZ and E.WITTEN: *Superstring Theory II* (Cambridge University Press, Cambridge, 1987).
- [7] S.-T.YAU: in *Proc. of Symposium on Anomalies, Geometry, Topology*, eds. W.A.Bardeen and A.R.White (World Scientific, Singapore, 1985).
- [8] M.DINE, N.SEIBERG, X.G.WEN and E.WITTEN: *Nucl. Phys.* **B278** (1986)769, *ibid.* **B289** (1987)319.
- [9] A.STROMINGER and E.WITTEN: *Comm. Math. Phys.* **101** (1986)341.
- [10] T.HÜBSCH: *Comm. Math. Phys.* **108** (1987)291, in “*Superstrings, Unified Theories and Cosmology*”, eds. G.Furlan, R.Jengo, J.C.Pati, D.W.Sciama, E.Sezgin and Q.Shafi (World Scientific, Singapore, 1987).
- [11] P.GREEN and T.HÜBSCH: *Comm. Math. Phys.* **109** (1987)99.
- [12] P.GREEN and T.HÜBSCH: *University of Maryland preprint* UMPP-87-158, to appear in *Comm. Math. Phys.* ; see also
T.HÜBSCH: in *Proc. of ‘Workshop on Particle Physics and Cosmology’* June–Aug. 1987, ICTP, Trieste, Italy.

- [13] P.S.ASPINWALL, B.R.GREENE, K.H.KIRKLIN and P.J.MIRON:
University of Oxford report, February 1987.
- [14] P.CANDELAS, A.M.DALE, C.A.LÜTKEN and R.SCHIMMRIGK:
University of Texas report UTTG-10-87(1987).
- [15] P.GREEN and T.HÜBSCH: *University of Maryland report* PP-87-202, to appear in
Comm. Math. Phys.
- [16] R.SLANSKI: *Phys. Rep.* **C79** (1981)1.
- [17] R.BOTT: *Mich. Math. J.* **6**(1959)211; For applications, see : Ref. [1,11] and [15].
- [18] P.GRIFFITHS and J.HARRIS: "*Principles of Algebraic Geometry*"
(John Wiley, New York, 1978).
- [19] P.CANDELAS, C.A.LÜTKEN and R.SCHIMMRIGK: *University of Texas report*
UTTG-11-87(1987).
- [20] R.SCHIMMRIGK: *Phys. Lett.* **193B** (1987)175.
- [21] C.T.C.WALL: *Invent. Math.* **1**(1966)355.
- [22] D.GEPNER: *University of Princeton report* PUPT-1066(1987).
- [23] C.SCHOEN: *Mathematical Sciences Research Institute report* MSRI-04620-87(1987).
- [24] R.HARTSHORNE: "*Algebraic Geometry*" (Springer-Verlag, New York, 1977);
F.HIRZEBRUCH: "*Topological Methods in Algebraic Geometry*"
(Springer-Verlag, New York, 1978);
S.IITAKA: "*Algebraic Geometry*" (Springer-Verlag, New York, 1982);
R.M.RANGE: "*Holomorphic Functions and Integral Representations in Several Complex Variables*" (Springer-Verlag, New York, 1986).
- [25] E.WITTEN: *Nucl. Phys.* **B258** (1985)75;
B.R.GREENE, K.H.KIRKLIN, P.J.MIRON and G.G.ROSS: *Nucl. Phys.* **B278** (1986)667;
P.CANDELAS: *University of Texas report* UTTG-05-87(1987).
- [26] K.KODAIRA: "*Complex Manifolds and Deformations of Complex Structures*"
(Springer, New York, 1986).
- [27] R.GODEMENT: *Theorie des Faisceaux* (Hermann, Paris, 1958); see also Ref. [18,24].
- [28] C.OKONEK, M.SCHNEIDER and H.SPINDLER: "*Vector Bundles on Complex Projective Spaces*", p.11 (Birkhauser, 1980); see however also :
R.BOTT: *Ann. Math.* **66**(1957)203.