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HEXAGON GAUGE ANOMALY AND SUPERMODULI
IN THE PATH INTEGRAL METHOD OF SUPERSTRING

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ABSTRACT

The hexagon gauge anomaly of the type-I superstring, which has been derived previously in the operator formalism, is recalculated in the framework of Polyakov's path integral method. It is shown that the existence of the conformal Killing spinor on the world surface is responsible for making the hexagon diagram anomalous. The role of supermoduli, which has no analogue in the operator method, is clarified. The hexagon anomaly is expressed as a surface integral in the moduli space.

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1. - INTRODUCTION

There are several distinct but mutually complementary ways of analyzing scattering amplitudes in string theories. The most conventional one is the so-called operator formalism which was developed in early seventies. Another way is the string field theory (second-quantized theory), the covariant formulation of which has been an important issue in the last few years but still leaves room for further development. The third method is the one based on Polyakov's path integral method [1,2] which will be our central concern in the present paper. This method is believed to be most suitable to investigate multiloop string amplitudes and in fact, we have witnessed in the last two years remarkable developments in the path integral method.

In their recent paper [3], the present authors applied the Polyakov's path integral method to the calculation of the hexagon gauge anomaly of the type-I superstring and have disentangled how the various geometrical notions related to super-Riemann surfaces conspire to produce the anomaly. Similar consideration was also made by Polchinski and Cai [4]. The purpose of the present paper is to give a full account of the calculation sketched in Ref. [3].

The gauge anomaly in the superstring was first investigated by Green and Schwarz [5] and later it has been re-analyzed from various standpoints, including the gravitational anomaly [6,7]. The calculations performed there are all based on the operator formalism.

In the present paper we will instead use the path integral method to calculate the hexagon gauge anomaly of the type-I superstring. This method has a nice feature that geometrical contents of the string theories are manifestly built in. We will clarify the roles played by supermoduli and conformal Killing spinors [8,9] in the anomaly calculation which have no counterpart in the conventional operator formalism.

General considerations of the path integral derivation of the anomaly were previously made in Ref. [10]. Miki and Yahihozawa [11] have investigated the anomalies in the heterotic string, using the technique of the conformal field theory. The relevant two-dimensional surface in their case is a torus, and in fact they have studied supermoduli and conformal Killing spinors thereupon. In our case, on the other hand, annulus and Möbius band are the relevant world surfaces contributing to the anomaly of the type-I superstring. Our analysis becomes more involved than the torus case because of the boundary conditions imposed on fermion fields.

In the heterotic string the anomaly can be expressed as a surface integral in the moduli space [7,8]. An analogous expression is expected to hold in the type-I superstring, and the expression was derived by Clavelli et al. [6] by using the operator formalism. We will show that it can also be derived in the path integral method, whose relation to the operator method will become clear only after the roles of supermoduli and conformal Killing spinor are clarified. We will use the superspace formalism since it makes our analysis more transparent.

The present paper is organized as follows. In Section 2, we give a quick review of Polyakov's path integral method in the superspace formulation. Section 3 deals with the boundary conditions. The involution method for supersurfaces (super-annulus, and super-Möbius band) is developed to express the boundary conditions in terms of superfields on supertorus. The supermoduli and superconformal Killing vector are worked out explicitly in Section 4. The vertex operator responsible for the gauge boson emission is given a superfield expression in Section 5. In Section 6, the hexagon gauge anomaly is calculated in the form of a surface integral in the moduli space. The final section is devoted to discussion and summary.

2. - PATH INTEGRAL METHOD OF SUPERSTRING

Let us start with a brief review of Polyakov's path integral method [1] in superspace [12,13], thereby explaining our notations. The supermanifold of world-sheet is parametrized by $z^M = (\sigma^m, \theta^\mu)$ where σ^m ($m = 1, 2$) are the co-ordinates of the string world surface and θ^μ ($\mu = 1, 2$) are the usual Grassmannian co-ordinates. The matter superfield $\Phi^I(z)$ can be expanded in powers of θ as

$$\Phi^I(z) = X^I(\sigma) + \theta \psi^I(\sigma) + \frac{1}{2} \theta \theta F^I(\sigma), \quad (2.1)$$

where $I = 1, 2, \dots, D$ ($D=10$). The auxiliary field F^I may be set to zero in accordance with the equation of motion which follows from the string action given presently.

The partition function is given by

$$Z = \sum \int \frac{\mathcal{D}E_m^A}{\Omega(\text{gauge})} \int \mathcal{D}\Phi^I e^{-S[\Phi, E]} \quad (2.2)$$

where the action is expressed as

$$S[\Phi, E] = \frac{1}{4\pi\alpha'} \int d^2x \, E \, D^\alpha \Phi^I D_\alpha \Phi^I. \quad (2.3)$$

The summation in Eq. (2.2) is taken over two-dimensional topologies and spin structures. We have used the notation $d^2z = d^2\sigma d^2\theta$ in Eq. (2.3). We will use the letters from the middle of the alphabet for the co-ordinate indices, while those from the beginning for the local Lorentz indices. The bosonic (fermionic) components will be given small Latin (Greek) letters. Capital letters represent both of them.

The superdeterminant of the supervierbein E_M^A is denoted by E . The covariant derivative is given by $D_A = E_A^M D_M = E_A^M \partial_M + \Omega_A^L L$, where Ω_A is the spin connection and L the Lorentz generator. The action (2.3) is invariant under superdiffeomorphism, local Lorentz and superconformal transformations. The path integral measure in Eq. (2.2) is divided by the gauge volume associated with these symmetries which is collectively denoted by $\Omega(\text{gauge})$ in Eq. (2.2).

The functional integration measure $\mathcal{D}E_M^A$ needs further clarification. The variables in the superspace formulation of the supergravity are highly redundant. The redundancy is removed by imposing constraints on the supertorsion tensor T_{AB}^C . Following Howe [12], we will impose the following conditions

$$T_{\alpha\beta}^\gamma = T_{\alpha\beta}^c = 0, \quad T_{\alpha\beta}^a = 2i(\gamma^a)_{\alpha\beta}. \quad (2.4)^*)$$

Equation (2.4) gives us 14 constraints in total. Four of them are used to determine the spin connection, and we have thus ten constraints. Therefore there are six independent superfields among 16 supervierbein fields E_M^A .

We will use the following six superfields

$$\delta H_\epsilon^\alpha, \quad (\gamma^3)_\alpha{}^\beta \delta H_\beta^\alpha, \quad \delta H_\alpha^\alpha, \quad (2.5)$$

to specify the variations δE_M^A . Here we have defined $\delta H_B^A = E_B^M \delta E_M^A$. The functional volume element $\mathcal{D}E_M^A$ in Eq. (2.2) is defined by the metric [13]

*) Our gamma matrices are such that γ^1 , γ^2 and $-i\gamma^1\gamma^2$ are the Pauli matrices.

$$\begin{aligned} \|\delta E\|^2 = \int d^2z \, E \left[(\delta H_\beta^\alpha)(\delta H_\alpha^\beta) + c_1 \left\{ (\gamma^3)_\alpha^\beta \delta H_\beta^\alpha \right\}^2 \right. \\ \left. + c_2 (\delta H_\alpha^\alpha)^2 \right] . \end{aligned} \quad (2.6)$$

Here c_1 and c_2 are arbitrary constants. Incidentally, the functional volume element $\mathfrak{D}\Phi^I$ is defined by

$$\|\delta\Phi\|^2 = \int d^2z \, E (\delta\Phi^I)^2 . \quad (2.7)$$

As we mentioned previously, the action (2.3) possesses the invariances under superdiffeomorphism, local Lorentz and superconformal transformations. We may use these invariances to remove the six degrees of freedom (2.5). Suppose we fix the supervierbein as $\hat{E}_M^A(z, T)$, where $T = (t_i, b_i)$ denotes collectively moduli (t_i) and supermoduli (b_i) parameters characterizing the gauge slice. The functional integrations with respect to the variations (2.5) may be decomposed into those of variations from $\hat{E}_M^A(z, T)$ corresponding to superdiffeomorphism, local Lorentz and superconformal transformations and the finite dimensional integration with respect to the moduli and supermoduli parameters. After carrying out the integrations over the first three sets of variations, we end up with the finite dimensional integration.

For more details on the fixing procedure, the reader is referred to Ref. [13] and we just quote the formula for the supervierbein path integral:

$$\begin{aligned} \int \frac{\mathfrak{D}E_M^A}{\Omega(\text{gauge})} \\ = \int \prod_i d\tau_i \frac{\text{sdet}'(P^\dagger P)^{1/2} \text{sdet} \langle \Psi^{(i)} | \hat{E}_\alpha^M \partial \hat{E}_M^A / \partial \tau_j \rangle}{\Omega(\text{SCK}) \text{sdet} \langle \Psi^{(i)} | \Psi^{(j)} \rangle^{1/2}} . \end{aligned} \quad (2.8)$$

The operator P transforms spin-1 superfields into spin-3/2 superfields according to the rule

$$(P\Xi)_\alpha{}^a = \frac{1}{2} (\gamma_c \gamma^a)_\alpha{}^\beta (D_\beta \Xi^c) . \quad (2.9)$$

The notation sdet' indicates that zero mode contributions have been omitted. The basis for $\text{Ker} P^\dagger$ is denoted by $\{\Psi^{(i)}\}$ and $\Omega(\text{SCK})$ is the volume of superconformal Killing vectors, i.e., the zero modes of the operator P .

The fixed supervierbein fields $\hat{E}_M^A(z, T)$ are taken in the following form

$$\hat{E}_m^a = \hat{e}_m^a(t_i) + i (\theta \gamma^a \chi_m) , \quad (2.10a)$$

$$\hat{E}_m^\alpha = \frac{1}{2} \chi_m^\alpha , \quad (2.10b)$$

$$\hat{E}_\mu^a = -i (\gamma^a \theta)_\mu , \quad (2.10c)$$

$$\hat{E}_\mu^\alpha = \delta_\mu^\alpha . \quad (2.10d)$$

Here $\hat{e}_m^a$ is the zweibein which contains (bosonic) moduli parameters $\{t_i\}$. The Rarita-Schwinger field χ_m^α gives us the (fermionic) supermoduli and is a solution to the equation

$$D_m \chi^m = 0 . \quad (2.11)$$

The N -point scattering amplitude in the path integral approach is expressed as an integral over moduli and supermoduli parameters

$$\begin{aligned} A(k_1, \dots, k_N) = & \frac{1}{Z} \sum \int \prod_i dT_i \int \mathcal{D}\Phi^i e^{-S[\Phi, \hat{E}]} \\ & \times \frac{\text{sdet}'(P^\dagger P)^{1/2} \text{sdet} \langle \Psi^{(i)} | \hat{E}_\alpha{}^M \partial \hat{E}_M^a / \partial T_j \rangle}{\Omega(\text{SCK}) \text{sdet} \langle \Psi^{(i)} | \Psi^{(j)} \rangle^{1/2}} \prod_{i=1}^N V(k_i) . \end{aligned} \quad (2.12)$$

Here $V(k)$ is a vertex operator and k_i are the momenta of the external particles.

3. - BOUNDARY CONDITIONS AND SPIN STRUCTURES

In the supersymmetric theories, the Lagrangian transforms as a total derivative under supersymmetry transformation. In case of a field theory with boundaries, one must impose boundary conditions on various fields to preserve the supersymmetry by eliminating the total derivative term. The boundary conditions have been worked out in Ref. [14] for general Riemann surfaces in the component field language. The analysis of Ref. [14] has been applied to annulus and Möbius bands [15,16], and the spin structures for these surfaces have been classified. It is known that the classification of the spin structures on a torus can be translated into that of the topologies of the supertorus. To carry out an analogous programme in the present case of annulus and Möbius bands, we introduce the involution technique in superspace.

3.1 Annulus

Let us begin with the case of the annulus. The annulus is parametrized by the region

$$0 \leq \sigma^1 \leq 1, \quad 0 \leq \sigma^2 \leq 1, \quad (3.1)$$

with boundaries $\sigma^1 = 0$ and $\sigma^1 = 1$. The point (σ^1, σ^2) is identified with $(\sigma^1, \sigma^2 + 1)$. The zweibein is given by

$$\hat{e}_m^a = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}, \quad (3.2)$$

where t is the moduli parameter ($0 < t < \infty$).

The boundary conditions for the bosonic fields $X^I(\sigma)$ and fermionic fields $\psi^I(\sigma)$ are

$$\partial_1 X^I(0, \sigma^2) = \partial_1 X^I(1, \sigma^2) = 0, \quad (3.3a)$$

$$\psi^I(0, \sigma^2) = \gamma^I \psi^I(0, \sigma^2), \quad \psi^I(1, \sigma^2) = \gamma^I \psi^I(1, \sigma^2). \quad (3.3b)$$

Here γ is $+1$ (-1) for the Ramond (Neveu-Schwarz) model. The Grassmannian supertransformation parameter ε must satisfy

$$\varepsilon = -\gamma^I \varepsilon, \quad \varepsilon = -\gamma^I \varepsilon \quad (3.4)$$

on the boundaries $\sigma^1 = 0$ and $\sigma^1 = 1$, respectively. Because of this constraint, only one of the two rigid supersymmetries is preserved in the open superstring.

In addition to the conditions on the boundaries, we have (anti-) periodicity conditions for fermions, i.e.,

$$\psi^I(\sigma^1, \sigma^2) = \lambda \psi^I(\sigma^1, \sigma^2 + 1), \quad (3.5)$$

where λ takes two values ± 1 . We have thus seen that there are four spin sectors for the annulus corresponding to $(r, s) = (\pm 1, \pm 1)$.

The conditions (3.3) and (3.5) for the component fields may be rephrased in terms of the condition for the superfield. A concise way to achieve this is to apply the involution method to the supertorus to obtain the superannulus. Suppose we have matter fields on the torus where (σ^1, σ^2) , $(\sigma^1 + 2, \sigma^2)$ and $(\sigma^1, \sigma^2 + 1)$ are identified. They satisfy (anti-) periodicity conditions

$$X^I(\sigma^1, \sigma^2) = X^I(\sigma^1 + 2, \sigma^2) = X^I(\sigma^1, \sigma^2 + 1), \quad (3.6a)$$

$$\psi^I(\sigma^1, \sigma^2) = \lambda \psi^I(\sigma^1 + 2, \sigma^2) = \lambda \psi^I(\sigma^1, \sigma^2 + 1), \quad (3.6b)$$

where $(r, s) = (\pm 1, \pm 1)$.

To deal with the supertorus, it is convenient to use the notation

$$\xi = \sigma^1 + \tau \sigma^2, \quad \zeta = \theta^1 + i \theta^2. \quad (3.7)$$

There are four types of supertori $ST_{(r,s)}(\tau, \delta)$ corresponding to the 2×2 spin structures on the torus $T(\tau)$, where τ and δ are the (complex) moduli and supermoduli parameters. The supertorus $ST_{(r,s)}$ is defined by the periodicity of the superfield $\Phi^I(z^M)$ on it

$$\Phi^I(z^M) = \Phi^I(z^{M'}) \quad (3.8)$$

with respect to $^*)$

$^*)$ The torus which is the double of the annulus (3.1) has the period in σ^1 twice as large as that considered here.

$$(\lambda, \mu) = (+, +)$$

$$(\xi', \zeta') = (\xi + \tau + \delta \zeta, \zeta + \delta), \quad (3.9)$$

$$(\xi', \zeta') = (\xi + 1, \zeta) ;$$

$$(\lambda, \mu) = (+, -), (-, +), (-, -)$$

$$(\xi', \zeta') = (\xi + \tau, \mu \zeta),$$

$$(\xi', \zeta') = (\xi + 1, \lambda \zeta). \quad (3.10)$$

The superannulus $SA_{(r,s)}(t,b)$ is obtained from the supertorus $ST_{(r,s)}(\tau, \delta)$ where $\tau = it/2$ and $\delta = (1+i)b$, by identifying the pair of conjugate points on it. The conjugate points are defined by the involution^{*}

$$z^M \longrightarrow \tilde{z}^M = (\tilde{\sigma}^m, \tilde{\theta}^\mu) \quad (3.11)$$

with

$$\tilde{\sigma}^m \equiv (-\sigma^1, \sigma^2), \quad \tilde{\theta} \equiv -\gamma^1 \theta. \quad (3.12)$$

The superboundaries of the SA are

$$\sigma^1 = 0, \quad \theta = -\gamma^1 \theta \quad (\theta_1 + \theta_2 = 0) \quad (3.13)$$

and

$$\sigma^1 = 1, \quad \theta = -\lambda \gamma^1 \theta \quad (\theta_1 + \lambda \theta_2 = 0). \quad (3.14)$$

They are the fixed points of the ST with respect to the involution (3.11):

$$z^M = \tilde{z}^M \quad (3.15)$$

and

^{*}) The involution in superspace has been considered previously in evaluating superstring tree amplitudes [17].

$$\tilde{z}^M = (\tilde{\sigma}^1 + 2, \tilde{\sigma}^2, \tilde{\theta}) \quad (3.16)$$

The boundary conditions (3.3) and (3.5) for the component fields can be translated into a single condition for the superfield, i.e., the invariance of $\Phi^I(z^M)$ under the involution (3.11) with (3.12):

$$\Phi^I(\tilde{z}^M) = \Phi^I(z^M) . \quad (3.17)$$

This can be easily seen by writing Eq. (3.17) explicitly,

$$\begin{aligned} \chi^I(\sigma^1, \sigma^2) + \theta \psi^I(\sigma^1, \sigma^2) + \frac{1}{2} \theta \theta F^I(\sigma^1, \sigma^2) \\ = \chi^I(-\sigma^1, \sigma^2) + \theta \tilde{\psi}^I(-\sigma^1, \sigma^2) - \frac{1}{2} \theta \theta F^I(-\sigma^1, \sigma^2) . \end{aligned} \quad (3.18)$$

To sum up, the matter fields on the annulus can be described equivalently in terms of superfield Φ^I on the supertorus with involution invariance (3.17).

3.2 The Möbius band

The zweibein for the Möbius band is taken to be the same as Eq. (3.2) and the parameter region is

$$0 \leq \sigma^1 \leq 1 , \quad 0 \leq \sigma^2 \leq 2 \quad (3.19)$$

The point (σ^1, σ^2) is identified with $(1-\sigma^1, \sigma^2+1)$. There is only one boundary ($\sigma^1=0$) and the classification of the spin structures differs from the case of the annulus. Fortunately, the involution technique makes the classification almost parallel to the annulus case: we will start with superfields on a supertorus and then we impose the involution invariance, deducing the matter fields on the Möbius band. In the Möbius case, however, the involution is defined separately in the Ramond and Neveu-Schwarz models.

In the Ramond case ($r=+1$), we prepare a torus where (σ^1, σ^2) , (σ^1+2, σ^2) and (σ^1+1, σ^2+1) are identified. The matter fields on the torus must satisfy

$$\chi^I(\sigma^1, \sigma^2) = \chi^I(\sigma^1+2, \sigma^2) = \chi^I(\sigma^1+1, \sigma^2+1) , \quad (3.20a)$$

$$\psi^I(\sigma^1, \sigma^2) = \psi^I(\sigma^1+2, \sigma^2) = -\psi^I(\sigma^1+1, \sigma^2+1) , \quad (3.20b)$$

where $s = \pm 1$.

The super-Möbius band $SM_{(+,s)}(t,b)$ is obtained from the supertorus $ST_{(+,s)}(\tau,\delta)$, where $\tau = it/2 + \frac{1}{2}$ and $\delta = (1+i)b$, by making the identification z^M with

$$\tilde{z}^M = (\tilde{\sigma}^m, \tilde{\theta}^\mu) . \quad (3.21)$$

Here we have introduced the quantities

$$\tilde{\sigma}^m = (1 - \sigma^1, \sigma^2 + 1) , \quad \tilde{\theta} = -\gamma^1 \theta . \quad (3.22)$$

The SM has one superboundary defined by the fixed point equation of the ST

$$z^M = (\tilde{\sigma}^1 - 1, \tilde{\sigma}^2 - 1, \Delta \tilde{\theta}) = (-\sigma^1, \sigma^2, -\Delta \gamma^1 \theta) \quad (3.23)$$

[recall (3.9) and (3.10)], which gives

$$\sigma^1 = 0 , \quad \theta = -\Delta \gamma^1 \theta \quad (\theta_1 + \Delta \theta_2 = 0) . \quad (3.24)$$

The matter superfield Φ^I on the super-Möbius band $SM_{(+,s)}$ is defined as that on the supertorus $ST_{(+,s)}$ satisfying the involution invariance, $\Phi^I(z^M) = \Phi^I(\tilde{z}^M)$. The components X^I and ψ^I of the Φ^I then satisfy the desired boundary conditions for the Möbius band

$$\partial_1 X^I(0, \sigma^2) = 0 , \quad (3.25)$$

$$\psi^I(0, \sigma^2) = \Delta \gamma^1 \psi^I(0, \sigma^2) . \quad (3.26)$$

Let us turn to the Neveu-Schwarz sector ($r=-1$). We start with the same torus as in the Ramond case, and the bosonic string field satisfies Eq. (3.20a). The (anti-) periodicity of the fermionic part is, however, not Eq. (3.20b), but

$$\psi^I(\sigma^1, \sigma^2) = -\psi^I(\sigma^1 + 2, \sigma^2) = -\Delta \gamma^3 \psi^I(\sigma^1 + 1, \sigma^2 + 1) . \quad (3.20c)$$

The supertorus $ST'_{(-,s)}^{*)}$ which is the double of the Neveu-Schwarz sector of the super-Möbius band $SM_{(-,s)}$ is defined by the periodicity of $\Phi^I(z^M)$ on it, $\Phi^I(z^M) = \Phi^I(z^{M'})$ with respect to

*) Strictly speaking, ST' is not a supertorus in the sense that ξ is involved in Eq. (3.27a).

$$(\xi', \zeta') = (\xi + \tau, -\lambda \bar{\xi}) , \quad (3.27a)$$

$$(\xi', \zeta') = (\xi + 1, -\xi) , \quad (3.27b)$$

instead of (3.10). In Eq. (3.27a), $\zeta' = -s\bar{\xi}$ corresponds to

$$\theta' = -\lambda \gamma^3 \theta . \quad (3.28)$$

The involution $z^M \rightarrow \tilde{z}^M$ is defined with

$$\tilde{\sigma}^m = (1 - \sigma^1, \sigma^2 + 1) , \quad \tilde{\theta} = -i \gamma^2 \theta . \quad (3.29)$$

The superboundary is defined by the fixed-point equation

$$\tilde{z}^M = (\tilde{\sigma}^1 - 1, \tilde{\sigma}^2 - 1, -\lambda \gamma^3 \tilde{\theta}) = (-\sigma^1, \sigma^2, -\lambda \gamma^1 \theta) , \quad (3.30)$$

which coincides with the Ramond case (3.23). Hence the Neveu-Schwartz SM has the same superboundary as the Ramond SM, (3.24).

The matter superfield Φ^I on the super-Möbius band $SM_{(-,s)}$ is defined as that on the supertorus $ST'_{(-,s)}$ satisfying the involution invariance (3.17). The components of the Φ^I then satisfy the boundary conditions for the Möbius band, (3.25) and (3.26) given above.

Finally, let us come to the conditions for χ^m . The Rarita-Schwinger field χ^m is a spinor and it has to be assigned an appropriate spin structure. It is fixed by the following consideration. i) Under supersymmetry transformation, the action changes by boundary terms. They have to be eliminated by imposing the condition

$$\chi^1 \psi^I = 0 \quad (3.31)$$

on the boundaries. ii) The product $\chi^m \psi^I$ must satisfy the same periodicity as X^I . The resulting conditions for χ^m are

$$\chi^1(0, \sigma^2) = \gamma^1 \chi^1(0, \sigma^2) , \quad \chi^1(1, \sigma^2) = \lambda \gamma^1 \chi^1(1, \sigma^2) , \quad (3.32)$$

$$\chi^m(\sigma^1, \sigma^2) = \lambda \chi^m(\sigma^1, \sigma^2 + 1) \quad (3.33)$$

for the annulus and

$$\chi^1(0, \sigma^2) = \lambda \gamma^1 \chi^1(0, \sigma^2), \quad (3.34)$$

$$\chi^m(\sigma^1, \sigma^2) = \begin{cases} \lambda \chi^m(\sigma^1+1, \sigma^2+1), & (\lambda = +) \\ \lambda \gamma^3 \chi^m(\sigma^1+1, \sigma^2+1), & (\lambda = -) \end{cases} \quad (3.35a)$$

$$(3.35b)$$

for the Möbius band.

4. - SUPERMODULI AND SUPERCONFORMAL KILLING VECTOR

Having established the boundary conditions, we are now in a position to work out necessary quantities to be substituted into the formula (2.8) for annulus and Möbius bands.

First of all, we have to derive the explicit form of the supermoduli which is contained in the supervierbein. The defining equation of the supermoduli (2.11) should be solved with the metric (3.2) under the boundary conditions (3.32) through (3.35). It is straightforward to see that there is no solution in the sectors $(r,s) = (-1,1)$, $(1,-1)$ or $(-1,-1)$. The boundary conditions do not admit any solution. For $(r,s) = (+1,+1)$, however, there is one and only one solution, which may be written in the following form

$$\chi^m = b \gamma^n \gamma^m \partial_n \eta, \quad (4.1)$$

where b is an anticommuting Grassmann number and η is a spin- $\frac{1}{2}$ spinor

$$\eta_\alpha = \frac{i}{2} \sqrt{t} \begin{bmatrix} \sigma^2 \\ -\sigma^2 \end{bmatrix}. \quad (4.2)$$

We will call b the supermoduli parameter. Here the factor $\sqrt{t}$ in Eq. (4.2) is to take into account the normalization of χ^m .

Another important quantity in Eq. (2.8) is the set of spin-3/2 superfields $\{\Psi^{(i)}\}$, namely, the zero mode of $P^\dagger$. More explicitly, we have to solve

$$(P^\dagger \Psi^{(i)})^a = D^\alpha \Psi_\alpha^{(i)a} = 0 , \quad (4.3)$$

$$(\gamma_a)_\alpha{}^\beta \Psi_\beta^{(i)a} = 0 . \quad (4.4)$$

There are two solutions and in fact we have found

$$\begin{aligned} \Psi_\alpha^{(1)a} = (\gamma_\theta \theta)_\alpha \left\{ \hat{e}^{am} \frac{\partial \hat{e}_m{}^\theta}{\partial t} + \hat{e}^{\theta m} \frac{\partial \hat{e}_m{}^a}{\partial t} \right. \\ \left. - \delta^{a\theta} \hat{e}_c{}^m \frac{\partial \hat{e}_m{}^c}{\partial t} \right\} , \end{aligned} \quad (4.5)$$

$$\Psi_\alpha^{(2)a} = (\gamma^n \gamma^a \partial_n \eta)_\alpha . \quad (4.6)$$

Let us turn to the superconformal Killing vector, the zero mode of P, i.e.,

$$(P \Xi)_\alpha{}^a = \frac{1}{2} (\gamma_c \gamma^a)_\alpha{}^\beta (D_\beta \Xi^c) = 0 . \quad (4.7)$$

The solution is obtainable rather trivially, and we have found

$$\Xi^a = u^a + i(\theta \gamma^a \lambda) . \quad (4.8)$$

Here, u^a is called conformal Killing vector and is proportional to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} . \quad (4.9)$$

The spinor λ in (4.8) corresponds to conformal Killing spinor and is proportional to

$$\hat{\theta}_\alpha \equiv \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \quad (4.10)$$

The existence of the superconformal Killing vector (4.8) tells us that our system is invariant under the translation in superspace along the direction dictated by (4.8).

5. - VECTOR BOSON EMISSION VERTEX IN SUPERSPACE

As we wish to calculate the gauge anomaly, we now discuss the vertex operator responsible for the vector boson emission, by using the superspace formulation. The vector boson is emitted and absorbed on the boundary, and considerations of the boundary conditions become vital. As we saw in Section 3, the supertransformation parameter is constrained by Eq. (3.4) and therefore the vertex operator should be described in the one-dimensional superspace as a line-integral along the "boundary of the supermanifold" [18], (3.13) and (3.14) for the superannulus and (3.24) for the super-Möbius band.

The vertex operator should be invariant under the superdiffeomorphism in the one-dimensional superspace and we are led to take as the vector boson emission vertex

$$V(k) = \zeta_I \int d\sigma^m \varepsilon_{mn} \int d\theta^\alpha \bar{E} (\tau^n)_\alpha{}^\beta D_\beta \Phi^I e^{ik \cdot \Phi}. \quad (5.1)$$

Here ζ_I is the polarization vector and $\bar{E}$ is the superdeterminant of the supervielbein in the one-dimensional superspace. The integral is along the superboundary given in Section 3. The superspace expression (5.1) has been proposed previously for the vertex in the Neveu-Schwarz sector [18]. The same expression should be used independently of the spin sector.

An important thing to be noticed here is that in the Ramond-Ramond sector, the supercovariant derivative and the superdeterminant $\bar{E}$, both depend on the supermoduli parameter. It turns out that the two sources of the supermoduli dependence cancel each other^{*}). The vertex operator for vector boson emission therefore does not contain the supermoduli in any spin sector. More explicitly, it is expressed in the case of annulus and Möbius bands as

$$V(k) = \int d(t\sigma^2) \int da \{ V_{II}(\sigma, \zeta, k) + a V_I(\sigma, \zeta, k) \}, \quad (5.2)$$

where

$$V_I(\sigma, \zeta, k) = 2 \zeta_I \left\{ \frac{1}{t} \partial_2 X^I + \frac{i}{2} k_J (\hat{\theta} \psi^J)(\hat{\theta} \psi^I) \right\} e^{ik \cdot X}, \quad (5.3)$$

$$V_{II}(\sigma, \zeta, k) = \zeta_I (\hat{\theta} \psi^I) e^{ik \cdot X}. \quad (5.4)$$

^{*}) The independence of the vertex on the supermoduli parameter does not appear to be a general feature shared by vertices of massive particle emission.

We have replaced $d\theta^\alpha$ in Eq. (5.1) by $d\theta^\alpha = da\hat{\theta}^\alpha$ for $(r,s) = (+,+)$ where a is a Grassmannian number. Note that the vertex operators (5.3) and (5.4) are analogous to those in the R_1 and R_2 pictures in the operator formalism [19].

We saw in Section 4 that both annulus and Möbius bands have a superconformal Killing vector in their Ramond-Ramond sector. This means that we can fix one of the positions of vertex operators arbitrarily on the boundary of the super-Riemann surface, while integrations over the remaining positions should be carried out. In the component language, one of the vertices is given by Eq. (5.4), whereas the rest by Eq. (5.3). The phenomenon that one of the vertices differs in form from others is peculiar to the Ramond-Ramond sector and is an immediate consequence of the existence of the superconformal Killing vector (4.8). The fixing of one of the positions may be handled in the same way as the Faddeev-Popov trick. After the fixing we get an invariance volume associated with the superconformal Killing vector, which is, however, cancelled by the volume $\Omega(\text{SCK})$ in Eq. (2.8).

Let us discuss the origin of the hexagon gauge anomaly in the path integral framework. To evaluate the matrix element of the gauge current divergence, one has to replace the polarization vector ζ_I by the momentum k_I in the vertex operator. It is not difficult to see that $V_I(\sigma, k, k)$ vanishes after integration over its position. The vertex operator $V_{II}(\sigma, k, k)$, on the other hand, does not always vanish and may be potentially anomalous. The appearance of the latter vertex operator is attributed to the existence of the conformal Killing spinor. Therefore, the anomaly would not show up unless there were conformal Killing spinors. As pointed out by Polchinski and Cai [4], this observation leads to the conclusion that there is no higher stringy-loop correction to the hexagon anomaly.

6. - HEXAGON ANOMALY

We are now well equipped to evaluate the hexagon gauge anomaly. The gauge anomaly could arise in parity-violating processes and hence in the Ramond-Ramond sector [20]. Let us apply formula (2.12) to annulus and Möbius bands in the Ramond-Ramond sector with the vertex operator given by Eq. (5.2). We have previously seen that the vertices (5.2) do not depend on the supermoduli and the action is the only source of the supermoduli dependence in the integrand of (2.12). In fact, the action in the superconformally flat metric (2.10) is expressed as

$$S[\Phi, \hat{E}] = S_0 - \frac{1}{2} \int d^2\sigma \sqrt{g} \, x^m \dot{x}_m, \quad (6.1)$$

where S_0 is the free part of the action

$$S_0 = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \left\{ \frac{1}{2} g^{mn} \partial_m X^I \partial_n X^I - \frac{i}{2} \psi^I \gamma^m \partial_m \psi^I \right\} \quad (6.2)$$

and j_m is the spinor current associated with the global supersymmetry

$$j_m = \frac{1}{2\pi\alpha'} (\gamma^n \gamma_m \psi^I) \partial_n X^I. \quad (6.3)$$

Thus the Grassmannian supermoduli integration gives us the following expression for the anomaly

$$\begin{aligned} T(k_1, \dots, k_6) &= \frac{1}{Z} \sum \int_0^\infty dt G(t, k_1, \dots, k_6) \\ &\times \frac{\text{sdet}'(P^\dagger P)^{1/2} \text{sdet} \langle \Psi^{(i)} | \hat{E}_\alpha^M \partial \hat{E}_M^a / \partial T_j \rangle}{\text{sdet} \langle \Psi^{(i)} | \Psi^{(j)} \rangle^{1/2}}. \end{aligned} \quad (6.4)$$

Here we have defined

$$\begin{aligned} G(t, k_1, \dots, k_6) &= \frac{1}{2} \int \prod_{i=1}^5 d(t\sigma_i^2) \int d^2\sigma \sqrt{g} \\ &\times \partial_n \eta(\sigma) \gamma^m \gamma^n \langle j_m(\sigma) \prod_{i=1}^5 V_I(\sigma_i, \zeta_i, k_i) V_{II}(\sigma_6, k_6, k_6) \rangle \end{aligned} \quad (6.5)$$

and the expectation value is defined as a path integral over the matter superfield

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi^I e^{-S_0} \mathcal{O}. \quad (6.6)$$

The position of the sixth vertex is fixed in Eq. (6.5) and we have replaced the polarization vector ζ_6 by the momentum k_6 to have the divergence of the potentially anomalous current.

We will evaluate the Green function in Eq. (6.5) by making use of the Ward identity associated with the (global) supersymmetry. Useful transformation properties of the vertex operators and supercurrent under supersymmetry are as follows:

$$\{Q_\alpha, \hat{j}_{m\beta}\} = 2(\gamma^n)_{\alpha\beta} \Theta_{nm}, \quad (6.7)$$

$$[Q_\alpha, V_I(\sigma, \zeta, k)] = \frac{1}{\zeta} (\gamma^2 \hat{\theta})_\alpha \partial_2 V_{II}(\sigma, \zeta, k), \quad (6.8)$$

$$[Q_\alpha, V_0(\sigma, k)] = -\frac{1}{2} (\gamma^2 \hat{\theta})_\alpha V_{II}(\sigma, k, k), \quad (6.9)$$

where Q_α and Θ_{nm} are the supercharges and the energy-momentum tensor and

$$V_0(\sigma, k) = \exp[ik \cdot X(\sigma)]. \quad (6.10)$$

We can replace the vertex operator $V_{II}(\sigma, k_6, k_6)$ in Eq. (6.5) by the commutator in Eq. (6.9). The supercharge is then shuffled in such a way that the Green function is expressed by the (anti-) commutators of Eqs. (6.7) and (6.8). The commutator (6.8) does not survive after the σ^2 -integration since it is a total derivative. We thus reach the following formula

$$\begin{aligned} G(t, k_1, \dots, k_6) &= i \int \prod_{i=1}^5 d(t\sigma_i^2) \int d^2\sigma \sqrt{g} \hat{\theta} \gamma^n \gamma^2 \gamma^m \partial_2 \eta(\sigma) \\ &\times \langle \Theta_{mn}(\sigma) \prod_{i=1}^5 V_I(\sigma_i, \zeta_i, k_i) V_0(\sigma_6, k_6) \rangle. \end{aligned} \quad (6.11)$$

Green functions in which the energy momentum tensor is inserted have been studied in recent literatures [21,22] on general Riemann surfaces. As for the annulus and Möbius bands, we may proceed in the following way to transform the Green function in Eq. (6.11). The symmetric and traceless properties of the energy momentum tensor enable us to replace $(\hat{\theta}_\gamma^n \gamma^\lambda \gamma^m \partial_\lambda \eta)$ in (6.11) by $2\delta^n(\hat{\theta}_\gamma^m \eta)$. Note that $(\hat{\theta}_\gamma^m \eta)$ is non-vanishing only for $m = 2$. Let us now perform the partial integration in Eq. (6.11), retaining the surface term carefully. Then we will have commutators of the vertex operators with the generator of translation in σ^2 -direction. We have thus end up with the following expression

$$G(t, k_1, \dots, k_6) = -\frac{2}{\sqrt{t}} \int \prod_{i=1}^5 d(t\sigma_i^2) \quad (6.12)$$

$$\times \left\{ \frac{\partial}{\partial t} + \sum_{i=1}^6 \frac{\sigma_i^2}{t} \frac{\partial}{\partial \sigma_i^2} + \frac{5}{t} \right\} \left\langle \prod_{i=1}^5 V_I(\sigma_i, \zeta_i, k_i) V_0(\sigma_6, k_6) \right\rangle.$$

The t -derivative in brackets is due to the surface term generated in the partial integration and should be taken with $t\sigma_i^2$ being fixed. The third term is due to the conformal dimensions of the vertex operators. The derivative terms in brackets in Eq. (6.12) may be put in the form of the total derivative

$$G(t, k_1, \dots, k_6) \quad (6.13)$$

$$= -\frac{2}{\sqrt{t}} \frac{\partial}{\partial t} \int \prod_{i=1}^5 d(t\sigma_i^2) \left\langle \prod_{i=1}^5 V_I(\sigma_i, \zeta_i, k_i) V_0(\sigma_6, k_6) \right\rangle.$$

The final step of our calculation is to evaluate the expectation value in Eq. (6.13). The zero mode path integral of the string fields requires separate consideration from others. The bosonic string variables X^I has a constant zero mode, whose path integral gives us, as usual, the momentum conservation law

$$(2\pi)^{10} \delta^{10}(\sum k_i) \left(\int d^2\sigma \sqrt{g} \right)^5. \quad (6.14)$$

The matter fermion, on the other hand, has ten zero modes

$$\psi^I = \frac{1}{\sqrt{2t}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (6.15)$$

($I = 1, \dots, 10$), where the boundary condition has been taken into account. A close look at Eq. (5.3) shows that the zero mode path integrals are non-vanishing if and only if the five vertex operators $V_I(\sigma_i, \zeta_i, k_i)$ ($i = 1, \dots, 5$) give us ten zero modes

through the second term of Eq. (5.3). In fact, we get the antisymmetric tensor structure

$$\frac{1}{t^5} \varepsilon(\zeta, k) = \frac{1}{t^5} \varepsilon_{I_1 J_1} \dots \varepsilon_{I_5 J_5} \zeta_1^{I_1} k_1^{J_1} \dots \zeta_5^{I_5} k_5^{J_5}. \quad (6.16)$$

Note that the integration over the X^I zero modes gave the moduli dependence t^5 , which is cancelled by $1/t^5$ arising from the integration over the ϕ^I zero modes.

The Green function (6.13) is thus reduced to

$$G(t, k_1, \dots, k_6) = \text{const.} \varepsilon(\zeta, k) (2\pi)^{10} \delta^{10}(\sum k_i) \times \frac{1}{\sqrt{t}} \frac{\partial}{\partial t} \int \prod_{i=1}^5 d(t\sigma_i^2) \langle \prod_{i=1}^6 V_0(\sigma_i, k_i) \rangle_0, \quad (6.17)$$

where $\langle \dots \rangle_0$ means that the zero mode path integrals are excluded. The evaluation of Eq. (6.17) is almost standard. The determinant of the bosonic and fermionic string variables gives us after zeta-function regularization

$$[\det'(\gamma \cdot \partial)]^{10} / [\det'(-\square)]^5 = t^{-5}. \quad (6.18)$$

The t -dependence in the measure of σ_i^2 -integration in Eq. (6.17) is cancelled altogether by (6.18). The final form of the Green function turns out to be

$$G(t, k_1, \dots, k_6) = \text{const.} \varepsilon(\zeta, k) (2\pi)^{10} \delta^{10}(\sum k_i) \times \frac{1}{\sqrt{t}} \frac{\partial}{\partial t} \int \prod_{i=1}^5 d\sigma_i^2 \exp[\pi\alpha' \sum k_i \cdot k_j N(\sigma_i, \sigma_j; t)], \quad (6.19)$$

where $N(\sigma_i, \sigma_j; t)$ is the Neumann function on either the annulus or the Möbius band.

The formula for the hexagon anomaly is completed by putting Eq. (6.19) and

$$\text{sdet} \langle \Psi^{(i)} | \hat{E}_\alpha^M \partial \hat{E}_M^a / \partial T_j \rangle = \text{const.} \frac{1}{t} , \quad (6.20)$$

$$\text{sdet} \langle \Psi^{(i)} | \Psi^{(j)} \rangle = \text{const.} \frac{1}{t} , \quad (6.21)$$

$$\text{sdet}'(P^\dagger P) = \text{const.} t \quad (6.22)$$

into Eq. (6.4), namely,

$$T(k_1, \dots, k_6) = \text{const.} \mathcal{E}(\zeta, k) \delta^{10}(\sum k_i) \quad (6.23)$$

$$\times \sum \int_0^\infty dt \frac{\partial}{\partial t} \int \prod_{i=1}^5 d\sigma_i^2 \exp [\pi \alpha' \sum k_i \cdot k_j N(\sigma_i, \sigma_j; t)] .$$

Suppose that the internal gauge group is $SO(N)$. The contribution of the annulus has the Chan-Paton factor N , while the Möbius band has the weight -2^5 . The latter comes from the difference in the integration region over σ_i^2 . The resulting expression for the anomaly is

$$T(k_1, \dots, k_6) = \text{const.} \mathcal{E}(\zeta, k) \delta^{10}(\sum k_i) \int_0^\infty dt \frac{\partial}{\partial t} \int_0^1 \prod_{i=1}^5 d\sigma_i^2$$

$$\times \left\{ N \exp [\pi \alpha' \sum k_i \cdot k_j N_A(\sigma_i, \sigma_j; t)] \right.$$

$$\left. - 32 \exp [\pi \alpha' \sum k_i \cdot k_j N_M(\sigma_i, \sigma_j; t)] \right\} , \quad (6.24)$$

where N_A and N_M are the propagators on annulus and Möbius bands, respectively.

7. - DISCUSSIONS AND SUMMARY

The formula (6.24) for the anomaly, being expressed as a surface integral in the t -space, clearly shows that the anomaly comes from the boundary of the moduli space. Our result, of course, agrees with the operator calculation [5,6]. The momenta of six external gauge bosons may be arranged so that the contribution from $t = \infty$ to Eq. (6.24) vanishes, thereafter the analytic continuation to the physical

region in the momentum space being made. Thus the contribution to the anomaly comes from $t = 0$, where the topology of the world-sheet changes.

The calculation presented in this paper in the path integral method has clarified the roles played by the supermoduli and superconformal Killing vector. We have seen that the potentially anomalous vertex (5.4) is caused to come in by the existence of the conformal Killing spinor. The anomaly would not exist, were it not for the conformal Killing spinor. As pointed out by Polchinski and Cai [4], this observation enables us to derive the Adler-Bardeen-like theorem in string theories.

We envisage a few other problems in which the technique developed in the present paper may be useful. It has been suggested that the boundary of the moduli space may be a source of conformal symmetry breaking [23]. This observation has emerged from the study of string theories in background fields. The requirement of the conformal invariance on the string loop level induces a correction to equations of motion of the background fields. The operator method invented in Ref. [24] to include the string loop correction may be rephrased in the path integral method by using a technique similar to ours.

It is also known that the boundary of the moduli space would jeopardize the BRST symmetry [25]. It has been pointed out that this anomaly should be cancelled by modifying vertex operators, giving due care to the mass renormalization [26]. This observation may be studied further by the method similar to ours. We will discuss these subjects elsewhere.

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REFERENCES

- [1] A.M. Polyakov, Phys. Lett. B103 (1981) 207, 211.
- [2] D. Friedan, in Recent Advances in Field Theory and Statistical Mechanics, Ed. Z. Zuber and R. Stora (North-Holland, Amsterdam, 1984), p. 839;
K. Fujikawa, Phys. Rev. D25 (1982) 2584;
O. Alvarez, Nucl. Phys. B216 (1983) 125.
- [3] T. Inami, H. Kanno and T. Kubota, Phys. Lett. B199 (1987) 389.
- [4] J. Polchinski and Y. Cai, Nucl. Phys. B296 (1988) 91.
- [5] M.B. Green and J.H. Schwarz, Nucl. Phys. B255 (1985) 93.
- [6] L. Clavelli, P.H. Cox and B. Harms, Phys. Rev. D35 (1987) 1908.
- [7] H. Suzuki and A. Sugamoto, Phys. Rev. Lett. 57 (1986) 1665;
H. Suzuki, Phys. Lett. 182B (1986) 37;
M. Hayashi, N. Kawamoto, T. Kuramoto and K. Shigemoto, Nucl. Phys. B294 (1987) 459; *ibid.* B296 (1988) 373.
- [8] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93;
E. Martinec, Nucl. Phys. B281 (1987) 157.
- [9] G. Moore, P. Nelson and J. Polchinski, Phys. Lett. B169 (1986) 47;
P. Nelson and G. Moore, Nucl. Phys. B274 (1986) 509;
E. D'Hoker and D.H. Phong, Nucl. Phys. B278 (1986) 225;
V.G. Knizhnik, Phys. Lett. B178 (1986) 21;
M. Bonini and R. Iengo, Phys. Lett. 191B (1987) 56.
- [10] K. Pilch, A.N. Schellekens and N.P. Warner, Nucl. Phys. B287 (1987) 362.
- [11] K. Miki, Nucl. Phys. B291 (1987) 349;
S. Yahikozawa, Nucl. Phys. B291 (1987) 369.
- [12] P.S. Howe, J. Phys. A12 (1979) 393.
- [13] E. Martinec, Phys. Rev. D28 (1983) 2604;
S. Chaudhuri, H. Kawai and S.H.H. Tye, Phys. Rev. D36 (1987) 1148.
- [14] P. Di Vecchia, B. Durhuus, P. Olesen and J.L. Petersen, Nucl. Phys. B207 (1982) 77.
- [15] T. Inami, H. Kanno and T. Kubota, Prog. Theor. Phys. 77 (1987) 26.
- [16] C.P. Burgess and T.R. Morris, Nucl. Phys. B291 (1987) 256, 285.
- [17] H. Itoyama and P. Moxhay, Nucl. Phys. B293 (1987) 685.
- [18] Y. Tanii and Y. Watabiki, preprint of Tokyo Institute of Technology TIT/HEP 111 (1987).
- [19] L. Brink and J.O. Winnberg, Nucl. Phys. B103 (1976) 445;
A. Neveu, J. Schwarz and C. Thorn, Phys. Lett. B35 (1971) 521;
J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
- [20] N. Seiberg and E. Witten, Nucl. Phys. B276 (1986) 272.
- [21] T. Eguchi and H. Ooguri, Nucl. Phys. B282 (1987) 308.
- [22] H. Sonoda, Nucl. Phys. B281 (1987) 546.

- [23] W. Fischler and L. Susskind, Phys. Lett. 171B (1986) 383; *ibid.* 173B (1986) 262.
- [24] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B293 (1987) 83.
- [25] P. Mansfield, Nucl Phys. B283 (1987) 551.
- [26] N. Seiberg, Phys. Lett. B187 (1987) 56;
A. Sen, preprint SLAC-PUB-4383 (1987).