THEORY OF RF ACCELERATION

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ABSTRACT
Formulae for RF acceleration and synchrotron motion are derived from basic principles in the case of an arbitrary RF voltage.

1. ENERGY GAIN AND TRANSIT TIME FACTOR

Particles experience the effect of RF fields when they cross accelerating gaps that basically produce an electric field \( \xi \) parallel to their trajectories. The gap is the space between two electrodes provided with a beam pipe, which for simplicity we take as a circular cylinder of radius \( a \).

Let \( \hat{V}(r) \) be the amplitude of the RF voltage impressed across the two electrodes. When a particle with electric charge \( e \) (which may be larger than an electron charge) crosses the gap at a distance \( r \) from the \( s \)-axis (see Fig. 1.1), it gains an energy

\[
\Delta E = e \int \delta_s(s,r,t) \, ds
\]

![Fig. 1.1 - Longitudinal cross section of an accelerating gap](image)

The time dependence of \( \delta_s \) is given by

\[
\delta_s(s,r,t) = \hat{\delta}(s,r) \sin(\omega_{RF}t)
\]

Traditionally, for circular accelerators the origin of time is taken at the zero crossing of the RF voltage with positive slope. The phase \( \phi \) of the RF voltage when a particle crosses the middle of the accelerating gap (at \( s = 0 \)) is called the phase of the particle with respect to the RF voltage. On the other hand, for circular accelerators in the Russian literature and for linacs, the origin of time is taken at the crest of the RF voltage. The phase \( \phi \) in that case is such that \( \phi = \frac{n}{2} + \omega \) (Strictly speaking, in the previous sentences, the term "RF voltage" should be understood as "RF voltage times the charge \( e \) of the particles"). If we neglect the change in velocity of the particle when crossing the gap, the time \( t \) when the particle is at position \( s \) in the gap reads
\[ t = \frac{\phi}{v_{RF}} + \frac{S}{v} \quad \text{and} \quad \omega_{RF} t = \phi + \frac{v_{RF}}{v} S \]

where \( v \) is the particle velocity in the middle of the gap.

For simplicity's sake we assume that the gap is symmetric with respect to the plane \( s = 0 \); then

\[ \Delta E = e \int \hat{\varepsilon}_S (s,r) \sin \left( \phi + \frac{\omega_{RF}}{v} s \right) \, ds = e \sin \phi \int \hat{\varepsilon}_S (s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) \, ds \quad (1.1) \]

By representing the fields for \( r \leq a \) as Fourier integrals along \( s \), one gets

\[ \int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) \, ds = \frac{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{r}{y} \right)}{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{a}{y} \right)} \int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,a) \cos \left( \frac{\omega_{RF}}{v} s \right) \, ds \]

With \( \tilde{V}(r) = \int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,r) \, ds = \tilde{V}(0) \cdot J_0 \left( \frac{\omega_{RF}}{C} \cdot r \right) \), this may be written as

\[ \int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) \, ds = \tilde{V}(r) \cdot T(r) = \frac{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{r}{y} \right)}{I_0 \left( \frac{\omega_{RF}}{v} \cdot \frac{a}{y} \right)} \cdot \tilde{V}(a) \cdot T(a) \]

where by definition

\[ T(r) = \frac{\int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,r) \cos \left( \frac{\omega_{RF}}{v} s \right) \, ds}{\int_{-\infty}^{+\infty} \hat{\varepsilon}_S (s,r) \, ds} \]

is the transit time factor at \( r \); it is the ratio of peak energy gained by a particle with velocity \( v \) to the same quantity if \( v \) were infinite.

\[ T(r) \text{ is simplest at } r = a, \text{ where } \hat{\varepsilon}_S \text{ is zero outside the gap. In many practical cases, a good approximation is obtained when } \hat{\varepsilon}_S (s,a) \text{ is considered to be constant in the gap; then} \]

\[ T(a) = \frac{\sin \left( \frac{\omega_{RF}}{v} \cdot \frac{a}{2} \right)}{\left( \frac{\omega_{RF}}{v} \cdot \frac{a}{2} \right)} \quad (1.2) \]
Finally,

\[ \Delta E = eV \sin \phi \]  \hspace{1cm} (1.3)

where

\[ V = \tilde{V}(a) \cdot \frac{I_0\left(\frac{\omega_{RF}}{V} \frac{r}{\gamma} \right)}{I_0\left(\frac{\omega_{RF}}{V} a \frac{r}{\gamma} \right)} = \tilde{V}(0) \cdot \frac{I_0\left(\frac{\omega_{RF}}{V} \frac{r}{\gamma} \right)}{I_0\left(\frac{\omega_{RF}}{V} \frac{a}{\gamma} \right)} \]

with \( eV > 0 \). Neglecting the second order variation in \( r \) due to \( I_0\left(\frac{\omega_{RF}}{V} \frac{r}{\gamma} \right) \), we are left with

\[ V = \tilde{V}(a) \cdot \frac{T(a)}{I_0\left(\frac{\omega_{RF}}{V} \frac{a}{\gamma} \right)} \quad \text{for all particles} \]  \hspace{1cm} (1.4)

It is seen that through the transit time factor and the Bessel function \( I_0\left(\frac{\omega_{RF}}{V} \frac{a}{\gamma} \right) \), the effective peak voltage \( V \) depends on the particle velocity \( v \). This effect will be neglected in what follows, so that all particles will be considered as experiencing the same peak voltage.

More precise (but more complicated) expressions for \( \Delta E \) can be found in Ref. 1.

2. HARMONIC NUMBER

For some reference particle (also called synchronous particle), the phase \( \phi \) is kept unchanged (mod 2\( \pi \)) at a value \( \phi_s \) when the particle returns to the same accelerating gap after one revolution along the ring. This requires that \( \omega_{RF} = h \omega_0 \) where \( \omega_0 = 2\pi/T_0 \) is the angular revolution frequency of the reference particle and \( h \) is an integer called harmonic number.

Then

\[ \omega_{RF}T_0 = 2\pi h \]  \hspace{1cm} (2.1)

When the ring is large, \( \omega_0 \) is small and \( h \) may be quite a big number.

3. FINITE DIFFERENCE EQUATIONS

For simplicity, let us assume that RF acceleration takes place in \( N \) identical cavities evenly spaced along the synchrotron ring. Let \( n \) be the number of accelerating cavity traversals by a particle.

Definition of variables (see Fig. 3.1)

- \( p_s, v_s \), momentum and velocity of the reference (synchronous) particle
- \( t_n \), time of \( n^{th} \) cavity traversal by the reference particle
\[ \delta p_n = p_n - p_s \]
\[ \delta \phi_n = \phi_n - \phi_s \]

In what follows, \( \delta \) represents a difference taken with respect to the reference particle \textit{at a given time}; \( d \) represents an increment \textit{during acceleration}.

![Diagram](image)

\textbf{Fig. 3.1 Definition of variables}

Besides the general coordinates \((R, \theta)\) whose origin is the accelerator centre, each bending magnet has its own local coordinates \((r, \phi)\) whose origin is the centre of the reference particle orbit in the magnet. Any integral with respect to \( \theta \) is taken in the bending magnets only.

\textit{Phase variation between adjacent cavities}

In order to keep the phase of the reference particle constant at every cavity traversal, the RF phase must be shifted by \( 2\pi h/N \) between adjacent cavities. The phase of any particle with respect to the RF voltage is then given by

\[
\phi(t) = \int \omega_{RF} \, dt \quad \text{h} \theta(t)
\]

(3.1)

where \( \theta(t) \) is the azimuthal position of the particle. With this relation \( \phi \) is not only defined during cavity traversals, but it is defined at any time. In particular, for the reference particle,

\[
\frac{d \phi_s}{dt} = \omega_{RF} - h\omega_0 = 0
\]

(3.2)

For a particle with an energy deviation \( \delta \gamma/\gamma_0 \) with respect to the reference particle, the phase is compared to \( \phi_s \). If \( T_r \) is the revolution period, the variation of \( \delta \phi_n \) from one cavity to the next is

\[
\delta \phi_{n+1} - \delta \phi_n = \phi_{n+1} - \phi_n = \omega_{RF} \frac{T_r}{N} - h \frac{2\pi}{N} = \frac{\omega_{RF}}{N} (T_r - T_0)
\]
But \( T_r = C/v \) where \( C \) is the orbit circumference and \( v \) is the particle velocity. For a relative momentum deviation \( \delta p/p \),

\[
\frac{\delta C}{C} = a \frac{\delta p}{p} \quad \text{by definition of the momentum compaction} \quad a; \quad a = \frac{1}{2\pi R} \int D_x(s) \, d\theta
\]

where \( D_x(s) \) is the radial dispersion.

\[
\frac{\delta v}{v} = \frac{1}{\gamma^2} \frac{\delta p}{p} \quad \text{by relativistic kinematics.}
\]

Therefore, if \( w_r \) is the angular revolution frequency of a particle,

\[
\frac{\delta w_r}{w_0} = \frac{\delta T_r}{T_0} = - \left( 1 - \frac{1}{\gamma^2} \right) \frac{\delta p}{p} = \eta \frac{\delta p}{p} \quad \text{where} \quad \eta = \frac{1}{\gamma^2} - a
\]

If \( \eta \) vanishes for some \( \gamma \), this particular energy is called the transition energy \( \gamma_{tr} \); when \( a \) is independent of \( \gamma \), \( a \) is equal to \( 1/\gamma^2 \). Finally,

\[
\left[ \frac{\delta \phi_{n+1} - \delta \phi_n}{\gamma} = \frac{\delta \phi_n}{\gamma} - \frac{\omega_{RF}}{N} \frac{\delta T_r}{T_0} = - \frac{2\pi \hbar}{N \gamma} \frac{\delta p_{n+1}}{p} = - \frac{2\pi \hbar}{N \gamma^2} \frac{\delta \gamma_{n+1}}{\gamma} \right]
\]

since \( \frac{\delta v}{\gamma} = \beta \frac{\delta p}{p} \).

**Energy variation between adjacent cavities**

Since all accelerating cavities are assumed to be identical, the total RF voltage produced along the ring is \( NV \). With (1.3) we have

\[
\Delta E = E_{n+1} - E_n = eV \sin \phi_n - \frac{e}{N} \oint \frac{\partial B_z}{\partial t} \cdot r \, d\theta \, dr
\]

In the righthand side, the first term represents an energy gain which is lumped in the accelerating cavities, whereas the second term represents an energy gain which is distributed all along the magnets. Although the second term is usually negligible with respect to the first one, its variation for particles with different energies must not be overlooked. For the reference particle

\[
\Delta E_s = E_{s,n+1} - E_{s,n} = eV \sin \phi_n - \frac{e}{N} \oint \frac{\partial B_z}{\partial t} \cdot r_s \, d\theta \, dr
\]

To first order,

\[
\frac{N}{2\pi} (\Delta E - \Delta E_s) = \frac{eNV}{2\pi} (\sin \phi_n - \sin \phi_s) = \frac{e}{2\pi} \oint \frac{\partial B_z}{\partial t} r_s \, d\theta \, dx
\]

*) This definition of \( \eta \) is the same as the one used in Ref. 2, but another definition of \( \eta \) which differs in sign is also used in the literature.
But
\[ \delta x = D_x(s) \cdot \frac{\delta p}{p} \quad \text{and} \quad p = -e B_z \cdot r = -e \langle B_z \rangle R \]
(3.7)

where \( \langle B_z \rangle \) is the average magnetic field along a closed orbit:
\[ \langle B_z \rangle = \frac{1}{2\pi R} \oint B_z \cdot r \, d\theta \]

With (3.7) the last term of (3.6) becomes
\[ -\frac{e}{2r} \frac{\delta p}{p} \frac{\delta B_z}{\delta t} r_s \int D_x(s) d\theta = -\frac{e}{2\pi} \frac{\delta p}{p} \frac{\delta B_z}{\delta t} R_s 2\pi R_s \alpha = -e \langle B_z \rangle \cdot \frac{\delta p}{p} \left[ \frac{\dot{p}}{p} - \frac{1}{\alpha} \frac{\ddot{R}}{R} \right] R_s \alpha \]
\[ = \delta p \left[ \alpha \frac{\dot{p}}{p} - \frac{\ddot{R}}{R} \right] R_s \]
(3.8)
because from (3.7),
\[ \frac{\dot{p}}{p} = \frac{1}{\langle B_z \rangle} \left[ \frac{\delta B_z}{\delta t} + \frac{\delta B_z}{\delta R} \frac{dR}{dt} \right] + \frac{1}{R} \frac{dR}{dt} = \frac{1}{\langle B_z \rangle} \left[ \frac{\delta B_z}{\delta t} \cdot \frac{1}{\langle B_z \rangle} \left( \frac{R}{\langle B_z \rangle} \cdot \frac{\delta B_z}{\delta R} + 1 \right) \right] \frac{\dot{R}}{R} \]
(3.9)

[average magnetic field index + 1] = \frac{1}{\alpha}.

Now we must remember that \( \Delta E, \Delta E_s \) are gained in different times \( T_r/N, T_0/N \). As next approximation, \( E \) and \( E_s \) are considered to be smooth functions of \( t \):
\[ \Delta E = \frac{T_r}{N} \frac{dE}{dt}, \quad \Delta E_s = \frac{T_0}{N} \frac{dE_s}{dt} \]

With (3.3),
\[ \Delta E - \Delta E_s = \frac{T_r}{N} \frac{dE}{dt} - \frac{T_0}{N} \frac{dE_s}{dt} = \frac{T_r}{N} \frac{d(E - E_s)}{dt} + \frac{T_r - T_0}{N} \cdot \frac{dE_s}{dt} = \frac{T_r}{N} \frac{d(E - E_s)}{dt} + \frac{\delta p}{p} \frac{T_0}{N} \frac{dE_s}{dt} \]

Using the kinematic relations \( dE = v \cdot dp = \omega R \cdot dp \), \( \delta E = v \cdot \delta p = \omega_0 R_s \delta p \):
\[ \frac{N}{2\pi} \left( \Delta E - \Delta E_s \right) = \frac{1}{\omega_r} \frac{d(\delta E)}{dt} - \frac{\delta p}{p} \frac{1}{\omega_0} \frac{dE_s}{dt} = \frac{d}{dt} \left( \frac{\delta E}{\omega_0} \right) + \frac{\delta p}{\omega_0} \frac{\delta E}{\omega_0} - \frac{\delta p}{p} \frac{R_s}{R} \frac{\dot{R}}{R} + \text{2nd order terms.} \]
\[ = R_s \delta p \left[ \frac{\dot{R}}{R} \right] + R_s \delta p \left[ a - \frac{\dot{R}}{R} \right] \frac{\dot{p}}{p} = R_s \delta p \left[ a \frac{\dot{p}}{p} - \frac{\dot{R}}{R} \right] \]

which is exactly equal to (3.8), i.e., to the last term of (3.6). Finally (3.6) reduces to
\[ \frac{d}{dt} \left( \frac{E - E_S}{\omega_0} \right) = \frac{eN}{2\pi} (\sin \phi - \sin \phi_s) \] (3.10)

The present derivation of Eq. (3.10) is the same as in Ref. 3 (p.156-163); this equation may also be derived by using the generalized angular momentum \( \vec{r} \times (\vec{p} + e \vec{A}) \) instead of \( \vec{B} \) (Ref. 4). Any correct derivation must take into account the electromotive force induced by a varying magnetic flux; it happened often in the past that Eq. (3.10) was either wrongly derived or wrongly stated.

The corresponding finite difference equation reads

\[ \left( \frac{E - E_S}{\omega_0} \right)_{n+1} - \left( \frac{E - E_S}{\omega_0} \right)_n = \frac{eV}{\omega_0} \left( \sin \phi_n - \sin \phi_s \right) \quad \text{where} \quad \frac{E - E_S}{\omega_0} = R_s (p - p_s) \] (3.11)

Finally, about the set of finite difference equations (3.4), (3.11), one should quote H. Hereward: "These equations are only roughly correct, and it is work to estimate how good they are" (Ref. 4, p. 11).

**Betatron electromotive force**

The betatron e.m.f. \( \xi \) along a closed orbit appears in (3-5) as

\[ \Delta E = eV \sin \phi + \frac{e \xi}{N} \] (3.12)

where

\[ \xi = -\frac{3}{\delta t} \int b(r, \theta) \cdot r \, d\theta \, dr \]

In this integral, the closed orbit should be considered to be fixed with respect to time. Computing the integral first with respect to \( r \), this may be rewritten as

\[ \xi = \frac{3}{\delta t} \int b(r, \theta) \cdot B_z \cdot (x = 0) \cdot r \, d\theta \]

where \( r \) is now taken on the closed orbit \( x=0 \), and \( b(r, \theta) \) is an effective magnet width inside the orbit. Using (3-7) it is seen that

\[ e \xi = \frac{3}{\delta t} \int b(r, \theta) \cdot p \, d\theta = \frac{3}{\delta t} \left[ p \int b(r, \theta) \cdot d\theta \right] \]

or

\[ e \xi = 2\pi \frac{3}{\delta t} \left[ p \cdot b \right] \quad \text{where} \quad b = < b(R) > = \frac{1}{2\pi} \int b(r, \theta) \cdot d\theta \] (3.13)
b = < b(R) > appears to be an average effective magnet width inside a closed orbit. Since the orbit must be considered as being fixed in time, the dependence of b on time can only be due to possible changes in the configuration of \( B_z \) when the magnetic field is increased; and these changes are kept as small as possible. By using (3.7) and (3.9) we obtain

\[
\begin{align*}
\frac{e\xi}{2\pi} &= b \cdot \frac{dp}{dt} + p \frac{\partial b}{\partial t} \\
\frac{1}{p} \frac{dp}{dt} &= \frac{1}{< B_z >} \frac{3}{\partial t} = \frac{1}{p} \frac{\partial}{\partial t} \frac{1}{a \cdot R} \quad (3.14)
\end{align*}
\]

and where ideally \( \frac{\partial b}{\partial t} \) should be negligible.

Since

\[
\Delta E \approx \frac{T}{N} \cdot \frac{dE}{dt} - \frac{T}{N} \cdot \omega \cdot R \frac{dp}{dt} = \frac{2\pi}{N} R \frac{dp}{dt},
\]

(3-12) becomes

\[
R \frac{dp}{dt} = \frac{eN}{2\pi} \sin \phi + \frac{e\xi}{2\pi} \quad (3.15)
\]

Finally, with (3-14),

\[
R \frac{dp}{dt} = \frac{eN}{2\pi} \sin \phi + b \left[ \frac{dp}{dt} - \frac{dR}{dt} \right] + p \frac{\partial b}{\partial t} \quad (3.16)
\]

In this relation the \( \frac{\partial b}{\partial t} \) term should be negligible, while the b-term is (at most) of the order \( b \frac{dp}{dt} \), which is a factor \( b/R \) smaller than the left-hand side. Therefore, when \( b \ll R \) (which is the case for all large synchrotrons), acceleration due to the betatron electromotive force is a small fraction of the acceleration produced by the RF voltage.

4. **DIFFERENTIAL EQUATIONS FOR AN ARBITRARY RF VOLTAGE**

If higher harmonics are added to the fundamental sinusoidal RF field, in Eq. (3.10) and (3.11) \( \sin \phi \) must be replaced by a more general function \( g(\phi) \) such that

\[
g(\phi + 2\pi) = g(\phi) \quad \text{and} \quad \int_{0}^{2\pi} g(\phi) \, d\phi = 0 \quad (4.1)
\]

hence

\[
g(\phi) = \sum_{n=1}^{\infty} \left( a_n \sin n\phi + b_n \cos n\phi \right) \quad (4.2)
\]
where one can take \( a_1 = 1, b_1 = 0 \), since the values of \( a_1, b_1 \) are defined by normalization and by the choice of the origin of time. Eq. (3.10) then becomes

\[
\frac{d}{dt} \left( \frac{\delta E}{\omega_0} \right) = \frac{eN}{2\pi} \left[ g(\phi) - g(\phi_0) \right]
\]  

(4.3)

which has to be combined with the differential form of (3.4):

\[
\frac{d}{dt} (\phi - \phi_0) = -\hbar \omega_0 \frac{n}{\beta z} \frac{\delta E}{E}
\]  

(4.4)

Instead of \( \phi \) and \( \delta E/\omega_0 \) as conjugate variables, we shall use \( \phi \) and \( \delta E/(\hbar \omega_0) \), so that the elementary phase space area will read

\[
\frac{\delta E}{\hbar \omega_0} \wedge \delta \phi = \frac{\delta E}{\hbar \omega_0} (\hbar \omega_0 \dot{t}) = \delta E \delta t
\]

With \( \phi \) and \( \delta E/(\hbar \omega_0) \) as conjugate variables, the system (4.3), (4.4) becomes

\[
\frac{d}{dt} \left( \frac{\delta E}{\hbar \omega_0} \right) = \frac{eN}{\hbar} \frac{\delta E}{2\pi} \left[ g(\phi) - g(\phi_0) \right]
\]  

(4.5)

\[
\frac{d}{dt} (\phi - \phi_0) = -\hbar^2 \omega_0^2 \frac{n}{\beta z E} \left( \frac{\delta E}{\hbar \omega_0} \right)
\]

where \( E = \gamma E_0 = \gamma m_0 c^2 \)

(4.6)

With (3.2), the system (4.5), (4.6) may be derived from the Hamiltonian

\[
H = -\frac{1}{2} \hbar^2 \omega_0^2 \frac{n}{\beta z \gamma E_0} \left( \frac{\delta E}{\hbar \omega_0} \right)^2 + \frac{eN}{\hbar 2\pi} \left[ \Gamma \phi + G(\phi) \right]
\]  

(4.7)

where

\[
\Gamma \equiv g(\phi_0) \quad \text{and} \quad G(\phi) = -\int g(\phi) d\phi = \sum_{n=1}^{\infty} \left( \frac{a_n}{n} \cos n\phi - \frac{b_n}{n} \sin n\phi \right)
\]  

(4.8)

H depends explicitly on time through parameters which vary slowly during acceleration.
5. **Hamiltonian with Reduced Variables**

The study of particle motion can be simplified by using reduced dimensionless variables $y$ and $t^*$ instead of $\delta E/(\hbar w_0)$ and $t$. Let

\[
\begin{align*}
\frac{dt}{dt^*} &= K_1 \\ \frac{\delta E}{\hbar w_0} &= K_2 \ y
\end{align*}
\]

where $K_1$, $K_2$ are slowly varying parameters. \( (5.1) \)

With the reduced variables and reduced Hamiltonian $H^*$, the equations of motion read

\[
\begin{align*}
\frac{dy}{dt^*} &= -\frac{K_1}{K_2} \frac{\partial H}{\partial y} = -\frac{\partial H^*}{\partial y} \\
\frac{d\phi}{dt^*} &= \frac{K_1}{K_2} \frac{\partial H}{\partial \phi} = \frac{\partial H^*}{\partial \phi}
\end{align*}
\]  

whence

\[
H^* = \frac{K_1}{K_2} H = -\frac{1}{2} \ h^2\omega_0^2 \ \frac{n}{\beta^2 \gamma E_0} \ K_1 \ K_2 \ y^2 + \frac{eN}{\hbar^2 \pi} \ K_1 \ K_2 \left[ \Gamma \phi + G(\phi) \right]
\]

By taking

\[
- \ h^2\omega_0^2 \ \frac{n}{\beta^2 \gamma E_0} \ K_1 \ K_2 = 1 \quad \text{and} \quad \frac{eN}{\hbar^2 \pi} \ K_1 \ K_2 = -\text{sgn}(n)
\]  

i.e.,

\[
\frac{1}{K_1} = -\text{sgn}(n) \ \hbar \omega_0 \left| \frac{n}{\beta^2 \gamma \ h^2 \pi E_0} \right|^{\frac{1}{2}}, \quad K_2 = \frac{E_0}{\hbar \omega_0} \left| \frac{\beta^2 \gamma}{n \ h^2 \pi E_0} \right|^{\frac{1}{2}}
\]  

\( (5.3) \)

\( (5.4) \)

$H^*$ becomes

\[
H^* = \frac{\psi^2}{2} - \text{sgn}(n) \left[ \Gamma \phi + G(\phi) \right]
\]  

\( (5.5) \)

**Fixed points.** From \((5.2)\) they correspond to \( \frac{\partial H^*}{\partial \phi} = 0 \) and \( \frac{\partial H^*}{\partial y} = 0 \)

i.e., with \((4.8)\): \( \Gamma + G'(\phi) = 0 \) or \( \Gamma \equiv g(\phi_s) = g(\phi) \) with \( y = 0 \).

Because of \((4.1)\), beside $\phi_s$ there will be in general another value of $\phi$ satisfying the condition $g(\phi) = \Gamma$. Let $\phi_0$ be any one of them; for small $\psi = \phi - \phi_0$,

\[
\Gamma \phi + G(\phi) = \left[ \Gamma \phi_0 + G(\phi_0) \right] - g'(\phi_0) \frac{\psi^2}{2} - g''(\phi_0) \frac{\psi^3}{3!} - \ldots
\]

\[
H^* = -\text{sgn}(n) \left[ \Gamma \phi_0 + G(\phi_0) \right] + \frac{\psi^2}{2} + \text{sgn}(n) \cdot g'(\phi_0) \frac{\psi^2}{2} + \text{sgn}(n) \cdot g''(\phi_0) \frac{\psi^3}{6}
\]  

\( (5.6) \)

If \( \text{sgn}(n) \cdot g'(\phi_0) > 0, \phi_0 \) is an elliptic fixed point; this is the case of $\phi_0 = \phi_s$ for $\phi_s$ being a stable fixed point.
In the figure, $\Gamma = \frac{1}{2}$

$G(\phi) = \cos \phi$

$g(\phi) = \sin \phi$

Fig. 5.1 Potential energy as a function of $\phi$. 
Fig. 5.2 Trajectories in synchrotron phase space, when \( n < 0 \); when \( n > 0 \), \( \phi_S \) and \( \phi_U \) are interchanged. The complete phase space is wrapped around a cylinder \( 0 \leq \phi \leq 2\pi h \).
If \( \text{sgn}(\eta) \cdot g'(\phi_0) < 0 \), \( \phi_0 \) is a hyperbolic fixed point; this is the case of the other fixed point \( \phi_0 = \phi_u \), which is unstable.

\( \phi_s \) is at a minimum of potential energy; \( \phi_u \) is at a maximum. When \( \eta \) changes sign, the two points \( \phi_s \) and \( \phi_u \) are interchanged (see Fig. 5.1). Therefore, when crossing the transition energy, the RF voltage must undergo a phase jump which puts the particles around the new stable fixed point.

**Separatrix.** The trajectory in phase space passing through the unstable fixed point \((\phi_u, y = 0)\) crosses the \( \phi \)-axis at another point \((\phi_e, y = 0)\). This trajectory is the boundary between trapped and untrapped motion (or between libration and rotation); it is called the **separatrix** (see Fig. 5.2). The phase space domain inside the separatrix is called **bucket**; its area \( A_\text{s} \) is the longitudinal acceptance of the accelerator.

From (5.5) the equation for the separatrix is

\[
\frac{y^2}{2} - \text{sgn}(\eta) \left[ \Gamma \phi + G(\phi) - \Gamma u - G(\phi_u) \right] = 0
\]  
(5.7)

Taking the derivative with respect to \( \phi \):

\[
y \cdot \frac{dy}{d\phi} - \text{sgn}(\eta) \left[ \Gamma - g(\phi) \right] = 0
\]

This equation is satisfied with \( y = 0 \) at \( \phi = \phi_u \) and with \( \frac{dy}{d\phi} = 0 \) at \( \phi = \phi_s \). Therefore \( y \) is maximum at \( \phi_s \) (this is also the case for any trajectory.)

**Bucket width.** The bucket width is \((\phi_e - \phi_u)\) where \( \phi_e \) is determined by the equation

\[
\Gamma\phi_e + G(\phi_e) = \Gamma\phi_u + G(\phi_u)
\]

(5.8)

**Bucket height.** With (5.1),

\[
\left( \frac{\delta E}{\hbar w_0} \right) = \frac{R}{h} \delta p = K_2 \hat{y}
\]

where \( \frac{y^2}{2} = -\text{sgn}(\eta) \left[ \Gamma\phi_u + G(\phi_u) - \Gamma\phi_s - G(\phi_s) \right] \) 

(5.9)

**Bucket area** (per bunch)

\[
A_\text{s} = K_2 A_\text{s}^* \quad \text{where} \quad A_\text{s}^* = 2 \int_{\phi_u}^{\phi_e} y \, d\phi = 2\sqrt{2} \int_{\phi_u}^{\phi_e} \Gamma\phi_u + G(\phi_u) - \Gamma - G(\phi) \, d\phi
\]

(5.10)

It is not invariant during acceleration.
Period $T_s$ of (large) synchrotron oscillations around the stable fixed point

The phase space trajectories are represented by (5.5) where $H^*$ is constant. Let $\phi_1$, $\phi_2$ be the two phases where $y = 0$; then (5.5) may be written as

$$\frac{y^2}{2} - \text{sgn}(n) \left[ \Gamma_{\phi} + G(\phi) \right] = -\text{sgn}(n) \left[ \Gamma_{\phi_1} + G(\phi_1) \right] = -\text{sgn}(n) \left[ \Gamma_{\phi_2} + G(\phi_2) \right]$$

From (5.2)

$$\frac{d\phi}{dt^*} = \frac{\partial H^*}{\partial \phi} = y$$

hence

$$dt^* = \frac{d\phi}{y}$$

and with (5.1),

$$T_s = |K_1| T_s^* \quad \text{where} \quad T_s^* = 2 \int_{\phi_1}^{\phi_2} \frac{d\phi}{y} = \sqrt{2} \int_{\phi_1}^{\phi_2} d\phi \left| \Gamma_{\phi_1} + G(\phi_1) - \Gamma_{\phi} - G(\phi) \right|^{-1}$$

(5.11)

For a general RF voltage this expression involves cumbersome elliptic integrals.

6. SMALL OSCILLATIONS AROUND THE STABLE FIXED POINT

From (5.6), the small amplitude trajectories around $\phi_s$ are represented by the ellipse equation

$$\frac{x^2}{2} + \left| g'(\phi_s) \right| \cdot \frac{y^2}{2} = \frac{C}{2} > 0 \quad , \quad \psi = \phi - \phi_s$$

(6.1)

It is apparent that all properties of small oscillations around $\phi_s$ involve the RF voltage only through its slope at $\phi_s$.

Period $T_{s0}$ of small synchrotron oscillations

The subscript 0 refers here to vanishingly small amplitudes. With (6.1) the general formula (5.11) simplifies to

$$T_{s0} = |K_1| T_{s0}^* \quad \text{where} \quad T_{s0}^* = 2 \sqrt{\frac{\psi}{\psi}} = 2 \sqrt{\frac{\psi}{\sqrt{C - |g'(\phi_s)| \psi^2}}} = \frac{2\pi}{\sqrt{|g'(\phi_s)|}}$$

This is independent of the amplitude $\hat{\psi}$ as long as the $\psi^3$ and higher order terms are missing in (5.6), which means as long as $g(\phi)$, i.e. the RF voltage, is a linear function of $\phi$. 
With (5.4) the synchrotron tune

\[ Q_s = \frac{1}{\omega_0} \frac{2\gamma}{T_s} = \frac{1}{\omega_0 |k_1|} \frac{2\gamma}{T_s} \]

is given by

\[ Q_{so}^2 = \left| \frac{\hbar n \beta^2 \gamma e N V g'(\phi_s)}{2\pi e E_0} \right| = \frac{1}{2\pi} \frac{\hbar n e N V g'(\phi_s)}{E_0} \]

where \( V_g'(\phi_s) = \frac{d}{d\phi} \) (RF voltage) at \( \phi_s \)

(6.2)

**Height of a trajectory in synchrotron phase space.**

For a trajectory of half width \( \delta \), the height is obtained from (6.1) as \( \gamma^2 = \left| g'(\phi_s) \right| \delta^2 \)

whence, with (5.1) and (5.4),

\[ \left( \frac{\delta \hat{E}}{\mu_0} \right) = K_2 \left( \frac{\delta \hat{y}}{\mu_0} \right) = \frac{E_0}{\hbar n \omega_0} \left| \frac{\beta^2 \gamma e N V g'(\phi_s)}{2\pi e E_0} \right| \left| \frac{1}{\gamma} \right| \]

**Longitudinal emittance. Bunch matching.**

The area of a bunch in synchrotron phase space is its longitudinal emittance \( E_s \); it is an invariant by Liouville's theorem. If we call "emittance of a single particle" the area \( 2nJ \) in phase space which is enclosed by the particle trajectory,

\[ 2\pi J = \int \frac{\delta E}{\mu_0} = \frac{E_0}{\hbar n \omega_0} \left| \frac{\beta^2 \gamma e N V g'(\phi_s)}{2\pi e E_0} \right| \left| \frac{1}{\gamma} \right| \left| \frac{\delta \hat{y}}{\mu_0} \right| \left[ eV_s \right] \]

(6.3)

The action \( J \) is an adiabatic invariant, i.e. it stays constant if the parameters in \( H \) are varied infinitely slowly (Ref. 5, p. 154; Ref. 6, p. 110; Ref. 7, p. 234). If at some time a bunch is matched (which means that its border in phase space is just the closed trajectory of the outermost particles) then its emittance \( E_s \) is equal to the single particle emittance of its outermost particles. After a change of the parameters in \( H \), the emittance \( E_s \) is unchanged but the action of the outermost particles has changed slightly and differently for each particle, which means that the bunch is no longer matched exactly: therefore the matching of a bunch can only be preserved in the adiabatic sense, i.e. if the parameters in \( H \) are varied very slowly.

**7. Motion in the vicinity of the fixed points**

Take the Hamiltonian (5.5) with reduced variables:

\[ H^* = \frac{\chi^2}{2} - \text{sgn}(\eta) \left[ \gamma\phi + G(\phi) \right] \]
Canonical equations:

\[ \frac{dy}{dt^*} = \text{sgn}(\eta) \left[ \Gamma - g(\phi) \right], \quad \frac{d\phi}{dt^*} = y \]

hence:

\[ dt^* = \frac{d\phi}{y} = \frac{d\phi}{\sqrt{2H^* + 2 \text{sgn}(\eta) \left[ \Gamma_0 + G(\phi) \right]}} \tag{7.1} \]

In the vicinity of a fixed point \( \phi_0 \),

\[ y^2 = 2H^* + 2 \text{sgn}(\eta) \left[ \Gamma_0 + G(\phi_0) \right] - \text{sgn}(\eta) g'(\phi_0) (\phi - \phi_0)^2 + \ldots \]

Motion around the stable fixed point \( \phi_s \)

\[ y^2 = C - |g'(\phi_s)| (\phi - \phi_s)^2 + \ldots \]

\[ y_{\text{max}}^2 \]

\[ dt^* = \frac{d\phi}{\sqrt{y_{\text{max}}^2 - |g'(\phi_s)| (\phi - \phi_s)^2}} \]

\[ t^* = - |g'(\phi_s)|^{-1} \text{arc cos} \left[ \frac{\sqrt{|g'(\phi_s)|}}{y_{\text{max}}} (\phi - \phi_s) \right] \]

or

\[ (\phi - \phi_s) = \frac{y_{\text{max}}}{\sqrt{|g'(\phi_s)|}} \cos \left[ \frac{\sqrt{|g'(\phi_s)|}}{y_{\text{max}}} t^* \right] \tag{7.2} \]

Motion around the unstable fixed point \( \phi_u \)

\[ y^2 = C + |g'(\phi_u)| (\phi - \phi_u)^2 + \ldots \]

\[ dt^* = \frac{d\phi}{\sqrt{C + |g'(\phi_u)| (\phi - \phi_u)^2}} \]
\[ C = 0 \, : \, \text{particle on the separatrix} \]
\[
t^* = \pm \left| g'(\phi_u) \right|^{-1} \log |\phi - \phi_u| \quad \text{when} \quad \phi \to \phi_u
\]  \hspace{1cm} (7.3)

Therefore the motion near \( \phi_u \) is very slow (it is a fixed point!) This relation may be inverted as
\[
\phi - \phi_u \sim e^{\pm \sqrt{|g'(\phi_u)|} t^*}
\]

\[ C > 0 \, : \, \text{particle outside the bucket} \]
\[
C = y_{\min}^2
\]
\[
t^* = \left| g'(\phi_u) \right|^{-1} \arg \text{sh} \left[ \frac{\sqrt{|g'(\phi_u)|}}{y_{\min}} (\phi - \phi_u) \right]
\]  \hspace{1cm} (7.4)

or
\[
\phi - \phi_u = \frac{y_{\min}}{\sqrt{|g'(\phi_u)|}} \text{sh} \left[ \sqrt{|g'(\phi_u)|} t^* \right]
\]

The motion is very slow when \( y_{\min} \to 0 \).

\[ C < 0 \, : \, \text{particle inside the bucket} \]
\[
C = - \left| g'(\phi_u) \right| \psi_{\min}^2 \quad \text{where} \quad \psi = \phi - \phi_u
\]
\[
t^* = \left| g'(\phi_u) \right|^{-1} \arg \text{ch} \left[ \frac{\phi - \phi_u}{\psi_{\min}} \right]
\]  \hspace{1cm} (7.5)

or
\[
\phi - \phi_u = \psi_{\min} \cdot \text{ch} \left[ \sqrt{|g'(\phi_u)|} t^* \right]
\]

When \( \psi_{\min} \to 0 \), the motion close to \( \phi_u \) becomes very slow, and the synchrotron period \( T_s \) becomes infinite. As shown by (7-3) to (7-5), all trajectories are slowed down in the vicinity of the unstable fixed point.
8. **STATIONARY BUCKET WITH A HARMONIC CAVITY**

When the beam is not accelerated but is simply kept bunched at a fixed energy, \( r = 0 \) and the bucket is called stationary. In this case, which corresponds to collider operation, the frequency of synchrotron oscillations as a function of phase amplitude \( \hat{\psi} \) is given by

\[
\omega_s = \frac{\omega_s^*}{|K_1|}
\]

where as a first approximation *(see Appendix A, Eq. (A.8)):

\[
\omega_s^* = \text{sgn}(\eta) \cdot \sum_{n=1}^{m} a_n \cos n\phi_s \cdot \frac{2}{\psi} J_1(n\hat{\psi}) + 0(\hat{\psi}^4)
\]  

(8.1)

when

\[
g(\phi) = \sum_{n=1}^{\infty} a_n \sin n\phi, \quad r = g(\phi_s) = 0, \quad \text{sgn}(\eta) \cdot g'(\phi_s) = \text{sgn}(\eta) \cdot \sum_{n=1}^{\infty} n a_n \cos n\phi_s > 0.
\]

Assuming that \( a_1 = 1 \) is the dominant term,

\[
\phi_s = 0 \quad \text{if} \quad \eta > 0
\]

\[
\phi_s = \pi \quad \text{if} \quad \eta < 0.
\]

With (8.1), the synchrotron frequency for vanishingly small amplitudes is

\[
\omega_{s0}^2 = \text{sgn}(\eta) \cdot \sum_{n=1}^{m} na_n \cos n\phi_s = \text{sgn}(\eta) \cdot g'(\phi_s) = |g'(\phi_s)|
\]  

(8.2)

in agreement with (6.2).

The relation (8.1) allows shaping the variation of \( \omega_s^2 \) with \( \hat{\psi} \). For example, if \( \omega_s^2 \) is to be proportional to \( \hat{\psi}^2 \), it is sufficient to take \( a_1 = 1 \) and \( \text{sgn}(\eta) \cdot na_n \cos n\phi_s = -1 \) for some \( n > 1 \); then

\[
\omega_s^2 = \frac{2}{\psi} J_1(\hat{\psi}) - \frac{2}{n\psi} J_1(n\hat{\psi}) + 0(\hat{\psi}^4) = \frac{n^2 - 1}{2} \left( \frac{\hat{\psi}}{2} \right)^2 + 0(\hat{\psi}^4)
\]  

(8.3)

If the sign of \( a_n \) is reversed so that \( \text{sgn}(\eta) \cdot na_n \cos n\phi_s = 1 \), then

\[
\omega_s^2 = \frac{2}{\psi} J_1(\hat{\psi}) + \frac{2}{n\psi} J_1(n\hat{\psi}) + 0(\hat{\psi}^4) = 2 \left[ 1 - \frac{n^2 + 1}{4} \left( \frac{\hat{\psi}}{2} \right)^2 + 0(\hat{\psi}^4) \right]
\]  

(8.4)

*) \( J_\nu(x) \) is the Bessel function of the first kind of order \( \nu \) and argument \( x \).
The latter case (8.4) corresponds to the usual operation of a Landau harmonic cavity, which increases the relative spread of synchrotron frequencies as a function of $\hat{\nu}$. In both cases the amplitude of the harmonic voltage is such that

$$\left| \frac{a_n}{a_1} \right| = \frac{1}{n}.$$ 

9. FORMULAE FOR A SINUSOIDAL RF VOLTAGE

Formulae are simpler when using the phase $\varphi$ measured from the crest of the RF voltage:

$$\hat{\varphi} = \frac{\pi}{2} \text{sgn}(\hat{\nu}) + \varphi$$

$$\hat{\varphi}_s = \frac{\pi}{2} \text{sgn}(\hat{\nu}) + \varphi_s, \quad \hat{\varphi}_u = \frac{\pi}{2} \text{sgn}(\hat{\nu}) - \varphi_s, \quad \text{sgn}(\varphi) = -\text{sgn}(\hat{\nu}) \cdot \text{sgn}(\hat{\nu}),$$

$$\sin \hat{\varphi}_s \cdot \text{sgn}(\hat{\nu}) = |\nu| = \cos \varphi_s.$$

From (4.2) and (4.8),

$$g(\varphi) = \sin \varphi = \text{sgn}(\hat{\nu}) \cdot \cos \varphi, \quad \Gamma = g(\hat{\varphi}_s) = \sin \hat{\varphi}_s = \text{sgn}(\hat{\nu}) \cdot \cos \varphi_s, \quad (9.1)$$

$$G(\varphi) = \cos \varphi = -\text{sgn}(\hat{\nu}) \cdot \sin \varphi$$

**Bucket width** $(\hat{\varphi}_e - \hat{\varphi}_u)$ where

$$\text{tg}(\hat{\varphi}_e - \hat{\varphi}_u - 3\varphi_s) = \frac{1}{10} \text{tg}^3 \varphi_s \left[ 1 + \frac{111}{140} \text{tg}^2 \varphi_s - 0(\text{tg}^4 \varphi_s) \right]^{-1}$$

(9.2)

**Bucket height** From (5.9),

$$\frac{\varphi_s^2}{2} = 2|\sin \varphi_s - \varphi_s \cos \varphi_s|$$

and, using (5.4):

$$\left( \frac{\delta \hat{\nu}}{\dot{\nu}_0} \right) = \frac{\hbar}{2\pi \hbar \nu_0} \frac{g_{\varphi}}{\hbar} = \frac{g_{\varphi}}{2\pi \hbar \nu_0} \left[ \text{tg}^2 \varphi_s \right]^{\frac{1}{2}} \left[ 1 - \varphi_s \text{cotg} \varphi_s \right]^{\frac{1}{2}} = \frac{\hbar}{2\pi \hbar} \frac{g_{\varphi}}{\hbar} \left[ \text{tg}^2 \varphi_s \right]^{\frac{1}{2}} \left[ 1 - \varphi_s \text{cotg} \varphi_s \right]^{\frac{1}{2}}$$

(9.3)

**Bucket area** From (5.10),

$$A_S = 2\pi \int_{\hat{\varphi}_u}^{\hat{\varphi}_e} d\varphi \left| \hat{\varphi}_u \sin \hat{\varphi}_s + \cos \hat{\varphi}_u - \varphi \sin \hat{\varphi}_s - \cos \varphi \right|^\frac{1}{2}$$

$$\hat{\varphi}_u (\text{for } \varphi < 0)$$

(9.4)
$A_s^*$ is steadily increasing with $|\varphi_s|$. Its maximum value is obtained for $\varphi_s = \pi/2$ (stationary bucket):

$$A_{s \text{ max}}^* = 2\sqrt{2} \int_0^{2\pi} d\varphi \sqrt{1 - \cos \varphi} = 16$$

Let

$$a(r) = \frac{A_s^*}{A_{s \text{ max}}^*} = \frac{A_s}{A_{s \text{ max}}} ; \quad a(r) = \frac{3}{10} \left| \varphi_s \right|^\frac{5}{2} \left[ 1 - \frac{1}{60} \varphi_s^2 + O(\varphi_s^4) \right]$$

(9.5)

The derivation of the series expansion for $a(r)$ is given in Appendix B.

With (5.4)

$$A_s = 16 \frac{E_0}{m_0 c} \left| \frac{\beta_s^2 v}{\gamma_n} \cdot \frac{eN}{2\pi E_0} \right|^\frac{1}{2} \cdot a(r) \text{ per bunch} \quad \left[ \text{eV.s} \right]$$

(9.6)

Remark: Instead of (9.3), the dimensionless quantity $\hat{\Delta}/m_0 c$ is often used. In these coordinates, we have:

**Bucket height**

$$\frac{\hat{\Delta}}{m_0 c} = 2 \left| \frac{\gamma_n}{\gamma_n} \cdot \frac{eN}{2\pi E_0} \right|^\frac{1}{2} \cdot \left[ 1 - \psi_s \cot \psi_s \right]^\frac{1}{4} \cdot \left[ \frac{\hat{\Delta}}{m_0 c} \right]$$

(9.7)

**Bucket area**

$$A_s \cdot \frac{\hbar}{m_0 c R} = 16 \left| \frac{\gamma_n}{\gamma_n} \cdot \frac{eN}{2\pi E_0} \right|^\frac{1}{2} \cdot a(r) \text{ per bunch} \quad \left[ \frac{\hat{\Delta}}{m_0 c} \cdot \text{RF Rad} \right]$$

(9.8)

**Period of (large) synchrotron oscillations in a stationary bucket**

Besides the trivial case of a linear RF voltage, a sinusoidal RF voltage is the only case where it is possible to compute simply the synchrotron period for any amplitude.

With $r = 0$ and $G(\psi) = \cos \psi$, the general equation (5.11) for the libration period reduces to

$$T_s^* = \sqrt{2} \int_{\psi_1}^{\psi_2} d\psi \left| \cos \psi_1 - \cos \psi \right|^{-\frac{1}{2}} = \sqrt{2} \int_{\psi_1}^{\psi_2} d\psi \left| \cos \psi - \cos \hat{\psi} \right|^{-\frac{1}{2}} = 4K \left( \frac{\hat{\psi}}{2} \right)$$

(9.9)

where $\psi = \phi - \psi_s$; $K(k)$ is the complete elliptic integral of the first kind with modulus $k$. 
Therefore

\[ \omega_s^2 = \left[ \frac{2}{\pi} K \left( \sin \frac{\phi}{2} \right) \right]^{-2} = 1 - \frac{1}{2} \sin^2 \frac{\phi}{2} - \frac{3}{32} \sin^4 \frac{\phi}{2} - \cdots \]  

(9.10)

\[ = 1 - \frac{1}{2} \left( \frac{\phi}{2} \right)^2 + \frac{7}{8.12} \left( \frac{\phi}{2} \right)^4 - \cdots \]

The approximate Eq. (8.1) would yield in this case

\[ \omega_s^2 = \frac{2}{\phi} J_1(\phi) = 1 - \frac{1}{2} \sin^2 \frac{\phi}{2} - \frac{1}{12} \sin^4 \frac{\phi}{2} - \cdots = 1 - \frac{1}{2} \left( \frac{\phi}{2} \right)^2 + \frac{1}{12} \left( \frac{\phi}{2} \right)^4 - \cdots \]

which shows that the error on the \( \phi^h \) term in Eq. (8.1) is rather small.

**Motion outside a stationary bucket**

\[ \Gamma = 0 \quad G(\phi) = \cos \phi \]

\[ \text{sgn} (\eta) \cos \phi_u = -1 \quad G(\phi) = \cos \phi_u \cos (\phi - \phi_u) \]

From (5-5),

\[ y^2 = 2H^* - 2 \cos (\phi - \phi_u) = (2H^* - 2) + 4 \sin^2 \left[ \frac{\phi - \phi_u}{2} \right] = y_{\text{min}}^2 + 4 \sin^2 \left[ \frac{\phi - \phi_u}{2} \right] \]

\[ dt^* = \frac{d\phi}{2\sqrt{\left[ \frac{y_{\text{min}}}{2} \right]^2 + \sin^2 \left[ \frac{\phi - \phi_u}{2} \right]}} \]

\[ \sqrt{1 - \cos^2 \left[ \frac{\phi - \phi_u}{2} \right]} \]

The motion outside a stationary bucket is a "rotation", the period of which is the time needed to increase \( \phi \) by \( 2\pi \):

\[ T^* = \frac{1}{\sqrt{1 + \left[ \frac{y_{\text{min}}}{2} \right]^2}} \int_{0}^{2\pi} \frac{d\left[ \frac{\phi - \phi_u}{2} \right]}{\sqrt{1 - \cos^2 \left[ \frac{\phi - \phi_u}{2} \right]}} \]

or

\[ T^* = \frac{2}{\sqrt{1 + \left[ \frac{y_{\text{min}}}{2} \right]^2}} \cdot \k \left[ \frac{1}{\sqrt{1 + \left[ \frac{y_{\text{min}}}{2} \right]^2}} \right] \]

(9.11)
It follows that
\[
T^* \rightarrow \begin{cases} 
2 \log \frac{8}{\gamma_{\text{min}}} & \text{for } \gamma_{\text{min}} \to 0 \\
\frac{2\pi}{\gamma_{\text{min}}} & \text{for } \gamma_{\text{min}} \to \infty 
\end{cases}
\]

When \( \frac{\gamma_{\text{min}}}{2} \ll 1 \), it plays the same role as \( \cot \frac{\phi}{2} \) in (9.9); both quantities represent half the minimum distance to \( \phi_{\text{m}} \), outside and inside the bucket respectively.

10. ADIABATIC DAMPING OF PHASE OSCILLATIONS.

From (6.3):
\[
\psi = \sqrt{2J} \left| \frac{2\pi \hbar^2 \omega_o^2 \eta}{E_0 \beta^2 \gamma \cdot eNV'(\psi)} \right|^{\frac{1}{4}} \tag{10.1}
\]

and
\[
\frac{\delta \hat{E}}{\hbar \omega_o} = \frac{2J}{\sqrt{2J}} \left| \frac{2\pi \hbar^2 \omega_o^2 \eta}{E_0 \beta^2 \gamma \cdot eNV'(\psi)} \right|^{-\frac{3}{4}} \tag{10.2}
\]

where
\[
\frac{\omega_o^2}{\beta^2} = \frac{c^2 \omega_o^2}{\nu_s^2} = \frac{c^2}{R^2}
\]

and
\[
2\pi J = \pi \psi \left( \frac{\delta \hat{E}}{\hbar \omega_o} \right)
\]
is an adiabatic invariant.

If \( Vg'(\psi_s) \) is kept constant during acceleration, the only quantity in (10.1) which varies (slowly) in a synchrotron is \( \frac{\eta}{\gamma} \).

Let us compute
\[
\frac{d}{d\gamma} \log \left| \frac{\eta}{\gamma} \right| = -\frac{2}{\gamma^2} - \frac{1}{a} = \frac{1}{\gamma} \left[ \frac{2}{\alpha \gamma^2} - 1 \right] = \frac{1}{\gamma} \left[ \frac{3 - \alpha \gamma^2}{\alpha \gamma^2 - 1} \right]
\]

\[
= \frac{1}{\gamma} \left[ \frac{3 \gamma_{\text{tr}}^2 - \gamma^2}{\gamma^2 - \gamma_{\text{tr}}^2} \right]
\]

\[
\frac{d}{d\gamma} \log \left| \frac{\eta}{\gamma} \right| > 0 \quad \text{when} \quad \gamma_{\text{tr}}^2 < \gamma^2 < 3 \gamma_{\text{tr}}^2
\]

\[
< 0 \quad \text{in all other cases}
\]
Therefore, when $\gamma^2 = 3 \gamma_{tr}^2$, $|\frac{n}{\gamma}|$ reaches a maximum where

$$\frac{n}{\gamma}_{\text{max}} = \frac{1}{\gamma} \left[ \frac{1}{\gamma^2} - \alpha \right]_{\text{max}} = -\frac{2}{(3 \gamma_{tr}^2)^{3/2}}$$

We notice that

$$\frac{n}{\gamma} = \frac{2}{(3 \gamma_{tr}^2)^{3/2}}$$

when $\gamma^2 = \frac{3}{4} \gamma_{tr}^2$

The variation of $|\frac{n}{\gamma}|$ i.e. of $\hat{\psi}$ as a function of $\gamma$, is shown in Fig. 10.1.

Fig. 10.1 Amplitude of phase oscillation as a function of $\gamma$.
In the figure, $\gamma_{tr} = 5$.

In case of a constant $|\varphi_s|$ during acceleration, the bucket width $|\varphi_e - \varphi_u|$ is constant. From (9-3) the bucket height is, for a sinusoidal voltage:

$$\left( \frac{\delta F}{h \omega_o} \right)_{\text{bucket}} = 2 \left| \frac{2 \pi \hbar^3 \omega_o^2 \eta}{E_o \beta^2 \gamma \cdot e \cdot N \cdot V'\left(\varphi_s\right)} \right|^{-\frac{1}{2}} \left[ 1 - \varphi_s \cot \varphi_s \right]^{\frac{1}{2}}$$
where \( g'(\varphi_s) = \cos \varphi_s = \sin \varphi_s \).

The ratio of particle height (10.2) to bucket height is thus:

\[
\frac{(\delta E)_{\text{particle}}}{(\delta E)_{\text{bucket}}} = \frac{\sqrt{\gamma}}{2} \left| \frac{2\pi h^3 \omega_0^2 \eta}{E_0 \beta^2 \gamma \cdot e \nu V g'(\varphi_s)} \right| \left[ 1 - \varphi_s \cot \varphi_s \right]^{-1}
\]

or, with (10.1),

\[
\frac{(\delta E)_{\text{particle}}}{(\delta E)_{\text{bucket}}} = \frac{\gamma}{2 \sqrt{1 - \varphi_s \cot \varphi_s}}
\]

Since this ratio behaves like \( \gamma \), it is maximum at injection when the injection energy is below \( \gamma_{tr} \frac{\sqrt{3}}{2} \) or above \( \gamma_{tr} \frac{\sqrt{3}}{2} \).

In these cases, if \(| \varphi_s | \) and \( V \) are kept constant during acceleration, particles captured at injection will stay in the bucket during the whole acceleration process.

**Remark:** In Fig. 10.1, \( \gamma \to 0 \) while \( \frac{\delta \hat{E}}{\hbar \omega_0} \to \infty \) when \( \gamma \to \gamma_{tr} \); this means that the adiabatic approximation breaks down in the vicinity of \( \gamma_{tr} \). A more refined treatment (see K. Johnsen, Ref. 2, p.178) shows that \( \gamma \) goes through a minimum while \( \frac{\delta \hat{E}}{\hbar \omega_0} \) goes through a maximum at transition.

**Separatrix crossing: golf-club**

Although the area \( 2\pi J \) which is enclosed by a particle trajectory in phase space is an adiabatic invariant, it follows from (9.6) that the bucket area \( A_s \) behaves like the bucket height (9.3) and is not an adiabatic invariant, as appears clearly in Fig. 10.1. Therefore, one must conclude that the separatrix is not a particle trajectory in phase-space, except when there is no acceleration (\( t=0 \)) and therefore no time variation of the parameters in the Hamiltonian (4.7). The reason is that, as shown by (7.3), the particle motion along the separatrix is extremely slow in the vicinity of \( \phi_u \), therefore violating the adiabaticity condition; this condition indeed requires that the variation of the parameters in the Hamiltonian (4.7) be negligible during a synchrotron period.

In order to arrive at a qualitative picture of the actual particle motion, we observe that the reduced Hamiltonian (5.5) does not depend on time. Therefore this is also true for the curves of constant \( H^* \) in the \((y, \phi)\) plane (see Fig. 5.2), in particular for the slope of the separatrix at \( \phi_u \). This would not be true in the \((\frac{\delta \hat{E}}{\hbar \omega_0}, \phi)\) phase space because from (5.1)
\[ \frac{\delta E}{h\nu_0} = K_2 \gamma \quad \text{or} \quad \gamma = \frac{1}{K_2} \cdot \frac{\delta E}{h\nu_0} \] (10.3)

and \( K_2 \) varies (slowly) with time. Since the elementary phase space area \( \frac{\delta E}{h\nu_0} \wedge \delta \phi \) is invariant, in the \((\gamma, \phi)\) plane the elementary area

\[ \delta y \wedge \delta \phi = \frac{1}{K_2} \cdot \frac{\delta E}{h\nu_0} \wedge \delta \phi \]

varies as \( K_2^{-1} \). In Fig. 10.1 it is seen that when \( \gamma < \gamma_{tr}, \frac{\sqrt{3}}{2} \) or \( \gamma > \gamma_{tr}, \frac{\sqrt{3}}{2} \), \( K_2^{-1} \) decreases with increasing \( \gamma \) (or time); therefore in the \((\gamma, \phi)\) plane, areas shrink with time and the motion appears to be damped. In particular, the separatrix is no longer a trajectory since it encloses a constant area in \((\gamma, \phi)\). Instead, one of the trajectories which leaves \( \Phi_u \) (or, strictly speaking, which tends to \( \Phi_u \) when \( t \to -\infty \)) will no longer come back to \( \Phi_u \) (strictly speaking, tend to \( \Phi_u \) when \( t \to +\infty \)), but it will spiral inwards around \( \Phi_S \); in other words, instead of being a center, \( \Phi_S \) has become a focus (see Fig. 10.2). This shows clearly the adiabatic damping of phase oscillations:

\( \dot{\phi} \) shrinks as \( K_2^{-1} \), \( \frac{\delta E}{h\nu_0} \) expands as \( K_2 \), and from (10.3) \( \dot{\gamma} \) also shrinks as \( K_2^{-1} \).

On the other end of the separatrix, the trajectory which tends to \( \Phi_u \) must then come from outside the fish-shaped bucket. As a result, in reduced coordinates the longitudinal acceptance of an accelerator is not a fish-shaped bucket, but rather has the shape of a golf-club. In the literature, this effect is mostly discussed for low-\( \beta \) linacs (Ref. 8; Ref. 9, p.27), for which the adiabaticity condition is worst fulfilled; since for a linac \( \gamma_{tr} = \infty \), the \((\gamma, \phi)\) plane corresponds to \( \gamma < \gamma_{tr} \) in Fig. 10.2.

When \( \gamma_{tr} < \gamma < \gamma_{tr}, \sqrt{3} \), \( K_2^{-1} \) increases with time; in the \((\gamma, \phi)\) plane areas expand with time and the motion appears to be anti-damped. In particular, one of the trajectories which tends to \( \Phi_u \) spirals outwards around \( \Phi_S \), and the handle of the golf-club is directed toward the negative \( y \)-axis.

In fact, the motion in the \((\gamma, \phi)\) plane depicted in Fig. 10.2 as a result of slow variation of parameters in the Hamiltonian, seems to be more generic (i.e. more common) in dynamical systems than the motion depicted in Fig. 5.2. Indeed (Ref. 10, p.29-31) it is exceptional that a trajectory leaving (or arriving at) a saddle point goes to (or comes from) another (or the same, as in Fig. 5.2) saddle point; it rather goes to an attractor or comes from a repeller, which in Fig. 10.2 is the stable fixed point \( \Phi_S \). In practice, the motion depicted in Fig. 10.2 becomes apparent mainly when \( K_2 \) varies fast with \( \gamma \), i.e. (see Fig. 10.1) at low \( \beta \) or near transition.
Fig. 10.2 - Actual trajectories in the reduced coordinate $y, \phi$ plane.
11. BACK TO FINITE DIFFERENCE EQUATIONS. STOCHASTICITY

For an arbitrary RF voltage, the finite difference equations (3.11) and (3.4) read in case of a synchrotron ($R_s = constant$):

\[
\begin{align*}
\delta p_{n+1} - \delta p_n &= \frac{eV}{v_s n} \left[ g(\phi_n) - g(\phi_s) \right] \\
\phi_{n+1} - \phi_n &= -\frac{2\pi n}{N} \cdot \frac{p_s n+1}{p_s} \cdot \delta p_{n+1}
\end{align*}
\] (11.1)

where $n$, $v_s$, $p_s$ are slowly varying parameters.

This mapping preserves area in the $\delta p_n$, $\phi_n$ plane. In contrast to differential equations, there is no (smooth) constant of motion for the finite difference equations.

**Fixed points (mod 2π):** If $k$ is any integer,

\[
\begin{align*}
\phi_n &= \phi_s + 2\pi n, \quad \frac{h_n}{N p_s} \cdot \delta p_n = k \\
\phi_n &= \phi_u + 2\pi n, \quad \frac{h_n}{N p_s} \cdot \delta p_n = k
\end{align*}
\]

is a stable fixed point

is an unstable fixed point.

The $k \neq 0$ case corresponds to working with the same RF frequency, but with an harmonic number ($h + N k$). Indeed, for a given $\omega_{RF}$ the synchronous revolution frequencies are such that

\[
\omega_0 = \frac{\omega_{RF}}{h}, \quad \frac{\delta \omega}{\omega_0} = -\frac{\delta h}{h}
\]

to which correspond the synchronous momenta

\[
\frac{\delta p}{p_s} = \frac{\delta \omega}{\omega_0} = -\frac{\delta h}{h} = -\frac{N k}{h}
\]

This means that the vertical distance between neighbouring buckets is \( \frac{\delta p}{p_s} = \frac{N}{|h|} \). In order to prevent stochastic effects (Ref. 11) from becoming important, the ratio $\xi$ between the full bucket height and the vertical distance between neighbouring buckets should be less than 1 (Chirikov's criterion). From (9.7),

\[
\xi = \frac{4}{p_s} \left| \frac{\frac{y}{h_n} e\pi \sin \varphi_s}{2\pi E_0} \right| \frac{1}{\sqrt{1 - \varphi_s \cotg \varphi_s}} = \frac{4}{N} \left| \frac{h_n}{p_s} \right| \frac{\frac{e\pi \sin \varphi_s}{2\pi E_0} \left| \sqrt{1 - \varphi_s \cotg \varphi_s} \right|}{h_n}
\]

so that Chirikov's criterion reads, with (6.2):
\[ \varepsilon = 4 \frac{Q_{SO}}{N} \cdot \sqrt{1 - \frac{\varphi_s}{\varphi_s} \cotg \varphi_s} < 1 \]

Therefore, differential equations are valid only when \( Q_{SO}/N \ll 1 \). For a finite \( Q_{SO} \), the motion near the bucket border becomes chaotic, making the bucket area shrink (Ref. 12); in practice, this effect is still very small for \( Q_{SO}/N < 0.1 \).

---

**Fig. 11.1** Phase space enlarged to several harmonic numbers at a fixed RF frequency (for \( n < 0 \))

**Period of small synchrotron oscillations**

In the close vicinity of \( \phi_s \), the system (11.1), (11.2) reduces to

\[
\begin{vmatrix}
2z \xi_n \\
N \phi_s \\
N \phi_s
\end{vmatrix}
\begin{vmatrix}
N \phi_s \\
N \phi_s \\
N \phi_s
\end{vmatrix}
\begin{vmatrix}
N \phi_s \\
N \phi_s \\
N \phi_s
\end{vmatrix}
\begin{vmatrix}
P_{n+1} - P_n = -K_0 (\phi_n - \phi_s) \\
\phi_{n+1} - \phi_n = P_{n+1} \\
P_n = \frac{2z \xi_n}{N \phi_s} \delta P_n
\end{vmatrix}
\]

where we have put

\[
P_n = \frac{2z \xi_n}{N \phi_s} \delta P_n
\]

and

\[
K_0 = \frac{2z \xi_n \cdot eV g'(\phi_s)}{N \phi_s \gamma_s} > 0
\]

With \( \mu \) defined by

\[
4 \sin^2 \frac{\mu}{2} = K_0
\]

the system (11.3) admits of the general solution

\[
\phi_n - \phi_s = \text{Re} \left[ a \ e^{i \mu} \right], \quad P_n = \text{Re} \left[ i a 2 \sin \frac{\mu}{2} \cdot e^{i(n-1)\mu} \right]
\]

(11.7)
This represents small synchrotron oscillations with angular frequency

$$\omega_{so} = \frac{N}{T_0} \mu = N \omega_0 \frac{\mu}{2\pi}.$$ 

From (11.5) and (11.6) the synchrotron tune for small oscillations is given by

$$4 N^2 \sin^2 \left( \frac{Q_{so}}{N} \right) = N^2 K_0 = \left[ 2 \pi \frac{N}{\gamma} \frac{eN \gamma'(\phi_s)}{p_s} \right]^2 = \frac{\frac{h}{p_s} eN \gamma'(\phi_s)}{E_0}.$$ 

(11.8)

The analogous formula (6.2) obtained with differential equations appears to be the limiting case of (11.8) when \( N \to \infty \), i.e. when particle acceleration is evenly distributed all around the ring with the total RF voltage \( NV \) remaining finite.

### Adiabatic damping of phase oscillations

From (11.4) an area element in the \( p_n', \phi_n' \) plane reads

$$\delta p_n' \wedge \delta \phi_n' = -\frac{2 \pi \hbar}{N p_s} \left( \delta p_n \wedge \delta \phi_n \right)$$

(11.9)

which shows that the mapping (11.3) does not preserve area in the \( p_n', \phi_n' \) plane. All successive points of a trajectory described by (11.7) lie on an ellipse with area \( \pi |a|^2 \sin \mu \); by (11.9) this area is related to the adiabatic invariant action \( J \) through

$$\pi |a|^2 \sin \mu = \left[ \frac{2 \pi \hbar}{N p_s} \right] \frac{h}{R_s} 2\pi J.$$

Since from (11.6)

$$\sin^2 \mu = \frac{K_0}{4} \left( 1 - \frac{K_0}{4} \right),$$

we obtain, using (11.5):

$$|a|^2 = \frac{2J}{R_s} \left| \frac{2 \pi \hbar}{N p_s} \right| \frac{K_0}{4} \left( 1 - \frac{K_0}{4} \right)^{-\frac{1}{2}} = \frac{2J}{R_s} \left| \frac{v_s}{N p_s} \frac{eN \gamma'(\phi_s)}{E_0} \right| \left( 1 - \frac{K_0}{4} \right)^{-\frac{1}{2}}$$

hence

$$|a| = \sqrt{2J} \left| \frac{2 \pi \hbar^3 n}{R_s^2 \gamma_e \gamma_s e N V g'(\phi_s)} \right| \left( 1 - \frac{K_0}{4} \right)^{-\frac{1}{2}}$$

(11.10)

This expression generalizes the expression (10.1) of \( \hat{\psi} \) to the case of finite difference equations.
12. PHASE DISPLACEMENT ACCELERATION

Empty bucket sweep

For a fixed harmonic number, the RF frequency determines the synchronous revolution frequency or, because

$$\frac{\delta p}{p_0} = \frac{\delta \omega}{\omega_0},$$

the synchronous momentum of the particles. Let \( \omega_1 \) and \( \omega_2 \) be two revolution frequencies located on both sides of the central revolution frequency \( \omega_0 \) of the stack, well outside the stack, with \( \omega_1 \) corresponding to a higher momentum than the stack (Fig. 12.1).

![Diagram of empty bucket sweep](image)

Fig. 12.1 Sweeping an empty bucket through the stack, from \( \omega_1 \) to \( \omega_2 \) \((\eta < 0)\). With \( \Gamma = \sin \phi < 0 \), particles move upwards around the bucket (when \( \Gamma \) changes sign, all \( \phi \)'s change sign).

![Diagram of RF variation](image)

Fig. 12.2 Variation of RF frequency with time \((\eta < 0)\).

If the RF frequency is varied from \( \omega_1 \) to \( \omega_2 \) (Fig. 12.2), an empty bucket is moved completely through the stack in the direction of decreasing \( p \); because phase space is incompressible, the average position of the stack is moved upwards by a quantity equal to \((\text{Bucket area/Horizontal axis period})\). Therefore, according to (9.8), the average momentum of the stack is increased by

$$\left< \frac{\Delta p}{m_0 c} \right> = \frac{16}{2\pi} \left| \frac{\gamma e N V}{\hbar n} \right| \frac{1}{2} \cdot a(r) \text{ per empty bucket sweep} \quad (12.1)$$
Since this average momentum increase is small, in order to maintain the beam at fixed radius the magnetic field $B_z$ must increase so slightly during an empty bucket sweep, that for all computations $B_z$ may be considered as constant in time. Therefore, the betatron electromotive force may be neglected in (3.15), which for any particle reduces to

$$
R \frac{dp}{dt} = \frac{eNV}{2\pi} \sin \phi
$$

(12.2)

In particular, the stable phase $\phi_s$ of the empty bucket is determined by

$$
\frac{dp_s}{dt} = \frac{eNV}{2\pi} \Gamma \quad \text{where} \quad \Gamma = \sin \phi_s < 0
$$

(12.3)

i.e. by the slope of the RF frequency with time, since for fixed $B_z$:

$$
\frac{d\omega_{RF}}{dt} = \frac{\omega_0}{h} = \frac{eNV}{R_s p_s} \cdot \frac{dp_s}{dt} = \frac{\eta}{R_s p_s} \frac{\nu_s}{\gamma} \frac{dp_s}{dt}
$$

Combined with (12.3) this yields

$$
\frac{d\omega_{RF}}{dt} = \frac{\eta}{R_s^2 m_0 \nu_s} \cdot \frac{eNV}{2\pi} \Gamma \quad \Gamma = \left( \frac{c}{R_s} \right)^2 \frac{\eta}{\gamma} \frac{eNV}{2m_0 F_0}
$$

(12.4)

For normal acceleration in a synchrotron, where $B_z$ increases noticeably with time, a similar formula applies but with $\eta$ replaced by $1/\gamma^2$.

The method of phase displacement (Ref. 13) allows acceleration of a stack by an empty bucket with a momentum height which is much smaller than the momentum spread of the stack; this is in contrast with normal acceleration, where the momentum height of the bucket is necessarily larger than the momentum spread of the bunch. Phase displacement acceleration has been successfully used in the ISR to accelerate coating beams from 26.6 to 31.4 GeV/c (Ref. 14); it necessitated around 200 sweeps of 3 seconds each with a total voltage NV of 12 kV.

**Momentum blow up**

For any particle, the change in momentum is given by (12.2). Using the reduced time (5.1) and the reduced Hamiltonian (5.5) yields

$$
dt = K_1 dt^* = K_1 \frac{d\phi}{y} \quad \text{where} \quad y = \pm \sqrt{2H^* + 2 \text{sgn}(\eta) \left[ R_\phi + \cos \phi \right]}
$$

Therefore, during an empty bucket sweep the total change in momentum of a particle which crosses the $\phi$-axis at $\phi_1$ (see Fig. 12.1) reads
\[
\delta p = \int_{\phi_1}^{-\text{sgn} (\eta) \cdot \text{sgn} (n)} \frac{eN \nu K_1 \sin \phi \cdot d\phi}{2 \pi R} \quad \text{where} \quad y = \sqrt{2H^* + 2 \text{sgn} (\eta) \left[ \Gamma \phi + \cos \phi \right] > 0}
\]

(12.5)

and \(0 < \phi_1 < 2\pi\) excluding the interval \((\phi_e, \phi_u)\).

As a first approximation we shall consider \(H^*\) to be a constant; with (5.3) this integral becomes

\[
\delta p = -2 \text{sgn} (\eta) \int_{\phi_1}^{-\text{sgn} (\eta) \cdot \text{sgn} (n)} \frac{h}{R} K_2 \sin \phi \cdot d\phi \quad \text{where} \quad y = \sqrt{-2 \text{sgn} (\eta) \left[ \Gamma (\phi_1 - \phi) + \cos \phi_1 - \cos \phi \right]}
\]

which can be rewritten as

\[
\delta p = -2 \text{sgn} (\eta) \cdot \frac{h}{R} K_2 \int_{\phi_1}^{-\text{sgn} (\eta) \cdot \text{sgn} (n)} \frac{\sin \phi \cdot d\phi}{\sqrt{-2 \text{sgn} (\eta) \left[ \Gamma (\phi_1 - \phi) + \cos \phi_1 - \cos \phi \right]}}
\]

(12.6)

The ensemble average of \(\delta p\) over the stack is given by (12.1) as

\[
\langle \delta p \rangle = \langle \delta p_{\text{ST}} \rangle \cdot \alpha (r) \quad \text{per empty bucket sweep}
\]

(12.7)

where \(\langle \delta p_{\text{ST}} \rangle\) is the average momentum increase due to a stationary bucket sweep:

\[
\langle \delta p_{\text{ST}} \rangle = \frac{16}{2\pi} \int_{\phi_1}^{\text{sgn} (\eta) \cdot \text{sgn} (n)} \frac{\sin \phi \cdot d\phi}{\sqrt{-2 \text{sgn} (\eta) \left[ \Gamma (\phi_1 - \phi) + \cos \phi_1 - \cos \phi \right]}}
\]

(12.8)

where using (5.4) and taking the average of the bucket area throughout the stack. Referred to \(\langle \delta p_{\text{ST}} \rangle\), the momentum change (12.6) of an individual particle reads

\[
S = \frac{\delta p}{\langle \delta p_{\text{ST}} \rangle} = -\frac{\pi}{4} \text{sgn} (\eta) \int_{\phi_1}^{-\text{sgn} (\eta) \cdot \text{sgn} (n)} \frac{\sin \phi \cdot d\phi}{\sqrt{-2 \text{sgn} (\eta) \left[ \Gamma (\phi_1 - \phi) + \cos \phi_1 - \cos \phi \right]}}
\]

(12.9)

The integral (12.9) converges when the upper limit of integration tends to \(\infty\); but it must be computed numerically. From (12.7) one must have

\[
\langle S \rangle = \alpha (r) \quad \text{where} \quad \alpha (r) = 0 \quad \text{for} \quad |r| > 1
\]

(12.10)
which has indeed been verified in the case of a uniform particle distribution in phase space (Ref.15). Detailed numerical computations (Ref. 15), confirmed by measurements, have also shown that:

\[
< (S - < S >)^2 > \approx \gamma \approx (\frac{\gamma}{\alpha})^3 \cdot \eta^{-1}
\]

for \( |\gamma| < 1 \)

(12.11)

Therefore, after the \( n^{th} \) empty bucket sweep, the mean square momentum spread of the stack is given by

\[
< \left( \frac{\delta p}{m_0 c} \right)^2 > = < \left( \frac{\delta p}{m_0 c} \right)^2 > + n \gamma^2 \left[ \frac{\Delta p_{ST}}{m_0 c} \right]^2 \]

(12.12)

*Choice of \( \gamma \)*

The total number \( n \) of empty bucket sweeps necessary to increase the momentum of the stack by an amount \( < \Delta p > \) reads, with (12.7):

\[
n = \frac{< \Delta p >}{< \Delta p >_{\text{sweep}}} = \frac{1}{\alpha(\gamma)} \cdot \frac{< \Delta p >_{\text{wanted}}}{< \Delta p >_{\text{sweep}}} \]

(12.13)

whereas the time needed for a single sweep is obtained from (12.3) as

\[
T_{\text{sweep}} = F \cdot \frac{1}{|\gamma|} \cdot \frac{2 \pi R}{enW} \cdot 2 \left[ (\delta p)_{\text{stack}} + M \cdot (\delta p)_{\text{bucket}} \right] \text{ per sweep} \]

(12.14)

where \( F \) and \( M \) are safety factors which are taken as (Ref. 16)

\[ 1 < F \leq 2 \quad \text{and} \quad 1 < M \leq 10 \]

With (9.7) and (12.8), the bucket height in (12.14) may be expressed as

\[
(\delta p)_{\text{bucket}} = \frac{1}{4} \cdot \frac{|\Delta p_{ST}|}{\sin \phi_s - \phi_s \cos \phi_s} \]

where \( \cos \phi_s = |\gamma| \)

(12.15)

In (12.14) this term is normally much smaller than \( (\delta p)_{\text{stack}} \).

From (12.12) and (12.13), in order to keep the momentum blow-up small, one should take

\[
\frac{\gamma^2}{\alpha(\gamma)}
\]

as small as possible. On the other hand, from (12.13) and (12.14), in order to keep
the total time $n \cdot T_{\text{sweep}}$ short, one should take $|\tau| \cdot a(\tau)$ as large as possible. Since $|\tau| \cdot a(\tau)$ reaches a maximum for $|\tau| \approx 0.4$, the best choice is certainly $|\tau| < 0.4$. These and other considerations (E. Ciapala: Ref. 2, p.217-220), based on the effects of RF phase noise on particle diffusion and on the effects of a change in the empty bucket area across the stack (due to a variation of $\gamma_{\text{tr}}$ across the machine aperture) (Ref. 17) have, in the case of the ISR led to the choice

$$-0.3 < \tau < -0.1 \quad \text{with mostly} \quad \tau = -0.2$$

13. LINEAR ACCELERATORS

A linear accelerator may be considered as the limiting case of a synchrotron where $R \to \infty$. The momentum compaction is zero:

$$a = 0 \quad \text{hence} \quad \gamma_{\text{tr}} = \infty$$

and, from (3-3),

$$\eta = \frac{1}{\gamma^2} > 0$$

Therefore a linac is always below transition.

In a linac, the distance $L$ between accelerating cavities becomes the cell length of the accelerating structure:

$$L = \frac{C}{N} = \frac{2\pi R}{N} \quad \text{is fixed while} \quad R \to \infty, \quad N \to \infty \quad (13.1)$$

$$\frac{h}{R} = \frac{\omega_{\text{RF}}}{\omega_0} = \frac{\omega_{\text{RF}}}{v_s} \quad \text{remains finite while} \quad R \to \infty, \quad h \to \infty \quad (13.2)$$

$$\theta = \frac{2\pi h}{N} = \frac{2\pi h}{2\pi R} = \frac{\omega_{\text{RF}}}{v_s} L = k_{\text{RF}} \cdot \frac{1}{\beta_s} \quad \text{where} \quad k_{\text{RF}} = \frac{\omega_{\text{RF}}}{c} \quad (13.3)$$

is the RF phase-shift that must be provided between adjacent cells in order to keep the the phase of the reference particle constant along the accelerating structure. (This phase shift should hopefully not be confused with the azimuthal variable $\theta$ defined in section 3).

In contrast to synchrotrons, $\omega_{\text{RF}}$ in linacs does not vary with time, and there are no harmonics on the RF frequency, i.e.

$$g(\phi) = \sin \phi$$
Moreover, linacs are usually designed so as to maintain the phase shift per cell \( \theta \) constant along the accelerating structure (which is then said to operate in the \( \theta \)-mode); in that case, as shown by (13.3), \( L/v_s \) is constant along the structure: the cell length increases with particle velocity.

Non-relativistic linacs

**Finite difference equations**

Dividing (3.11) by \( h \), and using (13.3) in (3.4) we obtain

\[
\begin{align*}
\delta E_{n+1} - \delta E_n &= eV g(\psi_n) - g(\phi_n) \quad (13.4) \\
\delta \phi_{n+1} - \delta \phi_n &= -\frac{\omega_{RF}}{v_s} \cdot L \cdot \frac{\eta}{\beta^2} \delta \gamma_{n+1} = -k_{RF} L \cdot \frac{\delta \gamma_{n+1}}{\beta^3} = k_{RF} L \cdot \delta \left( \frac{1}{\beta_{n+1}} \right) \quad (13.5)
\end{align*}
\]

where \( V \) is the peak RF voltage in a single cell.

This mapping preserves area in the \( (\delta E_n, \delta \phi_n) \) plane; in fact, the variable conjugate to \( \delta \phi_n \) is \( \frac{\partial}{\partial \omega_{RF}} = \frac{E_0}{\omega_{RF}} \delta \gamma_n \). The equations above can be written down at once when we observe that for any particle

\[
\begin{align*}
E_{n+1} - E_n &= eV \cdot g(\psi_n) \\
\psi_{n+1} - \psi_n &= \omega_{RF} \cdot \frac{L}{v_{n+1}} = \theta = k_{RF} L \left( \frac{1}{\beta_{n+1}^3} - \frac{1}{\beta_s^3} \right) \quad (13.6)
\end{align*}
\]

**Frequency of small phase oscillations**

It is given by (11.8), where

\[
2\pi \frac{Q_{SO}}{N} \omega_0 \cdot \frac{2\pi}{N} \omega_0 \cdot \frac{2\pi}{N} \frac{2\pi}{v_s} = \frac{\omega_0}{v_s} \frac{2\pi}{L}.
\]

Using (13.3) in (11.8) then yields

\[
4 \sin^2 \left( \frac{\omega_0}{v_s} \cdot \frac{\theta}{2} \right) = \frac{\eta}{\beta^2} \frac{eV g'(\psi_S)}{E_0}
\]

or

\[
\sin^2 \left( \frac{\omega_0}{v_s} \cdot \frac{\theta}{2} \right) = \frac{1}{4} \frac{1}{\beta_s^2} \cdot \frac{eV \cos \phi_S}{E_0} = \left( \frac{\theta}{2} \right)^2 \cdot \frac{1}{\beta_s^2} \cdot \frac{eV \cos \phi_S}{k_{RF} L E_0} \quad (13.8)
\]

where \( V/L \) is the average accelerating electric field on the linac axis.
Differential equations. Hamiltonian

Taking the distance $s$ along the linac axis as the independent variable, one immediately writes (13.4), (13.5) in differential form as

$$
\frac{d}{ds} (\delta y) = \frac{eV}{E_0 L} \left[ g(\phi) - g(\phi_s) \right] = -\frac{\partial H}{\partial \phi} \tag{13.9}
$$

where $\delta y = y - y_s$

$$
\frac{d}{ds} (\phi - \phi_s) = -k_{RF} \cdot \frac{\delta y}{\beta_s^3 \gamma^3} = \frac{\partial H}{\partial (\delta y)} \tag{13.10}
$$

Using (4.8) this system may be derived from the Hamiltonian

$$
H = -\frac{1}{2} \frac{k_{RF}}{\beta_s^3 \gamma^3} (\delta y)^2 + \frac{eV}{E_0 L} \left[ \Gamma \phi + G(\phi) \right] \tag{13.11}
$$

where the potential energy reads, for a sinusoidal RF voltage:

$$
\Gamma \phi + G(\phi) = \phi \sin \phi_s + \cos \phi
$$

$$
= \left( \frac{\pi}{2} + \phi \right) \cos \phi_s - \sin \phi = - (\sin \phi - \phi \cos \phi_s) + \frac{\pi}{2} \cos \phi_s
$$

Instead of the phase $\phi$ used in synchrotron theory, it is common in linac theory to use the phase $\phi$ measured from the crest of the RF voltage. Since $\text{sgn}(\phi_s) = -\text{sgn}(\eta) \cdot \text{sgn}(\gamma)$, it follows that $\phi_s$ is negative in a linear accelerator.

The variable conjugate to $\phi$ is $\delta y$ or, in the physical phase space, $\frac{\delta E}{\omega_{RF}} = \frac{E_0}{\omega_{RF}} \cdot \delta y$;

therefore area is preserved in the $(\delta y, \phi)$ plane.

Adiabatic damping of phase oscillations

The Hamiltonian (13.11) depends on $s$ through $\beta^3 \gamma^3$, even if $V/L$ and $\phi_s$ are assumed to be constant along the linac. Therefore $H$ is not exactly a constant of motion, and the curves $H = \text{constant}$ with $\beta^3 \gamma^3$ and $V/L$ taken as constants are only approximate trajectories.

In order to have a more accurate picture of the actual trajectories, it is better, as in Fig. 10.2, to use a representation in the reduced coordinate $(y, \phi)$ plane. From (5.1) and (5.4),

$$
y = \frac{1}{K_2} \cdot \frac{\delta E}{\omega_{RF}} = \frac{\beta_s^2 \gamma}{\eta} \left( \frac{eV}{E_0} \right)^{-\frac{1}{2}} \cdot \delta y = \left( \beta_s^3 \gamma^3 \frac{eV}{E_0 k_{RF} L} \right)^{-\frac{1}{2}} \cdot \delta y \tag{13.12}
$$
where again we have used (13.3). In the \((y, \phi)\) plane, the trajectories look like the bottom part \((y < y_{tr})\) of Fig. 10.2, where we recognize the familiar golf-club of low-\(\beta\) linacs (Ref. 8; Ref. 9, p.27).

**Relativistic linacs**

They are built for \(\delta_s = 1\), i.e. \(\gamma_s = \infty\). By (13.3) this entails that for \(\theta\) constant, \(L\) is constant along the linac: therefore the accelerating structure is exactly periodic, with geometrical period \(L\). Since \(\nu_s\) is constant, the synchronous particle is not accelerated, and

\[ \Gamma = g(\phi_s) = \sin \phi_s = 0 \]

Since \(\eta > 0\),

\[ \phi_s = 0 \]

at the stable fixed point.

Therefore the particles are accelerated inside a stationary bucket, but the reference energy is at infinity.

**Finite difference equations**

Since \(\gamma_s = \infty\), we can no longer use the system (13.4), (13.5); instead we must use the system (13.6), (13.7) which reads, for any particle:

\[
\begin{align*}
\gamma_{n+1} - \gamma_n &= \frac{eV}{E_0} g(\phi_n) \\
\phi_{n+1} - \phi_n &= k_{RF} L \left( \frac{1}{\gamma_{n+1}} - 1 \right) \quad \text{where} \quad \frac{1}{\gamma_{n+1}} = \left( 1 - \frac{1}{\gamma_{n+1}^2} \right)^{-1}
\end{align*}
\]  

(13.13) \hspace{2cm} (13.14)

This mapping preserves area in the \((\gamma_n, \phi_n)\) plane; but like the mapping (11.1), (11.2), it does not admit of any (smooth) constant of motion. In order to obtain an (approximate) constant of motion, we must again go over to differential equations.

**Differential equations. Hamiltonian**

In differential form, the system (13.13), (13.14) becomes

\[
\begin{align*}
\frac{d\gamma}{ds} &= \frac{eV}{E_0} g(\phi) = - \frac{\partial H}{\partial \phi} \\
\frac{d\phi}{ds} &= k_{RF} \left( \frac{1}{\gamma} - 1 \right) = \frac{\partial H}{\partial \gamma} \quad \text{where} \quad \frac{1}{\gamma} = \left( 1 - \frac{1}{\gamma^2} \right)^{-1}
\end{align*}
\]  

(13.15) \hspace{2cm} (13.16)
Using (4.8) this system may be derived from the Hamiltonian

$$H = k_{RF} (\beta - 1) \gamma + \frac{eV}{E_0} \cdot G(\phi)$$  \hspace{1cm} (13.17)

where, for a sinusoidal voltage:  \[ G(\phi) = \cos\phi = -\sin\psi \]

If, as is often the case, the average accelerating field V/L is maintained constant along the accelerator, H does not depend explicitly on s: it is therefore a constant of motion, and the particle trajectories in the \((\gamma, \phi)\) plane are the curves \(H = \text{constant}\). By putting with (13.3),

$$K = \frac{eV}{k_{RF} L \cdot E_0} = \frac{1}{\theta} \cdot \frac{eV}{E_0}$$  \hspace{1cm} (13.18)

it is seen that (13.17) is proportional to the reduced Hamiltonian

$$H^* = - (1 - \beta) \gamma + K \cdot \cos \phi = - \frac{1}{(1+\beta)} \gamma + K \cdot \cos \phi$$  \hspace{1cm} (13.19)

which depends on a single dimensionless parameter \(K\).

Fig. 13.1 - Longitudinal phase space of a relativistic linac \((\phi_s = 0, \gamma_s = \infty)\). In the figure, \(K = 0.2\); this value corresponds to the SLAC 3.2 km electron linac operated at 20 GeV \((\theta = 2\pi/3)\).
Particle trajectories $H^* = \text{constant}$ are shown in Fig. 13.1. When $\gamma \to \infty$, $\phi$ approaches an asymptotic value $\phi_\infty$ given by

$$H^* = K \cdot \cos \phi_\infty$$  \hspace{1cm} (13.20)

If $H^* > K$, there is no corresponding real trajectory in the $\gamma, \phi$ plane.
If $-K < H^* < K$, the trajectories are asymptotic to a line parallel to the $\gamma$-axis at $\phi_\infty$,
with $0 < \phi_\infty < \pi$.
If $H^* < -K$, $\gamma$ reaches a finite maximum at $\phi = \pi$.
All trajectories are symmetrical with respect to $\phi = 0$ and have period $2\pi$ in $\phi$. The particular curve which corresponds to $\phi_\infty = \pi$ separates the bounded and unbounded motions: it is thus the separatrix; at the same time it is an actual trajectory, because the bucket is stationary and $K$ is assumed to be constant.

In (13.15), (13.16), the variable conjugate to $\phi$ is $\gamma$, or, in the physical space, $E_{\text{RF}} = \frac{E_0}{\sqrt{\gamma}}$; therefore the particle motion preserves area in the $(\gamma, \phi)$ plane.

In order to reach high energies, particles are injected with initial conditions such that $\phi_\infty \approx \frac{\pi}{2}$.

* * *

REFERENCES


APPENDIX A  Synchrotron frequency in a stationary bucket with a harmonic cavity

\[ \phi_s = 0 \quad \text{if} \quad \eta > 0 \]

\[ \phi_s = \tau \quad \text{if} \quad \eta < 0 \]

\[ g(\psi) = \sum_{n=1}^{\infty} a_n \sin (n\psi) \quad G(\psi) = \sum_{n=1}^{\infty} \frac{a_n}{n} \cos (n\psi) \quad \text{with} \quad a_1 = 1 \]

Canonical equations:

\[ \begin{align*}
\frac{dy}{dt^2} &= \text{sgn} (\eta) \left( \tau - g(\psi) \right) \\
\frac{d\psi}{dt^2} &= y
\end{align*} \]

Differential equation for \( \psi \):

\[ \frac{d^2\psi}{dt^2} + \text{sgn} (\eta) \sum_{n=1}^{\infty} a_n \sin (n\psi) = 0 \]

For a stationary bucket, \( \tau = 0 \) and this equation reduces to

\[ \frac{d^2\psi}{dt^2} + \text{sgn} (\eta) \sum_{n=1}^{\infty} a_n \sin (n\psi_s) = 0 \]

Let \( \psi = \phi - \phi_s \); then

\[ \frac{d^2\psi}{dt^2} + \text{sgn} (\eta) \sum_{n=1}^{\infty} a_n \cos (n\phi_s) \sin (n\psi) = 0 \]

Putting

\[ c_n = \text{sgn} (\eta) \cdot a_n \cos (n\phi_s) \]

we finally have

\[ \frac{d^2\psi}{dt^2} + \sum_{n=1}^{\infty} c_n \sin (n\psi) = 0 \quad \text{with} \quad c_1 > 0 \]  \hspace{1cm} (A.1)

Depending on the origin of time, the solution of (A.2) may in specific cases appear as an even or an odd function of time. Let us assume that it is an even function of time, then we may write the solution of (A.2) in the form

\[ \psi = \sum_{m=0}^{\infty} a_m \cos (mr) \quad \text{where} \quad \tau = \omega^* t^* \]  \hspace{1cm} (A.3)

hence

\[ \frac{d^2\psi}{dt^2} = -\omega^* \sum_{m=0}^{\infty} m^2 a_m \cos (mr) \]  \hspace{1cm} (A.4)
Using the expansions
\[ e^{jz \sin \theta} = \sum_{p = -\infty}^{\infty} J_p(z) e^{j p \theta} \]
\[ e^{jz \cos \theta} = \sum_{p = -\infty}^{\infty} j^p J_p(z) e^{j p \theta} \]
we obtain
\[ e^{j n \psi} = \prod_{m=0}^{\infty} e^{j m \alpha_m \cos m \tau} \]
\[ = \prod_{m=0}^{\infty} \sum_{p_m = -\infty}^{\infty} j^{p_m} J_{p_m}(n \alpha_m) e^{j p_m \tau m} \]
\[ = \sum_{p_m = -\infty}^{\infty} \prod_{m=0}^{\infty} j^{p_m} J_{p_m}(n \alpha_m) e^{j \tau \sum_{m=0}^{\infty} p_m} \]

where \( p_m \) is a full set of integers \( \{ \text{from } -\infty \text{ to } +\infty \} \) corresponding to \( \tau \).

Combining (A.4) with (A.2) yields
\[ \omega^2 r^2 \sum_{r=0}^{\infty} r^2 a_r \cos (\tau r) = \sum_{n=1}^{\infty} c_n \sin (n \tau) \]
\[ = \sum_{n=1}^{\infty} c_n \text{Im} \sum_{m=0}^{\infty} \prod_{m=0}^{\infty} j^{p_m} J_{p_m}(n \alpha_m) \left[ \cos (\tau \sum_{m=0}^{\infty} p_m) + j \sin (\tau \sum_{m=0}^{\infty} p_m) \right] \]

Let
\[ \sum_{m=0}^{\infty} m p_m = r \quad \text{and} \quad \sum_{m=0}^{\infty} p_m = q \quad \text{(A.5)} \]

By changing \( p_m \) into \( -p_m \) we see that all sin-terms disappear; we are thus left with
\[ \omega^2 r^2 a_r = \sum_{q \text{ odd} = -\infty}^{\infty} \sum_{n=1}^{\infty} c_n j^{q-1} J_{p_0} (n \alpha_0) J_{p_1} (n \alpha_1) J_{p_2} (n \alpha_2) \ldots \]

It is easy to see that in the \((\psi, \dot{\psi})\) plane, a trajectory must be symmetrical with respect to both axes; this entails that in (A.3), \( m = 1, 3, \ldots \)
and (A.5) reduces to

\[ p_1 + 3p_3 + 5p_5 + \ldots = \pm r \]  

\[ p_1 + p_3 + p_5 + \ldots = q \]  

hence \[ \pm r - q = 2p_3 + 4p_5 + \ldots \] is even.

Since \( q \) must be odd, \( r \) must also be odd and we are left with

\[ \sum_{q \text{ odd}}^{+\infty} \sum_{n=1}^{+\infty} c_n j^{q-1} J_{p_1}(n_{a_1}) J_{p_3}(n_{a_3}) J_{p_5}(n_{a_5}) \ldots = \omega^{*2} r^2 \alpha_r \]

(\( r = 1, 3, \ldots \))

In the following, we neglect terms of order \( a_5 \) and higher; this amounts to taking \( p_5 = p_7 = \ldots = 0 \) and to putting \( J_{0}(n_{a_5}) = J_{0}(n_{a_7}) = \ldots = 1 \). Then (A.6) reduces to

\[ p_1 + 3p_3 = \pm r \]

\[ q = p_1 + p_3 = \pm r - 2p_3 \]

Changing \( p_m \) into \(-p_m\) changes \(+r\) into \(-r\) but leaves the summation in (A.7) unchanged. Therefore, it is sufficient to keep \(+r\) only if one introduces a factor 2 in the left-hand side of (A.7).

\[ r = 1 \]

\[ p_1 + 3p_3 = 1 \quad \text{with} \quad q = 1 - 2p_3 \]

Then (A.7) becomes successively

\[ \sum_{n=1}^{+\infty} c_n \sum_{p_3 = -\infty}^{+\infty} (-1)^{p_3} J_{1-5p_3}(n_{a_1}) J_{p_3}(n_{a_3}) = \omega^{*2} \alpha_1 \]

\[ \sum_{n=1}^{+\infty} c_n 2 \left[ J_{1}(n_{a_1}) J_{0}(n_{a_3}) - J_{2}(n_{a_1}) J_{1}(n_{a_3}) + J_{3}(n_{a_1}) J_{2}(n_{a_3}) - J_{4}(n_{a_1}) J_{3}(n_{a_3}) \right. \]

\[ + \left. J_{7}(n_{a_1}) J_{2}(n_{a_3}) - \ldots \right] = \omega^{*2} \alpha_1 \]

\[ \sum_{n=1}^{+\infty} n \left\{ \frac{J_{1}(n_{a_1})}{n_{a_1}} J_{0}(n_{a_3}) - \frac{J_{1}(n_{a_3})}{n_{a_1}} J_{2}(n_{a_3}) \right\} = 0 \left( \alpha_1^{*2}, \alpha_3^{*2} \right) \]

(A.8)
\[ r = 3 \]

\[ p_1 + 3p_3 = 3 \quad \text{with} \quad q = 3 - 2p_3 \]

Then \((A.7)\) becomes successively

\[
\sum_{n=1}^{\infty} c_n 2 \sum_{p=\infty}^{+\infty} (-1)^{1-p} J_{-3-3p} (n \alpha_1) J_{p} (n \alpha_3) = \omega^2 \cdot 3^2 \cdot \alpha_3
\]

\[
\sum_{n=1}^{\infty} c_n 2 \sum_{p=\infty}^{+\infty} J_{3p} (n \alpha_1) J_{p+1} (n \alpha_3)
\]

\[
= \sum_{n=1}^{\infty} c_n 2 \left[ J_0 (n \alpha_1) J_1 (n \alpha_3) - J_3 (n \alpha_1) J_0 (n \alpha_3) + J_3 (n \alpha_1) J_2 (n \alpha_3) - J_6 (n \alpha_1) J_1 (n \alpha_3) \right]
\]

\[
+ J_6 (n \alpha_1) J_3 (n \alpha_3) - \ldots \right] = 3^2 \cdot \omega^2 \cdot \alpha_3
\]

\[
\sum_{n=1}^{\infty} n c_n \left\{ \frac{2J_1 (n \alpha_3)}{n \alpha_3} \left[ J_0 (n \alpha_1) - J_6 (n \alpha_1) \right] - 2 \frac{J_3 (n \alpha_1)}{n \alpha_3} \left[ J_0 (n \alpha_3) - J_2 (n \alpha_3) \right] \right. 
\]

\[
+ 0 \left( \alpha_1^6, \alpha_3^2 \right) \left\} = 3^2 \cdot \omega^2 \right. \quad (A.9)
\]

Keeping only the first terms in \((A.9)\) and \((A.8)\) yields

\[
\left\{ \sum_{n=1}^{\infty} n c_n \left\{ J_0 (n \alpha_1) - 2 \frac{J_3 (n \alpha_1)}{n \alpha_3} + 0 (\alpha_1^6) + 0 (\alpha_3^2) \right\} \right. 
\]

\[
= 3^2 \cdot \omega^2 \quad (A.10)
\]

\[
\left\{ \sum_{n=1}^{\infty} n c_n \left\{ 2 \frac{J_1 (n \alpha_1)}{n \alpha_1} - \frac{n \alpha_3}{n \alpha_1} J_2 (n \alpha_1) + 0 (\alpha_1^6) + 0 (\alpha_1^2 \alpha_3) + 0 (\alpha_3^2) \right\} \right. 
\]

\[
= \omega^2 \quad (A.11)
\]
In particular, by letting $\alpha_1 \to 0$ in (A.11) we obtain

$$\omega_0^2 = \sum_{n=1}^{\infty} n c_n$$  \hspace{1cm} (A.12)

Combining (A.10) and (A.11) yields

$$\sum_{n=1}^{\infty} n c_n \left\{ 3 \frac{2 J_1(na_1)}{na_1} - \frac{\alpha_3}{\alpha_1} \frac{3}{2} J_2(na_1) - J_0(na_1) + 2 \frac{J_3(na_1)}{na_3} \right\} = 0$$

$$+ 0(a_1^6) + 0(a_1^3 a_3) + 0(a_3^2) = 0$$

$$\sum_{n=1}^{\infty} n c_n \left\{ 3 \frac{2 J_1(na_1)}{na_1} - J_0(na_1) - \frac{\alpha_3}{\alpha_1} 3 J_2(na_1) + \frac{\alpha_1}{\alpha_3} \frac{2}{2} J_3(na_1) \right\} = 0$$

$$+ 0(a_1^6) + 0(a_1^3 a_3) + 0(a_3^2) = 0$$

$$- \frac{\alpha_3}{\alpha_1} \sum_{n=1}^{\infty} n c_n \left[ 3 \frac{2 J_1(na_1)}{na_1} - J_0(na_1) \right] + \left( \frac{\alpha_3}{\alpha_1} \right)^2 \sum_{n=1}^{\infty} n c_n J_2(na_1)$$

$$= 0(a_1^6) + 0(a_1^3 a_3) + 0(a_3^2)$$  \hspace{1cm} (A.13)

Because

$$2^2 J_{\nu}(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$$

it is seen that:

$$A = 3^3 \cdot C = 0(a_1^6) = 0(a_1^2)$$ \hspace{1cm} while \hspace{1cm} $$B = 0(1)$$
Solve (A.13) for \( \frac{a_3}{a_1} \):

\[
\frac{a_3}{a_1} = \frac{-2C}{B + \sqrt{B^2 + 4AC}} \approx \frac{-C}{B + \frac{AC}{B}} = -\frac{BC}{B^2 + AC} \left[ 1 + 0(a_1^6) \right] = -\frac{C}{B} \left[ 1 + 0(a_1^6) \right]
\]

\[
= -\frac{BC}{B^2 + 3C^2} \left[ 1 + 0(a_1^6) \right] = -\frac{C}{B} \left[ 1 + 3 \left( \frac{3C}{B} \right)^2 \right]^{-1} \left[ 1 + 0(a_1^6) \right].
\]

Finally

\[
\alpha_3 = -\sum_{n=1}^{\infty} \frac{c_n}{n} \frac{2 J_3(na_1)}{2 \cdot J_1(na_1) - J_0(na_1)} \left[ 1 + 3 \left( \frac{3C}{B} \right)^2 \right]^{-1} \left[ 1 + 0(a_1^6) \right] \tag{A.14}
\]

The synchrotron frequency is then obtained as a function of

\[
\hat{\psi} = \psi(0) = \sum_{m=1,3,\ldots}^{\infty} \alpha_m
\]

by using (A.14) in (A.8) and remembering that \( \alpha_3 = 0(-a_1^3) \).

Remark: The series

\[
\psi(\tau) = \sum_{m=1,3,\ldots}^{\infty} \alpha_m \cos(m\tau)
\]

is equivalent to

\[
\psi\left( \tau - \frac{\pi}{2} \right) = \sum_{m=1,3,\ldots}^{\infty} \alpha_m \sin\left( m \frac{\tau}{2} \right) \sin(m\tau)
\]
APPENDIX B  Bucket Area \( a(\Gamma) \)

From (9-1),
\[
\Gamma = \sin \phi_s = \sin \phi_u \quad |\Gamma| = \cos \phi_s
\]

Put
\[
\psi = - \text{sgn}(n\Gamma) \cdot (\phi - \phi_u) > 0
\]

From (9-4) and (9-5),
\[
a = -\frac{2}{16} \text{sgn}(n\Gamma) \int_{\phi_u}^{\phi_e} \sqrt{2} \left\{ - \text{sgn}(n) \left[ \Gamma \phi_u + \cos \phi_u \right] + \text{sgn}(n) \left[ \Gamma \phi + \cos \phi \right] \right\} \cdot d\phi
\]

\[
= \text{sgn}(n) \cos \phi_u \left[ 1 - \cos \psi \right] - \left[ \Gamma \psi - \sin \phi_u \sin \psi \right]
\]

\[
= |\sin \phi_s| \left( 1 - \cos \psi \right) - \cos \phi_s (\psi - \sin \psi)
\]

Therefore
\[
a = \frac{2}{16} \int_0^{\psi_e - \psi_u} \sqrt{2} \left| \sin \phi_s \right| \left( 1 - \cos \psi \right) - \cos \phi_s (\psi - \sin \psi) \cdot d\psi
\]

or
\[
a = \frac{1}{2} \sqrt{\cos \phi_s} \int_{\psi=0}^{\psi=|\psi_e - \psi_u|} \frac{d\psi}{2} \sin \frac{\psi}{2} \sqrt{|\text{tg \phi_s}| - t} \quad (B.1)
\]

where
\[
t = \psi - \sin \psi
\]

As a series in \( \psi \),
\[
t = \frac{\psi}{3} + \frac{\psi^3}{2.5.9} + \frac{\psi^5}{2.4.5.7.9} + \frac{\psi^7}{3.4.2.5.2.7} + \frac{\psi^9}{3.3.4.2.8.7.9.11} + \frac{691 \psi^{11}}{3.5.4.3.7.2.11.13} + \ldots \quad (B.2)
\]

Inverting the series (B.2) one obtains
\[
\psi = \frac{3}{2} t \left[ 1 - \frac{2}{3.5} \left( \frac{3}{2} t \right)^2 + \frac{2.3}{5.7} \left( \frac{3}{2} t \right)^4 - \frac{22}{3.5.3} \left( \frac{3}{2} t \right)^6 + \ldots \right] \quad (B.3)
\]

\[
+ \frac{2.1103}{3.5.3.7.2.11} \left( \frac{3}{2} t \right)^8 - \frac{2^2 7.171}{5.7.2.11.13} \left( \frac{3}{2} t \right)^{10} + \ldots
\]
For $t = \tan \varphi_s$, this relation yields $\psi = \phi_c - \phi_u$ as a series in $\tan \varphi_s$; converted into a series in $\varphi_s$ it reads

$$
\phi_c - \phi_u = 3 \varphi_s + \frac{1}{2.5} \varphi_s^3 + \frac{29}{23.5^2.7} \varphi_s^5 + \frac{1317}{24^6.3^3.7} \varphi_s^7 + \frac{346943}{27^7.3^3.5^3.7^2.11} \varphi_s^9 + \cdots \quad (B.4)
$$

From (B.4) one can derive the faster converging expression (9.2).

From (B.3) one can write

$$
1 - \cos \frac{\psi}{2} = \frac{1}{2!} \left( \frac{3}{2} t \right)^2 - \frac{3.7}{5!} \left( \frac{3}{2} t \right)^4 + \frac{32.11.17}{7! 5!} \left( \frac{3}{2} t \right)^6 - \frac{3.1783}{8! 5} \left( \frac{3}{2} t \right)^8 - \frac{3^3(2791423)}{11! 5^2.7} \left( \frac{3}{2} t \right)^{10} - \cdots
$$

$$
= \sum_{n=2,4,\ldots} a_n t^n \quad (B.5)
$$

by definition of the $a_n$'s. Using this expansion transforms (B.1) into

$$
\alpha = \frac{1}{2} \sqrt{\cos \varphi_s} \left[ (1 - \cos \frac{\psi}{2}) \sqrt{|\tan \varphi_s| - t} \right]_{t=0}^{t=|\tan \varphi_s|} + \int_{0}^{1} (1 - \cos \frac{\psi}{2}) \frac{dt}{\sqrt{|\tan \varphi_s| - t}}
$$

$$
\alpha = \frac{1}{2} \sqrt{\cos \varphi_s} \sum_{n=2,4,\ldots} a_n \left[ \frac{t^n}{2 |\tan \varphi_s| - t} \right]_{t=0}^{t=|\tan \varphi_s|} + \int_{0}^{1} \frac{dt}{\sqrt{|\tan \varphi_s| - t}}
$$

$$
|\tan \varphi_s|^{n+1} \int_{0}^{x^2} (1 - x)^{-1} dx = |\tan \varphi_s|^{n+1} \left[ \frac{r(n+1) r(\frac{1}{2})}{2 \Gamma(n + \frac{1}{2})} \right]_{t=0}^{t=|\tan \varphi_s|}
$$

$$
= |\tan \varphi_s|^{n+1} \left( \frac{1}{2} \right)_n
$$
hence

$$a = \frac{1}{2} \sqrt{|\sin \varphi_s|} \sum_{n=2,4,\ldots}^{\infty} a_n \frac{(1)}{n} \left( \frac{1}{2} \right)_n$$

(B.6)

where the \( a_n \)'s are given by (B.5).

With \( \tan^2 \varphi_s = \frac{\sin^2 \varphi_s}{1 - \sin^2 \varphi_s} \), (B.6) can be converted into a series in \( \sin^2 \varphi_s \):

$$a = \frac{3}{10} \sqrt{\sin \varphi_s} \left( \frac{1}{2} \right) \left[ 1 + \frac{2}{5} \sin^2 \varphi_s + \frac{22.37}{7.13} \sin^3 \varphi_s + \frac{23.13.347}{5^2.7^2.11} \sin^4 \varphi_s 
+ \frac{2^4.(490487)}{5^3.7^3.112.13} \sin^5 \varphi_s + \frac{2^5.(10040329369)}{5^6.7^3.112^2.13^2.17.19} \sin^6 \varphi_s + \ldots \right]$$

(B.7)

and then to a series in \( \varphi_s^2 \):

$$a(\varphi_s^2) = \frac{3}{10} \sqrt{\frac{\varphi_s}{2}} \left[ 1 - \frac{1}{3.5} \left( \frac{\varphi_s}{2} \right)^2 + \frac{1607}{2.35.7^2.13} \left( \frac{\varphi_s}{2} \right)^4 + \frac{9.9051}{2.3^2.5^2.7^2.11} \left( \frac{\varphi_s}{2} \right)^6 
+ \frac{397918669}{2^3.3^2.5^3.7^3.112.13} \left( \frac{\varphi_s}{2} \right)^8 + \frac{24018007154899}{2^3.3^2.5^6.7^3.112^2.13^2.17.19} \left( \frac{\varphi_s}{2} \right)^{10} 
+ \ldots \right]$$

(B.8)

Cut after \( \left( \frac{\varphi_s}{2} \right)^4 \), this expression yields at the worst point for convergence \( \left( \frac{\varphi_s}{2} = \frac{\pi}{2} \right) \):

$$a \left( \frac{\pi}{2} \right) = 0.919262 \text{ instead of the correct value } 1.$$