CERN-TH.4993/88

ON THE CONJECTURED TRANSITION OF BLACK HOLES
INTO STRINGS

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Abstract

It is argued that if black holes change into strings in the last stage of their evaporation, as has been conjectured, they will go into massless strings directly rather than via an intermediate state containing a massive string.

Black holes are a firm prediction of general relativity and now there are even some observational indications that they exist in nature [1]. According to Hawking, black holes emit energy. Near the end of the life of a black hole the emission rate is very high. The last 10^{30} ergs are released in about 0.1 sec in an explosion equivalent to about 1 million 1 Mton hydrogen bombs [2]. According to the quasiclassical description what is left behind is a naked singularity. This, however, is considered unacceptable (cf. e.g. [3]).

The problem would be solved, if by some inverse of the supernova explosion the black hole could go over into ordinary matter before disappearing completely. Since the entropy of a black hole of mass M is $4\pi M^2$ [4] and the entropy of ordinary matter is usually proportional to its mass, it is easy to propose a process which is thermodynamically possible in the sense that, at given M , the entropy of ordinary matter is larger than the entropy of the black hole. Nevertheless, this proposal is not considered a plausible solution, because the corresponding transition probability is so small that the transition has no chance of happening in the time available. The situation may have changed with the advent of string theory and it has been suggested [3] that a very light black hole might have enough time to change into a bunch of strings. According to the string paradigm, the black hole, which consists of gravitons, is itself a bunch of strings. Thus the transition would be from a bunch of strings correlated in a way characteristic of a black hole into a bunch of strings more like ordinary matter.

The mass at which the transition occurs has been estimated using the statistical assumption, i.e. assuming that all string states allowed by energy conservation are so easily accessible that the black hole goes over into strings as soon as its entropy becomes smaller than that of a gas of strings at the same energy. The calculations have been done assuming that the final state is a heavy string with [3], [5], or without [5] a gas of massless strings in equilibrium with it. It is understood that at some later stage the heavy string decays into massless strings, but an intermediate state with a heavy string has been assumed in both analyses.

In the present paper we analyze the thermodynamics of the gas of non-interacting heterotic strings. We construct an analogue of the phase diagram and find that a priori a black hole could go over into a single heavy string, into a heavy string in equilibrium with light strings, or into a gas of massless (or nearly) strings. Using arguments involving the finite size of strings and angular momentum conservation, we find in the first two cases inconsistency with the statistical assumption. We conclude that a black hole, if it goes over into strings in the predicted mass range, will go over directly into massless (or nearly) strings without the intermediate state with a heavy string.

The energy levels of a single heterotic string [6] are

$$M_N = 2\sqrt{N} \quad N = 0, 1, \dots \quad (1)$$

The units are chosen so that $\hbar = c = k_B = \alpha' = 1$. The degeneracy of the N -th energy level of the string is given for large N by the asymptotic for-

mula¹

$$P(N) = 2^{-13/4} N^{-11/4} e^{M_N/T_H} \quad (2)$$

where $T_H = [\pi(2 + \sqrt{2})]^{-1}$ is the Hagedorn temperature. For low N one can calculate $P(N)$ directly from the generating function. For our work we have used formula (2) and exact values of $P(N)$ for $N = 1, \dots, 100$. Formula (2) underestimates $P(N)$. The error at $N = 100$ is about 20 per cent and slowly decreases with increasing N . The entropy of a string is the logarithm of the degeneracy. Thus for large N

$$S_N = M_N/T_H - 11 \ln M_N + 5.372 \quad (3)$$

The error for $N = 100$ is only 0.13 per cent for this formula. It is important that the so-called thermodynamic limit should not be taken here. In the thermodynamic limit only the leading term M_N/T_H is kept in formula (3). Consequently, the heat capacity of the string comes out infinite, which is too crude an approximation. For completeness we describe the thermodynamic limit in the Appendix. One can ascribe in the standard way a temperature to the transition of the string from state N to state $N-1$

$$T_N = (M_N - M_{N-1})/(S_N - S_{N-1}) \approx T_H/(1 - 11T_H/M_N) \quad (4)$$

¹ The exponent 13/4 was misprinted as 11/4 in [7] but the calculations there have been done with the correct value.

This temperature always exceeds the Hagedorn temperature. The high N approximation underestimates T_N at high N , the error at $N=100$ being 1.7 per cent of $T_{100} - T_H$.

Consider now a simple model for the string gas. This is a slight extension of the models used [3], [5] to describe transitions of black holes into the string gas. The strings are approximated by pointlike non-interacting particles with energy levels (1) and the corresponding degeneracies as given by the string model. They are confined within a space volume V . Following Accetta and Gleiser [5] we take this volume to be 9-dimensional. For the subsystem consisting of all the massless strings, the energy and entropy are correlated to the volume and temperature as for the photon gas, except that the space is 9-dimensional and the two polarization states of the photon are replaced by the 8064-fold degeneracy of the massless string, half the states being fermionic. Thus, using subscript g for quantities referring to the gas of massless strings,

$$E_g = 9 Q V T_g^{10} \quad (5)$$

$$S_g = 10 Q V T_g^9 \quad (6)$$

where $Q = 31\pi^5/15$. Note that the gas temperature T_g is in general different from the string temperatures T_N . In the following the indistinguishability and the translational motion of the heaviest string(s) in the system are neglected. This partly takes into account the conservation laws for momentum and for the position of the centre of mass [7].

Following ref. [3] we would like to draw a phase diagram for the string gas. Since, however, the thermodynamic limit cannot be taken, there are strictly speaking no phases. In order to find a replacement we propose the following construction. Let us consider a subsystem consisting of all the massless strings and of the heaviest string. The energy of this subsystem is

$$M = M_N + E_g \quad (7)$$

where M_N is the mass of the heaviest string. If there is more than one string with mass M_N , we include just one of them. Consider now the dependence of the entropy

$$S(N, V, M) = S_N + S_g \quad (8)$$

at fixed (V, M) on N or equivalently on M_N/M . There are three possibilities:

a) $M < M_1$, thus $N=0$ is imposed by energy conservation, or for increasing N the entropy decreases - the system is dominated by the gas of massless strings.

b) For increasing N entropy increases - the system is dominated by one massive string.

c) The entropy is not a monotonic function of N - in general both a massive string and the gas of massless strings are important.

In order to find in the V, M plane the regions corresponding to situations a), b), and c), let us first consider the function $V_N(M)$ defined by the relation

$$S(N, V_N, M) = S(N-1, V_N, M) \quad (9)$$

Consider transitions where N and E_g change at fixed M, V . Entropy increases in the transition $N \rightarrow N-1$ for $V > V_N$. It also increases in the transition $N-1 \rightarrow N$, if this is not forbidden by energy conservation, for $V < V_N$. The curves $V_N(M)$ for $N = 1, \dots, 6$ are shown in Fig 1. Each curve begins at $M = M_N$ and

$$V_N^{\min} = (0.9)^{10} (1/9Q) (M_N - M_{N-1}) T_N^{-10} \quad (10)$$

With increasing M each function V_N increases and for high M it is approximately given by the formula

$$V_N(M) = M/(9QT_N^{10}) \quad (11)$$

For $N > 1$, T_N is a decreasing function of N and consequently at sufficiently high M there is $V_{N+1} > V_N$. The only exception is $T_1 \approx T_5$. This is so because after the transition $1 \rightarrow 0$ the final string is included in the gas of massless strings and does not count as a heavy string any more. Consequently in the formula (4) for T_1 the entropy $S_0 \approx 9$ is replaced by zero.

According to the preceeding discussion, points in the V, M plane where $M < M_1$ or $V > V_N(M)$ for all N correspond to states of type (a). Points where $V > V_N(M)$ for all N correspond to states of type (b). All other points correspond to states of type (c).

In Fig 2 the analogue of the phase diagram for the string gas is shown. The region denoted b is delimited by the conditions $M > M_1$ and by the curve interpolating linearly between the points $(M_N, V_N^{\min})$. For $M = M_N$ $N=1,2,\dots$ in this region, the most probable state corresponds to no massless strings at all. At intermediate energies there is some gas of massless strings, but its energy is too low to excite a string. The region denoted a is above all the lines $V_N(M)$. There massless strings dominate. The region denoted c contains all the (c) states. As seen from Fig 1, it also contains some (b) states, but this is of no importance for our further discussion.

It is also possible to estimate the maximal volume allowed for states other than (a) as a function of the total mass M_{tot} of the string gas. It is well known that for $T_g < T_H$ massless strings dominate. Even at $T_g = T_H$ massive excitations contribute less than 0.3 per cent to the entropy [7]. Therefore the gas must be in an (a) state for

$$M_{\text{tot}} < 9 Q V T_H^{10} \quad (12)$$

Let us consider now the transition of a black hole into strings. According to ref [5] the transition happens at $M = 4.09 \times 10^{14}$ and $V = 1.44 \times 10^{21}$ (in our units). This result depends on $\gamma = M_{\text{Pl}} \sqrt{\alpha'}$ and in [5] $\gamma =$

10 has been chosen. The presence of a heavy string in equilibrium requires $T_g \approx T_H$. Thus from (5) $E_g = 4.07 \times 10^{14}$, which leaves for the heavy string $M_s = 2 \times 10^{12}$. Even that, however, seems too much. Using the mean square radius $R(N)$ of the bosonic string in 9-dimensions calculated for high N by Mitchell and Turok [8] as a rough estimate of the size of a heterotic string, it is natural to assume that the volume V should be larger than the volume of a 9-dimensional sphere of radius $R(N)$. Otherwise the string would not fit into the box. This gives the condition

$$M_s < 0.2 V^{2/9} \quad (13)$$

Combining it with (12) we find

$$m_s/M_{\text{tot}} < 7 \times 10^5 V^{-7/9} \quad (14)$$

For the situation found in ref. [5] this gives $M_s < 10^5$, which is a negligible fraction of the total energy of the string gas. Conversely, one could ask at what volume the massless strings could be in equilibrium with a string with mass $M_s = 4 \times 10^{14}$. From formula (12) one finds an upper limit $V = 1.42 \times 10^{21}$. That the gas in this volume could in a reasonable time reach equilibrium with a string filling on the average at this mass a volume $V = 8 \times 10^{68}$ seems most unlikely. Figure 2 illustrates the mismatch of the requirements that $M_s \approx M$ and that the string fits into volume V . We conclude that the standard study of string systems where a heavy string is in equilibrium with massless strings, when supplemented by the information about string sizes, leads to the conclusion that the final state is dominated by strings with $M_s/M \approx 0$.

Let us add some remarks. A related problem of the importance of fireball sizes for realistic applications of Hagedorn's bootstrap to problems in astrophysics was stressed long ago by Nahm [9]. Condition (14) implies that the phase with the heavy string, obtained in the thermodynamic limit as described in the Appendix, cannot be realized in nature. For the black-hole-to-string transitions one could abandon the statistical assumption and concentrate on string states, where only the highest frequency oscillators are excited. Such states correspond to small string sizes, but they are few. Thus in order to make a transition to such states, the black hole would have to go to entropies and consequently masses much smaller than previous estimates indicate.

Another possibility considered in ref [5] is that the black hole goes over into a heavy string unaccompanied by light strings. Then one finds [5] $M_s = 3 \times 10^{14}$. Here besides the problem of the large size of the string as compared to the size of the black hole, there is the problem with conservation laws. The black hole nearing the end of its life is usually assumed to have zero angular momentum (in 9 dimensions we take all the spin parameters [5] vanishing). However, heavy string states with zero angular momentum are rare. According to Curtright and Thorn [10] the degeneracy of the N-th massive excitation of the heterotic string for high mass is only

$$P_0(N) = A M^{-147,5} e^{\sqrt{M}/T_H} \quad (15)$$

with A constant, when just zero angular momentum states are included. Thus again, abandoning the statistical assumption and including only spin zero states, one could perhaps have a transition of a black hole into a

string, but this would have to happen at energies much lower than existing estimates indicate.

Finally, the black hole could go over into the string gas of light strings. Since, as seen from Fig. 2, the volume of the gas may be much smaller than the volume of a string of comparable mass, and since typical states of the gas have angular momentum zero, this transition seems to be the most probable one. At masses as high as had been suggested this is in fact the only possibility. The statistical argument does not fix the transition mass, because at any mass the entropy of the gas may exceed the entropy of the black hole, if the volume of the gas is sufficiently large. On the other hand, the transition of the black hole into a volume of gas is likely to decrease rapidly with increasing volume of the gas, because the black hole is very small. Thus the minimum volume at given mass, corresponding to $T_g \rightarrow T_H$ seems the most probable. This would give $M \approx 4 \times 10^{14}$ and $V \approx 1.4 \times 10^{21}$ close to the estimates from ref. [5]. The important and as yet unsolved problem is whether this transition is sufficiently probable to occur in the time available.

Let us summarize the results from our analysis. The most probable state of a system of pointlike, non-interacting, heterotic strings in a box of volume V may be: a single string, a heavy string in equilibrium with massless strings, or a gas of light mostly massless strings. Each of these at low mass has an entropy, which is higher than the entropy of a black hole of the same mass. In the first two cases the statistical assumption that all the string states are equally good final states for the transition of a black hole into strings is untenable. In the first case, very few strings have the

right angular momentum. In the second, an estimate of string sizes shows that very few states of the heavy string correspond to strings sufficiently small to fit into the box. In both cases, if the transition of the black hole into strings happens, the entropy (3) of the string is irrelevant and the mass at transition and consequently the mass of the final string must be much smaller than obtained from statistical arguments. The transition into the gas of light strings seems the most probable one, but we do not have sufficient proof that it will really occur.

ACKNOWLEDGEMENTS

The author thanks M. Reuter, L. Van Hove and G. Veneziano for helpful discussions and comments.

APPENDIX A

THERMODYNAMIC LIMIT

The thermodynamic limit corresponds to both M_{tot} and V tending to infinity at constant energy density

$$\rho = M_{\text{tot}}/V \quad (\text{A1})$$

According to formula (12) for

$$\rho < \rho_0 = 9 Q T_H^{10} \quad (\text{A2})$$

massless strings dominate. For $\rho > \rho_0$ an energetically possible state is a gas of massless strings with $T_g > T_H$, but a much more probable state is obtained reducing the energy density of the massless strings to ρ_0 , i.e. reducing T_g to T_H , and putting the remaining energy into one string with mass

$$M_s = (\rho - \rho_0)V \quad (\text{A3})$$

According to the results from ref. [7], in the thermodynamic limit this very heavy string is in stable equilibrium with the gas of massless strings. Since for N large $T_N \approx T_{N+1} \approx T_H$, the temperature of the heavy string is well defined and equal to the gas temperature of the massless strings. The massive strings with $0 < M < M_s$ are of little importance. The fraction of energy in the heavy string

$$M_s/M_{\text{tot}} = 1 - p_0/p \quad (\text{A4})$$

increases from zero at $p = p_0$ to one for $p \rightarrow \infty$. Of course for increasing p the situation changes at some point because of the gravitational collapse, but probably much before that point the simple picture of non-interacting strings becomes completely unrealistic. As explained in the text, the phase with the heavy string is unlikely to be realizable in nature.

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FIGURE CAPTIONS

Figure 1 Curves $V_N(M)$ for $N = 1, \dots, 6$.

Figure 2 Analogue of the phase diagram for pointlike non-interacting heterotic strings. a) Gas of massless strings. b) One heavy string. c) Coexistence region. For more precise definitions of the regions see text. ----- Estimated volume of a string of mass M .

Figure 1

Figure 2