THE GENERAL THEORY OF ALL SUM AND DIFFERENCE RESONANCES IN A THREE-DIMENSIONAL MAGNETIC FIELD IN A SYNCHROTRON

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THE GENERAL THEORY OF ALL SUM AND DIFFERENCE RESONANCES
IN A THREE-DIMENSIONAL MAGNETIC FIELD IN A SYNCHROTRON

G. Guignard
ABSTRACT

The problem of resonances for circulating proton beams is developed here for three-dimensional magnetic fields and for large variations of the betatron functions, in contrast to the two-dimensional case covered in the existing literature. The result is a general, non-linear theory yielding symplectic equations of motion under all conditions. The first part treats linear coupling, the second part sum and difference resonances of any order. Detailed analytical treatments of the three-dimensional magnetic field, of the perturbing Hamiltonian and of the invariants of the motion are included. The considerations are applied to the specific case of a detector solenoid to be installed in the CERN Intersecting Storage Rings.
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GENERAL INTRODUCTION

The usual parameters of a synchrotron, i.e. the average machine radius, the magnetic rigidity, the betatron functions and their derivatives, the phases and the tunes, are indicated in this report by the standard symbols $R, R_0, \beta_R, \beta_V, \alpha_R, \alpha_V, \mu_R, \mu_V$ and $Q_R, Q_V$. The independent variable $\theta$ used throughout is the axial distance divided by $R$ and the symbol $'$ indicates the derivative by $\theta$. The system of coordinates is given in Fig. 1.

Looking at Fig. 1, it is obvious that the radius of curvature $\rho$ is negative in the chosen coordinate system. Thus, the magnetic rigidity $B_\rho$, defined as the product of the amplitude of the guiding field on the equilibrium orbit by the radius of curvature, becomes negative. In order to avoid confusion in the text, we shall always put the modulus $|B_\rho|$ for the positive definition of this quantity, and simply $B_\rho$ for the negative one.

The use of a cartesian system implies that the three-dimensional perturbing field is only present where the bending is small.
PART 1

THE GENERAL THEORY OF LINEAR COUPLING
FOR THREE-DIMENSIONAL MAGNETIC FIELDS

1. INTRODUCTION

For reasons of simplicity, we begin with the theory of linear coupling. This permits us to familiarize the reader with the formalism and the mathematical development, before generalizing the theory in the second part of this report for all sum and difference resonances in a three-dimensional magnetic field.

The formalism is based on a perturbation treatment of the classical Hamiltonian. In this first part, the perturbation consists of the linear forces which couple both the transverse motions of protons in a synchrotron. First we have to define these forces and then the perturbing Hamiltonian. Knowing the solution of the unperturbed motion and applying the above-mentioned perturbation treatment, the equations of the perturbed motion can be solved.

Finally it is possible to define a general coefficient of linear coupling for a machine with any form factor. This coefficient will depend on the skew gradients and the longitudinal fields existing around the machine. Thus it characterizes the effects on a proton beam of elements like skew quadrupoles and solenoids. In the eventuality of a large, unwanted perturbation by such elements, this coefficient makes it possible to estimate which elements should be added in order to compensate the effect and what is the efficiency of this compensation.

This theory of linear coupling has been applied in the case of hypothetical solenoids in the ISR machine. These examples indicate the specific questions which have to be considered in a practical case.

2. PERTURBATION TREATMENT IN CLASSICAL DYNAMICS

In this section we summarize the treatment of perturbations in analytical mechanics.)

Let us assume that $H_0$ is the Hamiltonian of the unperturbed motion and $H_1$ is the Hamiltonian of the perturbation. Without making any approximations based on the smallness of the perturbing Hamiltonian, the general problem may be stated as follows. Assuming the general solution is known of the canonical equations for unperturbed motion,

$$\dot{q}_p = \frac{\partial H_0}{\partial p_p}, \quad \dot{p}_p = -\frac{\partial H_0}{\partial q_p}, \quad (1.2.1)$$
it is then necessary to set up a technique to find the motion for the perturbed Hamiltonian

\[ H = H_0(q,p,\theta) + H_1(q,p,\theta) , \]

(1.2.2)

\( \theta \) being the independent variable.

Let the solution of (1.2.1) be

\[ q_p = q_p(a_j,\theta) , \quad p_p = p_p(a_j,\theta) , \]

(1.2.3)

where \( a_j \) stands for \( 2N \) arbitrary constants. Solving (1.2.3), we have

\[ a_j = a_j(q_p, p_p, \theta) , \]

(1.2.4)

these \( 2N \) functions being determined by the form of the Hamiltonian \( H(q_p, p_p, \theta) \).

Consider now the perturbed motion. The perturbation problem is reduced to the study of the way in which the \( a_j \)'s vary with \( \theta \) in agreement with the form of the perturbing Hamiltonian \( H_1(q_p, p_p, \theta) \). Since the \( a_j \)'s are the constants of the unperturbed motion, we have

\[ a_j' = \frac{d a_j}{d\theta} = \left[ a_j, H_1 \right] = \sum_p \left( \frac{\partial a_j}{\partial q_p} \frac{\partial H_1}{\partial p_p} - \frac{\partial H_1}{\partial q_p} \frac{\partial a_j}{\partial p_p} \right) . \]

(1.2.5)

By virtue of (1.2.3) the right-hand side is a function of \( (a_j, \theta) \), and so we have a set of \( 2N \) equations to determine the functions \( a_j = a_j(\theta) \) and hence the perturbed motion.

The equations (1.2.5) can be put in a different form. Using (1.2.3), \( H_1 \) can be expressed as a function of \( (a_j, \theta) \):

\[ H_1(q_p, p_p, \theta) = U(a_j, \theta) . \]

(1.2.6)

Equation (1.2.5) can then be rewritten:

\[ \frac{d a_j}{d\theta} = \sum_m \left[ a_j, a_m \right] \frac{\partial U}{\partial a_m} , \]

(1.2.7)

where

\[ \left[ a_j, a_m \right] = \sum_p \left( \frac{\partial a_j}{\partial q_p} \frac{\partial a_m}{\partial p_p} - \frac{\partial a_m}{\partial q_p} \frac{\partial a_j}{\partial p_p} \right) . \]
So far everything is rigorous and (1.2.7) gives the exact solution of the motion. But if the derivatives of \( H_1 \) or of \( U \) are small, then the right-hand side of (1.2.7) is small and there may be the possibility of approximating the perturbed motion by keeping only the dominant terms of the function \( U \).

3. DESCRIPTION OF TRANSVERSE MOTIONS OF THE PROTONS

3.1 Unperturbed motion and the Hamiltonian

The equations of the unperturbed transverse motion are well known and can be written as follows:

\[
x'' + K_1(\theta)x = 0 \\
z'' + K_2(\theta)z = 0,
\]

(1.3.1)

where the derivatives are taken with reference to (w.r.t.) \( \theta \). The functions \( K_1 \) and \( K_2 \) are the forces exerted on the particles by the magnetic field gradients.

Knowing the equations (1.3.1), it is easy to define the Hamiltonian of the unperturbed motion:

\[
H_0 = \frac{1}{2} \left[ K_1 x^2 + K_2 z^2 + p_x^2 + p_z^2 \right].
\]

(1.3.2)

This is easily checked by applying the canonical equations (1.2.1) and verifying that the equations are equivalent to (1.3.1).

The general solutions of equations (1.3.1), by virtue of the Floquet's theorem\(^2\) are

\[
x = a_1 \ u(\theta) \ e^{iQ_H^0} + \bar{a}_1 \ \bar{u}(\theta) \ e^{-iQ_H^0} \\
z = a_2 \ v(\theta) \ e^{iQ_V^0} + \bar{a}_2 \ \bar{v}(\theta) \ e^{-iQ_V^0},
\]

(1.3.3)

where \( u \) and \( v \) are the Floquet's functions\(^2,3\)

\[
u(\theta) = \sqrt{\frac{B_V(\theta)}{2R}} \ \exp \left[ i \int_0^\theta \left( \frac{R}{B_V(\theta')} - Q_V \right) \, d\theta' \right],
\]

\[
u(\theta) = \sqrt{\frac{B_H(\theta)}{2R}} \ \exp \left[ i \int_0^\theta \left( \frac{R}{B_H(\theta')} - Q_H \right) \, d\theta' \right],
\]

(1.3.4)

\( \bar{u} \) and \( \bar{v} \) represent the complex conjugates of the functions \( u \) and \( v \),
a₁ and a₂ and their complex conjugates are the constants of the motion; they represent the complex amplitudes of the transverse oscillations of the protons.

3.2 Motions perturbed by linear coupling and the Hamiltonian

A three-dimensional magnetic field can couple both the transverse motions of the protons in the following way:

\[
x'' + K₁θx = \frac{R^2}{|B₀|} \frac{∂B_z}{∂z} x - \frac{R}{|B₀|} B₈z'
\]
\[
z'' + K₂θz = - \frac{R^2}{|B₀|} \frac{∂B_x}{∂x} x + \frac{R}{|B₀|} B₈x'
\]

where \(B_x, B_z, B_θ\) are the three field components, and \(|B₀|\) is the magnetic rigidity (always positive).

In order to simplify the notation, we can introduce the following functions characterizing the magnetic field:

\[
K(θ) = \frac{1}{2} \frac{R^2}{|B₀|} \left( \frac{∂B_x}{∂x} - \frac{∂B_z}{∂z} \right)
\]
\[
M(θ) = \frac{R}{|B₀|} B_θ.
\]

Remembering that

\[
\text{div} \ \vec{B} = 0
\]
\[
\frac{∂B_x}{∂x} + \frac{∂B_z}{∂z} + \frac{∂B_θ}{∂θ} = 0,
\]

then we can write the following equations:

\[
\frac{R^2}{|B₀|} \frac{∂B_x}{∂x} = K - \frac{1}{2} M'
\]
\[
- \frac{R^2}{|B₀|} \frac{∂B_z}{∂z} = K + \frac{1}{2} M',
\]

where the derivative of \(M\) is taken with \(θ\).

Putting the relations (1.3.6) and (1.3.8) in the motion equations (1.3.5), gives

\[
x'' + K₁x = - (K + \frac{1}{2} M')z - Mz'
\]
\[
z'' + K₂z = - (K - \frac{1}{2} M')x + Mx'.
\]
Two specific examples of magnetic elements which can give coupling are:

1) skew-quadrupole lenses for which we have

\[
M \equiv 0
\]

\[
K = \frac{R^2}{|B_0|} \frac{\partial B_x}{\partial x} = - \frac{R^2}{|B_0|} \frac{\partial B_z}{\partial z}; \tag{1.3.10}
\]

2) solenoids for which we have

\[
M = \frac{R}{|B_0|} B_S, \tag{1.3.11}
\]

and \(K\) describing the end effects in agreement with (1.3.6).

Let us put for instance\(\text{\textsuperscript{\text{\textcircled{5}}}c}\)

\[
\frac{\partial B_x}{\partial x} = - (1-a_s) B'_S, \quad \frac{\partial B_z}{\partial z} = - a_s B'_S, \tag{1.3.12}
\]

then \(a_s = \frac{1}{2}\) if the solenoid has no end plates and

\(a_s = 1\) if the solenoid has end plates with a horizontal slot.

The function \(K\) then has the following form:

\[
K = (a_s - \frac{1}{2}) M'. \tag{1.3.13}
\]

Knowing the motion equations (1.3.9) it is not difficult to find the associated Hamiltonian\(\text{\textsuperscript{\text{\textcircled{4}}}c}\):

\[
H = \frac{1}{2} \left[ K_1 X^2 + K_2 Z^2 + 2Kxz + \left( p_x - \frac{1}{2} M_x \right)^2 + \left( p_z + \frac{1}{2} M_z \right)^2 \right]. \tag{1.3.14}
\]

As before, we can check the form of \(H\) using the canonical equations (1.2.1) and comparing with the relations (1.3.9).

Subtracting \(H_0\) given by (1.3.2) from \(H\), we will obtain the perturbing Hamiltonian \(H_1\) associated with linear coupling:

\[
H_1 = Kxz - \frac{1}{2} M_x p_x + \frac{1}{2} M_x p_z + \frac{1}{8} M_x^2 + \frac{1}{8} M_z^2. \tag{1.3.15}
\]

4. GENERAL EQUATION OF THE PERTURBED MOTION OF PROTONS

Using the description given in Section 3 of the perturbation problem, we are now able to apply the treatment summarized in Section 2.

Firstly the solution of the unperturbed motion in a form equivalent to (1.2.3) is required. The expressions for \(x\) and \(z\) are already given in (1.3.3). In order to get the expressions in \(p_x\) and \(p_z\), we use the first set of canonical equations of (1.2.1):
\[ \frac{dx}{d\delta} = \frac{\partial H_z}{\partial p_x} = p_x \]  
\[ \frac{dz}{d\delta} = \frac{\partial H_z}{\partial p_z} = p_z. \]  

(1.4.1)

Taking the equations (1.3.3) and differentiating, we obtain the solution equivalent to (1.2.3):

\[ x = a_1 u e^{iQH\theta} + \bar{a}_1 \bar{u} e^{-iQH\theta} \]
\[ z = a_2 v e^{iQV\theta} + \bar{a}_2 \bar{v} e^{-iQV\theta} \]  
\[ p_x = a_1 (u' + iQH\bar{u}) e^{iQH\theta} + \bar{a}_1 (\bar{u}' - iQH\bar{u}) e^{-iQH\theta} \]
\[ p_z = a_2 (v' + iQV\bar{v}) e^{iQV\theta} + \bar{a}_2 (\bar{v}' - iQV\bar{v}) e^{-iQV\theta}. \]  

(1.4.2)

Solving (1.4.2), we have the equivalent of (1.2.4):

\[ a_1 = \frac{1}{W(u)} \left[ (u' - iQH\bar{u}) x - \bar{u} p_x \right] e^{-iQH\theta} \]
\[ \bar{a}_1 = -\frac{1}{W(u)} \left[ (u' + iQH\bar{u}) x - \bar{u} p_x \right] e^{iQH\theta} \]
\[ a_2 = \frac{1}{W(v)} \left[ (v' - iQV\bar{v}) z - \bar{v} p_z \right] e^{-iQV\theta} \]
\[ \bar{a}_2 = -\frac{1}{W(v)} \left[ (v' + iQV\bar{v}) z - \bar{v} p_z \right] e^{iQV\theta}, \]  

(1.4.3)

where \( W(u) \) and \( W(v) \) are the Wronskians associated with Floquet's functions\(^2\):

\[ W(u) = u (\bar{u}' - iQH\bar{u}) - (u' + iQH\bar{u}) \bar{u} = -i \]  
\[ W(v) = v (\bar{v}' - iQV\bar{v}) - (v' + iQV\bar{v}) \bar{v} = -i. \]  

(1.4.4)

Using the relations (1.2.7), we are now able to write the general equation of the perturbed motion:

\[ \frac{da_1}{d\delta} = \frac{1}{W(u)} \frac{\partial U}{\partial a_1} \]  
\[ \frac{d\bar{a}_1}{d\delta} = \frac{1}{W(u)} \frac{\partial U}{\partial \bar{a}_1}, \]  

(1.4.5)

and the equivalent equations for the vertical motion with \( a_2, \bar{a}_2 \).
By virtue of (1.4.4), these equations of motion become

\[
\frac{da_1}{d\theta} = i \frac{\partial U}{\partial a_1} \]
\[
\frac{\partial a_1}{d\theta} = -i \frac{\partial U}{\partial \overline{a}_1} \]
\[
\frac{da_2}{d\theta} = i \frac{\partial U}{\partial a_2} \]
\[
\frac{\partial a_2}{d\theta} = -i \frac{\partial U}{\partial \overline{a}_2} .
\] (1.4.6)

These are exact motion equations associated with the analytical forms (1.4.2) and with the perturbation \(U\) [or \(H_1\), from (1.2.6)].

5. **SOLUTION OF THE PERTURBATION EQUATIONS FOR LINEAR COUPLING**

5.1 Discussion of the assumptions

The problem is now reduced to the solution of the equations (1.4.6) for the particular case of linear coupling, which is described by the Hamiltonian \(H_1\) given in (1.3.14).

We can first express \(H_1\) as function of \(a_j\), by putting (1.4.2) in (1.3.14):

\[
U (a_1, \overline{a}_1, a_2, \overline{a}_2, \theta) = H_1 (x, z, p_x, p_z, \theta) .
\] (1.5.1)

The explicit form of \(U\) is given in Appendix 1. As the synchrotron is a periodic machine, it is possible to develop in Fourier series the function \(U\):

\[
U = \sum_{j,k,l,m}^{2j} \sum_{q=-\infty}^{+\infty} h_{jklmq}^{(2)} a_1^k \overline{a}_1^l a_2^m \overline{a}_2^m \exp \left\{ i \left[ (j-k)Q_{lq} + (j-m)Q_{lq} + q \theta \right] \right\} ,
\] (1.5.2)

the coefficients \(h_{jklmq}^{(2)}\) being explicitly given in Appendix 1.

Now we can make two assumptions, and consequently the solution will no longer be exact and will only be valid for small coupling, that is for a perturbation:

1) We can first neglect the \(M^2\) term in the Hamiltonian (1.3.14) compared with the \(K\) and \(M\) terms.

2) We can keep only the low-frequency part of the function (1.5.2), which gives the slow but important variations of the variables \(a_j\).
For this second assumption, it is necessary to define which specific resonance will be looked at. In our particular case of linear coupling, the Hamiltonian function $U$ (Appendix 1) indicates that four resonances can be excited:

$$\begin{align*}
Q_H + Q_V &= p \\
2Q_H &= p \\
2Q_V &= p \\
Q_H - Q_V &= p
\end{align*}$$

sum resonances

(1.5.3)

difference resonance ,

$p$ being an integer.

The mono-dimensional resonances vanish if the $M^2$ terms of the Hamiltonian are neglected. Thus, the first assumption cannot be accepted for the study of these mono-dimensional resonances. On the other hand, all the terms of the Hamiltonian contribute to the two-dimensional resonances and hence we can apply both assumptions mentioned above.

5.2 Solution of the equations for the resonances $2Q_H = p$ and $2Q_V = p$

Starting from the motion equations (1.4.6) and from the Hamiltonian (1.5.2) and keeping the low-frequency part in agreement with the second assumption of the Section 5.1, we can write the explicit equation (Appendix 1) for $2Q_H = p$:

$$\frac{da_1}{d\theta} = i(a_1 \lambda + \bar{a}_1 \lambda 2\bar{\kappa} e^{-i\theta}) ,$$

(1.5.4)

where $\kappa = \frac{h(z)}{\lambda}$ (see Appendix 1)

$\lambda = \frac{h(z)}{\eta}$ (see (A2.2) in Appendix 2)

$\Delta = 2Q_H - p = $ distance from the resonance.

Putting $a_1 = r_1 e^{i\phi_1}$ in the relation (1.5.4) and separating the imaginary part from the real part, we will get:

$$\frac{dr_1}{d\theta} = 2|\kappa| \sin (2\phi_1 + \phi_\kappa + \theta \Delta)$$

$$\frac{d\phi_1}{d\theta} = 2|\kappa| \cos (2\phi_1 + \phi_\kappa + \theta \Delta) + \lambda ,$$

(1.5.5)

with $\bar{\kappa} = |\kappa| e^{-i\phi_\kappa}$, $\lambda$ being real.

Putting $y = 2\phi_1 + \phi_\kappa + \theta \Delta$ in (1.5.5), it is possible to solve both equations. Defining $b = 4|\kappa|$ and $c = 2\lambda + \Delta$, the solution is

$$r_1 = r_{10} \exp \left[ \frac{b \theta}{c} \int_{\theta_0}^{\theta} \sin y \, d\theta \right] ,$$

(1.5.6)
where \( y = 2 \arctg \left[ \frac{\sqrt{c^2 - b^2}}{c - b} \tan \frac{\sqrt{c^2 - b^2}}{2} (\theta - \theta_0) \right] \) if \( |b| < |c| \)
\[
\begin{align*}
y &= 2 \arctg \left\{ \frac{\sqrt{b^2 - c^2}}{b - c} \exp \left[ \frac{\sqrt{b^2 - c^2}}{2} (\theta - \theta_0) \pm 1 \right] \exp \left[ \frac{\sqrt{b^2 - c^2}}{2} (\theta - \theta_0) \right] \right\} \quad \text{if } |b| > |c| \\
y &= 2 \arctg \left[ b(\theta - \theta_0) \right] \\
\end{align*}
\]
and
\[
\kappa = \frac{1}{32 \pi M} \int_0^{2\pi} M^2 \Theta_0(\theta) \exp \left\{ i \left[ 2 \int_0^\theta \left( \frac{R_{\Theta}^{(2)}(\theta')} - Q_\Theta \right) d\theta' + p\theta \right] \right\} d\theta .
\]

Putting (1.5.6) in the first equation (1.3.3) will give the explicit solution of the horizontal motion. The form of (1.5.6) shows that \( r \) remains finite only if \( 4|\kappa| < |2\lambda + \Delta| \).

The same type of solution can be applied to the vertical motion for the resonance \( 2Q^p = p \).

5.3 Solutions of the equations for the resonance \( Q^H + Q^v = p \)

Using again the equations (1.4.6) and (1.5.2), taking account of the two assumptions\(^*\) given in Section 5.1, the explicit equations are the following (Appendix 1):
\[
\frac{d\tilde{a}_1}{d\theta} = i \kappa a_2 \exp^{-i\tilde{\Delta}}
\]
\[
\frac{d\tilde{a}_2}{d\theta} = i \kappa a_1 \exp^{-i\tilde{\Delta}}
\]
with \( \kappa = h_{1010}^{(2)} \) (see Appendix 1).
\( \Delta = Q^H + Q^v - p \) = distance from the resonance.

Defining the new variable \( \tilde{a}_2 = a_2 \exp^{i\tilde{\Delta}} \) and putting it in (1.5.7), we have
\[
\frac{d\tilde{a}_1}{d\theta} = i \kappa \tilde{a}_2
\]
\[
\frac{d\tilde{a}_2}{d\theta} = -i \kappa a_1 - i \Delta \tilde{a}_2 .
\]

Instead of (1.5.8), it is possible to write a second order equation for \( \tilde{a}_2 \) by differentiating:
\[
\frac{d^2\tilde{a}_2}{d\theta^2} + i \Delta \frac{d\tilde{a}_2}{d\theta} - \kappa \tilde{a}_2 = 0 .
\]

\(^*\) In Appendix 2 a more general solution, valid for all the terms of the Hamiltonian, is given.
The general solution of (1.5.9) is then:

$$\bar{a}_2 = A_+ e^{\frac{i\omega_0}{\omega_+}} + A_- e^{\frac{i\omega_0}{\omega_-}},$$  

(1.5.10)

where $$\omega_\pm = -\frac{\lambda}{2} \pm \sqrt{(\frac{\lambda}{2})^2 - |\kappa|^2}.$$  

Knowing the explicit form of $$\bar{a}_2$$, the expressions for $$a_1$$ and $$a_2$$ are,

$$a_1 = \kappa \left( \frac{A_+}{\omega_+} e^{\frac{i\omega_0}{\omega_+}} + \frac{A_-}{\omega_-} e^{\frac{i\omega_0}{\omega_-}} \right),$$

$$a_2 = \bar{A}_+ e^{\frac{i\omega_0}{\omega_+}} + \bar{A}_- e^{\frac{i\omega_0}{\omega_-}},$$  

(1.5.11)

where $$A_+, A_-$$ are complex constants of the motion,

$$\omega_\pm$$ are the frequencies given in (1.5.10), and

$$\kappa$$ is given by

$$\kappa = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\frac{\rho_H}{\rho_V}} \left[ k + \frac{1}{2} \frac{MR}{R_V^2} \left( \frac{\rho_H}{\rho_H} - \frac{\rho_V}{\rho_V} \right) - \frac{1}{2} \frac{MR}{R_H^2} \left( \frac{1}{\rho_H} - \frac{1}{\rho_V} \right) \right]$$

$$\exp \left[ i \left( \int_0^\theta \left( \frac{R}{\rho_H(\theta')} - Q_H \right) d\theta' + \int_0^\theta \left( \frac{R}{\rho_V(\theta')} - Q_V \right) d\theta' + p\theta \right) \right] d\theta.$$

(1.5.12)

Putting (1.5.11) in the equations (1.3.5) will give the explicit solution of the perturbed motion. The initial conditions for $$x, x', z, z'$$ will define the constants $$A_+$$ and $$A_-$$.

Depending on the amplitude of $$|\kappa|$$, the motion can be stable or unstable. It is stable if $$\omega_\pm$$ (1.5.10) is real. In other words, this means that

if $$|\kappa| < \frac{|A|}{2}$$ the motion is stable,

(1.5.13)

if $$|\kappa| > \frac{|A|}{2}$$ the motion is unstable.

On the resonance ($$\lambda \equiv 0$$), the motion is never stable for any small perturbation $$\kappa$$.

5.4 Solutions of the equations for the resonance $$Q_H - Q_V = p$$

Using (1.4.6), (1.5.2), and the assumptions given in Section 5.1\(^*\), the explicit equations are (Appendix 1):

\(^*\)In Appendix 2 a more general solution, valid for all the terms of the Hamiltonian, is given.
\[
\frac{da_1}{d\theta} = i \kappa a_2 e^{-i\Delta} \\
\frac{da_2}{d\theta} = i \kappa a_1 e^{i\Delta},
\]  
(1.5.14)

with \(\kappa = h_{1001-p}^{(2)}\) (see Appendix 1),

\[\Delta = Q_H - Q_V - p = \text{distance from the resonance},\]

Defining the new variable \(a_2^- = a_2 e^{-i\Delta}\), putting it in (1.5.14), and then differentiating, the equation for \(a_2^-\) becomes a second order equation:

\[
\frac{d^2 a_2^-}{d\theta^2} + i \Delta \frac{da_2^-}{d\theta} + \kappa \overline{a_2^-} = 0.
\]  
(1.5.15)

The general solution of (1.5.15) is then

\[
a_2^- = A_+ e^{i\omega_+ \theta} + A_- e^{i\omega_- \theta},
\]  
(1.5.16)

where

\[
\omega_{\pm} = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + |\kappa|^2}.
\]

The expressions for \(a_1\) and \(a_2\) are then

\[
a_1 = \kappa \left( A_+ \frac{e^{i\omega_+ \theta}}{\omega_+} + A_- \frac{e^{i\omega_- \theta}}{\omega_-} \right),
\]  
(1.5.17)

\[
a_2 = \left( A_+ \frac{e^{i\omega_+ \theta}}{\omega_+} + A_- \frac{e^{i\omega_- \theta}}{\omega_-} \right) e^{i\Delta},
\]

where \(A_+, A_-\) are complex constants of the motion,

\(\omega_{\pm}\) are the frequencies given in (1.5.16), and

\(\kappa\) is given by

\[
\kappa = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\frac{\rho}{\rho_H \rho_V}} \left[ \kappa + \frac{1}{2} \text{MR} \left( \frac{\alpha_H}{\rho_H} - \frac{\alpha_V}{\rho_V} \right) - \frac{1}{2} \text{MR} \left( \frac{1}{\rho_1} + \frac{1}{\rho_V} \right) \right]
\]

\[\exp \left[ i \left( \int_0^\theta \left( \frac{R}{\rho_H(\theta')} - Q_H \right) d\theta' - \int_0^\theta \left( \frac{R}{\rho_V(\theta')} - Q_V \right) d\theta' \right) \right] d\theta.
\]  
(1.5.18)

Both (1.5.17) and (1.3.3) will give the explicit solution of the perturbed motion, the constants \(A_+\) and \(A_-\) being defined by the initial conditions for \(x, x', z, z'\).

For this case, the motion is always stable, because the frequencies in (1.5.16) are real for any \(\Delta\) and \(|\kappa|\).
6. **Linear Coupling Coefficients**

6.1 **Definition of linear coupling coefficients**

The most important resonances excited by linear coupling are the two-dimensional ones, which depend linearly on $K$ and $M$. The characteristic parameter $\kappa$ of the perturbed motion is given for these resonances in (1.5.12) and (1.5.18), respectively. It is possible to summarize both expressions as follows:

For $\Delta = Q_H + nQ_V - p$, $n = \pm 1$

$$\kappa = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\beta_H \beta_V} \left[ K + \frac{1}{2} MR \left( \frac{a_H}{\beta_H} - \frac{a_V}{\beta_V} \right) - \frac{i}{2} MR \left( \frac{1}{\beta_H} - \frac{n}{\beta_V} \right) \right]$$

$$\exp \left\{ i \left[ \int_0^\theta \left( \frac{R}{\beta_H(\theta')} - Q_H \right) d\theta' + n \int_0^\theta \left( \frac{R}{\beta_V(\theta')} - Q_V \right) d\theta' + p \theta \right] \right\} d\theta. \tag{1.6.1}$$

$\kappa$ is the sum of two terms which are basically orthogonal in the complex plane, if we do not consider the phases associated with the exponentials. The first term represents the skew quadrupole field effects and the second one represents the longitudinal field effects (see Section 3.2). So we can split $\kappa$ in two parts and define two (or three) linear coupling coefficients:

$$C = C_q + C_b = 2\kappa$$

$$C_q = \frac{1}{2\pi R} \int_0^{2\pi} \sqrt{\beta_H \beta_V} \left[ K + \frac{1}{2} MR \left( \frac{a_H}{\beta_H} - \frac{a_V}{\beta_V} \right) \right]$$

$$\exp \left\{ i \left[ (\mu_H - Q_H\theta) + n(\mu_V - Q_V\theta) + p\theta \right] \right\} d\theta \tag{1.6.2}$$

$$C_b = -\frac{1}{4\pi} i \int_0^{2\pi} \sqrt{\beta_H \beta_V} M \left( \frac{1}{\beta_H} - \frac{n}{\beta_V} \right)$$

$$\exp \left\{ i \left[ (\mu_H - Q_H\theta) + n(\mu_V - Q_V\theta) + p\theta \right] \right\} d\theta .$$

The coefficient $C_q$ for skew fields contains not only the pure skew term $K$ in agreement with (1.3.10) or (1.3.12), but also a skew term due to the longitudinal changes of the $\beta$-functions.

These definitions (1.6.2) of coupling coefficients are consistent with the previous definitions\(^4\) given in the sinusoidal approximation. If we put $\beta_H = \beta_V = \text{constant}$, $a_H = a_V = 0$ and $\mu = Q\theta$ in (1.6.2), we find, using the definitions of $k$ (1.3.10) and $M$ (1.3.6),
\[ |C_q| = \frac{R_2}{Q} \left( \frac{1}{|b_p|} \frac{\partial b_x}{\partial x} \right) \]

\[ |C_b| = \frac{1}{|b_p|} b_s . \]

(1.6.3)

In our definition (1.6.2), we are taking account of the amplitude modulation around the machine and of the phases. The phases \( \omega_H \) and \( \omega_V \) are very important. Their presence means that \( C_q \) and \( C_b \) are not always orthogonal. It is therefore possible to compensate the \( C_b \) term with the \( C_q \) term provided the phases are well chosen.

6.2 Calculation of the linear coupling coefficients

To avoid beam blow-up, we would like to have \(|\kappa| = 0\). First we evaluate the coupling coefficients with the existing fields and then, if necessary, with compensating fields in order to suppress the perturbation.

\[ |C| = 2|\kappa| = \sqrt{C_q^2 + C_b^2 + \bar{C}_q \bar{C}_b + \bar{C}_b C_q} . \]

(1.6.4)

To find the amplitudes of the coefficients \( C_q \), \( C_b \) and \( C \) for a given practical problem, it is necessary to approximate the integrals (1.6.2) with summations, splitting the magnetic elements in pieces. Assuming we have \( N_e \) elements cut in \( N_c \) parts, we can calculate any of these amplitudes as indicated below.

The functions \( K, M \) being defined by (1.3.6), the amplitudes and phases of the coupling coefficients for one piece of element can be deduced from (1.6.2):

\[ |C_q|_{jk} = \left| \frac{g}{2\pi R^2} \left( \sqrt{\frac{\partial b_x}{\partial y}} \right)_{jk} \left[ K_{jk} + \frac{1}{2} R M_{jk} \left( \frac{\omega_H}{\beta_H} - \frac{\omega_V}{\beta_V} \right)_{jk} \right] \right| \]

\[ \psi_{qj} = \left( m_H + n m_V \right)_{jk} - \left( Q_H + n Q_V \right)_{jk} + \phi_{jk} \pm \pi \]

\[ |C_b|_{jk} = \left| \frac{g}{4\pi R^2} M_{jk} \left( \sqrt{\frac{\beta_H}{\beta_H}} - n \sqrt{\frac{\beta_H}{\beta_V}} \right)_{jk} \right| \]

\[ \psi_{bj} = \psi_{qj} \pm \frac{\pi}{2} \]

\[ |C|_{jk} = \sqrt{|C_q|_{jk}^2 + |C_b|_{jk}^2} \]

\[ \psi_{jk} = \psi_{qj} \pm \arctg \left( \frac{|C_b|_{jk}}{|C_q|_{jk}} \right) , \]

with

\[ n = \frac{1}{2} \]

depending on the resonance \( Q_H + n Q_V = p \)

\[ j = 1, \ldots, N_e \quad \text{No. of elements} \]

\[ k = 1, \ldots, N_c \quad \text{No. of cuts} \]

\[ \ell = \text{length of one piece of elements} . \]

The signs appearing in \( \psi_q \) and \( \psi \) depend on the signs of \( C_q \) and \( C_b \), and the phase shift \( \pi \) is present in \( \psi_q \) only for negative \( C_q \).
Knowing the amplitudes and phases, we can sum the contributions of each piece in each element:

\[ C = \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} |C|_{jk} e^{i\psi_{jk}} \]

\[ |C|^2 = \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} |C|_{jk}^2 + \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} \sum_{n=k+1}^{N_c} 2|C|_{jk}|C|_{jn} \cos(\psi_{jk} - \psi_{jn}) + \]

\[ + \sum_{j=1}^{N_e-1} \sum_{m=j+1}^{N_c} \sum_{k=1}^{N_c} \sum_{n=1}^{N_c} 2|C|_{jk}|C|_{mn} \cos(\psi_{jk} - \psi_{mn}). \]  

(1.6.6)

The relations (1.6.6) are written for \( C \) but are equally valid for \( C_q \) and \( C_b \). Using (1.6.5) and (1.6.6) \( |C_q|, |C_b| \) or \(|C|\) can be found.

A computer program has been written to calculate these coupling coefficients and to determine the necessary field levels in elements such as skew quadrupoles or solenoids to minimize \(|C|\).

7. APPLICATION TO THE CERN STORAGE RINGS

Some skew field always exists in the ISR due to random tilts of the magnet units and for this reason skew quadrupoles have been included in the lattice\(^7\). A project to install a solenoid at one intersection to analyse secondary particles is now under way. This will be the first axial field element in the ISR machine. To check experimentally the effects of a solenoid on the beams, a test solenoid was installed.

It is interesting to apply the present theory of linear coupling to both of these solenoids. As current ISR working lines are close to the diagonal, only the difference resonance \( Q_{h} - Q_{v} = 0 \) is of importance.

Table 1 summarizes the amplitudes of the coupling coefficients associated with the detector solenoid in intersection 1. The characteristics of this solenoid are the following:

- Length = 1.8 m
- \( \int B_s \, dl = 2.7 \) Tm

and both ends with slots so that \( a_s = 1 \), as in (1.3.12).

The \(|C|\) amplitudes have been calculated for four currently used working lines\(^8,9\), for a steel low-\( B_{v} \) section\(^10\) and for a superconducting low-\( B_{v} \) section\(^11\). Both low-\( B_{v} \) sections are planned for the same intersection as the solenoid.
TABLE 1
Linear coupling coefficients for the detector solenoid in the ISR at 26 GeV/c

| Working conditions | $\beta_H$ (m) | $\beta_V$ (m) | $\alpha_H$ | $\alpha_V$ | $|C_q|$ (10^{-3}) | $|C_b|$ (10^{-3}) | $|C|$ (10^{-3}) |
|--------------------|---------------|---------------|------------|------------|----------------|----------------|----------------|
| FP                 | 22.6          | 14.6          | 0.232      | -0.131     | 1.69           | 4.96           | 3.65           |
| TW                 | 24.0          | 18.4          | 0.474      | -0.217     | 2.3            | 4.89           | 4.55           |
| 8C                 | 22.7          | 14.5          | 0.236      | -0.130     | 1.71           | 4.96           | 3.65           |
| ELSA               | 20.9          | 12.2          | 0.189      | -0.106     | 1.90           | 5.02           | 3.33           |
| steel low-$\beta_V$ | 47.5       | 2.6           | -0.214     | 0.100      | 12.64          | 10.52          | 2.11           |
| supercon. low-$\beta_V$ | 4.7      | 0.3           | 0.134      | 0.219      | 6.87           | 5.05           | 1.84           |

Note: the values of $\beta_H$, $\beta_V$, $\alpha_H$ and $\alpha_V$ are given at the centre of the solenoid.

Two interesting points arise from these results:

1) Both low-$\beta_V$ sections give large $|C_q|$ and $|C_b|$ but also low $|C|$.
   This means that the coefficients $C_b$ and $C_q$ are roughly collinear and opposed.

2) The $|C_b|$ amplitude for the superconducting low-$\beta_V$ section is roughly half that for the steel section, although the $\beta_H/\beta_V$ ratios look very similar. This is due to the fact that the $\beta_V$ is increasing far more rapidly inside the solenoid in the superconducting section, so that after integration the average ratio $\beta_H/\beta_V$ is smaller.

Table 2 summarizes the amplitudes of the coupling coefficients associated with the test solenoid. The characteristics of this solenoid are the following:

\[
\text{Length} = 0.3 \text{ m} \\
\int B_z \, d\ell = 0.34 \text{ Tm}.
\]

The $|C|$ amplitudes have been calculated for the 8C working line, but for different end conditions (solenoid without end plates, with two slots and with one slot only).
TABLE 2
Linear coupling coefficients for the test solenoid in the ISR at 26 GeV/c

Working line: 8C  \( \beta_H = 57.1 \)  \( \beta_V = 12.4 \)  \( \alpha_H = 0.253 \)  \( \alpha_V = 0.061 \)
(values given at the centre of the solenoid)

| End plates | \( |C_q| \)  \((10^{-3})\) | \( |C_b| \)  \((10^{-3})\) | \( |C| \)  \((10^{-3})\) |
|-----------|-----------------|-----------------|-----------------|
| see (1.3.12) | \( a_{s1} = a_{s2} = 0.5 \) | 0.03 | 0.71 | 0.71 |
| | \( a_{s1} = a_{s2} = 1 \) | 0.46 | 0.71 | 0.30 |
| | \( a_{s1} = 1 \)  \( a_{s2} = 0.5 \) | 22.43 | 0.71 | 22.43 |

Two points are of interest in this table:

1) With two slots, the test solenoid also excites co-linear and opposed coefficients, \( C_b \) and \( C_q \).

2) With one slot only, the \( C_q \) dominates (~30 times larger than \( C_b \)). For comparison, the normal machine with the 8C line at 26 GeV/c has a residual \( C_q \) of ~1-2 \( \times 10^{-3} \).

Both these tables illustrate the main points:

1) A solenoid clearly contributes to the \( C_b \) coefficient but it also contributes to the \( C_q \) term in two different ways: firstly because the \( \beta \)-functions change inside the solenoid (\( \alpha \)-contribution) and secondly because of the skew fields in the end plates (\( K = M' \) contribution). Finally, \( C_q \) can be as important as \( C_b \) for such an element.

2) It can happen that the slots of the end plates partially compensate the effects of the centre of the solenoid.

3) It is possible to compensate the effects of a longitudinal field with skew fields by virtue of the phase terms. In other words, a set of skew quadrupoles can compensate both the random tilts of the main magnets and the solenoids' contribution.

8. CONCLUSIONS

An analytical formalism able to treat the problem of the linear coupling of the transverse motions in a synchrotron has been developed. This formalism enabled us to give the explicit solutions of the motion equations in case of linear coupling for the different resonances excited. Linear coupling coefficients for a machine with a high form factor were defined. These coefficients characterize the perturbed motions and make it possible to analyse the effects of skew and longitudinal fields. This is of use when introducing new elements and when studying compensation schemes.
Finally it is interesting to see the relation of this theory with the work already existing. Basically, three methods have been used. The first one employs transfer matrices and does not give the differential equations and their explicit solutions. The second one uses the Hamiltonian formalism, but was only applied to a two-dimensional transverse magnetic field. The third one starts by directly substituting the solutions of the unperturbed motion in the differential equations and then to deduce the equations for the constants. This method was successful in the case of pure skew quadrupole fields and in the sinusoidal approximation, but it was not successful in the case of longitudinal field and large B-variations. For example, in Ref. 13 Kolomensky derives the equations (1.8.1), which are similar to equations (1.5.14) in this report.

\[
\frac{da_1}{d\varphi} = Q_z a_2 e^{-i\varphi\Delta},
\]

\[
\frac{da_2}{d\varphi} = Q_x a_1 e^{i\varphi\Delta}
\]

with \(Q_z \neq -Q_x\).

The problem which arises is that the equations (1.8.1) admit the possibility of having coupling much stronger one way than the other, which is non-symplectic and forbidden. This was realized by Kolomensky who calculated \(Q_z + Q_x\) and showed that this is small if \(\Delta\) is small (in the vicinity of the resonance). This is restrictive and not a typical ISR operating condition. In the present report in (1.5.14) we have the identity \(\kappa = (-\kappa)\), which means the equations are symplectic under all conditions. Finally the comparison of \(\kappa\) with \(Q_z\) and \(Q_x\) of Ref. 13 suggests that some terms are missing in \(Q_z\) and \(Q_x\).
1. **INTRODUCTION**

The formalism described in the first part of this report and used for the treatment of linear coupling, can be applied in the more general case, where the transverse components of the magnetic field are non-linearly dependent on the transverse coordinates. This statement implies that the longitudinal component of the magnetic field is also non-linearly dependent on the transverse coordinates.

This generalization of the theory is based on an analytical description of the three-dimensional magnetic field, which makes it possible to write the perturbing Hamiltonian. Using the same mathematical development as in Part 1, it is then possible to determine the equations of the perturbed motion by applying the same assumptions as mentioned in the linear coupling theory.

As in the case of the linear coupling, a general complex parameter has been defined, which is equivalent to the linear coupling coefficient and generalizes the parameter $\kappa$ introduced in Ref. 19. This parameter depends on the multipole components of the magnetic field and is valid for a machine with any form factor. Thus it characterizes the effects on a proton beam of any magnetic element.

Finally, the invariants of the perturbed motion were deduced from the motion equations. As these invariants have the same analytical form as those used in Ref. 19 with extended definitions of the parameters $h_{jkm}$, it is possible to introduce generalized expressions of the bandwidth and of the growth rate of the transverse amplitudes for the sum resonances excited by a three-dimensional field. It is also possible to use the same analysis for the difference resonances which were not treated in Refs. 19 and 20, and to define a bandwidth for difference resonances.

These expressions give the possibility of evaluating the importance of a three-dimensional magnetic field perturbation, of judging the necessity to compensate it and finally of calculating at what distance from a resonance the working point should be, in order to keep the beam blow-up inside given limits.

2. **Simplified Description of a Three-Dimensional Magnetic Field**

2.1 **Analytical form of the potential vector**

The magnetic field is defined by the following relations:
\[ \vec{B} = \nabla \times \vec{A} \]
\[ \text{div} \, \vec{B} = 0 \]
\[ \nabla \times \vec{B} = \mu_0 \vec{j}, \quad \text{(2.2.1)} \]

where \( \vec{j} \) is the current density and \( \vec{A} \) the potential vector associated with the field. For a two-dimensional magnetic field\(^{19,20}\), the potential vector has only one component and the longitudinal field component, \( B_0 \), is zero, which gives

\[ \begin{align*}
A_\theta & \neq 0 \quad A_x = A_z = 0 \\
B_0 &= 0 \quad B_x = \frac{\partial A_\theta}{\partial z} \quad B_z = -\frac{\partial A_\theta}{\partial x} \\
\frac{\partial B_x}{\partial z} &= -\frac{\partial B_z}{\partial z} \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} 
\end{align*} \quad \text{(2.2.2)} \]

For a three-dimensional field, none of the components of \( \vec{A} \) in general vanish and the relations (2.2.1) cannot be simplified as in (2.2.2). The only simplification which is possible comes from the fact that \( \vec{j} = 0 \):

\[ \frac{\partial B_x}{\partial s} = \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial s} \quad \text{(2.2.3)} \]

At first we use an expression for \( \vec{A} \) which has the advantage of being similar to the linear case and the two-dimensional theory, but is somewhat restrictive concerning end effects\(^*\) [see Appendix 3, equations (A3.13) and (A3.14)]:

\[ \begin{align*}
A_\theta &= + \frac{B_0}{R^2} \, D(x, z, \theta) \\
A_x &= + \frac{B_0}{R} \, P_x(x, z, \theta) \\
A_z &= + \frac{B_0}{R} \, P_z(x, z, \theta) 
\end{align*} \quad \text{(2.2.4)} \]

where \( D(x, z, \theta) \) is a polynomial development corresponding to the standard two-dimensional multipole terms\(^{19,20}\),

\( P_x(x, z, \theta) \) and \( P_z(x, z, \theta) \) are also polynomial developments,

\( \frac{B_0}{R} \) and \( \frac{B_0}{R^2} \) are normalization factors taking account of the energy \( (P_0 = -eB_0) \)

and of the fact that \( \theta \) is the independent variable. These factors are taken with their signs, i.e. they are negative (see Fig. 1).

The form (2.2.4) for \( \vec{A} \) satisfies in all cases the relation \( \text{div} \, \vec{B} = 0 \) (Appendix 3).

In order to satisfy the specific relations(2.2.3), the polynomials \( P_x \) and \( P_z \) have to be the partial derivatives of another polynomial \( P_0 \):

\[ \text{*) A general approach which avoids this restriction is outlined in Section 7.} \]
\[ P_x(x, z) = \frac{\partial P_0(x, z)}{\partial x}, \quad P_z(x, z) = -\frac{\partial P_0(x, z)}{\partial z}, \quad (2.2.5) \]

where \( P_0(x, z) \) must be a polynomial development of the type given in Appendix 3. The form of the polynomial \( D(x, z) \) is well known\(^9\), so that we can write

\[
D(x, z, \theta) = -\sum_{N=2}^{\infty} \text{Re} \left[ \frac{1}{N!} \left( k_z^{(N-1)}(\theta) + i k_x^{(N-1)}(\theta) \right) w^N \right],
\]

\[ (2.2.6) \]

\[
P_0(x, z, \theta) = -\sum_{N=2}^{\infty} \frac{1}{N!} \left[ F_z^{(N-1)}(\theta) \ t_1^{(N)} + F_x^{(N-1)}(\theta) \ t_2^{(N)} \right],
\]

where \( w = x + iz, t_1^{(N)} = \text{even powers in } z \) of \((x + z)^N, t_2^{(N)} = \text{odd powers in } z \) of \((x + z)^N\).

\( k_z^{(N-1)}, k_x^{(N-1)} \) and \( F_z^{(N-1)}, F_x^{(N-1)} \) are parameters characterizing respectively the transverse and the longitudinal fields for the 2N-pole term and giving their azimuthal variation. They will be defined later in terms of field derivatives.

Putting (2.2.6) and (2.2.5) in (2.2.4) will give the following analytical expression for the potential vector:

\[
A_0 = -\frac{B_0}{R^2} \sum_{N=2}^{\infty} \frac{1}{N!} \left[ k_z^{(N-1)}(\theta) w_1^{(N)} - k_x^{(N-1)}(\theta) w_2^{(N)} \right],
\]

\[
A_x = -\frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \left[ F_z^{(N-1)}(\theta) \ t_1^{(N-1)} + F_x^{(N-1)}(\theta) \ t_2^{(N-1)} \right],
\]

\[ (2.2.7) \]

\[
A_z = +\frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \left[ F_x^{(N-1)}(\theta) \ t_1^{(N-1)} + F_z^{(N-1)}(\theta) \ t_2^{(N-1)} \right],
\]

where \( t_1^{(N)}, t_2^{(N)} \) are respectively the even and the odd powers in \( z \) of \((x + z)^N\)

\( w_1^{(N)}, i w_2^{(N)} \) are respectively the even and the odd powers in \( z \) of \((x + iz)^N\).

These expressions (2.2.7) are the starting point on which is based the three-dimensional theory which follows.

2.2 Consequent analytical form of the magnetic field

The magnetic field components can always be developed in series around the vacuum chamber axis \( x = z = 0 \). In general this development will be
\[ B_x = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n+1} \frac{\partial^n B_x}{\partial x^{(n-j+1)} \partial z^{(j-1)}} \bigg|_{x=z=0} x^{(n-j+1)} z^{(j-1)} \]

\[ B_z = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n+1} \frac{\partial^n B_z}{\partial x^{(n-j+1)} \partial z^{(j-1)}} \bigg|_{x=z=0} x^{(n-j+1)} z^{(j-1)} \]  \hspace{1cm} (2.2.8)

\[ B_0 = B_0(x=z=0) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n+1} \frac{\partial^n B_0}{\partial x^{(n-j+1)} \partial z^{(j-1)}} \bigg|_{x=z=0} x^{(n-j+1)} z^{(j-1)} , \]

where the dipole transverse fields are not considered, assuming that they are treated separately.

Alternatively it is possible to write the expressions for the field components by putting the relations (2.2.7) for the potential vector in the first equation (2.2.1). The comparison of these expressions with (2.2.8) gives the expression for the longitudinal field component at the origin and for the field derivatives as function of the parameters \( K_z, K_x, F_z, F_x \) (Appendix 3):

For \( B_0(x=z=0) \)

\[ B_0(x=z=0) = \frac{|B_0|}{R} 2F_X^{(1)} . \]

For \( k_2 \) even, \( k_1 + k_2 = n, n = 1, \ldots , \infty \)

\[ \frac{\partial^n B_x}{\partial x^{k_1} \partial z^{k_2}} \bigg|_{x=z=0} = \binom{n}{k_2} \frac{|B_0|}{R^2} \left[ (-1)^{k_2/2} \frac{d}{d\theta} \left( k_x^{(n)} \right) - \frac{dF_X^{(n)}}{d\theta} \right] \]  \hspace{1cm} (2.2.9)

\[ \frac{\partial^n B_z}{\partial x^{k_1} \partial z^{k_2}} \bigg|_{x=z=0} = \binom{n}{k_2} \frac{|B_0|}{R^2} \left[ (-1)^{k_2/2} \frac{d}{d\theta} \left( k_z^{(n)} \right) - \frac{dF_z^{(n)}}{d\theta} \right] \]

\[ \frac{\partial^n B_0}{\partial x^{k_1} \partial z^{k_2}} \bigg|_{x=z=0} = \binom{n}{k_2} \frac{|B_0|}{R} 2F_X^{(n+1)} , \]

and similar expressions (Appendix 3) for \( k_2 \) being odd, where the sign is different and the parameters \( (k_x,F_x) \) and \( (k_z,F_z) \) are exchanged. Here the modulus of \( B_0 \) has been introduced.

It is easy to see that the parameter \( F_z^{(1)} \) can be arbitrarily taken as zero (Appendix 3). Secondly, it is now convenient to change the names of the parameters \( k_x^{(1)} \) and \( F_x^{(1)} \) to be in agreement with Part 1. Thus we put
\[ K_X^{(1)}(\theta) = K(\theta) \quad F_X^{(1)}(\theta) = \frac{1}{2} M(\theta) \]
\[ K_z^{(1)}(\theta) = \frac{R^2 \partial B_x}{|B_\theta| \partial z} \quad F_z^{(1)}(\theta) = 0 \quad (2.2.10) \]

Now the longitudinal field component and the first field derivatives calculated from (2.2.9) become

\[
B_y(x=z=0) = \frac{|B_\theta|}{R} M(\theta) \]
\[
\left. \frac{\partial B_x}{\partial x} \right|_{x=z=0} = \frac{|B_\theta|}{R^2} \left[ K(\theta) - \frac{1}{2} M'(\theta) \right] \quad (2.2.11) \]
\[
\left. \frac{\partial B_z}{\partial z} \right|_{x=z=0} = -\frac{|B_\theta|}{R^2} \left[ K(\theta) + \frac{1}{2} M'(\theta) \right] \]
\[
\left. \frac{\partial B_x}{\partial z} \right|_{x=z=0} = \left. \frac{\partial B_z}{\partial x} \right|_{x=z=0} = \frac{|B_\theta|}{R^2} K_z^{(1)}(\theta) \quad . \]

Solving (2.2.11) for \( K, K_z \) and \( M \) we get

\[
M(\theta) = 2 F_X^{(1)}(\theta) = \frac{R}{|B_\theta|} B_y(x=z=0) \]
\[
K(\theta) = K_X^{(1)}(\theta) = \frac{R^2}{2|B_\theta|} \left[ \left. \frac{\partial B_x}{\partial x} \right|_{x=z=0} - \left. \frac{\partial B_z}{\partial z} \right|_{x=z=0} \right] \quad (2.2.12) \]
\[
K_z^{(1)}(\theta) = \frac{R^2 \partial B_x}{|B_\theta| \partial z} \left. \right|_{x=z=0} \quad , \]

which are exactly the definitions (1.3.6) given in the linear coupling theory (Part 1). The relations (2.2.12) define the coefficients associated with the quadrupole term (N=2).

To have an explicit description of the other parameters \( K_{X,z}(\theta) \) and \( F_{X,z}(\theta) \) of (2.2.7), it is necessary to solve the relations (2.2.9) (see also Appendix 3) for \( K_{X,z} \) and \( F_{X,z} \):
For $k_2$ even, $1 \leq k_2 \leq (N-2)$; $N \geq 3$

\[
P^{(N-1)}_{x}(\theta) = \frac{R}{2|B_0|} \frac{k_2!(N-k_2-2)!}{(N-2)!} \frac{\partial (N-2)B_0}{\partial x (N-k_2-2) \partial z k_2} \bigg|_{x=z=0} \tag{2.2.13}
\]

\[
K^{(N-1)}_{x}(\theta) = (-1) \frac{k_2+2}{2|B_0|} \frac{R^2}{(N-k_2-2)!} \frac{(k_2+1)!(N-k_2-2)!}{(N-1)!} \left[ \frac{\partial (N-1)B_z}{\partial x (N-k_2-2) \partial z (k_2+1)} \bigg|_{x=z=0} - \left( \frac{N-k_2-1}{k_2+1} \right) \frac{\partial (N-1)B_x}{\partial x (N-k_2-1) \partial z k_2} \bigg|_{x=z=0} \right],
\]

For $k_2$ odd, $1 \leq k_2 \leq (N-2)$; $N \geq 3$

\[
P^{(N-1)}_{z}(\theta) = \frac{R}{2|B_0|} \frac{k_2!(N-k_2-2)!}{(N-2)!} \frac{\partial (N-2)B_0}{\partial x (N-k_2-2) \partial z k_2} \bigg|_{x=z=0} \tag{2.2.14}
\]

\[
K^{(N-1)}_{z}(\theta) = (-1) \frac{k_2+1}{2|B_0|} \frac{R^2}{(N-k_2-2)!} \frac{(k_2+1)!(N-k_2-2)!}{(N-1)!} \left[ \frac{\partial (N-1)B_z}{\partial x (N-k_2-2) \partial z (k_2+1)} \bigg|_{x=z=0} - \left( \frac{N-k_2-1}{k_2+1} \right) \frac{\partial (N-1)B_x}{\partial x (N-k_2-1) \partial z k_2} \bigg|_{x=z=0} \right],
\]

where $N$ corresponds to the $2N$-pole term.

$K_{x,z}$ and $P_{x,z}$ depend on $\theta$ since the field derivatives are also dependent on $\theta$. They can be calculated from magnetic field measurements through (2.2.12), (2.2.13) and (2.2.14). This set of equations defines all the parameters used in (2.2.7), and the three components of the potential vector $\vec{A}$.

3. TRANSVERSE MOTIONS IN A THREE-DIMENSIONAL FIELD

3.1 General description of the motion

The transverse motions are characterized by the Hamiltonian function. By analogy with the Hamiltonian (1.3.13) associated with linear coupling forces, it is possible to write the form of the general Hamiltonian of charged particles in a three-dimensional magnetic field. This Hamiltonian should have the form of $H_4$, in which the two-dimensional momentum vector $(p_x, p_z)$ is replaced by the generalized momentum vector $(p_x - \frac{R}{|B_0|} A_x, p_z - \frac{R}{|B_0|} A_z)$ and to which the "scalar potential" $-\frac{R^2}{|B_0|} A_0$ is added:
\[ H = \frac{1}{2} \left[ K_1 x^2 + K_2 z^2 - \frac{2 R^2}{|B_0|} A_0^2 + \left( p_x - \frac{R}{|B_0|} A_x \right)^2 + \left( p_z - \frac{R}{|B_0|} A_z \right)^2 \right], \quad (2.3.1) \]

where \(|B_0|\) is positive in agreement with the General Introduction.

The difference in the treatment of \(A_0\) and \((A_x, A_z)\) is due to the fact that we are looking for the Hamiltonian of the transverse motions of protons, without considering the longitudinal motion.

The transverse motion equations are then given by the canonical equations (1.2.1), in which \(H\) is replaced by (2.3.1) and \(B_0\) has its negative sign (see Fig. 1):

\[
\begin{align*}
\frac{dx}{d\theta} &= \frac{\partial H}{\partial p_x} = p_x + \frac{R}{B_0} A_x \\
\frac{dp_x}{d\theta} &= -\frac{\partial H}{\partial x} = -K_1 x - \frac{R^2}{B_0} \frac{\partial A_0}{\partial x} - \frac{R}{B_0} \left( p_x + \frac{R}{B_0} A_x \right) \frac{\partial A_x}{\partial x} - \frac{R}{B_0} \left( p_z + \frac{R}{B_0} A_z \right) \frac{\partial A_x}{\partial z} \\
\frac{dz}{d\theta} &= \frac{\partial H}{\partial p_z} = p_z + \frac{R}{B_0} A_z \\
\frac{dp_z}{d\theta} &= -\frac{\partial H}{\partial z} = -K_2 z - \frac{R^2}{B_0} \frac{\partial A_0}{\partial z} - \frac{R}{B_0} \left( p_x + \frac{R}{B_0} A_x \right) \frac{\partial A_x}{\partial z} - \frac{R}{B_0} \left( p_z + \frac{R}{B_0} A_z \right) \frac{\partial A_z}{\partial z}.
\end{align*}
\]

(2.3.2)

The first equation (2.3.2) can be solved for \(p_x\) and then differentiated with respect to \(\theta\):

\[
\frac{dp_x}{d\theta} = \frac{d^2 x}{d\theta^2} - \frac{R}{B_0} \frac{\partial A_x}{\partial \theta}. \quad (2.3.3)
\]

Doing the same for the third equation (2.3.2) and replacing \(p_x'\) and \(p_z'\) by their expressions like (2.3.3) into the second and fourth equations (2.3.2) will give the usual motion equations in the transverse coordinates:

\[
\begin{align*}
\frac{d^2 x}{d\theta^2} + K_1 x &= \frac{R^2}{B_0} \left( \frac{\partial A_x}{\partial \theta} - \frac{\partial A_0}{\partial x} \right) - \frac{R}{B_0} \frac{dx}{d\theta} \frac{\partial A_x}{\partial x} - \frac{R}{B_0} \frac{dx}{d\theta} \frac{\partial A_x}{\partial z} \\
\frac{d^2 z}{d\theta^2} + K_2 z &= \frac{R^2}{B_0} \left( \frac{\partial A_z}{\partial \theta} - \frac{\partial A_0}{\partial z} \right) - \frac{R}{B_0} \frac{dz}{d\theta} \frac{\partial A_z}{\partial z} - \frac{R}{B_0} \frac{dz}{d\theta} \frac{\partial A_z}{\partial x}.
\end{align*}
\]

(2.3.4)

These equations are general and give the equations (1.3.5) in the particular case of linear coupling. It is now clear from the form of (2.3.4) why we introduced the seemingly arbitrary factors \(R^2/B_0\) and \(R/B_0\) in the generalized momentum vector used in (2.3.1). In (2.3.4) these factors come from the energy normalization \((p_\theta = -eB_0)\) and from the use of \(\theta\) as independent variable \((ds = Rd\theta)\).
3.2 Description of the motion in the linear case

One particular case which is worth mentioning separately, is that in which the forces are linear. This corresponds to the quadrupole term in the field expansion, i.e. to \( N = 2 \).

Using (2.3.1), (2.2.12) and (2.2.7), it is possible to write the Hamiltonian in the linear case:

\[
H = \frac{1}{2} \left[ K_1 x^2 + K_2 z^2 + 2K_{xz} xz - K_z^{(1)} (x^2 - z^2) + \left( \frac{p_x}{M} - \frac{1}{2} \frac{M'}{M} z \right)^2 + \left( \frac{p_z}{M} + \frac{1}{2} \frac{M'}{M} x \right)^2 \right]. \tag{2.3.5}
\]

The motion equations can be deduced from (2.3.4), (2.2.12) and (2.2.7):

\[
\frac{d^2 x}{dt^2} + K_1 x = K_z^{(1)} x - (K + \frac{1}{2} M') z - M' \tag{2.3.6}
\]
\[
\frac{d^2 z}{dt^2} + K_2 z = - K_z^{(1)} z - (K - \frac{1}{2} M') x + M' x.
\]

These are the linear equations of the transverse motions. If we now compare them with (1.3.9) or if we compare the Hamiltonian (2.3.5) with (1.3.13), we notice that we have now one more term due to \( K_z^{(1)} \). This term has not been considered in Part 1, because it changes the focusing properties in each transverse plane, but it does not couple the transverse motions.

4. PERTURBING HAMILTONIAN OF THE TRANSVERSE MOTIONS

4.1 Perturbing Hamiltonian as function of the constants of the unperturbed motion

Having \( H \) (2.3.1) and \( H_0 \) (1.3.2), a simple subtraction will give the perturbing Hamiltonian \( H_1 \):

\[
H_1 = + \frac{R^2}{B_0} A_0 + \frac{R}{B_0} \left( p_x A_x + p_z A_z \right) + \frac{1}{2} \left( \frac{R A}{B_0} \right)^2 + \frac{1}{2} \left( \frac{RA}{B_0} \right)^2. \tag{2.4.1}
\]

We considered here that the parameters \( K_1 \) and \( K_2 \) describe the main gradients whereas the potential vector \( A \) introduced in Section 2 defines the perturbing magnetic field containing multipole terms of any order \( N \geq 2 \).

We would like to introduce now our first assumption in order to simplify the future developments.

Let us assume that the perturbing field is small enough for us to neglect the square terms \( A_x^2 \) and \( A_z^2 \) with respect to the linear ones.

This assumption simplifies \( H_1 \):

\[
H_1 = + \frac{R^2}{B_0} A_0 + \frac{R}{B_0} \left( p_x A_x + p_z A_z \right). \tag{2.4.2}
\]
It is now necessary to write $H_1$ as function of $(x, p_x, z, p_z, \theta)$ using (2.2.7). This is done in Appendix 4. Then, in agreement with the formalism summarized in Section 2 of Part 1, $H_1$ has to be expressed as function of constants $a_j$ of the unperturbed motion, using the relations (1.4.2) (See Appendix 4).

Finally we can write the following:

$$H_1 = \sum_{N=2}^{\infty} H_1^{(N)}(x, p_x, z, p_z, \theta)$$

or

$$U(a_j, \theta) = \sum_{N=2}^{\infty} U^{(N)}(a_1, \bar{a}_1, a_2, \bar{a}_2, \theta),$$

where $H_1^{(N)}$ or $U^{(N)}$ represents the Hamiltonian of the 2N-pole perturbation. $U^{(N)}$ has the following form:

$$U^{(N)}(a_j, \theta) = \sum_{j, k, l, m=0}^{N} h_j^{(N)}(\theta) \frac{1}{a_1 \bar{a}_1 a_2 \bar{a}_2} \exp\left\{i \left[ (j-k)Q_h + (k-m)Q_v \right] \theta \right\},$$

with $j + k + l + m = N$.

For $(k + m)$ even,

$$h_j^{(N)}(\theta) = \frac{1}{j! k! l! m!} \left[ - (-1)^{\frac{j+m}{2}} \sum_{z}^{(N-1)} i z^j u z^k \right.$$  

$$- j F_z^{(N-1)} u (j-1) z^k z^{m-1} (u' + iQ_h u) -$$  

$$- k F_z^{(N-1)} u z^j (k-1) v z^{m-1} (v' - iQ_h \bar{u}) +$$  

$$+ \delta F_z^{(N-1)} u z^j z^{(k-1)} v (v' - iQ_v v) +$$  

$$+ m F_z^{(N-1)} u z^j z^{(k-1)} (m-1) (v' - iQ_v \bar{v}) \left. \right].$$
For \((k + m)\) odd,

\[
\begin{align*}
\mathcal{H}_{jkkm}^{(N)}(\theta) &= \frac{1}{j! \cdot k! \cdot m!} \left[-(-1)^{\frac{k+m+1}{2}} \mathcal{K}^{(N-1)}_{x} u_{x}^{k} v_{x}^{m} - \right. \\
&\quad - jF^{(N-1)}_{x} u_{x}^{j-1} v_{x}^{k-1} v^{m} (u' + iQ_{u} u) - \\
&\quad - kF^{(N-1)}_{x} u_{x}^{i-1} v_{x}^{k} v^{m} (u' - iQ_{u} u) + \\
&\quad + kF^{(N-1)}_{x} u_{x}^{i-1} v_{x}^{k-1} v^{m} (v' - iQ_{v} v) + \\
&\quad + mF^{(N-1)}_{x} u_{x}^{i-1} v_{x}^{k} v^{m-1} (v' - iQ_{v} v) \right].
\end{align*}
\]

(2.4.6)

To complete this, we need the explicit form of the Floquet functions

\[
\begin{align*}
u &= \sqrt{\frac{\beta_{H}(\theta)}{2R}} \exp \left[ i \int_{0}^{\theta} \left( \frac{R}{\beta_{H}(\theta')} - Q_{H} \right) d\theta' \right] \\
v &= \sqrt{\frac{\beta_{V}(\theta)}{2R}} \exp \left[ i \int_{0}^{\theta} \left( \frac{R}{\beta_{V}(\theta')} - Q_{V} \right) d\theta' \right] \\
u' + iQ_{u} u &= \sqrt{\frac{\beta_{H}(\theta)}{2R}} \left[ - \frac{R \mathcal{R}_{H}(\theta)}{\beta_{H}(\theta)} + i \frac{R}{\beta_{H}(\theta)} \right] \exp \left[ i \int_{0}^{\theta} \left( \frac{R}{\beta_{H}(\theta')} - Q_{H} \right) d\theta' \right] \\
v' + iQ_{v} v &= \sqrt{\frac{\beta_{V}(\theta)}{2R}} \left[ - \frac{R \mathcal{R}_{V}(\theta)}{\beta_{V}(\theta)} + i \frac{R}{\beta_{V}(\theta)} \right] \exp \left[ i \int_{0}^{\theta} \left( \frac{R}{\beta_{V}(\theta')} - Q_{V} \right) d\theta' \right].
\end{align*}
\]

(2.4.7)

This expression for the perturbing Hamiltonian associated with a three-dimensional magnetic field can be compared with that given in Ref. 19 (pp. 72-74) for a two-dimensional field. Both expressions are very similar in their form, but the \(\mathcal{H}_{jkkm}^{(N)}\) coefficients defined here contain, in addition to the term \(\mathcal{K}^{(N-1)}\) which also appears in Ref. 19, some terms \(\mathcal{F}^{(N-1)}\) associated with the longitudinal component of the field. Note that the expressions (2.4.5) and (2.4.6) have opposite signs to those of Ref. 19, because \(B_{y}\) appears in modulus inside \(\mathcal{K}_{x,z}^{(N)}\) and \(\mathcal{F}_{x,z}^{(N)}\), but with its sign (-ve) in Ref. 19.

4.2 Perturbing Hamiltonian associated with a specific resonance of any order

A specific resonance is defined by the following relation between the transverse tunes:

\[
n_{1}Q_{H} + n_{2}Q_{V} = p \quad n_{1}, n_{2}, p \text{ integers } \geq 0 .
\]

(2.4.8)
This equation is the condition that a resonance may exist in the two-dimensional transverse oscillations of the protons. The quantity \( N = |n_1| + |n_2| \) is called the order of the resonance.

To determine the excitation of a resonance, it is necessary to analyse the frequency spectrum of the perturbing Hamiltonian (2.4.3). As the magnetic imperfections have the circumference periodicity, a Fourier's development of \( H_1 \) is possible:

\[
\hat{h}_{jkkm}(\theta) = \sum_{q=-\infty}^{\infty} \hat{h}_{jkkmq} e^{i q \theta},
\]

(2.4.9)

with

\[
\hat{h}_{jkkmq} = \frac{1}{2\pi} \int_0^{2\pi} \hat{h}_{jkkm}(\theta) e^{-i q \theta} d\theta.
\]

Putting (2.4.9) in (2.4.3) and (2.4.4) gives

\[
U(a_j, \theta) = \sum_{N=2}^{\infty} \sum_{j,k,m} \sum_{q=-\infty}^{\infty} \hat{h}_{jkkmq} \frac{1}{a_1 a_2 a_3} \exp \left\{ i \left[ (j-k)Q_1 + (k-m)Q_2 + q \right] \theta \right\}.
\]

(2.4.10)

It is now possible to introduce a second assumption:

The perturbing Hamiltonian may be restricted to its low-frequency part, which gives the slow but important variations of the variables \( a_j \).

This low-frequency part of \( U \) includes on the one hand all the terms with zero frequency, i.e. \( j = k, k = m, q = 0 \), and on the other hand the terms with very low frequency associated with the condition (2.4.8).

Providing the intersection of several resonance lines is not considered, the low-frequency part of the perturbing Hamiltonian (2.4.10) may be written as (Appendix 4)

\[
U(a_1, a_2, a_3, \theta) = \sum_{\nu=1}^{N/2} \sum_{q+s=\nu} \hat{h}_{qqss}(\nu) (a_1 a_2 a_3)^s +
\]

(2.4.11)

\[
+ \hat{h}_{jkkm-p} \frac{1}{a_1 a_2 a_3} \exp \left\{ i \left[ (j-k)Q_1 + (k-m)Q_2 - p \right] \theta \right\} +
\]

\[
+ \hat{h}_{jkkm+p} \frac{1}{a_1 a_2 a_3} \exp \left\{ -i \left[ (j-k)Q_1 + (k-m)Q_2 - p \right] \theta \right\}.
\]
with $\frac{N}{2} = \text{the smallest integer } \geq \frac{N}{2}$

$$h^{(N)}_{kjm,p} = h^{(N)}_{jkkm-p}$$

$$N = |n_1| + |n_2| ,$$

and where the indices $j$, $k$, $\ell$ and $m$ can take the following values (Appendix 4) depending on the signs of $n_1$ and $n_2$ (2.4.8):

**TABLE 3**

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$j$</th>
<th>$k$</th>
<th>$\ell$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>$</td>
<td>n_1</td>
<td>$</td>
<td>0</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>$</td>
<td>n_1</td>
<td>$</td>
<td>0</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>0</td>
<td>$</td>
<td>n_1</td>
<td>$</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>0</td>
<td>$</td>
<td>n_1</td>
<td>$</td>
</tr>
</tbody>
</table>

It is obvious, looking at (2.4.11), that the perturbing Hamiltonian is now characterized by two parameters, which are $h^{(2\nu)}_{\text{qqs0}}$ and $h^{(N)}_{jkkm-p}$. Using (2.4.5), (2.4.6), (2.4.7) and (2.4.9), these parameters can be expressed in terms of the field parameters $K_{x,z}$ and $F_{x,z}$:

$$h^{(2\nu)}_{\text{qqs0}} = \frac{1}{\pi^2 (v+1) R^\nu (q1s)^2} \int_0^{2\pi} \beta_h q(\theta) \beta_v (\theta) \left[ (-1)^{(s+1)} K_z^{(2\nu-1)} (\theta) + F_z^{(2\nu-1)} (\theta) \right] R \left( \frac{\alpha_h (\theta)}{\beta_h (\theta)} - \frac{\alpha_v (\theta)}{\beta_v (\theta)} \right) d\theta$$

(2.4.12)

$$\kappa = h^{(N)}_{jkkm-p} = \frac{1}{\pi^2 (N/2+1) R^N/2 |n_1|! |n_2|!} \int_0^{2\pi} \beta_h |n_1|^{1/2} (\theta) \beta_v |n_2|^{1/2} (\theta) \left[ \left( \frac{|n_2|+z}{2} \right)^{K_{N-1}} (\theta) + \right. $$

$$\left. \Re \left( \frac{\alpha_h (\theta)}{\beta_h (\theta)} - \frac{\alpha_v (\theta)}{\beta_v (\theta)} \right) \right] d\theta$$

$$\exp \left[ i \left[ n_1 \int_0^\theta \left( \frac{R}{\beta_h (\theta')} - Q_h \right) d\theta' + n_2 \int_0^\theta \left( \frac{R}{\beta_v (\theta')} - Q_v \right) d\theta' + p\theta \right] \right] d\theta$$

for $|n_2|$ even

(2.4.13)
\[ \kappa = h^{(N)}_{jkm-p} = \frac{1}{\pi_2^{(N/2+1)} \pi^{N/2} \left| n_1 \right|! \left| n_2 \right|!} \int_0^{2\pi} \left[ \frac{\left| n_1 \right|^{1/2}(\theta) \left| n_2 \right|^{1/2}(\theta)}{\kappa_{X}} \right]^{\left( N-1 \right)}(\theta) + \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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The distance $\Delta$ from the resonance comes from the phase in (2.4.11), in which the indices $j, k, \ell, m$ can only take the values given in Table 5. The other parameters are given in (2.4.12), (2.4.13) and (2.4.14) and the remaining indices $j, k, \ell, m$ must have the above mentioned values.

In general the equations (2.5.1) are not simple, as the amplitudes $a_j$ and their complex conjugates appear simultaneously on the right hand side. In order to avoid this difficulty, it is possible to change the variables by putting

$$
a_1 = r_1 e^{i\phi_1} \quad \bar{a}_1 = r_1 e^{-i\phi_1}\]
$$
$$
a_2 = r_2 e^{i\phi_2} \quad \bar{a}_2 = r_2 e^{-i\phi_2}.
$$

Introducing (2.5.2) in (2.5.1), the equations of the perturbed motion become

$$
\frac{dr_1}{dq} + ir_1 \frac{d\phi_1}{dq} = i \sum_{\nu=1}^{N/2} \sum_{q+s=\nu} q h_{qqss}^{(2\nu)} r_1^{(2\nu-1)} r_2^{2s} + i \kappa r_1^{(j+k-1)} r_2^{(k+m)} (\cos \psi + i \sin \psi) + i \kappa r_1^{(j+k-1)} r_2^{(k+m)} (\cos \psi - i \sin \psi),
$$

and something similar for $(r_2, \phi_2)$. The identity $\kappa = |\kappa| e^{i\phi_\kappa}$ was used and the phase $\psi$ is defined as follows:

$$
\psi = (j-k)\phi_1 + (j+k)m + \phi_\kappa + \theta \Delta. \quad (2.5.4)
$$

By virtue of the limited choice of the values for the indices $j, k, \ell, m$, we have

$$
j-k = n_1 \quad j+k = |n_1|\]
$$
$$
\ell-m = n_2 \quad \ell+m = |n_2|.
$$

Putting (2.5.5) in (2.5.3) and (2.5.4) and then dividing the equations (2.5.3) into real and imaginary parts, gives the equations for $(r_1, \phi_1, r_2, \phi_2)$:

---

*) The real amplitudes $r_1$ and $r_2$ defined in this paper are $\sqrt{2}$ times smaller than the amplitudes $r_1$ and $r_2$ used in Refs. 19 and 20.
\[
\frac{dr_1}{d\theta} = n_1 |\kappa| r_1^{|n_1|-1} r_2^{|n_2|} \sin \psi \\
\frac{dr_2}{d\theta} = n_2 |\kappa| r_1^{|n_1|} r_2^{|n_2|-1} \sin \psi \\
\frac{d\phi_1}{d\theta} = \sum_{\nu=1}^{N/2} \sum_{q+s=p} q h_{qss0}^{(2\nu)} r_2^{2(q-1)} r_1^{2s} + |n_1| |\kappa| r_1^{|n_1|-2} r_2^{|n_2|} \cos \psi \\
\frac{d\phi_2}{d\theta} = \sum_{\nu=1}^{N/2} \sum_{q+s=p} s h_{qss0}^{(2\nu)} r_1^{2q} r_2^{2(s-1)} + |n_2| |\kappa| r_1^{|n_1|-2} r_2^{|n_2|} \cos \psi ,
\]

with \( \psi = n_1\phi_1 + n_2\phi_2 + \phi + \theta \), \( \kappa \) and \( h_{qss0}^{(2\nu)} \) being defined in (2.4.12), (2.4.13) and (2.4.14).

The equations (2.5.6) are identical to those previously obtained\(^{19,21}\) for a two-dimensional field. The generalization made here for a three-dimensional field appears only in the definitions of \( h_{qss0}^{(2\nu)} \) and \( \kappa \). Also, the equations (2.5.6) are valid for both sum and difference resonances, which is not the case in Refs. 19 and 21.

5.2 Invariants of the perturbed motion

Combining the two first equations (2.5.6) gives

\[
\frac{n_2}{r_1} \frac{dr_1}{d\theta} - \frac{n_1}{r_2} \frac{dr_2}{d\theta} = 0 .
\]

(2.5.7)

It is also possible, starting from the equations (2.5.6), to write the Hamiltonian associated with the variables \((r_1^2, \phi_1, r_2^2, \phi_2)\):

\[
G(r_1^2, \phi_1, r_2^2, \phi_2, \theta) = \sum_{\nu=1}^{N/2} \sum_{q+s=p} h_{qss0}^{(2\nu)} (r_1^2)^q (r_2^2)^s + z|\kappa| (r_1^2)^{n_1/2} (r_2^2)^{n_2/2} \cos \psi ,
\]

(2.5.8)

remembering that \( r^2 \) is related to the emittance by

\[
r_1^2 = \frac{1}{2} \text{Re}_H \\
r_2^2 = \frac{1}{2} \text{Re}_V .
\]

(2.5.9)

To check the form of \( G \), it is only necessary to write the canonical equations (1.2.1) in the variables \((r_1^2, \phi_1, r_2^2, \phi_2)\) and to check them against (2.5.6).

The integration of (2.5.7) will give the first invariant, whereas the Hamiltonian (2.5.8) and the associated canonical equations will give the second invariant\(^{19}\).
\[ \frac{r_1^2}{n_1} - \frac{r_2^2}{n_2} = A \]  
\[ (Q_H - \frac{p}{n_1+n_2}) r_1^2 + (Q_V - \frac{p}{n_1+n_2}) r_2^2 + G = C. \]  
(2.5.10)

These motion invariants are valid for any sum and difference resonance of order \( N \) (2.4.8).
We find the same invariants as defined in Ref. 19, the differences being contained inside
the parameters \( h^{(2\nu)} \) and \( |\kappa| \) of the Hamiltonian \( G \) (2.5.8).

As was done in Refs. 19 and 20, it is easy to rewrite these invariants:

\[
\begin{align*}
A_1 &= r_2^2 - \frac{r_1^2}{n_1} \quad n_1 \neq 0 \\
C_1 &= r_1^2 \left( n_1 Q_H^* + n_2 Q_V^* - p \right) + n_1 G \\
A_2 &= r_1^2 - \frac{r_2^2}{n_2} \quad n_2 \neq 0 \\
C_2 &= r_2^2 \left( n_1 Q_H^* + n_2 Q_V^* - p \right) + n_2 G 
\end{align*}
\]  
(2.5.11)

the function \( G \) being given in (2.5.8) and the parameters contained in \( G \) being defined in
(2.4.12), (2.4.13) and (2.4.14).

5.3 Equations and invariants in the linear case

The linear case is associated with \( N = 2 \). Putting that particular value in (2.5.1)
gives the equations of the motion mentioned in Appendix 1. Depeding then on the choice
of \( n_1 \) and \( n_2 \), we will find exactly the equations described in Part 1 in (1.5.4), (1.5.7)
and (1.5.14). Finally, using the general expression (2.4.14) with \( N = 2 \), it is possible
to deduce again the relation (1.6.1) valid for the sum and difference resonances of
second order.

Concerning the monodimensional resonances of second order, one point should be noted.
Using (2.4.13) we now get :

\[
\kappa = -\frac{1}{8\pi R} \int_0^{2\pi} \beta_H(\theta) K_z^{(1)}(\theta) \exp \left\{ i \left[ 2 \int_0^{\theta} \left( \frac{R}{R(\theta')} - Q_H \right) d\theta' + p \theta \right] \right\} d\theta. \]  
(2.5.12)

This expression of \( \kappa \) is not identical to the one given in (1.5.6). The reason of this
is that we did not consider the term \( K_z^{(1)} \) in Part 1, so that only the second order term \( M^2 \)
was contributing to the monodimensional resonances.
It is interesting to write the invariants for the linear case, because they are not given in Part 1.

For \( 2Q_H = p \)
\[
\begin{align*}
\Delta + 2h^{(z)}_{11000} & \quad r_1^2 = \text{constant} \\
\Delta + 2h^{(z)}_{00110} & \quad r_1^2 + 4|\kappa| r_1^2 \cos \psi = \text{constant}
\end{align*}
\]

For \( 2Q_V = p \)
\[
\begin{align*}
\Delta + 2h^{(z)}_{00110} & \quad r_2^2 = \text{constant} \\
\Delta + 2h^{(z)}_{00110} & \quad r_2^2 + 4|\kappa| r_2^2 \cos \psi = \text{constant}
\end{align*}
\]

For \( Q_H + Q_V = p \)
\[
\begin{align*}
\Delta + h^{(z)}_{11000} & \quad r_2^2 - r_1^2 = \text{constant} \\
\Delta + h^{(z)}_{00110} & \quad r_1^2 + h^{(z)}_{00110} r_2^2 + 2|\kappa| r_1 r_2 \cos \psi = \text{constant} \\
\Delta + h^{(z)}_{00110} & \quad r_2^2 + h^{(z)}_{11000} r_1^2 + 2|\kappa| r_1 r_2 \cos \psi = \text{constant}
\end{align*}
\]

For \( Q_H - Q_V = p \)
\[
\begin{align*}
\Delta + h^{(z)}_{11000} & \quad r_2^2 + r_1^2 = \text{constant} \\
\Delta + h^{(z)}_{11000} & \quad r_1^2 + h^{(z)}_{00110} r_2^2 + 2|\kappa| r_1 r_2 \cos \psi = \text{constant} \\
\Delta - h^{(z)}_{00110} & \quad r_2^2 - h^{(z)}_{11000} r_1^2 - 2|\kappa| r_1 r_2 \cos \psi = \text{constant}
\end{align*}
\]

with \( \psi = n_1 \phi_1 + n_2 \phi_2 + \theta \Delta \), the other parameters being given in (2.4.12), (2.4.13) and (2.4.14).

It comes from (2.5.13) that the amplitudes are basically limited only in the case of the difference resonance, which is always stable as seen in Part 1 (Section 5.4). In the cases of the monodimensional resonances and of the sum resonance, the invariants (2.5.13) give the following stability criteria:

\[
\left| \frac{4|\kappa|}{\Delta + 2h^{(z)}_{11000}} \right| \leq 1 \quad \text{for } 2Q_H = p
\]

\[
\left| \frac{2|\kappa|}{\Delta + h^{(z)}_{11000} + h^{(z)}_{00110}} \right| \leq 1 \quad \text{for } Q_H + Q_V = p
\]

and something similar to the first one for \( 2Q_V = p \).

6. ANALYTICAL DESCRIPTION OF THE RESONANCES

6.1 Description of the sum resonances

The sum resonances \( (n_1, n_2 \geq 0) \) can be described by means of the resonance curves already defined in Refs. 19 and 20, and by the bandwidth, introduced in Ref. 19, which is directly associated with the resonance curves. These curves are deduced by analysing the second invariant of the perturbed motion. As the invariants (2.5.11), valid for
a three-dimensional magnetic field, have the same analytical form as those defined in Ref. 19 for a two-dimensional field, the results obtained in this reference can be taken in the present case.

A typical example of such curves for a sum resonance is given in Fig. 2. There are four different curves\(^{19}\) giving \(c_2\) as a function of \(x\), where

\[
c_2 = \frac{\Delta}{2|\kappa| \left( n_1^{(1-n_2/2)} n_2^{n_2/2} \left( \frac{RE_0}{2} \right)^{(N-2)/2} \right)} \quad \text{for } n_1 \neq 0
\]

\[
x = \frac{r_1}{r_{10}} \geq 1
\]

(2.6.1)

\[
c_2 = \frac{\Delta}{2|\kappa| \left( n_1^{n_1/2} n_2^{(1-n_1/2)} \left( \frac{RE_0}{2} \right)^{(N-2)/2} \right)} \quad \text{for } n_1 \neq 0
\]

\[
x = \frac{r_2}{r_{20}} \geq 1
\]

\(\Delta\) is the distance from the resonance, \(\epsilon_{H_0}\) and \(\epsilon_{V_0}\) are the initial emittances and \(x\) represents the amplitude blow-up. Thus these resonance curves limit the perturbed amplitudes except in the band \(\Delta c_2\) (Fig. 2). As \(c_2\) is proportional to \(\Delta\), the interval \(\Delta c_2\) defines a bandwidth inside which the amplitudes can become infinite. Reference 19 gives the expression of the bandwidth for a sum resonance as a function of \(|\kappa|\)*

\[
\Delta e = \Delta (n_1 Q_{H} + n_2 Q_{V} - p) \quad \text{with } n_1 \geq 0, n_2 \geq 0, n_1 + n_2 = N
\]

(2.6.2)

\[
\Delta e = 2|\kappa| \left( \frac{N}{2} \right)^{(N-2)/2} \epsilon_{H_0}^{(n_1-2)/2} \epsilon_{V_0}^{(n_2-2)/2} (n_1^2 \epsilon_{H_0}^2 + n_2^2 \epsilon_{V_0}^2),
\]

\(|\kappa|\) being now defined by (2.4.13) and (2.4.14)

\[
|\kappa| = \frac{R^{(2-N/2)}}{|BP| \left( \frac{2}{N/2} \right) n_1! n_2!} |d_p| \quad \text{for } n_2 \text{ even}
\]

(2.6.3)

\[
|\kappa| = \frac{R^{(2-N/2)}}{|BP| \left( \frac{2}{N/2} \right) n_1! n_2!} |f_p| \quad \text{for } n_2 \text{ odd}
\]

*) In order to be consistent with Ref. 19, the notation \(\Delta e\) has been used for the bandwidth. It should be clear to the reader where \(\Delta\) is used to indicate an increment and where \(\Delta = n_1 Q_{H} + n_2 Q_{V} - p\).
\[ c_2 \propto n_1 Q_H + n_2 Q_V - p \]

\[ c_2^{++} (1+\delta) \]

\[ \Delta c_2 \]

\[ x = \frac{r}{r_0} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \]

\[ x \gg 1 \]

\[ c_2^{++} \]

\[ c_2^{--} \]

\[ c_4, c_6, \ldots, c_2 \frac{N}{2} \text{ are zero} \]

\[ n_1 \text{ and } n_2 \text{ are positive} \]

Fig. 2 Resonance curves for sum resonances
with

\[ d_p = \frac{1}{2\pi} \int_0^{2\pi} \left( n_1 / \beta_H \right)^{n_1/2} \left( n_2 / \beta_V \right)^{n_2/2} \left( B_0 / R \right)^2 \left( \frac{n_2 + 1}{2} \right)^{K(N-1)} \left( -1 \right)^{n_2} \left( R F \right)^{K(N-1)} \left( n_1 \frac{\alpha_H}{\beta_H} - n_2 \frac{\alpha_V}{\beta_V} \right) \exp \left\{ i \left[ n_1 u_H + n_2 u_V - (n_1 Q_H + n_2 Q_V - p) \theta \right] \right\} d\theta \]

for \( n_2 \) even

\[ f_p = \frac{1}{2\pi} \int_0^{2\pi} \left( n_1 / \beta_H \right)^{n_1/2} \left( n_2 / \beta_V \right)^{n_2/2} \left( B_0 / R \right)^2 \left( \frac{n_2 + 1}{2} \right)^{K(N-1)} \left( -1 \right)^{n_2} \left( R F \right)^{K(N-1)} \left( n_1 \frac{\alpha_H}{\beta_H} - n_2 \frac{\alpha_V}{\beta_V} \right) \exp \left\{ i \left[ n_1 u_H + n_2 u_V - (n_1 Q_H + n_2 Q_V - p) \theta \right] \right\} d\theta \]

for \( n_2 \) odd.

These expressions for \( d_p \) and \( f_p \) generalize for a three-dimensional field, the expression given in Ref. 19 (p. 95).

In order to be complete, it is necessary to finally mention the equations\(^{19,21}\) giving the maximum amplitude growth due to oscillations in \( Q_H \) and \( Q_V \) across the resonance:

\[ \left( \frac{\varepsilon_{V_O}}{\varepsilon_{H_O}} \right)^{n_2/2} \int_{x_1}^{x_2} \frac{dx_1}{x_1^{(n_1-1)}} \left[ \frac{\varepsilon_{V_O}}{\varepsilon_{H_O}} + n_2 / n_1 (x_1^2 - 1) \right]^{n_1/2} \]

\[ = \pi \Delta \xi \]

\[ \left( n_1 + n_2 \frac{\varepsilon_{H_O}}{\varepsilon_{V_O}} \right) \left( n_1 |\Delta Q_{H*rev}| + n_2 |\Delta Q_{V*rev}| \right)^{1/2} \]

for \( n_1 \neq 0 \)

\[ \left( \frac{\varepsilon_{H_O}}{\varepsilon_{V_O}} \right)^{n_1/2} \int_{x_1}^{x_2} \frac{dx_2}{x_2^{(n_2-1)}} \left[ \frac{\varepsilon_{H_O}}{\varepsilon_{V_O}} + n_1 / n_2 (x_2^2 - 1) \right]^{n_2/2} \]

\[ = \pi \Delta \xi \]

\[ \left( n_2 + n_1 \frac{\varepsilon_{V_O}}{\varepsilon_{H_O}} \right) \left( n_2 |\Delta Q_{H*rev}| + n_1 |\Delta Q_{V*rev}| \right)^{1/2} \]

for \( n_2 \neq 0 \),

where \( \Delta Q_{rev} \) is the tune change in one revolution.

The expressions (2.6.5) assume that the change of the phase \( \psi \) in (2.5.6) is mainly due to the tune changes, i.e. that the trapping of particles does not occur. A criterion (2.6.2) in (2.6.5) gives the amplitude growth for a sum resonance excited by a three-dimensional magnetic field.
6.2 Description of the difference resonances

The difference resonances were not treated in detail in Refs. 19, 20 and 21. Therefore, it is necessary to analyse the invariants (2.5.11) in the case where \( n_1 \) and \( n_2 \) have opposite signs. This has been done (Appendix 5) and resonance curves have also been defined (Appendix 5). A typical example of such curves for a difference resonance is given in Fig. 3. As in Fig. 2, there are four different curves, but with quite different shapes. The definitions of \( c_2 \) and \( x \) are the same as in (2.6.1), but with the modulii of \( n_1 \) and \( n_2 \), and with other limits on \( x \) (Appendix 5).

\[
c_2 = \frac{\Delta}{2 |\kappa| |n_1| (1-|n_2|/2) |n_2|/2} \left( \frac{R_{n_2}}{2} \right)^{(N-2)/2} \quad \text{for } n_1 \neq 0
\]

\[
c_2 = \frac{\Delta}{2 |\kappa| |n_1|/2 |n_2| (1-|n_1|/2) \left( \frac{R_{n_1}}{2} \right)^{(N-2)/2}} \quad \text{for } n_2 \neq 0
\]

(2.6.6)

\[0 \leq x = \frac{\Sigma_1}{\Sigma_2} \leq \sqrt{1 + \frac{n_1}{n_2} \frac{\varepsilon_{v_2}}{\varepsilon_{n_2}}}
\]

\[0 \leq x = \frac{\Sigma_2}{\Sigma_1} \leq \sqrt{1 + \frac{n_2}{n_1} \frac{\varepsilon_{v_2}}{\varepsilon_{n_2}}}
\]

These limits on \( x \) result from the first invariant (2.5.11) (Appendix 5). Another consequence of the form of the first invariant is that the perturbed amplitudes are always limited by the resonance curves (Fig. 3), which was not the case for a sum resonance (Fig. 2). Although there is no interval inside which the amplitudes can become infinite, it is nevertheless possible to define bands \( \Delta c_2 \) inside which the amplitude variations can reach the limits \( x = 0 \) or \( x = x_{\text{max}} \) (see Fig. 3). We have thus two possibilities to define a bandwidth, either at \( x = 0 \) or \( x = x_{\text{max}} \). To avoid this difficulty, we can define a band which is the average of both (Fig. 3):

\[
\Delta c_2 = \frac{1}{2} \left[ c_2^{--}(x=0) - c_2^{++}(x=0) + c_2^{++}(x=x_{\text{max}}) - c_2^{--}(x=x_{\text{max}}) \right]
\]

\[
= c_2^{--}(x=0) + c_2^{++}(x=x_{\text{max}}^+).
\]

(2.6.7)

In this way a so-called bandwidth for difference resonances can be defined*):

\[
\Delta e = \Delta(n_1 \Omega_H + n_2 \Omega_V - p) \quad n_1 \geq 0 \quad n_2 \leq 0 \quad N = n_1 + |n_2| \quad \text{or} \quad n_1 \leq 0 \quad n_2 \geq 0 \quad N = |n_1| + n_2,
\]

(2.6.8)

with the following explicit expression (Appendix 5):

\[
\Delta e = 2 |\kappa| \left( \frac{R}{2} \right)^{(N-2)/2} \left( \frac{n_1}{2} \right)^{(N-2)/2} \left( \frac{n_2}{2} \right)^{(N-2)/2} \left( \frac{\varepsilon_{n_1}}{\varepsilon_{v_1}} + \frac{n_2}{\varepsilon_{n_2}} \right).
\]

(2.6.9)

* See footnote on p. 36.
\[ c_2 \propto n_1 Q_H + n_2 Q_Y - p \]

\[ c_2^{--}(1-\delta_1) = c_2^{--}(1+\delta_2) \]

\[ \Delta c_2(x=0) \]

\[ x = \frac{r}{r_0} = \sqrt{\frac{e}{E_0}} \]

\[ 0 \leq x \leq x_{max} \]

\[ c_4, c_6, \ldots, c_2 \bar{N}_2 \] are zero

\[ n_1 \text{ and } n_2 \text{ have opposed signs} \]

**Fig. 3** Resonance curves for difference resonances
\[ |\kappa| = \frac{R^{(2-N/2)}}{|B_p| \cdot \sqrt{2^{N/2}}|n_1| \times |n_2|} |d_p| \quad \text{for } n_2 \text{ even} \]

\[ |\kappa| = \frac{R^{(2-N/2)}}{|B_p| \cdot \sqrt{2^{N/2}}|n_1| \times |n_2|} |f_p| \quad \text{for } n_2 \text{ odd} , \]

with

\[ d_p = \frac{1}{2\pi} \int_0^{2\pi} \beta_H |n_1|/2 \beta_V |n_2|/2 \left| \frac{|B_p|}{R^2} \right| \left[ (-1)^{\frac{|n_2|+2}{2}} K_{z}^{(N-1)} + R_{F_z}^{(N-1)} \left| n_1 \frac{\alpha_H}{\beta_H} \right| - \left| n_2 \frac{\alpha_V}{\beta_V} \right| \right] - i R_{F_x}^{(N-1)} \left( \frac{n_1}{\beta_H} - \frac{n_2}{\beta_V} \right) \exp \left\{ i \left[ n_1 \nu_H + n_2 \nu_V - (n_1 Q_H + n_2 Q_V - p)\theta \right] \right\} d\theta \]

for \( n_2 \text{ even} \)

\[ f_p = \frac{1}{2\pi} \int_0^{2\pi} \beta_H |n_1|/2 \beta_V |n_2|/2 \left| \frac{|B_p|}{R^2} \right| \left[ (-1)^{\frac{|n_2|-1}{2}} K_{x}^{(N-1)} + R_{F_x}^{(N-1)} \left| n_1 \frac{\alpha_H}{\beta_H} \right| - \left| n_2 \frac{\alpha_V}{\beta_V} \right| \right] - i R_{F_x}^{(N-1)} \left( \frac{n_1}{\beta_H} - \frac{n_2}{\beta_V} \right) \exp \left\{ i \left[ n_1 \nu_H + n_2 \nu_V - (n_1 Q_H + n_2 Q_V - p)\theta \right] \right\} d\theta \]

for \( n_2 \text{ odd} . \)

These expressions of the bandwidth for difference resonances are very close to the expressions valid for the sum resonances. The only difference is in the factor \((|n_1| \epsilon_{\nu_0} + |n_2| \epsilon_{\beta_0})\), replacing the factor \((n_2^2 \epsilon_{\nu_0} + n_2^2 \epsilon_{\beta_0})\), so that for the same order of resonance the sum resonances have always larger bandwidths than the corresponding difference resonances. In any case, the bandwidth of difference resonances, as defined above, gives only the width of an interval surrounding the resonance, in which the perturbed amplitudes can reach the limits, which are always finite.

Let us finally compare the coefficient \( f_p \) in the case where \( n_1 = 1 \) and \( n_2 = \pm 1 \) with the linear coupling coefficients defined in Part 1 (1.6.2). Putting these values for \( n_1 \), \( n_2 \) and \( N = 2 \) in (2.6.11), and introducing the definitions (2.2.10) of \( K_{X}^{(1)} \) and \( F_{X}^{(1)} \), gives the following identity:

\[ f_p(n_1=1, n_2=\pm 1, N=2) = \frac{|B_p|}{R} C \quad . \]

C is dimensionless. Thus, for given transverse field gradients and longitudinal fields, C is smaller if the energy \((p_q = -eB_p)\) increases, but the coefficient \( f_p \) (2.6.12) is independent of the energy. In the general case, both coefficients \( d_p \) and \( f_p \) introduced in (2.6.4) and in (2.6.11) are independent of the energy for given three-dimensional magnetic fields, and they have the dimension of the field differentials.
6.3 Criterion concerning the distance of the working point from the resonance line

All the sum and difference resonances excited by a three-dimensional field are characterized by the total bandwidth defined either in (2.6.2) or in (2.6.9). This parameter gives a measure of the importance of a resonance and thus makes it possible to compare different resonances. Nevertheless, it seems interesting to formulate a criterion giving the distance of the working point \((Q_w, Q_v')\) from the resonance line, which has to be maintained during machine operation in order to prevent serious beam blow-up.

We will first treat the case of sum resonances with the assumption that \(c_s, c_\delta, ... c_{2(N/2)}\) are negligible. Looking at Fig. 2, it is obvious that the amplitude growth \(x\) is strictly limited by the curve \(c_2^-\) (assuming \(c_2^x = 0\)), if the distance \(\Delta c_2\) from the resonance line \((c_2 = 0)\) is large enough. The criterion becomes: in order that \(x\) does not exceed \(1 + \delta\), the distance \(\Delta c_2\) from the resonance has to stay larger or equal to \(c_2^-\) \((x = 1 + \delta)\). The analytical calculation has been made (Appendix 6) and the two following results have been found:

\[
\delta e_1 \geq \Delta e \left( \frac{1}{2} + \frac{1}{\delta} \frac{n_1 e V_o}{n_1^2 e V_o + n_2^2 e H_0} \right),
\]  
(2.6.13)

with \(\frac{n_1}{n_1^0} \leq 1 + \delta\).

\[
\delta e_2 \geq \Delta e \left( \frac{1}{2} + \frac{1}{\delta} \frac{n_2 e H_0}{n_1^2 e V_o + n_2^2 e H_0} \right),
\]  
(2.6.14)

with \(\frac{n_2}{n_2^0} \leq 1 + \delta\),

where \(\delta > 0; \delta\) small

\(\Delta e\) is the bandwidth for a sum resonance (2.6.2).

In order to have a unique criterion, it is possible to say that the distance \(\delta e\) from the resonance has to be larger than the average of both \(\delta e_1\) and \(\delta e_2\). Thus \(\delta e\) is given by

\[
\delta e \geq \frac{\Delta e}{2} \left( \frac{1}{2} + \frac{1}{\delta} \frac{n_1 e V_o + n_2 e H_0}{n_1^2 e V_o + n_2^2 e H_0} \right),
\]  
(2.6.15)

with \(\frac{r}{r_0} \leq 1 + \delta;  \delta > 0, \delta\) small; \(n_1, n_2 > 0\).

Let us now consider the case of difference resonances with the same assumption that \(c_s, c_\delta, ... c_{2(N/2)}\) are negligible. Looking at Fig. 3 it is clear that the amplitude oscillation \(x\) is strictly limited by the curves \(c_2^-\) and \(c_2^+\), if the distance \(\Delta c_2\) is large enough. The criterion now becomes: in order that \(x\) stays in the interval \([1 - \delta_1, 1 + \delta_2]\), the distance \(\Delta c_2\) has to be larger or equal to \(c_2^-\) \((x = 1 + \delta_2) = c_2^+\) \((x = 1 - \delta_1)\). The analytical calculation has been made (Appendix 6) and the following two results have been found:
\[ \delta e_1 > \frac{\Delta e}{|n_1|e_{v_o} + |n_2|e_{v_H}} \left[ \frac{|n_1|e_{v_H}}{\delta} - \frac{1}{2} \left( n_1^2 e_{v_o} - n_2^2 e_{v_H} \right) \right], \quad (2.6.16) \]

with \( 1 - \delta_1 \leq \frac{r_1}{r_{10}} \leq 1 + \delta_2 \);

\[ \delta e_2 > \frac{\Delta e}{|n_1|e_{v_o} + |n_2|e_{v_H}} \left[ \frac{|n_2|e_{v_H}}{\delta} - \frac{1}{2} \left( n_2^2 e_{v_H} - n_1^2 e_{v_o} \right) \right], \quad (2.6.17) \]

with \( 1 - \delta_1 \leq \frac{r_2}{r_{20}} \leq 1 + \delta_2 \),

where \( \delta_1 + \delta_2 = 2\delta; \quad \delta_1, \delta_2 \geq 0; \quad \delta_1, \delta_2 \) small

\( \delta e \) is the bandwidth for a difference resonance (2.6.9).

In order to have a unique criterion, it is again possible to take the average of \( \delta e_1 \) and \( \delta e_2 \), which, in this case, gives a simple result:

\[ \delta e \geq \frac{1}{2} \frac{\Delta e}{\delta}, \quad (2.6.18) \]

with \( 1 - \delta_1 \leq \frac{r_2}{r_{20}} \leq 1 + \delta_2 \),

where \( \delta_1 + \delta_2 = 2\delta; \quad \delta_1, \delta_2 \geq 0; \quad \delta_1, \delta_2 \) small

\( n_1 \geq 0, n_2 \leq 0 \) or \( n_1 \leq 0, n_2 \geq 0 \).

We should note here that the tolerance \( \delta \) on the amplitude is associated with the motion of one isolated particle.

7. EXTENSION OF THE PRECEDING THEORY WITH A GENERAL APPROACH TO THE FIELD DESCRIPTION

7.1 General approach to the three-dimensional magnetic field description

The constraints on the magnetic field are given in (2.2.1) and (2.2.3):

\[ \text{div } \vec{B} = 0 \]

\[ \text{rot } \vec{B} = 0 \quad (\text{in free-space}). \quad (2.7.1) \]

In order to satisfy the second constraint, it is necessary and sufficient \(^{22}\) that \( \vec{B} = \vec{\text{grad}} S \). In general, the scalar potential \( S \) can be written as follows:

\[ S = - \sum_N \frac{B_0}{R_N} \sum_{k_1, k_2} \binom{N}{k_1, k_2} d_N^{(N)}(0) x^{k_1} z^{k_2}. \quad (2.7.2) \]
Hence $S$ is a polynomial development in $x, z$ with coefficients $d$ which are functions of $\theta$. The question is now to satisfy the first constraint (2.7.1), which means

$$\text{div } B = \text{div grad } S = 0 . \quad (2.7.3)$$

Putting (2.7.2) in (2.7.3) gives the following recurrence relation for the coefficients $d^{(N)}$:

$$R^2 d^{(N)}_{k_1+2, k_2} + R^2 d^{(N)}_{k_1, k_2+2} + d''^{(N)}_{k_1 k_2} = 0 . \quad (2.7.4)$$

The next problem is to find relations between the coefficients $d^{(N)}(\theta)$ and the field derivatives. The magnetic field components are developed in series in (2.2.8). Alternatively it is possible to write the expressions for the field components by taking the gradient of $S$ (2.7.2). The comparison of these expressions for the components give

$$d^{(N)}_{k_1+1, k_2}(\theta) = \frac{R}{|B_\theta|} \frac{(N-k_2-1)!: k_2!}{(N-1)!} \left. \frac{\partial^{(N-1)} B_x}{\partial x^{k_1} \partial z^{k_2}} \right|_{x=z=0}$$

$$d^{(N)}_{k_1, k_2+1}(\theta) = \frac{R}{|B_\theta|} \frac{(N-k_2-1)!: k_2!}{(N-1)!} \left. \frac{\partial^{(N-1)} B_z}{\partial x^{k_1} \partial z^{k_2}} \right|_{x=z=0} \quad (2.7.5)$$

$$d'_{k_1 k_2}^{(N)}(\theta) = \frac{R^2}{|B_\theta|} \frac{(N-k_2)! k_2!}{N!} \left. \frac{\partial'^N B_\theta}{\partial x^{k_1} \partial z^{k_2}} \right|_{x=z=0} .$$

By equating the first two relations (2.7.5), it is possible to show the following dependence between the field derivatives:

$$\left. \frac{\partial^{(N-1)} B_x}{\partial x^{k_1} \partial z^{k_2+1}(k_2+1)} \right|_{x=z=0} = \frac{(N-k_2-1)!}{(k_2+1)!} \left. \frac{\partial^{(N-1)} B_z}{\partial x^{k_1+1} \partial z^{k_2}} \right|_{x=z=0} \quad (2.7.6)$$

In the first sections, it was found to be necessary to have the vector potential $\vec{A}$ in order to calculate the Hamiltonian $H$. $\vec{A}$ is related to the scalar potential $S$ by

$$\text{rot } \vec{A} = \text{grad } S . \quad (2.7.7)$$

Also writing the vector $\vec{A}$ as a polynomial development we obtain...
\[ A_x = \frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \sum_{k_1, k_2} \binom{N-1}{k_2} a_{k_1 k_2}^{(N-1)} x^{k_1 k_2} \]

\[ A_z = \frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \sum_{k_1, k_2} \binom{N-1}{k_2} b_{k_1 k_2}^{(N-1)} x^{k_1 k_2} \]  
\[ (2.7.8) \]

\[ A_\theta = \frac{B_0}{R^2} \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{k_1, k_2} \binom{N}{k_2} c_{k_1 k_2}^{(N)} x^{k_1 k_2} . \]

Putting (2.7.8) in (2.7.7) gives the following constraints on the coefficients \( a, b, c \), which are functions of \( \theta \):

\[ b_{k_1 k_2}^{(N-1)} - c_{k_1 k_2+1}^{(N)} = - R \frac{d^{(N)}}{d k_1 + 1, k_2} \]

\[ a_{k_1 k_2+1}^{(N-1)} - b_{k_1+1, k_2}^{(N-1)} = - \frac{1}{R} \frac{d^{(N-2)}}{d k_1 k_2} \]  
\[ (2.7.9) \]

\[ c_{k_1+1, k_2}^{(N-1)} - a_{k_1 k_2}^{(N-1)} \] = - R \frac{d^{(N)}}{d k_1, k_2+1} .

It is easy to see that the relations (2.7.9) are not independent because of the recurrence condition (2.7.4). In other words, it is only possible to use two of the equations (2.7.9). Let us choose the last two in order to eliminate the coefficients \( b \) and \( c \) in the vector potential (2.7.8):

\[ A_x = \frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \sum_{k_1, k_2} \binom{N-1}{k_2} a_{k_1 k_2}^{(N-1)} x^{k_1 k_2} \]  
\[ (2.7.10) \]

\[ A_z = \frac{B_0}{R} \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \sum_{k_1, k_2} \binom{N-1}{k_2} \left[ a_{k_1-1, k_2+1}^{(N-1)} + \frac{1}{R} \frac{d^{(N-2)}}{d k_1-1, k_2} \right] x^{k_1 k_2} \]

\[ A_\theta = \frac{B_0}{R^2} \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{k_1, k_2} \binom{N}{k_2} \left[ a_{k_1-1, k_2}^{(N-1)} - R \frac{d^{(N)}}{d k_1-1, k_2+1} \right] x^{k_1 k_2} . \]
This is the general form of the vector potential to be compared with (2.2.7). The relations (2.7.10) satisfy the constraints (2.7.1) for any functions $a^{(N)}(\theta)$. Hence the vector potential is not fully determined by the field in free-space where (2.7.1) is true. The field is indeed not affected by the so-called gauge transformation

$$
\hat{A}_1 = \hat{A} - \text{grad} f,
$$

(2.7.11)

where $f$ is an arbitrary function of the coordinates.

The Lorentz condition restricts the gauge and is valid in the whole volume, equally where the equations are inhomogeneous, since the fields sought are those excited by the current distributions and, in general, by the magnetization distributions associated with the iron. The Lorentz gauge condition finally gives the following equations \(^{23}\) for static fields:

$$
\text{div} \, \hat{A} = 0
$$

(2.7.12)

$$
\text{div} \, \text{grad} \, \hat{A} = - \mu_0 (\hat{j} + \text{rot} \, \hat{M}) = - \mu_0 \hat{j},
$$

where $\hat{j}$ is the current density, and

$$
\hat{M} = \chi_m (H) \hat{H},
$$

(2.7.13)

with $\chi_m$ = magnetic susceptibility, which is a function of $\hat{H}$, where $\hat{H}$ is given by

$$
\hat{H}(\mathbf{r}) = \int \frac{I(r) \, d\hat{k} \times (\hat{k} - r)}{4\pi |\hat{k} - r|^3}.
$$

(2.7.14)

The magnetic field intensity $\hat{H}$ given in (2.7.14) is that produced by a current line with the following definitions: $I =$ local current in the line, $d\hat{k} =$ infinitesimal vector tangential to the line, $\hat{k} =$ radius vector for the point where $\hat{H}$ is calculated and $r =$ radius vector for the point on the current line where $d\hat{k}$ is defined.

The generalized current density $\hat{J}$ introduced in (2.7.12) satisfies the continuity equation \(^{21}\), which becomes for constant charge density:

$$
\text{div} \, \hat{J} = \text{div} \, (\hat{j} + \text{rot} \, \hat{M}) = 0.
$$

(2.7.15)

### 7.2 Discussion of the consequences of the Lorentz gauge condition

The Lorentz gauge condition should give us the possibility to define unequivocally the vector potential $\hat{A}$, i.e. to choose the functions $a^{(N-1)}$ appearing in (2.7.10).

The first consequence of (2.7.12) comes from $\text{div} \, \hat{A} = 0$. Using (2.7.10) in this constraint gives a recurrence relation for the coefficients $a^{(N-1)}$

$$
R^2 a^{(N-1)}_{k_1 + 2, k_2} + R^2 a^{(N-1)}_{k_1, k_2 + 2} + a^{(N-3)}_{k_1, k_2} = 0,
$$

(2.7.16)
which is equivalent to the one established for the \( d \)'s in (2.7.4). Nevertheless a recurrence relation is not enough for determining entirely the functions \( a^{(N-1)}(3) \). In order to do this, it is necessary to find some new relation between the \( \vec{A} \) components.

The second consequence of (2.7.12) is to furnish such a relation. If we consider the second relation (2.7.12) in the volume where the generalized current density \( \vec{J} \) exists, we can deduce from the properties of the current a relation between the \( \vec{A} \) components. As the vector \( \vec{A} \) is continuous, this relation is still true in the volume where \( \vec{J} \) is zero and where \( \vec{A} \) has the form (2.7.10). The explicit solution of (2.7.12) for \( \vec{A} \) shows\(^{23}\) that each element of current \( d\vec{J} \) gives a contribution \( d\vec{A} \) to the vector potential which is parallel to \( d\vec{J} \). The direct consequence of this is that if one of the components of \( \vec{J} \) is zero in the whole volume, the corresponding component of \( \vec{A} \) will also be zero.

Let us illustrate the last remarks with examples. Considering first the case of a two-dimensional field, let us show how we can derive the known results\(^{19,24}\). In the middle of an infinite long magnet, we have

\[
J_x = J_z = 0 \quad J_\theta \neq 0.
\]  
(2.7.17)

The consequence of (2.7.17) is that \( A_x = A_z = 0 \). Applying this result in (2.7.10), we get

\[
da^{(N-1)}_{k_1 k_2}(\theta) = 0 \quad d^{(N-2)}_{k_1-1, k_2}(\theta) = 0,
\]  
(2.7.18)

which means that \( \delta^{(N)}_{k_1 k_2} = 0 \). Using this last relation in (2.7.4) gives the well known multipole coefficients

\[
da^{(N)}_{k_1 k_2} = \begin{cases} 
\frac{k_x}{2} \, \frac{1}{R} \, k^{(N-1)}_x(\theta) & \text{for } k_2 \text{ even} \\
\frac{k_z}{2} \, \frac{1}{R} \, k^{(N-1)}_z(\theta) & \text{for } k_2 \text{ odd}.
\end{cases}
\]  
(2.7.19)

Let us now consider a solenoid, which has a cylindrical symmetry. Then, using (2.7.15), we get the following relations:

\[
J_\theta \equiv 0 \quad \frac{\partial J_x}{\partial x} = -\frac{\partial J_z}{\partial z}.
\]  
(2.7.20)

The second relation (2.7.20) gives an analogous relation for \( \nabla \cdot \vec{A} \) and \( \nabla \times \vec{A} \) by virtue of (2.7.12). Knowing the form of \( \vec{A} \) (2.7.10), we can write

\[
da^{(N-1)}_{k_1 k_2} = -a^{(N-1)}_{k_1-2, k_2+2} - \frac{1}{R} \, d^{(N-2)}_{k_1-2, k_2+1}.
\]  
(2.7.21)
Hence (2.7.21) fully determines the vector potential $\vec{A}$. If we now assume that $a_{k_1^2,k_2^2} = a_{k_1-2,k_2+2}$, we find again the form of $\vec{A}$ given in Section 2. Indeed (2.7.21) becomes

$$a_{k_1,k_2}^{(N-1)} = -\frac{1}{2\pi} d_{k_1,k_2}^{(N-2)}.$$  

(2.7.22)

Comparing then (2.7.10) with (2.2.7), it is easy to verify that

$$d_{k_1,k_2}^{(N-2)} = \begin{cases} + 2 R \left( N-1 \right) \left( \theta \right) & \text{for } k_2 \text{ even} \\ + 2 R \left( N-1 \right) \left( \theta \right) & \text{for } k_2 \text{ odd} \end{cases}$$

(2.7.23)

$$d_{k_1+1,k_2-1}^{(N-2)} - d_{k_1-1,k_2+1}^{(N-2)} = \begin{cases} k_2 + \frac{2}{R} \left( N-1 \right) \left( \theta \right) & \text{for } k_2 \text{ even} \\ k_2 - \frac{2}{R} \left( N-1 \right) \left( \theta \right) & \text{for } k_2 \text{ odd} \end{cases} \times (-1)^{k_2}.$$

(2.7.24)

Hence, the Lorentz gauge condition gives the possibility of choosing the correct vector $\vec{A}$. In a given specific case, the necessary relation between the $\vec{A}$ components has to be worked out using (2.7.12), (2.7.13) and (2.7.14). This can nevertheless be difficult, because the magnetization distributions have to be considered. Let us note that a simulation computer program can be used for complex magnets in order to fit $\vec{A}_x (2.7.10)$ and thus to get the functions $a(\theta)$, using also the recurrence (2.7.16).

7.3 Consequent perturbing Hamiltonian

The perturbing Hamiltonian has been given in (2.4.2). It is now necessary to introduce for $\vec{A}$ the general expressions (2.7.10). Following exactly what has been done in Section 4 and in Appendix 4, it is sufficient to give here the new form of the coefficients $h_{jk\ell m}^{(N)}$ introduced in (2.4.4):

$$h_{jk\ell m}^{(N)} (\theta) = \sum_{j} \frac{1}{k!} \frac{1}{m!} \left\{ \left[ a_{j+1,k-1,\ell+m}^{(N-1)} - R d_{j+1,k-1,\ell+m+1}^{(N)} \right] u^{j} \hat{u}^{k} \hat{v}^{\ell} \overline{v}^{m} + \right.$$  

$$+ j a_{j+1,k-1,\ell+m}^{(N-1)} u^{j-1} \hat{u}^{k} \hat{v}^{\ell} \overline{v}^{m} (u' + iQ_u u) +$$

$$\left. + k a_{j+1,k-1,\ell+m}^{(N-1)} u^{j} \hat{u}^{k} \hat{v}^{\ell} \overline{v}^{m} (u' - iQ_u u) + \right.$$  

$$\left. + \ell\left[ a_{j+1,k-1,\ell+m}^{(N-1)} + \frac{1}{R} d_{j+1,k-1,\ell+m-1}^{(N-2)} \right] u^{j} \hat{u}^{k} \hat{v}^{\ell} \overline{v}^{m+1} (u' + iQ_u v) + \right.$$  

$$\left. + \left[ a_{j+1,k-1,\ell+m}^{(N-1)} + \frac{1}{R} d_{j+1,k-1,\ell+m-1}^{(N-2)} \right] u^{j} \hat{u}^{k} \hat{v}^{\ell} \overline{v}^{m} (v' - iQ_v \overline{v}) \right\}.$$  

(2.7.25)

valid for any $j$, $k$, $\ell$, $m$, but with $j+k+\ell+m = N$. 
Still following Section 4, the perturbing Hamiltonian associated with a specific resonance is characterized by two parameters $h_{q_0s_0}^{(2v)}$ and $\kappa$ introduced in (2.4.12), (2.4.13) and (2.4.14). The new form of these parameters is

$$h_{q_0s_0}^{(2v)} = \frac{1}{\pi^{(q_0s)!}} \int_0^{2\pi} d\theta \left\{ \frac{a_{2q-1,2s}^{(2v-1)}(\theta) - R d_{2q-1,2s}^{(2v)}(\theta)}{a_{2q-1,2s}^{(2v-1)}(\theta) - R d_{2q-1,2s}^{(2v)}(\theta)} - 2R \left( a_{2q-1,2s}^{(2v-1)}(\theta) + \frac{d_{2q-1,2s}^{(2v-2)}(\theta)}{R} \right) \alpha_1(\theta) \beta_d(\theta) \right\}$$

$$\kappa = h_{j_{km-n}}^{(N)} = \frac{1}{(2\pi R)^{N/2} |n_1|! |n_2|!} \int_0^{2\pi} d\theta \left\{ a_{|n_1|-1,|n_2|}^{(N-1)}(\theta) - R d_{|n_1|-1,|n_2|}^{(N)}(\theta) \right\}$$

$$\exp \left\{ i \int_0^{2\pi} d\theta \left[ \frac{R}{\beta_H(\theta)} - C_H \right] + n_2 \int_0^{2\pi} d\theta \left[ \frac{R}{\beta_V(\theta)} - C_V \right] \right\}$$

These more general definitions of $h_{q_0s_0}^{(2v)}$ and $\kappa$ can be directly used in the analysis of the perturbed motion (Section 5) and in the analytical description of the resonances (Section 6). All the results of these sections remain fully valid.

8. APPLICATION TO THE CERN STORAGE RINGS

This general theory for any order resonances can be applied to the detector solenoid referred to in Part 1 and which will be installed in the ISR. It is effectively possible to calculate for any sum and difference resonance the coefficients $d_r$ and $f_p$ (2.6.4) and (2.6.11), the bandwidth (2.6.2) and (2.6.9) and the necessary distance of the working point from the resonance line (2.6.15) and (2.6.18), provided that the coefficients of the vector potential for the detector solenoid are known. Given the values of these parameters, it is easy to compare them with the values measured and calculated for the existing fields in the ISR, and to see if the resonance excitation due to this solenoid is important or not.
There are two ways of obtaining the coefficients of the vector potential. The first one consists of using a computer program calculating the magnetic field inside a given volume, the characteristics of the magnetic element being known. As the detector solenoid has end plates with slots, there is no cylindrical symmetry and the computer program must be able to work in three dimensions. Since such a general program does not yet exist at CERN\textsuperscript{\textdagger}, only the second possibility, which is to measure the magnetic field of the solenoid, is really available. Thus, in order to calculate the higher order effects of the solenoid, it is necessary to wait until the field has been mapped.

It is nevertheless interesting to show how it is possible to use in the linear case the results obtained in Sections 5 and 6, the principles being identical for higher orders.

Let us treat again the case of the difference resonance \( Q_H - Q_V = 0 \). From Part 1 we know that the ISR have at present a linear coupling coefficient \( C \):

\[
|C| \cong 2 \times 10^{-1} .
\]

Thus the bandwidth of this difference resonance (2.6.9) is

\[
\Delta e = |C| \left( \frac{\varepsilon_{V_0}}{\varepsilon_{H_0}} + \sqrt{\frac{\varepsilon_{H_0}}{\varepsilon_{V_0}} + 1} \right) \cong 4.3 \times 10^{-3} ,
\]

if we assume \( \varepsilon_{H_0}/\varepsilon_{V_0} \cong 2.25 \).

The ISR working lines are designed for \( |Q_H - Q_V| \cong 10^{-2} \). Thus, using the relation (2.6.18) gives the tolerance \( \delta \) on the amplitude changes:

\[
\delta = \frac{\Delta e}{2 \delta e} = \frac{4.3 \times 10^{-3}}{2 \times 10^{-2}} \cong 0.21 .
\]

This indicates that the actual setting of the working lines correspond to a maximum amplitude change of \( \sim 20 \% \) for a single particle. The maximum change in the beam height is much smaller, because it is necessary to average on the distributed particles and on the sinusoidal variation of the actual amplitude.

Another interesting application of the theory comes from the use of the invariants (2.5.11). In the case of \( Q_H - Q_V = 0 \), these invariants are given in (2.5.13). Assuming the zero harmonics of the coefficients \( h(\varphi) \) are negligible and putting the extreme values \( \pm 1 \) for \( \cos \varphi \), we get

\[
r_1^2 + r_2^2 = \text{constant}
\]

\[
r_1^2 \Delta \pm |C| r_1 r_2 = \text{constant}
\]

\[
r_2^2 \Delta \pm |C| r_1 r_2 = \text{constant} .
\]

\textsuperscript{\textdagger}The three-dimensional magnet computation program GRUND3D from Rutherford Laboratory has since become available at CERN.
Let us now apply the relations (2.8.4) to the case where the coherent oscillation of a kicked pulse is measured. Kicking a pulse in the horizontal plane and looking at the Q-filter outputs gives the following signals (Fig. 4): at the time of the kick \( t = 0 \), the vertical amplitude is zero \( (r_2 = 0) \) and the horizontal amplitude is maximum \( (r_1 = r_{1,\max}) \), later \( (t > 0) \) both the amplitudes oscillate and the horizontal amplitude reaches a minimum \( (r_1 = r_{1,\min}) \), when \( r_2 \) is maximum. Let us introduce the initial conditions in (2.8.4):

\[
\begin{align*}
    r_1^2 + r_2^2 &= r_{1,\max}^2 \\
    r_1^2 \Delta + |C| r_1 r_2 &= r_{1,\max}^2 \Delta \\
    r_2^2 \Delta + |C| r_1 r_2 &= 0
\end{align*}
\]  

(2.8.5)

Eliminating \( r_1 \) or \( r_2 \) with the first invariant, we get the following two relations:

\[
\begin{align*}
    \sqrt{r_{1,\max}^2 - r_1^2} \Delta - |C| r_1 &= 0 \\
    r_2 \Delta - |C| \sqrt{r_{1,\max}^2 - r_2^2} &= 0
\end{align*}
\]  

(2.8.6)

As the extreme values of \( \cos \psi \) were taken in (2.8.4), \( r_1 \) and \( r_2 \) correspond to the extreme values \( r_{1,\min} \) and \( r_{2,\max} \), respectively (see Fig. 4). The solutions of the relations (2.8.6) are consequently

\[
\begin{align*}
    \frac{r_{1,\min}}{r_{1,\max}} &= \frac{|\Delta|}{\sqrt{\Delta^2 + |C|^2}} \\
    \frac{r_{2,\max}}{r_{1,\max}} &= \frac{|C|}{\sqrt{\Delta^2 + |C|^2}}
\end{align*}
\]  

(2.8.7)

Looking at Fig. 4, it is obvious that the three parameters \( r_{1,\min} \), \( r_{1,\max} \) and \( r_{2,\max} \) appear directly on the Q-filter outputs. Thus, the relations (2.8.7) give two methods for measuring \( |C| \), provided that \( \Delta = Q_x - Q_y \) is known (e.g. by calibration of one set of quadrupoles, see Ref. 24). The first one consists of looking at the Q-filter output from the plane in which the kick was applied (H-plane in our example) and then to use the first relation in (2.8.7). The second one is to look at the Q-filter output from both planes and then to use the second relation in (2.8.7) involving \( r_{1,\max} \) and \( r_{2,\max} \) (Fig. 4). For the second method it is nevertheless possible to look at the vertical output only, since on the diagonal \( (\Delta = 0) \) the second relation in (2.8.7) gives \( r_{2,\max} = r_{1,\max} \). The first method requires only one measurement at any \( \Delta \neq 0 \), whereas the second method needs two measurements, one at any \( \Delta \neq 0 \) and one at \( \Delta = 0 \).
Fig. 4 Q-filter outputs for coupling measurements at the ISR
During coupling measurements made in the ISR in May 1975\(^{13}\), pictures of the vertical filter output only have been taken. This restricts us to using the second method described above. The measurements were made for different values of \(|\Delta|\). Three examples of the obtained results are given in Fig. 5. The points shown are the direct measurements of \(r_{z,\text{max}}\) as a function of \(|\Delta|\) and also the calculated values of \(r_{z,\text{max}}\) using (2.8.7) in which we put the value of \(|C|\) deduced from a best fit to the measurements. The agreement between measured and calculated values is fairly good, and the values obtained for \(|C|\) are the following:

\[
\begin{align*}
R1 & \quad 26 \text{ GeV/c} \quad (8C26) \quad |C| = (2.1 \pm 0.3) \times 10^{-3} \\
R2 & \quad 26 \text{ GeV/c} \quad (8C26) \\
& \quad \text{TF at -100 %} \\
& \quad \text{Model solenoid OFF} \quad |C| = (6.0 \pm 0.35) \times 10^{-3} \quad (2.8.8) \\
R2 & \quad 26 \text{ GeV/c} \quad (8C26) \\
& \quad \text{TF at -100 %} \\
& \quad \text{Model solenoid ON} \quad |C| = (6.0 \pm 0.35) \times 10^{-3} .
\end{align*}
\]

TF is the notation for a set of ISR quadrupoles used here in order to increase the form factor, 8C26 indicates the working line used\(^3\), and the model solenoid has already been mentioned in Part 1. Looking at the value of \(|C|\) for the model solenoid (Table 2 of Part 1) with both end plates (0.30 \(10^{-3}\)), it is not surprising that \(|C|\) apparently does not change when the solenoid is switched on.

The coupling measurement using the amplitude of the signals (Fig. 5) seems more accurate than the measurement using the period of the signals\(^{14}\).

9. **CONCLUSIONS**

An analytical formalism for the treatment of non-linear perturbations of the transverse motions in a three-dimensional magnetic field has been developed. This formalism enabled us to give the general form of the perturbing Hamiltonian associated with a resonance \(n_1 Q_1 + n_2 Q_2 = p\) of any order, excited by a three-dimensional field. General expressions for the equations and for the invariants of the perturbed motion were deduced. Bandwidths for all sum and difference resonances have been defined. Other general complex parameters like \(\kappa, d_p^p, f_p^p\) characterizing the perturbed motions in a three-dimensional field were also introduced. All these parameters make it possible to analyse the effects of transverse and longitudinal fields for a machine with any form factor. This is of use when introducing new elements for comparison with the existing machine. Other possible applications of this theory are the formulation of a criterion for the distance of the working point from the resonance line and the use of the motion invariants for measuring the amplitude of \(\kappa\).

Finally looking at the relation of this theory with the work already existing\(^{18,19,20,21}\) where the sum resonances excited by a two-dimensional magnetic field were treated, it appears that this is a generalization to the case of a three-dimensional magnetic field for difference resonances as well as sum resonances.
1. R1 on BC.26
   Best fit gives:
   \[ C = (2.1 \pm 0.3) \times 10^{-3} \]

2. R2 on BC.26
   Test solenoid at 100 %
   TF at -100 %
   End plates on
   Best fit gives:
   \[ C = (6.0 \pm 0.5) \times 10^{-3} \]

3. R2 on BC.26
   Test solenoid OFF
   TF at -100 %
   Best fit gives:
   \[ C = (6.0 \pm 0.35) \times 10^{-3} \]

Fig. 5 Coupling measurement in the ISR.
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*) All ISR Performance Reports have been referred to as Private communication.
The Hamiltonian of the perturbation due to linear coupling

The relation (1.3.14) gives the form of the Hamiltonian of the perturbation due to linear coupling. In order to have the expression of $H_1$ as function of $(a_j, \theta)$, we have to put the expressions (1.4.2) in (1.3.14). Doing so we get:

$$U(a_1, a_2, a_3, \theta) = \sum_{j, k, l, m}^{2} h^{(z)}_{jklm} a^j_a^1 a^k_a^2 a^l_a^3 a^m_a^3 \exp \left\{ i \left[ (j-k)Q_H + (l-m)Q_V \right] \theta \right\}, \quad \text{(Al.1)}$$

with the following definitions:

$$h^{(z)}_{2000} = \frac{M^2}{8} \bar{u}^2 \quad h^{(z)}_{0200} = \frac{M^2}{8} \bar{v}^2$$

$$h^{(z)}_{0200} = \frac{M^2}{8} \bar{u}^2 \quad h^{(z)}_{0002} = \frac{M^2}{8} \bar{v}^2$$

$$h^{(z)}_{1100} = \frac{M^2}{4} \bar{u} \bar{v} \quad h^{(z)}_{0011} = \frac{M^2}{4} \bar{v} \bar{v} \quad \text{(Al.2)}$$

$$h^{(z)}_{1010} = K \bar{u} \bar{v} + \frac{M}{2} \left[ \bar{u} (v' + iQ_v) - v (u' + iQ_u) \right]$$

$$h^{(z)}_{0101} = K \bar{u} \bar{v} + \frac{M}{2} \left[ \bar{u} (v' + iQ_v) - v (u' + iQ_u) \right]$$

$$h^{(z)}_{1001} = K \bar{u} \bar{v} + \frac{M}{2} \left[ \bar{u} (v' + iQ_v) - v (u' + iQ_u) \right]$$

$$h^{(z)}_{0010} = K \bar{u} \bar{v} + \frac{M}{2} \left[ \bar{u} (v' + iQ_v) - v (u' + iQ_u) \right],$$

$u$ and $v$ being the Floquet's functions (1.3.4).

As the synchrotron is a periodic machine, it is possible to develop the function $U$ in Fourier's series or, which is equivalent, the coefficients $h^{(z)}_{jklm}$:

$$h^{(z)}_{jklm}(\theta) = \sum_{q=-\infty}^{+\infty} h^{(z)}_{jklmq} e^{iq \theta}, \quad \text{(Al.3)}$$

with $h^{(z)}_{jklmq} = \frac{1}{2\pi} \int_{0}^{2\pi} h^{(z)}_{jklm}(\theta) e^{-iq \theta} d\theta$. 
Putting (A1.3) in (A1.1) will give the expression (1.5.2).

As we have seen in Section 3, we have to keep the low-frequency part of the Hamiltonian function \( U \) corresponding to the resonance we want to look at:

\[
n_1 Q_{1H} + n_2 Q_{1V} = p,
\]

\[n_1, n_2, p \text{ being integers and, in the present case, } |n_1| + |n_2| = 2.
\]

This low-frequency part of \( U \) can be written as follows:

\[
U(a_j, \theta) = h^{(2)}(a_1 a_1, a_2 a_2) + h^{(2)}(a_1 a_1, a_2 a_2) e^{i(n_1 Q_{1H} + n_2 Q_{1V} - \pi) \theta} + h^{(2)}(a_1 a_1, a_2 a_2) e^{i(n_1 Q_{1H} + n_2 Q_{1V} - \pi) \theta} + h^{(2)}(a_1 a_1, a_2 a_2) e^{i(n_1 Q_{1H} + n_2 Q_{1V} - \pi) \theta},
\]

where \( j, k, \ell, m \) can take the following values:

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( j )</th>
<th>( k )</th>
<th>( \ell )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The form of \( U \) as given in (A1.5) and (A1.6) is just adequate for being used in the motion equations (1.4.6). For example, the equation for \( a_1 \) is:

\[
\frac{da_1}{d\theta} = i h^{(2)}(a_1) + i k h^{(2)}(a_1, a_2) e^{i(n_1 Q_{1H} + n_2 Q_{1V} - \pi) \theta} + i j h^{(2)}(a_1, a_2) e^{-i(n_1 Q_{1H} + n_2 Q_{1V} - \pi) \theta}.
\]

For simplifying, we will define the parameter \( \kappa \):

\[
\kappa = h^{(2)}(a_1 a_1, a_2 a_2),
\]

with

\[
\kappa = \frac{1}{2\pi} \int_{0}^{2\pi} h^{(2)}(a_1 a_1, a_2 a_2) e^{ip\theta} d\theta.
\]

These last two relations are used in Sections 5.2, 5.3 and 5.4.
APPENDIX 2

Solutions of the equations for the resonances

\[ Q_H + Q_V = p \quad \text{and} \quad Q_H - Q_V = p, \quad \text{valid for all the terms of the Hamiltonian} \]

In Section 5, solutions are given with the assumption that the \( M^2 \) terms in the Hamiltonian (1.3.14) can be neglected. But explicit solutions can be established without this assumption.

Let us begin with the resonance \( Q_H + Q_V = p \). Using the motion equations (A1.7) and keeping now all the terms, the explicit equations are

\[
\begin{align*}
\frac{da_1}{d\theta} = & \quad i \lambda_1 a_1 + i \kappa a_2 e^{-i\theta \Delta} \\
\frac{da_2}{d\theta} = & \quad i \lambda_2 a_2 + i \kappa a_1 e^{-i\theta \Delta},
\end{align*}
\]

(A2.1)

with \( \kappa = h^{(2)}_{1010-p} \) (see (1.5.12))

\[
\lambda_1 = h^{(2)}_{11000} = \frac{1}{16\pi R} \int_0^{2\pi} \beta_H M^2 d\theta
\]

\[
\lambda_2 = h^{(2)}_{00110} = \frac{1}{16\pi R} \int_0^{2\pi} \beta_V M^2 d\theta.
\]

(A2.2)

The contribution of the \( M^2 \) terms in the Hamiltonian is given by \( \lambda_1 \) and \( \lambda_2 \), which are real coefficients. In order to solve (A2.1), we make the following change in the variables:

\[
a_1 = b_1 e^{i\lambda_1 \theta} \quad a_2 = b_2 e^{i\lambda_2 \theta}.
\]

(A2.3)

Putting (A2.3), which is the solution of (A2.1) for \( \kappa = 0 \), in (A2.1) gives the equations for \( b_1 \) and \( b_2 \)

\[
\frac{db_1}{d\theta} = i \kappa b_2 e^{-i(\Delta + \lambda_1 + \lambda_2) \theta}
\]

\[
\frac{db_2}{d\theta} = i \kappa b_1 e^{-i(\Delta + \lambda_1 + \lambda_2) \theta}.
\]

(A2.4)

Equations (A2.4) have exactly the same form as (1.5.7). Hence the solutions can be written by analogy...
\[ b_1 = \frac{1}{\kappa} \left( \frac{\Lambda_+}{\omega_+} i \omega \theta + \frac{\Lambda_-}{\omega_-} i \omega \theta \right) \]
\[ b_2 = \frac{1}{\Lambda_+} i \omega \theta \frac{\omega_+}{e} + \frac{1}{\Lambda_-} i \omega \theta \frac{\omega_-}{e}, \]

with \[ \omega_{\pm} = -\frac{\Delta + \lambda_1 + \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta + \lambda_1 + \lambda_2}{2}\right)^2 - |\kappa|^2}. \]

Using (A2.3), the explicit solutions for \[ Q_H + Q_V = p \] are
\[ a_1 = \frac{1}{\kappa} \left( \frac{\Lambda_+}{\omega_+} i \omega \theta + \frac{\Lambda_-}{\omega_-} i \omega \theta \right) \]
\[ a_2 = \frac{1}{\Lambda_+} i \omega \theta \frac{\omega_+}{e} + \frac{1}{\Lambda_-} i \omega \theta \frac{\omega_-}{e}, \]

with \[ \omega_{1\pm} = -\frac{\Delta - \lambda_1 + \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta + \lambda_1 + \lambda_2}{2}\right)^2 - |\kappa|^2} \]
\[ \omega_{2\pm} = -\frac{\Delta + \lambda_1 - \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta + \lambda_1 + \lambda_2}{2}\right)^2 - |\kappa|^2}. \]

Let us now consider the resonance \[ Q_H - Q_V = p. \] Appendix 1 again gives the explicit equations for all the terms of the Hamiltonian
\[ \frac{d}{d\theta} a_1 = i \lambda_1 a_1 + i \kappa a_2 e^{-i\Delta} \]
\[ \frac{d}{d\theta} a_2 = i \lambda_2 a_2 + i \kappa a_1 e^{i\Delta}, \]

with \[ \kappa = h^{(2)}_{1001-p} \] (see (1.5.18))
\[ \lambda_1, \lambda_2 \] have already been given in (A2.2).

The same change (A2.3) in the variables gives the following equations:
\[ \frac{d}{d\theta} b_1 = i \kappa b_2 e^{-i(\Delta + \lambda_1 - \lambda_2) \theta} \]
\[ \frac{d}{d\theta} b_2 = i \kappa b_1 e^{i(\Delta + \lambda_1 - \lambda_2) \theta}. \]

These equations have the same form as (1.5.14) and the solutions can be directly written:
\[
\begin{align*}
\alpha_1 &= \frac{\lambda_+}{\omega_+} \left( \frac{\lambda_1}{\omega_1} e^{i \omega_1 \theta} + \frac{\lambda_2}{\omega_2} e^{i \omega_2 \theta} \right) \\
\alpha_2 &= \lambda_+ e^{-i \omega_1 \theta} + \lambda_- e^{-i \omega_2 \theta},
\end{align*}
\]  
(A2.9)

with \(\omega_\pm = \frac{\Delta + \lambda_1 - \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta + \lambda_1 - \lambda_2}{2}\right)^2 + |\kappa|^2}\).

Using (A2.3), the explicit solutions for \(Q_H - Q_V = p\) are

\[
\begin{align*}
\alpha_1 &= \frac{\lambda_+}{\omega_+} \left( \frac{\lambda_1}{\omega_1} e^{i \omega_1 \theta} + \frac{\lambda_2}{\omega_2} e^{i \omega_2 \theta} \right) \\
\alpha_2 &= \lambda_+ e^{i \omega_1 \theta} + \lambda_- e^{i \omega_2 \theta},
\end{align*}
\]  
(A2.10)

with \(\omega_{1z} = -\frac{\Delta - \lambda_1 - \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta - \lambda_1 - \lambda_2}{2}\right)^2 + |\kappa|^2}\)

\(\omega_{2z} = -\frac{\Delta + \lambda_1 + \lambda_2}{2} \pm \sqrt{\left(\frac{\Delta + \lambda_1 + \lambda_2}{2}\right)^2 + |\kappa|^2}\).

Both results, (A2.6) and (A2.10), show that the inclusion of the \(N^2\) terms from the Hamiltonian only slightly modifies the frequencies of the two modes and that this effect is different from the horizontal and vertical planes.
APPENDIX 3

Analytical treatment of the three-dimensional magnetic field

Starting from the simplified form of $\mathbf{A}$ given in (2.2.4), the field components can be written as follows:

\[
B_x = \frac{B_0}{R^2} \left[ \frac{3P_x}{3y} - \frac{3D}{3z} \right]
\]

\[
B_\theta = \frac{B_0}{R} \left[ \frac{3P_x}{3z} - \frac{3P_\theta}{3x} \right]
\]  \hspace{1cm} (A3.1)

\[
B_z = -\frac{B_0}{R^2} \left[ \frac{3P_x}{3\theta} - \frac{3D}{3x} \right]
\]

As (A3.1) is the explicit form of $\mathbf{B} = \nabla \times \mathbf{A}$, the relation $\text{div} \mathbf{B} = 0$ is automatically verified (div $\nabla \times \mathbf{A} = 0$). The other relation which has to be true is

\[
\omega_0 = \nabla \times \mathbf{B} = 0.
\]  \hspace{1cm} (A3.2)

As we are firstly interested by the $(x,z)$ dependence of $P_x$ and $P_z$, let us consider the $J_\theta$ component. Using (A3.1) in (A3.2) gives the following relation for $J_\theta$:

\[
\omega_0 J_\theta = \frac{B_0}{R^2} \left[ \frac{3^2 P_z}{3z^2} - \frac{3^2 D}{3x^2} - \frac{3^2 P_x}{3x^2} - \frac{3^2 D}{3x^2} \right] = 0.
\]  \hspace{1cm} (A3.3)

Assuming that $D(x,z,\theta)$ is the polynomial development corresponding to the standard two-dimensional multipole terms\(^1\),\(^2\) given explicitly in (2.2.6), the next identity is true:

\[
-\frac{3^2 D}{3z^2} = \frac{3^2 D}{3x^2}.
\]  \hspace{1cm} (A3.4)

Both (A3.3) and (A3.4) imply

\[
\frac{3^2 P_z}{3z^2} = -\frac{3^2 P_x}{3x^2}.
\]  \hspace{1cm} (A3.5)

As a consequence of (A3.5), $P_x$ and $P_z$ are the derivatives of one polynomial $P_\theta$:

\[
P_x = \frac{3P_\theta}{3x}, \quad P_z = -\frac{3P_\theta}{3z}.
\]  \hspace{1cm} (A3.6)

The negative sign associated with $P_z$ has been chosen so that the $B_\theta$ component in (A3.1) is not zero. The equations (A3.6) and (A3.5) give

\[
\frac{3^2 P_\theta}{3z^2} = \frac{3^2 P_\theta}{3x^2}.
\]  \hspace{1cm} (A3.7)
The comparison of (A3.7) with (A3.4) shows that $P_0$ cannot have the same form as $D$. Nevertheless, it is possible to build the polynomial $P_0$, using the same principles applied in building the polynomial $D$. In order to show that, let us write $D$ in another form giving explicitly its properties:

$$D = -\sum_{N=2}^{\infty} \frac{1}{N!} \left[ k_z^{(N-1)} w_1^{(N)} - k_x^{(N-1)} w_2^{(N)} \right],$$

(A3.8)

where $w_1^{(N)} = \text{even powers in } z \text{ of } (x + iz)^N$  

$\text{and } w_2^{(N)} = \text{odd powers in } z \text{ of } (x + iz)^N$.

The following properties can be easily established:

$$\frac{\partial w_1^{(N)}}{\partial x} = N w_1^{(N-1)} \quad \frac{\partial w_2^{(N)}}{\partial x} = N w_2^{(N-1)}$$

(A3.9)

$$\frac{\partial w_1^{(N)}}{\partial z} = -N w_1^{(N-1)} \quad \frac{\partial w_2^{(N)}}{\partial z} = N w_1^{(N-1)}.$$

The unique negative sign appearing in (A3.9) is necessary in order to satisfy (A3.4). Hence, $P_0$ must have a form equivalent to (A3.8), but with functions like $w_1$ and $w_2$ satisfying relations similar to (A3.9) without this negative sign. Such polynomial $P_0$ can be written as follows:

$$P_0 = -\sum_{N=2}^{\infty} \frac{1}{N!} \left[ F_z^{(N-1)} t_1^{(N)} + F_x^{(N-1)} t_2^{(N)} \right],$$

(A3.10)

where $t_1^{(N)} = \text{even powers in } z \text{ of } (x + z)^N$  

$t_2^{(N)} = \text{odd powers in } z \text{ of } (x + z)^N$.

The following relations can be easily established:

$$\frac{\partial t_1^{(N)}}{\partial x} = N t_1^{(N-1)} \quad \frac{\partial t_2^{(N)}}{\partial x} = N t_2^{(N-1)},$$

$$\frac{\partial t_1^{(N)}}{\partial z} = N t_1^{(N-1)} \quad \frac{\partial t_2^{(N)}}{\partial z} = N t_1^{(N-1)}.$$

(A3.11)

By virtue of (A3.11), the polynomial $P_0$ satisfies (A3.7). Putting (A3.10) in (A3.6) and using the properties (A3.11) will give the explicit form of $P_x$ and $P_z$. Having $D$ (A3.8) and $P_x$ and $P_z$, it is possible to give the potential vector $A$ (2.2.7).

So far only the relation $J_\theta = 0$ (A3.2) has been used in order to determine the $(x, z)$ dependence of $P_x$ and $P_z$. It is now interesting to use the other relations $J_\chi = 0$ and $J_z = 0$ in order to see what are the restrictions imposed on the dependence with $\theta$ of the coefficients $k_x$, $k_z$ (A3.8) and $F_x$, $F_z$ (A3.10).
Starting from $D$, $P_x$, and $P_z$, it is possible to write the field components (A3.1) and then the vector $\mathbf{J}$ as a function of $K_x$, $K_z$, $F_x$ and $F_z$. Putting $J_x = 0$ and $J_z = 0$ gives the following equations:

\[
\begin{align*}
2R^2 F_z(N-1) + F_z''(N-3) & \quad t_1^{(N-5)} - K_z' (N-3) w_1^{(N-5)} = 0 \\
2R^2 F_x(N-1) + F_x''(N-3) & \quad t_2^{(N-5)} + K_x' (N-3) w_2^{(N-5)} = 0 \\
2R^2 F_z(N-1) + F_z''(N-3) & \quad t_2^{(N-3)} - K_z' (N-3) w_2^{(N-3)} = 0 \\
2R^2 F_x(N-1) + F_x''(N-3) & \quad t_1^{(N-3)} - K_x' (N-3) w_1^{(N-3)} = 0 .
\end{align*}
\] (A3.12)

As $t_1^{(n)} \neq w_1^{(n)}$ and $t_2^{(n)} \neq w_2^{(n)}$ for any $n > 1$, (A3.12) implies the following:

\[
\begin{align*}
\text{For } N \geq 5 & \quad K_z' (N-3) = K_x' (N-3) = 0 \\
F_z''(N-3) & = -2R^2 F_z(N-1) \\
F_x''(N-3) & = -2R^2 F_x(N-1)
\end{align*}
\] (A3.13)

\[
\begin{align*}
\text{For } N = 4 & \quad K_x'(1) = 0 \\
F_z''(1) & = -2R^2 F_z(3) + K_z'(1) \\
F_x''(1) & = -2R^2 F_x(3) .
\end{align*}
\] (A3.14)

Equation (A3.14) implies for the linear coupling theory that $K' = 0$ and $M'' = 0$ (see (1.3.6) in Part 1 and (2.2.10) in Part 2).

Having the vector $\mathbf{A}$ as a function of the parameters $K_x$, $K_z$, $F_x$, $F_z$, it is then necessary to find relations between these parameters and the magnetic field derivatives. In general, the field components can be developed as was done in (2.2.8). Alternatively it is possible to use (A3.1) and the expressions (2.2.6) for $D$ and $P_0$:

\[
B_x = \frac{B_0}{R^2} \sum_{n=1}^{\infty} \frac{1}{n!} \left[ -F_x(n) w_1(n) - F_z(n) w_2(n) + F_x' (n) t_1^{(n)} + F_z' (n) t_2^{(n)} \right] .
\] (A3.15)

Similar expressions for $B_z$ and $B_\theta$ can be written. Comparing all these expressions with (2.2.8) gives
\[ B_\theta(x=z=0) = -2 \frac{B_\theta}{R} f^{(1)}_x \]

\[ \frac{\partial^m B_\theta}{\partial x^{k_1} \partial z^{k_2}} = \begin{cases} 
-2 \frac{B_\theta}{R} \binom{m}{k_2} f^{(m+1)}_x & k_2 \text{ even} \\
-2 \frac{B_\theta}{R} \binom{m}{k_2} f^{(m+1)}_z & k_2 \text{ odd} 
\end{cases} \]

\[ \frac{\partial^m B_x}{\partial x^{k_1} \partial z^{k_2}} = \begin{cases} 
-\frac{B_\theta}{R^2} \binom{m}{k_2} \left[ (-1)^{k_2/2} k_x^m - f'_x^{(m)} \right] & k_2 \text{ even} \\
-\frac{B_\theta}{R^2} \binom{m}{k_2} \left[ (-1)^{(k_2-1)/2} k_z^m - f'_z^{(m)} \right] & k_2 \text{ odd} 
\end{cases} \]  
(A3.16)

\[ \frac{\partial^m B_z}{\partial x^{k_1} \partial z^{k_2}} = \begin{cases} 
-\frac{B_\theta}{R^2} \binom{m}{k_2} \left[ (-1)^{k_2/2} k_x^m - f'_x^{(m)} \right] & k_2 \text{ even} \\
-\frac{B_\theta}{R^2} \binom{m}{k_2} \left[ (-1)^{(k_2+1)/2} k_z^m - f'_z^{(m)} \right] & k_2 \text{ odd} 
\end{cases} \]

with \( \binom{m}{k_2} = \frac{m!}{(m-k_2)!k_2!} \), \( m = k_1 + k_2 \geq 1 \).

Here \( B_\theta \) is considered as being negative (Fig. 1). Taking the modulus of \( B_\theta \), we get the equations (2.2.9). It is obvious from (A3.16) that \( f^{(1)}_z \) does not appear in the field expansion. Thus the choice of \( f^{(1)}_z \) is free and the simplest one is \( f^{(1)}_z = 0 \).

In the case where \( N = 2 \), it is very easy to solve the relations (A3.16) for \( k_x^{(1)} \) and \( k_z^{(1)} \), and this had been done in Section 2. For any \( N \geq 3 \) it is also easy to solve the first relations (A3.16) for \( f^{(N-1)}_x \) and \( f^{(N-1)}_z \). Solving (A3.16) for \( k_x^{(N-1)} \) is rather more complicated. We have to choose a value of \( k_2 \leq N-2 \). Depending on the parity of \( k_2 \), the relations (A3.16) will give \( k_x^{(N-1)} \) or \( k_z^{(N-1)} \). Let us take for instance \( k_2 \) even:

\[ \frac{R^2}{|B_\theta|} \frac{\partial (N-1) k_x^{(N-1)}}{\partial x^{(N-k_2-1)} \partial z^{k_2}} = \binom{N-1}{k_2} \left[ (-1)^{k_2/2} k_x^{(N-1)} - f'_x^{(N-1)} \right]. \]  
(A3.17)

Then \((k_2+1)\) is necessarily odd, so that we get:

\[ \frac{R^2}{|B_\theta|} \frac{\partial (N-1) k_z^{(N-1)}}{\partial x^{(N-k_2-2)} \partial z^{(k_2+1)}} = -\frac{N-k_2-1}{k_2+1} \binom{N-1}{k_2} \left[ (-1)^{k_2/2} k_z^{(N-1)} + f'_x^{(N-1)} \right]. \]  
(A3.18)

Multiplying (A3.17) by \(-\frac{N-k_2-1}{k_2+1}\) and adding the result to (A3.18) gives something proportional to \( k_z^{(N-1)} \). The explicit result is written in (2.2.13). If we then take \( k_2 \) odd and we repeat the same calculation, we get \( k_z^{(N-1)} \) written in (2.2.14).
APPENDIX 4

Analytical treatment of the perturbing Hamiltonian

The perturbing Hamiltonian $H_1$ is written in (2.4.2) and the potential vector components are given in (2.2.7). In the expressions (2.2.7) appear the polynomials $(x + iz)^N$ and $(x + z)^N$. Using Newton's formula for the coefficients of these polynomials and putting (2.2.7) in (2.4.2) gives for $H_1$

$$H_1 = \sum_{N=2}^{\infty} H_1^{(N)}$$

$$H_1^{(N)} = \sum_{k_1, k_2, k_3, k_4} V^0_{k_1 k_2} x^{k_1} z^{k_2} + \sum_{k_3, k_4} V^x_{k_3 k_4} x^{k_3} z^{k_4} p_x +$$

$$+ \sum_{k_5, k_6, k_7, k_8} V^z_{k_5 k_6} x^{k_5} z^{k_6} p_z ,$$

(A4.1)

where

$$V^0_{k_1 k_2} = \begin{cases} \frac{-1}{N!} (-1)^{k_2/2} \binom{N}{k_2} k_2^{(N-1)} & \text{if } k_2 \text{ even} \\ \frac{-1}{N!} (-1)^{(k_2+1)/2} \binom{N}{k_2} k_2^{(N-1)} & \text{if } k_2 \text{ odd} \end{cases}$$

$$V^x_{k_3 k_4} = \begin{cases} \frac{1}{(N-1)!} \binom{N-1}{k_4} f^{(N-1)}_z & \text{if } k_4 \text{ even} \\ \frac{1}{(N-1)!} \binom{N-1}{k_4} f^{(N-1)}_x & \text{if } k_4 \text{ odd} \end{cases}$$

$$V^z_{k_5 k_6} = \begin{cases} \frac{1}{(N-1)!} \binom{N-1}{k_6} f^{(N-1)}_x & \text{if } k_6 \text{ even} \\ \frac{1}{(N-1)!} \binom{N-1}{k_6} f^{(N-1)}_z & \text{if } k_6 \text{ odd} . \end{cases}$$

The Hamiltonian $H_1$ has to be expressed as a function of constants $a_j$ of the unperturbed motion using (1.4.2). In these relations $x$ and $z$ are sums of two terms, so that we get polynomials in (A4.1) for $x^{k_1}$, $z^{k_2}$, ....... Using again Newton's formula and putting (1.4.2) in (A4.1) the perturbing Hamiltonian becomes
\[ H_1^{(N)} = u^{(N)}(a_j) = \sum_{j,k,\ell,m}^{N} V_{k_1 k_2}^{\theta} (j+k)(\ell+m) \begin{pmatrix} \ell-m \\ k \end{pmatrix} \begin{pmatrix} j+k \\ a_1 a_2 \end{pmatrix} \begin{pmatrix} \ell-m \\ u v \end{pmatrix} \exp \left\{ i \left[ (j-k)Q_H \theta + (\ell-m)Q_V \theta \right] \right\} + \]

\[ + \sum_{j',k',\ell',m'}^{N-1} V_{k_3 k_4}^{\theta} (j'+k')(\ell'+m') \begin{pmatrix} \ell'-m' \\ k' \end{pmatrix} \begin{pmatrix} j'+k' \\ a_1 a_2 \end{pmatrix} \begin{pmatrix} \ell'-m' \\ u v \end{pmatrix} \exp \left\{ i \left[ (j'-k')Q_H \theta + (\ell'-m')Q_V \theta \right] \right\} + \]

\[ \left[ a_1 (u'+iQ_H) e^{iQ_H \theta} \right] \exp \left\{ i \left[ (j'-k')Q_H \theta + (\ell'-m')Q_V \theta \right] \right\} + \]

\[ + \sum_{j'',k'',\ell'',m''}^{N-1} V_{k_5 k_6}^{\theta} (j''+k'')(\ell''+m'') \begin{pmatrix} \ell''+m'' \\ k'' \end{pmatrix} \begin{pmatrix} j''+k'' \\ a_1 a_2 \end{pmatrix} \begin{pmatrix} \ell''+m'' \\ u v \end{pmatrix} \exp \left\{ i \left[ (j''-k'')Q_H \theta + (\ell''-m'')Q_V \theta \right] \right\} + \]

where c.c. means complex conjugate.

It is possible to simplify considerably the expression \((A4.2)\). The first sum clearly contains terms of power \(N\) in \(a_j\) and the introduction of the explicit form of \(p_x\) and \(p_z\) \((1.4.3)\) in the two other sums gives also terms of power \(N\) in \(a_j\). This means that it is possible to group the three sums in one. A second simplification comes from the introduction of the equations \((A4.1)\) for the \(V\) functions in \((A4.2)\), which removes most of the factorials. Doing these operations finally gives the form \((2.4.4)\) where the coefficients \(h_{jk\ell m}^{(N)}(\theta)\) have been introduced:

\[ U^{(N)}(a_j,\theta) = \sum_{j,k,\ell,m}^{N} h_{jk\ell m}^{(N)}(\theta) \begin{pmatrix} j+k \ell m \\ a_1 a_2 a_2 \end{pmatrix} \exp \left\{ i \left[ (j-k)Q_H + (\ell-m)Q_V \right] \theta \right\}. \]

\[ \boxed{ (A4.3) } \]

As mentioned in Section 4 of Part 2, it is necessary to analyse the frequency spectrum of \(H_1\), in order to determine the excitation of a resonance. The Fourier development of \(H_1\) gives \((2.4.10)\)

\[ U(a_j,\theta) = \sum_{N=2}^{\infty} \sum_{j,k,\ell,m}^{N} \sum_{b=0}^{\infty} h_{jk\ell m b}^{(N)} \begin{pmatrix} j+k \ell m \\ a_1 a_2 a_2 \end{pmatrix} \exp \left\{ i \left[ (j-k)Q_H + (\ell-m)Q_V + b \right] \theta \right\}. \]

\[ \boxed{ (A4.4) } \]

The second assumption made in Section 4 of Part 2 implies that we look for the low-frequency part of \(U\). The condition to be verified in this case is the following:
(j-k)Q_H + (k-m)Q_V + b = 0 .  \tag{A4.5}

Clearly the first terms we have to keep are all the zero harmonics associated with any \( N \geq 2 \)

\[ b = 0 \quad j = k \quad \kappa = m . \tag{A4.6} \]

Putting \( j = q \) and \( \kappa = s \), we finish with the following zero harmonics for a given \( v = q + s \):

\[ \sum_{q,s} h^{(2v)}_{qqss0} (a_1 a_1) q (a_2 a_2) s . \tag{A4.7} \]

The other terms at low frequency are associated with an isolated resonance line defined by

\[ n_1 Q_H + n_2 Q_V = p . \tag{A4.8} \]

The condition (A4.5) becomes on the resonance line

\[ (j-k)Q_H + (k-m)Q_V + b = \pm (n_1 Q_H + n_2 Q_V - p) = 0 , \tag{A4.9} \]

where \( j, k, \kappa, m \) are integers \( \geq 0 \)

\( n_1, n_2, p \) are integers (with any sign).

Thus (A4.9) gives

\[ b = -p \quad (j-k) = n_1 \quad (k-m) = n_2 \tag{A4.10} \]

and similar relations with opposed signs.

The values of \( j, k, \kappa \) and \( m \) clearly depend upon the signs of \( n_1 \) and \( n_2 \):

\[
\begin{align*}
\text{if } n_1 \geq 0 \quad &j = n_1 ; \quad \text{if } n_2 \geq 0 \quad &\kappa = n_2 ; \\
\text{if } n_1 \leq 0 \quad &k = |n_1| ; \quad \text{if } n_2 \leq 0 \quad &m = |n_2| .
\end{align*} \tag{A4.11}
\]

The moduli are necessary since \( j, k, \kappa \) and \( m \) are always positive. The possible values for indices \( j, k, \kappa \) and \( m \) are summarized in Table 3 (Section 4 of Part 2).

The two signs in (A4.9) give rise to two terms in the Hamiltonian \( U \) (A4.4) which are associated with the conditions in (A4.10):

\[
\begin{align*}
\hbar^{(N)} &j_k\kappa_m \quad a_1 a_1 a_2 a_2 \exp \left\{ i \left[ (j-k)Q_H + (k-m)Q_V - p \right] \theta \right\} + \\
\hbar^{(N)} &k_j\kappa_m \quad a_1 a_1 a_2 a_2 \exp \left\{ -i \left[ (j-k)Q_H + (k-m)Q_V - p \right] \theta \right\} ,
\end{align*} \tag{A4.12}
\]

where \( j, k, \kappa \) and \( m \) take the values given in Table 3, calculated from (A4.11).

Taking all the zero harmonics (A4.7), from \( v = 1 \) until \( v \) is close to \( N/2 \) (\( N \) being the order of the resonance), and the two low-frequency terms (A4.12), which are associated with the resonance, gives the low-frequency part of \( U \) as written in (2.4.11).
Invariants and resonance curves for difference resonances

Here we analyse the invariants for difference resonances, following the same treatment as made in Ref. 19 for the sum resonances. The second invariant (C₁ or C₂) in (2.5.11) can be written as follows:

\[ k_2 r^2 + k_4 r^4 + \ldots + k_{2N-1} r^{2(N-2)} + \cos \varphi k_{2N-1} r |n| (E-r^2)^{N-n}/2 = C, \quad (A5.1) \]

where \( r \) stands for \( r_1 \) or \( r_2 \),
\( |n| \) stands for \( |n_1| \) or \( |n_2| \), \( n_1 \) and \( n_2 \) having opposed signs,
\( N/2 \) = the smallest integer \( \geq N/2 \),
\( k_{2N-1} \) = 2 \( |\varphi| \) \( |n| \) \( N-n|/n \),
\( E \) = \( A \frac{|n|}{N-n} \), \( A \) being the first invariant (A₁ or A₂) in (2.5.11),
\( k_j \) = parameters depending on the \( h^{(2\varphi)} \) coefficients,
\( k_2 \) is linearly depending on the distance \( \Delta \) from the resonance.

The extreme values of the amplitude \( r \) are given by the solution of the following two equations:

\[ F_2(k_2, r) = k_2 r^2 + k_4 r^4 + \ldots + k_{2(N-2)} r^{2(N-2)} + k_{2N-1} r |n| (E-r^2)^{N-n}/2 = C, \quad (A5.2) \]

where it is always possible to assume \( k_{2(N-2)} \geq 0 \) and \( k_{2N-1} \geq 0 \).

For \( r_0 \) being fixed, the extreme values of the constant \( C \) are given by the two quantities \( F_2(k_2, r_0) \). Thus, for known initial conditions, the extreme values of \( r \) are determined by four equations:

\[ F_{++} = F_+(k_2, r_0) - F_+(k_2, r) = 0, \quad (A5.3) \]

and three others giving \( F'_+, F'_- \) and \( F''_- \).

It is useful to introduce a new variable, which is a normalized amplitude

\[ x = \frac{r}{r_0}, \quad \text{(for } r_0 \neq 0). \quad (A5.4) \]

We then write the functions \( F_+, F'_-, F'_+ \) explicitly, starting from (A5.2) and using the variable \( x \) (A5.4). Dividing the obtained relations by \( k_{2N-1} r_0^N \geq 0 \), we get something of the form

\[ F''_+ = c_2(1-x^2) + c_4(1-x^4) + \ldots + (E_1-x^N E_2) = 0, \quad (A5.5) \]
with
\[ c_2 = \frac{k_2}{k_{2N-1} r_0^{N-2}}, \quad c_4 = \frac{k_4}{k_{2N-1} r_0^{N-4}}, \quad \ldots \]
\[ E_1 = \left( \frac{E}{r_0^2} - 1 \right)^{N-n}/2, \quad E_2 = \left( \frac{E}{r_0^2 x^2} - 1 \right)^{N-n}/2, \]

and three other relations for \( F_+^n, F_-^n \) and \( F_-^n \).

Finally we solve the four equations (A5.5) with respect to \( c_2 \). The case where \( c_4, c_6, \ldots, c_2(N/2) \) are equal to zero is of special interest and these solutions are given below:
\[ c_2^{++} = \frac{x^N E_2 - E_1}{1 - x^2}, \quad c_2^{+-} = \frac{-x^N E_2 - E_1}{1 - x^2} \]
\[ c_2^{-+} = \frac{-x^N E_2 + E_1}{1 - x^2}, \quad c_2^{--} = \frac{x^N E_2 + E_1}{1 - x^2} \quad (A5.6) \]

The four curves \( c_2 = c_4(x) \) are so-called resonance curves and they limit the amplitude of the oscillation. The region of \( x \) inside which the resonance curves are defined is given by the first invariant \( A \). It is clear from (2.5.11) that
\[ \left| \frac{n_1}{n_2} \right| A_1 - r_1^2 = \left| \frac{n_1}{n_2} \right| r_1^2 \geq 0 \]
\[ \left| \frac{n_2}{n_1} \right| A_2 - r_2^2 = \left| \frac{n_2}{n_1} \right| r_2^2 \geq 0, \quad (A5.7) \]

which, in the notation adopted in this Appendix, becomes
\[ E - r^2 \geq 0 \quad \text{or} \quad r^2 \leq E. \quad (A5.8) \]

The relations (A5.7) provide
\[ r_1^2 \leq \frac{1}{2} \left| \frac{n_1}{n_2} \right| Re_{V_0} + \frac{1}{2} Re_{B_0} \]
\[ r_2^2 \leq \frac{1}{2} \left| \frac{n_2}{n_1} \right| Re_{B_0} + \frac{1}{2} Re_{V_0}, \quad (A5.9) \]

which give the limits in (2.6.6), remembering that \( r_1^2 = \frac{1}{2} Re_{B_0} \) and \( r_2^2 = \frac{1}{2} Re_{V_0} \).

Equation (A5.9) gives the upper limit for \( r_1, r_2 \) or for \( x \). The lower limit is clearly zero, since \( r_2 \) is zero when \( r_1 \) is maximum and reciprocally (A5.7).

In order to have an idea about the shape of the resonance curves inside the interval \( [0, x_{\text{max}}] \), it is helpful to calculate the values of \( c_2 \) in the following three particular points:
\[
\begin{aligned}
&c_{2-}^-(x = 0) = c_{2+}^+(x = 0) = E_1 \\
&c_{2+}^-(x = 0) = c_{2-}^+(x = 0) = -E_1
\end{aligned}
\]

\[
\begin{aligned}
&c_{2+}^-(x \lesssim 1) = c_{2-}^+(x \gtrsim 1) = +\infty \\
&c_{2+}^+(x \lesssim 1) = c_{2-}^-(x \gtrsim 1) = -\infty \\
&c_{2+}^+(x = 1) = \frac{1}{2} E_1 \left| n \right| - \frac{|N-n|}{E/r_0^2 - 1} \\
&c_{2-}^+(x = 1) = \frac{1}{2} E_1 \left| n \right| - \frac{|N-n|}{E/r_0^2 - 1}
\end{aligned}
\]

\[
\begin{aligned}
&c_{2+}^+(x = x_{\text{max}}) = c_{2-}^-(x = x_{\text{max}}) = \frac{E_1}{E/r_0^2 - 1} \\
&c_{2-}^-(x = x_{\text{max}}) = c_{2+}^+(x = x_{\text{max}}) = -\frac{E_1}{E/r_0^2 - 1}
\end{aligned}
\]

E₁ being defined in (A5.5) and E in (A5.1).

By virtue of (A5.8), E and \(E/r_0^2 - 1\) are positive. This results in the resonance curves \(c_{2+}^-\) and \(c_{2-}^+\), as well as \(c_{2+}^+\) and \(c_{2-}^-\), being symmetrical with respect to the x-axis. The curves \(c_{2+}^+\) and \(c_{2-}^-\) cross the x-axis in the interval \([0, x_{\text{max}}]\), and the curves \(c_{2+}^-\) and \(c_{2-}^+\) have asymptotes in \(x = 1\). These considerations enable us to draw a picture of the resonance curves for different resonances (Fig. 3). This figure shows that there are two bands \(\Delta c_2\) inside which the amplitude variations can reach the limits \(x = 0\) or \(x = x_{\text{max}}\). Looking at (A5.10), these bands are

\[
\begin{aligned}
&\Delta c_2(x = 0) = 2E_1 \\
&\Delta c_2(x = x_{\text{max}}) = \frac{2E_1}{E/r_0^2 - 1}
\end{aligned}
\]

The whole analysis made here is valid either for the invariant \(C_1\) associated with \(r_1\) or for the invariant \(C_2\) associated with \(r_2\). The following interesting relations exist:

\[
\begin{aligned}
&\Delta c_2(x_1 = 0) = \Delta c_2(x_2 = x_{1,\text{max}}) \\
&\Delta c_2(x_1 = x_{1,\text{max}}) = \Delta c_2(x_2 = 0)
\end{aligned}
\]

(A5.12)
Thus, the general definition of the bandwidth, valid simultaneously for \( r_1 \) and \( r_2 \), must be a combination of the two bandwidths (A5.11). The simplest combination is the average

\[
\Delta c_2 = \frac{E_1 E}{E-r_0^2},
\]

which gives for \( \Delta e \), by virtue of the \( c_2 \) definition (A5.5),

\[
\Delta e = \Delta(n_1 Q_H + n_2 Q_V - p) = k_{2N-1} r_0^{(N-2)} \frac{E_1 E}{E-r_2^2}.
\]

Putting the explicit forms of \( k_{2N-1} \), \( E \) and \( E_1 \) introduced in (A5.1) and (A5.5) gives the expression (2.6.9).
APPENDIX 6

Distance of the working point from the resonance line

In Section 6.3, a criterion concerning the distance of the working point from the resonance line is formulated.

For the sum resonances, this criterion points out that the distance $\Delta c_2$ from the resonance line has to stay larger or equal to $c_2^+(x = 1 + \delta)$, in order that $x$ does not exceed $1 + \delta$ (Fig. 2).

Let us calculate here explicitly $c_2^+(x = 1 + \delta)$. From Ref. 19, we have in the assumption that $c_4, c_6, \ldots c_{2(N/2)}$ are zero

$$c_2^+ = -\frac{E_1 + x^N E_2}{1 - x^2} ,$$

with

$$E_1 = \left(\frac{nA}{(N-n) r_0^2} + 1\right)^{(N-n)/2} \quad \quad E_2 = \left(\frac{nA}{(N-n) r_0^2 x^2} + 1\right)^{(N-n)/2} .$$

Introducing a new parameter $E_3$,

$$E_3 = \frac{nA}{(N-n) r_0^2} ,$$

where $A$ is the first invariant, it is then possible to rewrite (A6.1):

$$c_2^+ = -\frac{E_1 + x^N (E_3 + x^2)^{(N-n)/2}}{1 - x^2} .$$

Putting into (A6.3) the equivalence

$$x = 1 + \delta \quad \quad \delta \geq 0 \quad \quad \delta \text{ small}$$

and keeping the terms of 1st order in $\delta$ gives the following expressions for $c_2^+$:

$$c_2^+(1+\delta) = E_1 \left[\frac{1}{\delta} + \frac{1}{2} \left(\frac{n}{E_3} + \frac{N-n}{E_3} + 1\right)\right] .$$

Equation (A6.5) is thus the wanted limit on the distance from the sum resonance. The definition (A5.5) of $c_2$ enables us to give this limit in terms of $(Q_H, Q_V)$ for $r_1$ and for $r_2$:
\[ \delta e_1 \geq 2|x| n_1 \left( \frac{R e_{v_0}}{n_2} \right)^{\frac{(N-2)/2}{2}} c_2^+(x = 1 + \delta) \]  
\[ \delta e \geq 2|x| n_1^{1/2} n_2^{1/2} \left( \frac{R e_{v_0}}{2} \right)^{\frac{(N-2)/2}{2}} c_2^+(x = 1 + \delta) . \]  

(A6.6)

It is not difficult to express \( E_1 \) and \( E_3 \) as functions of the emittances:

\[ \begin{align*}
E_1 &= \left( \frac{n_1 e_{v_0}}{n_2 e_{v_0}} \right)^{n_2/2} \quad \text{for} \quad x = \frac{r_1}{r_1}, \\
E_3 &= \frac{n_1 e_{v_0}}{n_2 e_{v_0}} - 1 \\
E_1 &= \left( \frac{n_2 e_{v_0}}{n_1 e_{v_0}} \right)^{n_1/2} \quad \text{for} \quad x = \frac{r_2}{r_2}. \\
E_3 &= \frac{n_2 e_{v_0}}{n_1 e_{v_0}} - 1 .
\end{align*} \]  

(A6.7)

Putting (A6.7) and (A6.5) in (A6.6) gives the limits explicitly written in (2.6.13) and (2.6.14).

For different resonances, the criterion points out that the distance \( \Delta c_2 \) from the resonance line has to stay larger or equal to \( c_2^+(x = 1 + \delta_2) = c_2^-(x = 1 - \delta_1) \), in order that \( x \) stays in the interval \( [1 - \delta_1, 1 + \delta_2] \) (Fig. 3), \( \delta_1 \) and \( \delta_2 \) being \( \geq 0 \).

Equation (A5.6) gives us the expressions of \( c_2^+ \) and \( c_2^- \) assuming that \( c_u, c_6, \ldots, c_2(N/2) \) are zero. Introducing a new parameter \( E_3 \),

\[ E_3 = \frac{|n| A}{|N-n| r_0^2}, \]  

(A6.8)

where \( A \) is the first invariant, it is then possible to rewrite

\[ \begin{align*}
c_2^+ &= - E_1 + x \left[ \frac{|n| (E_3 - x^2)|N-n|/2}{1 - x^2} \right] \\
c_2^- &= E_1 + x \left[ \frac{|n| (E_3 - x^2)|N-n|/2}{1 - x^2} \right].
\end{align*} \]  

(A6.9)

Note that the definitions of \( E_1 \) (A5.5) and \( E_3 \) (A6.8) used here are not identical to the ones introduced in (A6.1) and (A6.2) for sum resonances. Some signs are opposed and \( n \) and \( N-n \) are always taken in modulus in (A6.9).

It is now necessary to introduce \( x = 1 + \delta_2 \) in \( c_2^+ \) (A6.9) and \( x = 1 - \delta_1 \) in \( c_2^- \) (A6.9) and to equate both functions. Keeping the linear terms in \( \delta_1 \) and \( \delta_2 \), we get

\[ \frac{a + b \delta_2}{2 \delta_2} = \frac{a - b \delta_1}{2 \delta_1} , \]  

(A6.10)
with
\[
a = 2E_1, \\
b = E_1 \left[ |n| - \frac{|N-n|}{E_3 - 1} \right].
\]

The solution of (A6.10) is the following:
\[
\delta_1 = \delta \left[ 1 - \frac{1}{a/2b\delta - 1} \right], \\
\delta_2 = \delta \left[ 1 + \frac{1}{a/2b\delta - 1} \right],
\]
with \(2\delta = \delta_1 + \delta_2\).

In order to have the wanted limit, (A6.11) is put into (A6.10):
\[
c_2^+ (x = 1 + \delta_2) = c_2^- (x = 1 - \delta_1) = \frac{1}{2} \left( \frac{a}{\delta} - b \right)
\]
\[
= E_1 \left[ \frac{1}{2} - \frac{1}{2} \left( |n| - \frac{|N-n|}{E_3 - 1} \right) \right].
\]

Equation (A6.12) is thus the limit on the distance from the difference resonance.

The definitions of \(c_2\), \(E_1\) and \(E_3\) give this limit in terms of \((Q_n, Q_p)\) for \(r_1\) and for \(r_2\):
\[
\delta e_1 \geq 2|\kappa||n_1| \left( \frac{R}{2} \right)^{(N-2)/2} \varepsilon_{H_0} \left( |n_1| - 2 \right)/2 \varepsilon_{V_0} \left| n_2 \right|/2 \left[ \frac{1}{\delta} - \frac{1}{2} \left( |n_1| - \frac{|n_1|^2 \varepsilon_{H_0}}{|n_1| \varepsilon_{V_0}} \right) \right]
\]
\[
\delta e_2 \geq 2|\kappa||n_2| \left( \frac{R}{2} \right)^{(N-2)/2} \varepsilon_{H_0} \left( |n_1| - 2 \right)/2 \varepsilon_{V_0} \left( |n_2| - 2 \right)/2 \left[ \frac{1}{\delta} - \frac{1}{2} \left( |n_2| - \frac{|n_2|^2 \varepsilon_{V_0}}{|n_2| \varepsilon_{H_0}} \right) \right].
\]

Introducing the definition of the bandwidth \(\Delta e\), these expressions become those given in (2.6.16) and (2.6.17).