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Sum rules for two-particle operators and double beta decay

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Abstract

Sum rules for the double Gamow-Teller and Fermi operators are derived. These sum rules are exact when additional symmetries, isospin and $SU(4)$ invariance, hold. Moreover, they represent a useful approximation independent of further assumptions for the cases of practical importance, nuclei with $N-Z \gg 1$. The 2ν mode of double beta decay exhausts only $\approx 10^{-4}$ of the G-T sum rule, which makes the difficulties associated with evaluation of the double beta decay matrix elements understandable. Examples of the double strength distribution calculated within QRPA are shown and the problem of experimental determination of the double strength is briefly discussed.

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Double beta decay in both 0ν and 2ν modes is the process by which a nucleus (N, Z) in its ground state undergoes a transition to the ground or low lying state of the nucleus $(N-2, Z+2)$ by a simultaneous change of two neutrons into two protons. Study of the 0ν mode of the double beta decay furnishes unique information about the mass and charge conjugation properties of the neutrino. In order to plan and correctly interpret the results of experiments on double beta decay, one has to know the various nuclear matrix elements associated with the decay processes.

It has been pointed out recently [1-5] that the matrix elements for the 2ν and to some extent also for the $0\nu \beta\beta$ decay are suppressed when evaluated in the QRPA with a consistent treatment of the particle-hole and particle-particle components of the neutron-proton interaction. To illustrate the difficulties in evaluating such small nuclear matrix elements we consider here a more general situation, namely the problem of how large the total double Gamow-Teller and Fermi strengths are, and how they are distributed among the states in the final nuclei.

Let us recall how a similar problem can be formulated for the usual single-particle Gamow-Teller and Fermi operators

$$\vec{Y}_{\pm} = \sum_i \vec{\sigma}_i t_i^{\pm}, \quad T_{\pm} = \sum_i t_i^{\pm}. \quad (1)$$

The total Gamow-Teller (Fermi) strength for the β^- and β^+ transitions is simply the norm of the state $\vec{Y}_{\pm} | 0 \rangle$ ($T_{\pm} | 0 \rangle$). The difference of the β^- and β^+ strengths is then (summing over the angular momentum projections in the GT case)

$$\langle 0 | [\vec{Y}_-, \vec{Y}_+] | 0 \rangle = \langle 0 | -6T_z | 0 \rangle = 3(N-Z), \quad (2a)$$

$$\langle 0 | [T_-, T_+] | 0 \rangle = \langle 0 | -2T_z | 0 \rangle = (N-Z). \quad (2b)$$

Equation (2a) represents the famous Gamow-Teller sum rule. It is obvious that the sum rule is independent of the structure of the ground state $| 0 \rangle$ and that it is obeyed as long as nuclei are assumed to be made of nucleons only. In nuclei with a large neutron excess, $N-Z \gg 1$, the β^+ transitions are Pauli blocked and thus one expects the β^+ strength to be much smaller than the β^- strength. In that case the sum rules (2a) and (2b) give not only a lower limit, but also an estimate of the total β^- strength.

The sum rules (2) say nothing, however, about the distribution of the strength among the states of the final nuclei. The Fermi strength distribution is trivial to evaluate if one assumes isospin conservation. In that case the Fermi β^+ strength vanishes entirely (for $T = -T_z > 0$) and the Fermi β^- strength is concentrated in a single state, the isobar analogue state. Similarly, if one assumes the validity of the Wigner SU(4) supermultiplet symmetry, the Gamow-Teller β^+ strength vanishes and the β^- strength is concentrated in a single state with $S = 1$, $T_f = T - 1$. The validity of the isospin conservation, and therefore also of the selection rules for the Fermi transitions, is known to be very good. The SU(4) is not expected to be a good symmetry in nuclei. Nevertheless, experiments show that the "giant GT" state dominates the β^- strength distribution and exhausts most (typically $\approx 60\%$) of the sum rule for Gamow-Teller transitions [6]. At the same time both β^+ and β^- transitions to low lying states are

suppressed by a factor of ≈ 100 when expressed in units of the sum rule (2a). Thus, it appears that the SU(4) selection rules offer useful guidance, even though the giant GT state *does not* actually exhaust the sum rule, it is quite broad, and in many cases a substantial part ($\approx 25\%$) of the observed strength is spread in the energy interval below it.

Sum rules for the double operators, as in the case of the single-particle operators of eqs. (2), can be obtained from the corresponding commutators. The commutators of the double operators, being three body operators, are more complicated now, and their expectation values are no longer exactly conserved quantities. For the Fermi operator we obtain

$$[T_-^2, T_+^2] = 4T_z - 4T_-T_+T_z - 4T_zT_+T_- . \quad (3)$$

Taking the expectation value of (3) in a state of isospin T and its third component $T_z = (Z-N)/2$, we find

$$\langle 0 | [T_-^2, T_+^2] | 0 \rangle = 2(N-Z) \left(2(T^2 - T_z^2) + 2T - 1 \right) = 2(N-Z)(N-Z-1) . \quad (4)$$

where we obtained the last expression by assuming $T = -T_z$. Equation (4) is the sum rule for the double Fermi operator. If isospin is exactly conserved, the strength is concentrated in a single state, the double analogue state, and for nuclei with $N-Z \geq 0$ the T_-^2 strength vanishes.

The double Gamow-Teller operator can be a scalar, vector, or second rank tensor in ordinary space. For application to double beta decay, as we are typically interested in the $0^+ \rightarrow 0^+$ transitions, we shall consider the scalar operator. The commutator equals

$$\begin{aligned} [\vec{Y}_- \cdot \vec{Y}_-, \vec{Y}_+ \cdot \vec{Y}_+] &= -2 \sum_{\alpha} (Y_-^{\alpha} T_z Y_+^{\alpha} + T_z Y_-^{\alpha} Y_+^{\alpha} + Y_+^{\alpha} Y_-^{\alpha} T_z + Y_+^{\alpha} T_z Y_-^{\alpha}) \\ &+ i \sum_{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma} (Y_-^{\alpha} \Sigma^{\gamma} Y_+^{\beta} + \Sigma^{\gamma} Y_-^{\alpha} Y_+^{\beta} + Y_+^{\beta} Y_-^{\alpha} \Sigma^{\gamma} + Y_+^{\beta} \Sigma^{\gamma} Y_-^{\alpha}) , \end{aligned} \quad (5)$$

with $\vec{\Sigma} = \sum_i \vec{\sigma}_i$.

Taking the expectation value of the commutator (5) in a state with $T_z = (Z-N)/2$ we find

$$\begin{aligned} \langle 0 | [\vec{Y}_- \cdot \vec{Y}_-, \vec{Y}_+ \cdot \vec{Y}_+] | 0 \rangle &= 6(N-Z)(N-Z+1) \\ &+ 4(N-Z) \langle 0 | \vec{Y}_+ \cdot \vec{Y}_- | 0 \rangle + 2 \langle 0 | \vec{\Sigma} \cdot (\vec{Y}_- \times \vec{Y}_+) - (\vec{Y}_+ \times \vec{Y}_-) \cdot \vec{\Sigma} | 0 \rangle . \end{aligned} \quad (6)$$

For Pauli blocked nuclei the second line of (6) vanishes. It also vanishes if one assumes the validity of SU(4) symmetry and considers the simplest SU(4) representation for the even-even nucleus. Thus, for the double Fermi and Gamow-Teller operators we cannot obtain exact sum rules, equivalent to (2), without additional assumptions about nuclear symmetries, but we can obtain approximate sum rules. It is important to remember that amplitudes of "wrong" isospin or of the "wrong" SU(4) representation enter quadratically into these sum rules, so that small deviations from exact symmetry should not affect the sum rules very much.

The distribution of the double Gamow-Teller strength can be again evaluated if one assumes the validity of SU(4). (For notation and techniques needed to evaluate matrix elements in SU(4) see [7].)

For even-even nuclei with $(N-Z) \geq 1$, the ground state belongs to the supermultiplet $[y, y, 0]$ and has $S=0$, $T=y=(N-Z)/2$. The ground state is the only state of the nucleus (N, Z) belonging to that supermultiplet. In the odd-odd nucleus $(N-1, Z+1)$ there are two states belonging to the supermultiplet $[y, y, 0]$, the isobar analogue state with $T=y$, $S=0$ and the GT state with $T=y-1$, $S=1$. There are no states belonging to $[y, y, 0]$ in the odd-odd nucleus $(N+1, Z-1)$. In the next even-even nucleus $(N-2, Z+2)$ the states belonging to $[y, y, 0]$ are: the double analogue $S=0, T=y$, and two double GT states with $T=y-2, S=0$ and $T=y-2, S=2$.

The double GT operator $\vec{Y}_+ \cdot \vec{Y}_+$ populates both $S=0$ states in the nucleus $(N-2, Z+2)$ with strengths

$$| \langle S=0, T=y-2 | \vec{Y}_+ \cdot \vec{Y}_+ | 0 \rangle |^2 = 12y \frac{4(y^2-1)}{2y-1}, \quad (7)$$

$$| \langle S=0, T=y | \vec{Y}_+ \cdot \vec{Y}_+ | 0 \rangle |^2 = 12y \frac{3}{2y-1}. \quad (8)$$

These two states exactly exhaust the sum rule (6), and there is no double β^+ strength in $SU(4)$ for nuclei with $N-Z > 0$. Note that for the cases of practical interest the double GT state with $T=y-2$ contains essentially the whole strength.

In order to establish a link to the double beta decay we summarize here the definitions of the relevant nuclear matrix elements. The decay rate of the 2ν mode of $\beta\beta$ decay is proportional to the square of the amplitude [8-11]

$$M_{GT}^{2\nu} = \sum_m \frac{\langle 0_f^+ | \vec{Y}_+ | 1_m^+ \rangle \cdot \langle 1_m^+ | \vec{Y}_+ | 0_i^+ \rangle}{E_m - (M_i + M_f)/2}. \quad (9)$$

In a crude, but nevertheless useful approximation one can replace the energy denominator in (9) by an estimated "typical" value and obtain the closure matrix element

$$M_{GT}^{2\nu} = \frac{1}{\Delta E} M_{Clos}; \quad M_{Clos} = \langle 0_f^+ | \vec{Y}_+ \cdot \vec{Y}_+ | 0_i^+ \rangle. \quad (9a)$$

Thus the scalar double Gamow-Teller operator determines the rate of the 2ν $\beta\beta$ decay, at least in the closure approximation.

For the 0ν mode of $\beta\beta$ decay the rate formula contains terms proportional to the neutrino mass square, $\langle m_\nu \rangle^2$, and terms proportional to the strength of weak interaction which couples to the right-handed lepton current [8-11]. Here, for simplicity, we shall consider only the $\langle m_\nu \rangle^2$ part, and assume that neutrino masses are less than a few MeV. In that case the nuclear structure part of the process is determined by the matrix elements

$$M_{GT}^{0\nu} = \langle 0_f^+ | R \sum_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j t_i^+ t_j^+ \frac{\Phi(r_{ij})}{r_{ij}} | 0_i^+ \rangle, \quad (10a)$$

$$M_F^{0\nu} = \langle 0_f^+ | R \sum_{ij} t_i^+ t_j^+ \frac{\Phi(r_{ij})}{r_{ij}} | 0_i^+ \rangle, \quad (10b)$$

where $R = r_0 A^{1/3}$ is the nuclear radius and the summation is over all nucleon pairs. The factor $\Phi(r)/r$ in (10) comes from the neutrino propagator, $\Phi(r) \approx 1$. As we are interested in the qualitative behavior of these matrix elements we shall assume that one can replace the propagator by a constant and again characterize the nuclear structure part of the problem by the ground state to ground state matrix elements of the double Gamow-Teller and Fermi operators. The idea that the matrix elements (10a) and (9a) are proportional to each other was proposed by Primakoff and Rosen [12] already in 1959 and found to be well satisfied in the shell model calculations [8]. Later QRPA results [1-5] found strong cancellations in the sum (9) and suggest that the proportionality is badly violated in some cases. In particular, note that while the Fermi analogue of the matrix element (9a) vanishes due to the isospin selection rule, the matrix element $M_F^{0\nu}$ is nonvanishing due to the presence of the radial factor $\Phi(r)/r$.

Thus, the matrix elements which characterize both modes of double beta decay are, at least crudely, the quantities which determine the extreme low energy tail of the strengths that obey the sum rules (4) and (6). These sum rules, therefore, offer a convenient unit by which one can judge the degree of suppression of the matrix elements obtained by detailed calculation.

In the past calculations the "real" $\beta\beta$ amplitudes were the goal, therefore, no attempt was made to study the more general distribution of the "double Gamow-Teller" or "double Fermi" strengths even though the relation between the ordinary single-particle Gamow-Teller strength and the factors entering the sum in eq. (9) was noted. In all cases the calculated closure matrix elements represent a small fraction of the sum rule (6). Taking the shell model values of ref. [8] as an example we find that the ground state transitions correspond to 0.001 - 0.005 of the sum rule for the heavier $\beta\beta$ decay candidates, and 0.0004 of the sum rule for ^{48}Ca . The recent shell model results of ref. [13] give similar, although somewhat larger, matrix elements. Note that the shell model calculations [8,13] by necessity use a very limited number of single-particle orbits and consequently do not obey either the single-particle sum rules (2) or its double analogues (4) and (6). In fact, the matrix elements based on the shell model appear to be too large when compared to those derived from the geochemically determined lifetimes of ^{82}Se and ^{130}Te [14] and from the laboratory experiment on ^{82}Se [15]. From these experimental values we find that the fraction of the sum rule (6) is ≈ 0.0005 for ^{82}Se and ≈ 0.00002 for ^{130}Te . (In order to extract the closure matrix elements from experimental lifetimes one has to estimate the corresponding average energy denominators.)

The calculation based on QRPA [1-5] usually gives matrix elements smaller than the shell model values [8,13]. The results, however, depend very sensitively on the value of the particle-particle interaction constant g_{pp} , which is not known with high accuracy. For the "most likely" values of g_{pp} the amplitudes (9) vary between 0 and the shell model value. Thus, it appears that for the cases of interest the ground to ground state double Gamow-Teller strength represents about 10^{-4} of the sum rule (6). It is, therefore, not surprising that the calculation of double beta decay lifetimes is so difficult; we are dealing with very hindered processes.

QRPA obeys automatically sum rule (2) for single-particle operators. For the two-particle operators which populate two-phonon states of QRPA the sum rules (4) and (6) are obeyed in the case of Pauli blocked nuclei with $(N-Z) \gg 1$.

In Fig. 1 we show an example of the QRPA calculation of the double strengths for the nucleus ^{76}Ge which is Pauli blocked; the programs and parameters of ref. [3] were used. The double sum rules (4) and (6) are obeyed in this case. The double β^+ strengths are not displayed, as they are negligibly small. One can see in Fig. 1 that the double β^- Fermi strength is concentrated in the double analogue state; the small additional peaks reflect isospin nonconservation introduced by the approximate nature of the QRPA. The double β^- GT strength shows some concentration with the average energy of ≈ 18 MeV. As expected the average energy equals approximately twice the energy of the single giant GT state. The spreading of the double β^- GT strength is considerable. The asymmetric shape appears to be real, not just an artifact of QRPA.

In order to illustrate the limitations of the validity of this approach and of the QRPA treatment we show in Figs. 2a and 2b the double strengths of the nucleus ^{54}Fe which is not Pauli blocked as the single-particle transition $\pi f_{7/2} \rightarrow \nu f_{5/2}$ is allowed. Even in this case, however, the double β^+ Fermi strength is negligible.

Fig. 2a. shows the double β^- GT strength concentrated in two peaks which correspond approximately to the $\nu f_{7/2} \rightarrow \pi f_{7/2}$ and $\nu f_{7/2} \rightarrow \pi f_{5/2}$ transitions. The distance between the double GT and the double IA resonances is much larger here than in ^{76}Ge , in agreement with the general trend for the single GT vs. IA resonance [16]. The double β^+ GT strength of ^{54}Fe , also shown in Fig. 2a, is sizable. As expected in this case the simple expression $6(N-Z)(N-Z+1)$ for the double GT sum rule is not applicable.

The double β^- Fermi strength, shown in Fig. 2b, should be concentrated in the double analogue state of ^{54}Fe , i.e., in the ground state of ^{54}Ni . Again, the QRPA leads to some spreading.

We shall discuss now possible ways to determine experimentally the double charge changing strength. Since the realization that study of the (p,n) , and more recently also (n,p) reactions furnishes information on the single charge changing Gamow-Teller strength, a whole field of nuclear study has been opened, complementing the traditional studies of the low energy tails of these strengths in β^- and β^+ decays. In analogy, a possible way to study the double charge-changing strength is through reactions based on strong interaction in which the projectile loses two units of charge and the target gains them, or vice versa.

What is the motivation for such program? The relation to double beta decay has been stressed already. Knowing the strength distribution would be a very useful test of the calculations aiming at the "real" double beta decay amplitudes (9) and the matrix elements (10a,b). But knowing the strength distribution would be desirable on more general grounds. As the study of the two-phonon vibrational states represents a much more severe test of the various nuclear models than the study of just the first 2^+ or 3^- states, study of the double strength represents a similar extension for charge-changing modes. In particular, it would be possible to see whether the ideas of SU(4) symmetry restoration (see, e.g., [17] and references therein) is a viable one.

An example of a projectile which can lose two units of charge is the pion through the double charge exchange reaction $\pi^+ \rightarrow \pi^-$. On the basis of PCAC the dominant GT part of the double beta decay is proportional to the pion double charge exchange amplitude taken for *soft* pions. However, the

fact that the soft pions have a vanishing energy, instead of an energy $\omega \approx m_\pi$ as is the case of physical pions, makes a fundamental difference. For physical pions the intermediate states with one pion are favored. The two body operator, responsible for the transition from the initial to the final nucleus involves then pion propagator and therefore cannot have the simple $T_+ T_+$ or $\vec{Y}_+ \cdot \vec{Y}_+$ form. On the contrary, it is more advantageous to have no pions in the intermediate state. The intermediate states with one pion, occurring in double charge exchange of physical pions, represent only a minor correction [18].

In order to retain the link between double charge exchange and double beta decay the charge exchange process should be soft. Double charge exchange of a hadronic probe induced by two successive spin-isospin interactions is an example. Since a single charge exchange, such as (p, n) , has proven so successful in study of the giant GT states, one can hope that the double charge exchange of a hadronic probe will allow one to study the double GT states. A possible candidate is the (p, Δ^-) reaction shown schematically in Fig. 3. The momentum transferred to the nuclear system, $\vec{q} = \vec{p}_p - \vec{p}_\Delta$ should be as small as possible. This implies that the Δ^- should be emitted in the forward direction, and the proton energy should be large when compared to the proton rest mass. (Note, however, that the smallness of $|\vec{q}|$ does not guarantee that there is no momentum transfer at each successive step.) To give a concrete example, for protons of 3 GeV energy the momentum transfer is ≈ 100 MeV/c. Unfortunately, observation of the (p, Δ^-) reaction requires the detection of two particles, π^- and n , and represents therefore a formidable task.

Another possible probe would be a reaction involving the target nucleus (Z_t, A_t) and the projectile (Z_p, A_p) ,

$$(Z_p, A_p) + (Z_t, A_t) \rightarrow (Z_p - 2, A_p) + (Z_t + 2, A_t), \quad (11)$$

in which the transferred nucleon pair is dominantly in a relative s -state and which is dominantly a one-step process. Reaction (11) could allow determination of the double β^- strength of the target, provided the projectile is left in a well defined state. The difficulties with this proposal are obvious. The projectile would have to undergo a "double β^+ " transition and we argued that such transitions are usually suppressed in stable nuclei. Thus, one needs projectiles with $Z > N$ and the only stable one is ${}^3\text{He}$ where, however, the final $3n$ system is unbound. A way around it is to use a radioactive beam, e.g., ${}^{14}\text{O}$ or ${}^{18}\text{Ne}$, since in even-even projectiles with $Z = A/2 + 1$ both the initial and final nuclei belong to the same $\text{SU}(4)$ multiplet. Beams of these and similar radioactive nuclei are available at low intensities, for example, at GANIL [19].

While experimental determination of the double strength is undoubtedly not imminent, it is worthwhile to keep this possibility in mind and hope that the ingenuity of our experimental colleagues will overcome the formidable difficulties associated with such a project.

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Figure captions

Fig. 1. Double β^- strengths for ^{76}Ge versus the excitation energy of the final nucleus ^{76}Se . The discrete lines obtained in QRPA have been replaced by a continuous distribution by assigning a width to each of them. The GT strength is denoted by the full line, the Fermi strength by the dashed line.

Fig. 2. Double strength for ^{54}Fe . In part a) the GT strength is shown; the double β^- is shown by the full line and the double β^+ strength by the dashed line. In part b) the double β^- Fermi strength is shown.

Fig. 3. Schematic diagram describing the (p, Δ^-) reaction. The dashed line represents the spin-isospin interaction.

Fig. 1

Fig. 2

Fig. 3