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CALCULATION AND OPTIMIZATION OF STRAY FIELDS OF SEPTUM DIPOLE MAGNETS

A.J.T. Holmes

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CALCULATION AND OPTIMIZATION OF STRAY FIELDS OF SEPTUM DIPOLE MAGNETS

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ABSTRACT

A theoretical treatment is described of the external stray field of C-shaped septum magnets, such as those designed for the beam extraction systems of the 400 GeV CERN Super Proton Synchrotron. A special conformal transformation of the magnetic plane yields analytic expressions for the four components of the stray field: the septum-shape field (due to the form of the septum conductor), the edge-effect field (due to the mechanical clearance between septum and yoke), the cooling-duct field (due to the presence of these ducts in the septum), and the magnetomotive field (caused by the ampere-turn losses in the yoke). These expressions can be computed by numerical iteration. The septum-shape field turns out to be opposite in sign to the other three, making possible a criterion which creates a minimal stray field for a given magnetic induction. Plots of calculated and measured stray fields are presented for four prototype septum magnets whose total induction is between 0.38 and 1.41 T.

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1. INTRODUCTION

The layout of the energizing coil in the magnet aperture of the C-shaped septum magnet discussed here is shown in Fig. 1. The achievement of a low external stray field for this type of magnet is technically difficult, and a theoretical treatment of this problem is described below. Firstly, a special conformal transformation is made, yielding analytic expressions which describe the several components of the stray field. These expressions are compared with field measurements which have been made on the MST and MSE extraction septum magnets constructed for the 400 GeV accelerator project. Secondly, these equations are used to derive a criterion for the septum shape which creates a minimal stray field for a given magnetic induction.

Fig. 1 General form of a septum magnet

2. CONFORMAL TRANSFORMATION OF THE MAGNETIC PLANE

2.1 Notation

The units used throughout the paper are MKSA. The complex numbers and operators are identified by underlining and their conjugates by asterisks. Quantities which depend on x and y , the coordinates of the $\underline{Z}$ plane, or have a special significance in this plane, have a subscript z , and a similar convention applies for quantities which depend on u and v , the $\underline{W}$ plane coordinates, where the subscript is w .

2.2 The transformation equation

The dimensions of the magnet yoke and coils in the $\underline{Z}$ plane are shown in Fig. 2. The outer dimensions of the yoke are taken to infinity to simplify the transformation, and in addition the return conductor is assumed to be at $x = -\infty$. An additional constraint which is placed on the transformation is that it must lead to analytic equations which describe the field.

Fig. 2 The geometry of the septum conductor in the $\underline{Z}$ plane

The Christoffel-Schwartz equation is applied to the iron boundary to transform it to the real axis of the $\underline{T}$ plane. Hence

$$\frac{d\underline{Z}}{d\underline{T}} = f[(\underline{T} + 1)(\underline{T} - 1)]^{\frac{1}{2}} \cdot \underline{T}^{-1},$$

where the two exterior right angles of the yoke at $y = \pm G/2$ appear at $\underline{T} = \pm 1$ and the return conductor is at $\underline{T} = 0$. The unknown factor f can be determined from the gap height of G , and is G/π :

$$\therefore \underline{Z} = \frac{G}{\pi} \left\{ (1 - \underline{T}^2)^{\frac{1}{2}} + \frac{1}{2} \ln \left[\frac{(1 - \underline{T}^2)^{\frac{1}{2}} - 1}{(1 - \underline{T}^2)^{\frac{1}{2}} + 1} \right] \right\}. \quad (1)$$

This $\underline{T}$ -plane solution is very difficult to use, however, and a better solution is obtained if a second transformation to the $\underline{W}$ plane is made where

$$\underline{W} = u + iv = (1 - \underline{T}^2)^{\frac{1}{2}}$$

$$\therefore \underline{Z} = \frac{G}{\pi} \left[\underline{W} + \frac{1}{2} \ln \left(\frac{\underline{W} - 1}{\underline{W} + 1} \right) \right] \quad (2)$$

and

$$\frac{d\underline{Z}}{d\underline{W}} = \frac{G\underline{W}^2}{\pi(\underline{W}^2 - 1)}. \quad (3)$$

Fig. 3 The geometry of the septum conductor in the $\underline{w}$ plane

A sketch of the magnet form in the $\underline{w}$ plane is made in Fig. 3 and, as can be seen, the septum surrounds the return conductor completely, thus containing the field. However, the septum outside the yoke does not surround the return conductor completely, allowing some of the field to leak out. It is this aperture in the septum which is responsible for the field created by the septum form.

2.3 Derivation and transformation of the field equation

If the C-magnet yoke has a longitudinal dimension much greater than any of the dimensions of its cross-section, which is the case for most magnets, then the fields in and around the aperture can be considered as two-dimensional fields generated by current elements of infinite length. In general the complex field $\underline{B}_z$ has two components B_x, B_y which can be described by a vector potential A_z which has only a single component normal to the $\underline{z}$ plane:

$$\begin{aligned} \underline{B}_z &= B_x + iB_y = \frac{\partial A_z}{\partial y} - \frac{i\partial A_z}{\partial x} \\ &= -i \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) A_z \\ &= \underline{D}_z A_z . \end{aligned}$$

The complex operator $\underline{D}_z$ may also be expressed as

$$\underline{D}_z = -i \left[\frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} + i \left(\frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v} \right) \right]$$

which, on using the Cauchy-Riemann relations, yields

$$\begin{aligned} \underline{D}_z &= -i \left(\frac{\partial u}{\partial x} - \frac{i \partial v}{\partial x} \right) \left(\frac{\partial}{\partial u} + \frac{i \partial}{\partial v} \right) \\ &= \frac{\underline{D}_w}{(dZ/dW)^*} . \end{aligned}$$

Now

$$\begin{aligned} A_z(x, y) &= A_z[x(u, v), y(u, v)] \\ &= A_w(u, v) , \end{aligned}$$

thus

$$\underline{B}_z = \frac{\underline{B}_w}{(dZ/dW)^*} . \quad (4)$$

The above method of relating $\underline{B}_z$ to $\underline{B}_w$ does not rely on the existence of a complex potential, hence it has a better physical basis¹⁾.

The next step is to determine the spatial dependence of the two-dimensional field in the $\underline{W}$ plane for a current element ΔI in the $\underline{Z}$ plane. It is not necessary to transform the elemental area, as currents which pass through conformally mapped areas are identical¹⁾. A two-dimensional field $\Delta \underline{B}_{sw}$, where the subscript s refers to an element of the septum conductor, has a strength equal to $\Delta I \mu_0 / (2\pi |\underline{a} - \underline{r}|)$. The two vertical lines indicate the modulus, and $\underline{r}$ and $\underline{a}$ are the positions of the field calculation point and current element, respectively, both in the $\underline{W}$ plane. The direction of this field is normal to the vector $(\underline{r} - \underline{a})$ and can be expressed by $i(\underline{r} - \underline{a})/|\underline{r} - \underline{a}|$. However, it is convenient if the main dipole field has a positive y component for a defined positive current I and is normal to the iron boundary (the v-axis). This may be accomplished by introducing a normalizing factor of -1 and placing an image current element at $-\underline{a}$ in the iron half-plane. The total field from both elements is

$$\begin{aligned} \Delta \underline{B}_{sw} &= \frac{-\mu_0 \Delta I}{2\pi |\underline{a} - \underline{r}|} \frac{(\underline{r} - \underline{a})i}{|\underline{r} - \underline{a}|} - \frac{\mu_0 \Delta I}{2\pi |-\underline{a} - \underline{r}|} \frac{(\underline{r} + \underline{a})i}{|\underline{r} + \underline{a}|} \\ &= \frac{-\mu_0 i \Delta I}{2\pi} [(\underline{r}^* - \underline{a}^*)^{-1} + (\underline{r}^* + \underline{a}^*)^{-1}] . \end{aligned} \quad (5)$$

In a similar manner the field created by the inverse current in the return conductor and its image, located at $\underline{a}^* = \pm 1$, is

$$\Delta \underline{B}_{rw} = \frac{\mu_0 \Delta I}{2\pi} [(\underline{r}^* - 1)^{-1} + (\underline{r}^* + 1)^{-1}] . \quad (6)$$

Summing the two components and using Eq. (4) gives

$$\Delta \underline{B}_z = \Delta \underline{B}_{sz} + \Delta \underline{B}_{rz} = \frac{-\mu_0 \Delta I}{2G} \left[\frac{(\underline{r}^* + 1)(\underline{a}^* - 1)}{\underline{r}^{*2}(\underline{r}^* - \underline{a}^*)} - \frac{(\underline{r}^* + 1)(\underline{a}^* + 1)}{\underline{r}^{*2}(\underline{r}^* - \underline{a}^*)} \right] . \quad (7)$$

3. STRAY FIELDS

The stray field consists of four components, each of which can be considered separately. These components are: firstly, the septum shape field $\underline{B}_{fz}$ created by the over-all form of the septum conductor; secondly, the edge effect field $\underline{B}_{ez}$ caused by the small mechanical clearance between the septum and the yoke; thirdly, the cooling duct field $\underline{B}_{cz}$ generated by the presence of these ducts in the septum; and lastly, the magnetomotive loss field $\underline{B}_{mz}$ caused by the ampere-turn losses in the yoke. Each of these components can be derived using the transformation above.

3.1 Stray field due to septum form, $\underline{B}_{fz}$

3.1.1 The exact field equation

The field $\Delta \underline{B}_{fz}$ created by an element of the septum conductor carrying a current $jdh\,d\ell$, where h and ℓ are parallel to the y and x axes, respectively, is given by Eq. (7), replacing ΔI by $jdh\,d\ell$. The total field $\underline{B}_{fz}$ is found by integrating over the cross-section of the septum. This is a complicated procedure and the first step is to integrate the elements lying along a strip located at $x = \ell$ and running between $y = \pm(G/2 + H)$, where H is the height of the septum "noses" (see Fig. 2). If $\underline{A}$ and $\underline{A}^*$ are the complex coordinates of the ends of this strip in the $\underline{W}$ plane then the field $\underline{B}_{fz}$ is

$$\underline{B}_{fz} = \int_{-F_z}^L \left[\int_{\underline{a}=\underline{A}^*}^{\underline{a}=\underline{A}} \left(\frac{\Delta \underline{B}_{fz}}{\partial h \partial \ell} \right) dh \right] d\ell ,$$

where L and $-F_z$ are the boundaries of the septum on the x -axis. However, the part of the septum that lies within the magnet aperture (where $H = 0$) has different upper and lower limits for the inner integral. The path of integration in this case forms a closed contour around the singularity at $\underline{r}^* = \underline{a}^*$ as the limits of the integral both lie on the u -axis. Rearrangement gives

$$\underline{B}_{fz} = \int_0^L \int_{\underline{A}}^{\underline{A}} \frac{\Delta \underline{B}_{fz}}{\partial h \partial \ell} \frac{\partial h}{\partial \underline{z}^*} \frac{\partial \underline{z}^*}{\partial \underline{a}^*} d\underline{a}^* d\ell + \int_{-F_z}^0 \oint \frac{\Delta \underline{B}_{fz}}{\partial h \partial \ell} \frac{\partial h}{\partial \underline{z}^*} \frac{\partial \underline{z}^*}{\partial \underline{a}^*} d\underline{a}^* d\ell . \quad (8)$$

The exterior stray field is determined by the first integral in the equation which can be integrated easily, as $\partial h / \partial \underline{z}^*$ is i and $\partial \underline{z}^* / \partial \underline{a}^*$ is $G \underline{a}^{*2} / (\underline{a}^{*2} - 1)$. The interior field is derived from the second integral by applying Cauchy's integral formula for a closed contour around the singularity $\underline{a} = \underline{r}$. The image current contribution to the integral is zero in this case, as the image singularity at $-\underline{a} = \underline{r}$ lies outside the contour. On integration the field becomes

$$\underline{B}_{fz} = \int_0^L \frac{\mu_0 j}{2\pi \underline{r}^{*2}} \left[\ln(\underline{a}^* - 1) - \ln(\underline{a}^* + 1) - 2\underline{r}^* \underline{a}^* + \underline{r}^2 \{ \ln(\underline{r}^* + \underline{a}^*) - \ln(\underline{r}^* - \underline{a}^*) \} \right]_{\underline{A}}^{\underline{A}^*} d\ell + \int_{-F_z}^0 i j \mu_0 d\ell . \quad (9)$$

The field described by Eq. (9) must be integrated over the septum thickness to obtain the final expression for the field. However, only the first integral, which gives the stray field, presents any difficulty in integration, as the limits $\underline{A}$ and $\underline{A}^*$ are complicated functions of ℓ . The second integral, which describes the internal dipole field of the magnet, is based on the assumption that the septum conductor fills the magnet aperture exactly (non-ideal septa are discussed in Section 3.2). Thus the two field components become

$$B_{fy} = \text{Im} \left(\int_0^L \frac{dB_f}{d\ell} d\ell + \mu_0 j F_z \right), \quad B_{fx} = \text{Re} \left(\int_0^L \frac{dB_f}{d\ell} d\ell \right).$$

The integrand $dB_f/d\ell$ is extremely complicated, as can be seen in Eqs. (3) and (9) and the integration of these functions would be very difficult. However, as the field correction caused by a finite septum thickness is expected to be small, because $L \ll 1$ for most real septa, then the above integrands can be expanded by Maclaurin's series. In this case the derivative becomes

$$\left(\frac{dB_f}{d\ell} \right)_\ell = \left(\frac{dB_f}{d\ell} \right)_{L/2} + \frac{(\ell - L/2)}{1!} \left(\frac{d^2 B_f}{d\ell^2} \right)_{L/2} + \frac{(\ell - L/2)^2}{2!} \left(\frac{d^3 B_f}{d\ell^3} \right)_{L/2} + \dots$$

Integrating the above equation with respect to ℓ over the range $\ell = 0$ to L gives

$$\int_0^L \frac{dB_f}{d\ell} d\ell = L \left(\frac{dB_f}{d\ell} \right)_{L/2} + \frac{L^3}{24} \left(\frac{d^3 B_f}{d\ell^3} \right)_{L/2} + \dots \quad (10)$$

The second term in Eq. (10) can be considered as a correction component. The third derivative of B_f can be obtained from Eq. (9):

$$\begin{aligned} \frac{d^2 B_f}{d\ell^2} &= \frac{d}{d\mathbf{a}^*} \left(\frac{dB_f}{d\ell} \right) \frac{d\mathbf{a}^*}{d\mathbf{z}^*} \frac{d\mathbf{z}^*}{d\ell} \Big|_{\mathbf{A}^*, -\mathbf{A}}^{\mathbf{A}, -\mathbf{A}^*} = \frac{\mu_0 j}{2G} \frac{(\mathbf{a}^* - 1)(\mathbf{r}^* + 1)}{(\mathbf{r}^* - \mathbf{a}^*) \cdot \mathbf{r}^{*2}} \Big|_{\mathbf{A}, -\mathbf{A}^*}^{\mathbf{A}^*, -\mathbf{A}} \\ \frac{d^3 B_f}{d\ell^3} &= \frac{d}{d\mathbf{a}^*} \left(\frac{d^2 B_f}{d\ell^2} \right) \frac{d\mathbf{a}^*}{d\mathbf{z}^*} \frac{d\mathbf{z}^*}{d\ell} \Big|_{\mathbf{A}, -\mathbf{A}^*}^{\mathbf{A}^*, -\mathbf{A}} = \frac{\pi}{2G^2} \mu_0 j \frac{(\mathbf{r}^{*2} - 1)(\mathbf{a}^{*2} - 1)}{(\mathbf{r}^* - \mathbf{a}^*)^2 \cdot \mathbf{a}^{*2} \cdot \mathbf{r}^{*2}} \Big|_{\mathbf{A}, -\mathbf{A}^*}^{\mathbf{A}^*, -\mathbf{A}} = 0. \end{aligned}$$

On substitution of $\underline{A}$ and $\underline{A}^*$ the third derivative becomes zero. Thus the two equations below describe the field up to the fifth-order correction:

$$B_{fy} = L \left[\text{Im} \left(\frac{dB_f}{d\ell} \right)_{L/2} \right] + \mu_0 j F_z \quad (11)$$

$$B_{fx} = L \left[\text{Re} \left(\frac{dB_f}{d\ell} \right)_{L/2} \right]. \quad (12)$$

3.1.2 The simplified field equation

A simpler version of the above field equations can be derived from Eq. (9) by expanding the logarithmic terms and substituting into Eq. (11) to obtain

$$B_{fz} = \frac{\mu_0 j L}{2\pi \mathbf{r}^{*2}} \left\{ \left[\frac{\mathbf{a}^{*3}}{3} (1 + \mathbf{r}^{*-1}) \right]_{\underline{A}}^{\underline{A}^*} + \left[\frac{\mathbf{a}^{*3}}{3} (\mathbf{r}^{*-1} - 1) \right]_{-\underline{A}^*}^{-\underline{A}} \right\}.$$

The dipole field is neglected, as only the stray field is required. If Eq. (2) is also expanded the value of $\underline{A}^3$ can be expressed in terms of $\underline{Z}$. Thus

$$\underline{Z} = i\left(\frac{G}{2} + H\right) + \frac{L}{2} = \frac{Gi}{2} - \frac{GA^3}{3\pi}$$

and similarly for A^* . Substitution in the above equation leads to

$$\underline{B}_{fz} = \frac{2jLH\mu_0}{Gr^{*3}} . \quad (13)$$

This equation is usually within 15% of the values given by Eq. (9), particularly on the u-axis. As can be seen, it is mostly the cross-section of the projecting noses on the septum which creates the stray field.

3.2 Stray fields caused by negative currents

The effects of the zero current zones within the boundary of the septum described above can be simulated by adding together the $\underline{B}_{fz}$ field and the fields of an array of negative current elements which correspond to the positions and sizes of the zero current zones.

Two types of zones are considered here, the spaces at the top and bottom edges of the septum (see Fig. 2) and the cooling ducts inside the septum itself.

3.2.1 The edge effect field, $\underline{B}_{ez}$

Usually in real magnets the top or bottom of the septum is pressed against the yoke, resulting in good electrical contact (if the septum is not insulated) while the other edge has double the mean spacing between the conductor and yoke. This asymmetry will result in an asymmetrical field distribution which varies from side to side along the septum because of its mechanical undulations in the gap. Hence the average field is symmetrical.

The electrical contact between septum and yoke allows a current to flow in the iron. The value of this current depends on the lamination thickness τ , septum width F_z , and resistivity ratio between the conductor and yoke, and is shown in Appendix 1 to be

$$I_p = 0.2712 \frac{F_z \tau j \rho}{\rho_I} .$$

This current is zero if the coil is insulated.

As both the negative cavity current and I_p are confined to very thin layers on each side of the septum, they can be considered together to give an average line current density $j_e (A \cdot m^{-1})$ which is equal on each side of the septum, hence giving a symmetrical field

$$j_e = \frac{I_p}{2F} - \frac{j_e}{2} = j \left(0.2712 \frac{\tau \rho}{\rho_I} - E \right) / 2 , \quad (14)$$

where E is the total space between septum and yoke.

In $\underline{W}$ space these line currents will lie on the u-axis and the images on the negative u-axis, hence the upper and lower currents are superimposed and can be considered as a single current equal to $2j_e$. Using Eqs. (4) and (5), replacing ΔI by $2j_e df$, and taking df as parallel to the x-axis gives:

$$\underline{B}_{esz} = \frac{-j_e \mu_0 i}{G} \left[\int_{a=-F_i}^{a=F_i} \frac{(r^{*2} - 1) df}{r^{*2}(r^* - \underline{a}^*)} - \int_{a=-F_o}^{a=F_o} \frac{(r^{*2} - 1) df}{r^{*2}(r^* - \underline{a}^*)} \right] ,$$

where F_i and F_o are the inner and outer limits of the septum on the u-axis. Rearrangement of the integration limits and integration gives

$$B_{esz} = \frac{\mu_0 j_e i}{\pi} \left\{ \ln \left[\frac{(r^* + F_i)(r^* - F_o)}{(r^* - F_i)(r^* + F_o)} \right] + \frac{1}{r^*} \ln \left[\frac{(1 + F_o)(1 - F_i)}{(1 - F_o)(1 + F_i)} \right] \right\}.$$

The field created by the return of these line currents can be found from Eq. (6) as the returns are located at $a^* = \pm 1$:

$$B_{erz} = \frac{j_e \mu_0 i F_z}{G} \left(\frac{r^* + 1}{r^{*2}} + \frac{r^* - 1}{r^{*2}} \right) = \frac{2j_e F_z \mu_0 i}{G r^*}.$$

The total field is the sum of the two components, and using Eq. (2) to simplify the terms gives

$$B_{ez} = \frac{i \mu_0 j_e}{\pi} \left[\ln \frac{(r^* + F_i)(r^* - F_o)}{(r^* - F_i)(r^* + F_o)} - \frac{2(F_i - F_o)}{r^*} \right]. \quad (15)$$

There is no explicit dependence on F_z , the septum thickness in the yoke, and F_i is virtually unity if F_z exceeds $G/4$. Hence the edge effect stray field tends to a maximum value irrespective of septum thickness for septa whose thickness exceeds $G/4$.

A simplified form of Eq. (15) can be derived by expanding the logarithm to give

$$B_{ez} = \frac{4i \mu_0 j_e}{\pi} \left(\frac{F_i^3 - F_o^3}{3r^{*3}} + \frac{F_i^5 - F_o^5}{5r^{*3}} + \dots \right).$$

Normally the septum extends to the corner of the yoke making F_o equal to zero, in which case for $F_z > G/4$ a reasonable approximation for B_{ez} is

$$B_{ez} = \frac{1.8 \times 4i \mu_0 j_e}{3\pi r^{*3}}. \quad (16)$$

The factor of 1.8 is introduced to allow for the effects of the higher-order terms. If the septum does not extend to the corner of the yoke, the value of F_o approaches unity so that B_{ez} tends to zero and hence can be neglected in comparison with the other fields.

3.2.2 The cooling duct field B_{ez}

The cooling ducts can be considered as an array of point-like current elements opposed to the main current in the septum. These holes, which are usually arranged parallel to the x- and y-axes, are imaged in the iron in the $\underline{z}$ plane to give an array which extends to infinity in the y direction, and the field of such an array has been discussed by many authors. Umstätter²⁾ has applied this concept to septum magnets, but the drawback to this approach is that it neglects effects caused by the shape of the iron boundary which perturbs the infinite line of images, and also this approach demands that the cooling duct spacing across the gap be a rational fraction of the gap width.

A more accurate method is to consider only the real cooling ducts and their single images in the $\underline{w}$ plane. As these ducts are usually symmetric about the u- and v-axes, only a single quadrant need normally be considered and the current return of these ducts is at $a^* = \pm 1$. Thus the problem is reduced to finding the field of a series of elements in one

quadrant and then summing over the other quadrants. From Eq. (5) the field of a set of current elements of area S_n , current density $-j$, at positions $\underline{a}_{nk}$ ($k = 1,4$), is

$$\underline{B}_{csz} = \sum_{n=1}^N \sum_{k=1}^4 \frac{jS_n \mu_0 i}{2G} \frac{(\underline{r}^{*2} - 1)}{(\underline{r}^* - \underline{a}_{nk}^*) \underline{r}^{*2}} . \quad (17)$$

The field of the current return can be found in a similar manner, save that it is not necessary to commute through the four quadrants. Equation (6) enables $\underline{B}_{crz}$ to be written as:

$$\underline{B}_{crz} = \sum_{n=1}^N \frac{-2jS_n \mu_0 i}{2G} \frac{1}{\underline{r}^{*2}} [(\underline{r}^* + 1) + (\underline{r}^* - 1)] = \sum_{n=1}^N \frac{-2jS_n \mu_0 i}{G \underline{r}^*} . \quad (18)$$

The total field is given by the sum of $\underline{B}_{crz}$ and $\underline{B}_{csz}$, which gives the following field:

$$\underline{B}_{cz} = \frac{\mu_0 i j}{2G} \sum_{n=1}^N \sum_{k=1}^4 \frac{S_n (\underline{a}_{nk}^* \underline{r}^* - 1)}{\underline{r}^{*2} (\underline{r}^* - \underline{a}_{nk}^*)} .$$

It is possible to expand the sum over k to yield the following expression:

$$\underline{B}_{cz} = \frac{\mu_0 i j}{2G \underline{r}^{*2}} \sum_{n=1}^N S_n \left[\frac{(\underline{a}_n^{*2} - 1) \underline{r}^*}{(\underline{r}^{*2} - \underline{a}_n^{*2})} + \frac{(\underline{a}_n^2 - 1) \underline{r}^*}{(\underline{r}^{*2} - \underline{a}_n^2)} \right] . \quad (19)$$

A simpler form for the cooling duct field may be obtained from the above equation if the cooling ducts lie fairly close to the entrance of the magnet aperture. In this case it can be shown that $\underline{a}_{vn}$ is much less than either $\underline{a}_{un}$ or $\underline{r}_u$. Thus

$$\underline{B}_{cz} = \frac{\mu_0}{G} \sum_{n=1}^N \frac{jS_n (\underline{a}_{un}^2 - 1)}{\underline{r}^* (\underline{r}^{*2} - \underline{a}_{un}^2)} .$$

The position of the cooling ducts on the $\underline{Z}$ plane y -axis can be related to the $\underline{W}$ plane $\underline{a}_{un}$ coordinate by the equation below, which is only valid close to the $\underline{Z}$ plane origin. The x coordinate is neglected in this case.

$$\underline{y}_n = 2 - 1.5 \underline{a}_{un}^2 , \quad (20)$$

$$\therefore \underline{B}_{cz} = \frac{\mu_0}{3G} \sum_{n=1}^N \frac{jS_n (1 - 2\underline{y}_n)}{\underline{r}^* (\underline{r}^{*2} + 0.66\underline{y}_n - 1.33)} .$$

The field described by Eq. (20) has short- and long-range components. The former decays according to the factor: $\exp(-2\pi \times G/d_c)$, where d_c is the duct separation for equal spacing²⁾. This part is described by the last two terms in the denominator. The latter component can be separated out by ignoring these two terms to obtain:

$$\underline{B}_{cz} = \frac{\mu_0 j}{3G \underline{r}^{*3}} \left[\sum_{n=1}^N (1 - 2\underline{y}_n) S_n \right] . \quad (21)$$

This field decays as r^{*3} like B_{fz} and B_{ez} , but it is the terms in the summation which are the most interesting because if the ducts have equal areas and there are an even number of ducts across the gap (i.e. $2N$), then B_{cz} is zero when $\sum_{n=1}^N y_n = N/2$, which gives an equidistance spacing d_c of $G/2N$. The short-range field is unaffected by this and remains non-zero. However, if there is an odd number of holes, one hole being on the median line or u -axis, the field can be described by Eq. (21) where $y_1 = 0$ and $S_1 = \frac{1}{2}S_{n \neq 1}$. The odd field is zero when $\sum_{n=1}^N y_n = (2N - 1)/4$, which gives an equidistance spacing d_c of $(2N - 1)G/4N(N - 1)$.

In the even case, a zero long-range field is obtained when the ducts form a uniform infinite line of images in the Z plane, which is equivalent to a spacing of $G/2N$. However, the odd case is different as the spacing above does not give this uniform line of images. This deviation from uniformity is probably caused by shape of the yoke. Also B_{cz} is always finite for the case of a single duct.

If the spacing and duct size are not uniform the images rapidly become out of step after a few reflections, causing bunching which leads to high fields. The field strength of this long-range component increases very rapidly with departure from uniformity and can easily dominate the other stray fields.

3.3 Stray fields due to magnetomotance loss in iron, B_{mz}

This field can be considered as a perturbation field opposed to the ideal dipole field of a C-shaped magnet with infinite permeability. Thus the field strength can be calculated by considering this loss of magnetomotance (ampere-turns) in the yoke as equivalent to the field generated by a current flowing round the "backleg" of the yoke. This current has a value equal to the average of the integral of the magnetic field H_I along the field lines in the yoke if the current is assumed to flow in a single-turn coil.

The magnetomotance loss in the iron is a complicated function of the induction in iron B_I , the geometrical field line length f , and the relative permeability μ which is a function of B_I . An approximate method for calculating this loss is presented in Appendix 2, which relies on the magnetic energy in the yoke being at a minimum for a given flux ϕ flowing through the yoke. The equivalent current I_m is given by

$$I_m = - \frac{2}{\lambda \mu_0 b \alpha (P + P_a) (\alpha \mu_0 / n \gamma)^{1/(n-1)}} \left[\left(\frac{n-1}{2n-1} \right) f^{(n-1)/(2n-1)} + f \gamma \lambda^n \left(\frac{\alpha}{n \gamma} \right)^{n/n-1} \right]^{-bP+c}_c, \quad (22)$$

where the symbols are defined in the appendix.

The position of this equivalent current at the inner end of the magnet aperture is not very important, so long as it is on the midplane. Thus to simplify matters this current will be placed at the position of the return conductor in the W plane (i.e. at $a^* = \pm 1$).

The field can hence be derived from Eq. (6) where ΔI has been replaced by I_m :

$$\begin{aligned} B_{mz} &= \frac{2\mu_0 i I_m}{G} \left[\frac{1}{4} \frac{(r^* + 1)}{r^{*2}} + \frac{1}{4} \frac{(r^* - 1)}{r^{*2}} \right] \\ &= \frac{\mu_0 i I_m}{G r^*}. \end{aligned} \quad (23)$$

These field equations are the simplest of the four components and are valid only for the exterior of the magnet. They can be applied to the aperture field if the value of P in Eq. (22) is reduced according to the position in the gap. All the other field components can be applied to the interior field.

The equations also show that B_{mz} goes to infinity at the origin of the W plane because the field lines approach the corner of the yoke radially. Saturation limits the real field to about 2 tesla, and hence for a typical stray field of 0.01 tesla the field equations are valid outside a radius of $(0.01/2) \cdot G/2$ around the corners.

4. COMPENSATION OF STRAY FIELD EFFECTS

Of the four stray field components described in the previous sections, only the field created by the septum form or more precisely the septum "noses" is totally under the control of the designer. The parameters of the other fields are fixed by other considerations, such as the need for a certain gap width or a certain flow of cooling water. Fortunately, the latter fields are normally negative in sign ($B_y < 0$) and the former field B_{fy} is invariably positive. Hence there exists the possibility of compensating these negative fields to yield a low net stray field.

From Eqs. (13), (16), (21), and (23), it can be seen that the magnetomotive loss field B_{mz} decays as $1/r^*$, whereas the other three fields decrease as $1/r^{*3}$. Hence at large distances it is the former field which dominates. However, it is the region close to the septum which is the most important for stray field compensation, and hence the optimization technique must not allow the effects at large distances to dominate.

A method of satisfying this condition is to have a "nose" area such that the stray field at the magnet aperture mouth is positive (if the short-range field of the ducts is neglected), becomes negative further away, and peaks at a negative value equal to the initial positive value. The initial positive field B^+ at the gap mouth is

$$B^+ = \frac{2\mu_0}{G} \left\{ 1.2^{-3} jLH + 1.2^{-2} \frac{Gj_e}{\pi} + 1.2^{-3} \frac{j}{6} \left[\sum_{n=1}^N S_n(1 - 2y_n) \right] - 2.4^{-1} I_m \right\} ,$$

where $r_u = 1.2$ and $r_v = 0$ in Eqs. (13), (16), (21), and (23).

Differentiation of these equations with respect to r_u enables the position and field at the negative peak to be calculated:

$$r_u^- = \left(\frac{6\beta}{I_m} \right)^{\frac{1}{2}} ,$$

where

$$\beta = jLH + 1.2 \frac{Gj_e}{\pi} + \frac{j}{6} \sum_{n=1}^N S_n(1 - 2y_n) ,$$

$$\therefore B^- = 2\mu_0 I_m^{\frac{3}{2}} \beta^{-\frac{1}{2}} \left(6^{-\frac{3}{2}} - \frac{6^{-\frac{1}{2}}}{2} \right) / G = -B^+ .$$

Solving this equation for the minimum value of B^- gives

$$\beta = 0.960 I_m \quad (24)$$

$$B^+ = |B^-| = 0.278 \frac{\mu_0 I_m}{G} . \quad (25)$$

The above expression shows that firstly there is a different optimum area LH for each total induction because of the non-linear dependence of I_m on the induction B_I . However, if the magnet only operates over a restricted range of fields, an optimum area can be found which should reduce the peak stray field to the value expressed by Eq. (25). Equation (25) shows that the optimum stray field is inversely proportional to the induction at low fields but then rises when the maximum of the $\mu(B_I)$ curve is surpassed.

5. ANALYSIS AND COMPARISON WITH MEASUREMENTS

Measurements have been made of the stray fields of the prototype septum magnets which are being designed for the 400 GeV accelerator project³⁾, and parameters necessary to calculate the stray field are listed in Table 1 below.

Table 1

Magnet parameters (MKSA units)

Magnet Name	P0	P1	P2	P4
Total induction (T)	0.961	1.407	0.377	0.377
Current density ($A m^{-2}$)	0.692×10^8	1.10×10^8	1.02×10^8	1.02×10^8
Edge current (A)	-1.4×10^3	1.7×10^3	-1.4×10^3	-1.4×10^3
Equivalent current (A)	35.6	117.2	25.6	25.6
Magnet aperture (m)	2.0×10^{-2}	2.0×10^{-2}	2.0×10^{-2}	2.0×10^{-2}
Septum thickness (m)	1.6×10^{-2}	1.6×10^{-2}	0.4×10^{-2}	0.4×10^{-2}
Nose area (m^2)	1.0×10^{-6}	1.1×10^{-6}	1.0×10^{-6}	0.6×10^{-6}
$\sum_{n=1}^N S_n(1 - 2y_n)$ (m^2)	0.34×10^{-6}	-0.32×10^{-6}	-0.27×10^{-6}	-0.27×10^{-6}
Optimum area (m^2)	0.99×10^{-6}	1.22×10^{-6}	0.62×10^{-6}	0.62×10^{-6}

The P0, P1, and P4 magnets were designed to have the optimum "nose" area from experimental measurements on earlier magnets. If these parameters are introduced into the equations describing the stray field, a field map can be made using iterative techniques and a computer. The y component field of P1 and P4 on the midplane of the magnet has been drawn in Figs. 4 and 5 and shows both the calculated and measured fields. Another method of presenting the stray field is to plot B_y as function of y at various points along the x-axis. This field shows clearly the short-range field of the cooling ducts and also the rapid decay of this field component. Figures 6 and 7 show this method of field plotting for the magnets P0 and P2.

The calculated optimum area in Table 1 is in moderate agreement with the actual "nose" area for those magnets which have been compensated. The P1 magnet "nose" area is slightly too small, and this is borne out by the negative bias of the stray field.

Fig. 4 Stray field B_y of the P1 magnet on the mid-plane

Fig. 5 Stray field B_y of the P4 magnet on the mid-plane

Fig. 6 The y-axis variation of the stray field B_y of the P0 magnet

Fig. 7 The y-axis variation of the stray field B_y of the P2 magnet

6. CONCLUSION

Analytic expressions have been presented which enable the components of the stray field of C-shaped magnets to be calculated. These expressions are complicated but can be handled by numerical iteration using a computer. They have been used to make theoretical field maps of the extraction septum magnets of the 400 GeV accelerator at CERN, which are in good agreement with measurement.

Two points of special interest arise from these field equations: firstly, the existence of a strong, long-range field component created by the cooling duct cavities in the septum; and secondly, the possibility of minimizing the external stray field by varying the "nose" area of the septum conductor. The theoretical values for this area agree reasonably well with the experimental values.

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APPENDIX 1

THE PARALLEL CURRENT IN THE YOKE

The lamination thickness τ is in the x' direction and the origin of coordinates is the centre of the lamination at its junction with the septum conductor. The x' -axis is parallel to the septum in the direction of current flow and the y' -axis perpendicular to the septum. The $x'y'$ subplane is hence perpendicular to the $\underline{z}$ plane. Laplace's equation for the electric potential V in the lamination gives

$$\nabla^2 \cdot V = \frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2} = 0 ,$$

where V is independent of z' . If V consists of separable functions, then

$$\begin{aligned} V &= X(x') \cdot Y(y') \\ &= (A' e^{ky'} + B' e^{-ky'}) (C' \cos kx' + D' \sin ky') . \end{aligned}$$

The boundary conditions are that $V = 0$ when y' is infinite, and that $V = 0$ on the median line of the lamination for all y' (when $x' = 0$). Thus

$$V = M \sin kx' e^{-ky'} . \quad (\text{A1.1})$$

As the lamination is insulated from its neighbours then the electric field must be normal to the insulated surface. Hence

$$\frac{\partial V}{\partial x'} = 0 \text{ for all } y' \text{ at } x' = \pm \frac{\tau}{2} . \quad (\text{A1.2})$$

From Eqs. (A1.1) and (A1.2) k can be determined to be

$$k = \frac{(2n-1)\pi}{\tau} .$$

If uniform contact is assumed with the septum along the bottom edge of the lamination (i.e. from $-\tau/2$ to $\tau/2$), then

$$V = \frac{2V_0 x'}{\tau} = \sum_{n=1}^{\infty} M_n \sin \left[\frac{(2n-1)\pi x'}{\tau} \right] .$$

Using Fourier analysis to derive M_n gives the total potential distribution:

$$V = - \frac{8V_0}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x'}{\tau} \exp \frac{-\pi y'(2n-1)}{\tau} .$$

The current density components along the two axes are defined by

$$\begin{aligned}\bar{J}_x &= -\frac{1}{\rho_I \tau} \int_{-\tau/2}^{\tau/2} \frac{\partial V}{\partial x'} dx'; \quad \bar{J}_y = -\frac{1}{\rho_I \tau} \int_{-\tau/2}^{\tau/2} \frac{\partial V}{\partial y'} dx' = 0, \\ \therefore \bar{J}_x &= \frac{1}{\rho_I} \frac{8V_0}{\tau \pi^2} \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \exp \left[\frac{-\pi y'}{\tau} (2n-1) \right] \\ I_p &= F_z \int_0^{\infty} \bar{J}_x dy' = \frac{16F_z V_0}{\tau \rho_I \pi^3} \sum_{n=1}^{\infty} \frac{\tau}{(2n-1)^3} \\ &= 8.411 \frac{2V_0 F_z}{\rho_I \pi^3}.\end{aligned}$$

The value of V_0 can be derived from the septum resistance to be

$$\begin{aligned}2V_0 &= j\rho\tau \\ \therefore I_p &= 0.2712 \frac{\rho F_z \tau j}{\rho_I}.\end{aligned}\tag{A1.3}$$

Equation (A1.3) gives the current which flows in the yoke when one side of the septum is in uniform electrical contact with the iron laminations.

APPENDIX 2

THE MAGNETOMOTANCE LOSS

The problem of calculating the field distribution in the iron yoke from the Laplace equation and hence determining the magnetomotance loss is extremely complicated. However, a much simpler method, which is also fairly accurate, is to use the minimum energy principle to determine the field distribution in one dimension across the yoke. This method also permits the variation of μ with induction to be included.

The magnetic energy in the yoke per unit length can be expressed by the equation below, where $\vec{B}_I$ and $\vec{H}_I$ are the vector fields on the median line of the yoke:

$$E_I = \frac{2P}{(P + P_a)} \int_0^P f \vec{B}_I \vec{H}_I dp ,$$

where P is the width of the yoke on the median line, P_a is the width of the magnetic aperture, f is the geometrical field line length, and p is the distance from the end of the aperture across the iron thickness (Fig. 1). The terms involving P and P_a in front of the integral allow for the variation in the square of the magnetic induction along the field line, providing this variation is less than a factor of two.

The energy content may be minimized according to the total flux per unit length, ϕ remaining constant. Thus

$$\phi \hat{e} = \int_0^L \vec{B}_I dp , \quad (A2.1)$$

where $\hat{e}$ is the unit vector normal to the median line of the yoke (the y direction in the Z plane). On this line $\hat{e}$, $\vec{B}_I$, and $\vec{H}_I$ are all parallel, and hence using the Euler condition and a Lagrangian multiplier λ^{n-1} (λ and n are constant) gives

$$\frac{f \partial}{\partial B_I (B_I H_I)} = \lambda^{n-1} , \quad (A2.2)$$

where the fields are scalars and f is assumed independent of B_I . This relationship minimizes $E_I(P + P_a)/2P$ or E_I as the other terms are constants.

It is now necessary to create a function for μ the relative permeability, such that it agrees with the tabulated values of μ and also gives integrateable functions. A suitable function is

$$\mu = \frac{\alpha B_I^2}{1 + \gamma B_I^n} , \quad (A2.3)$$

where α , γ , and n are constants. Substituting Eq. (A2.3) into (A2.2) gives the following expression for the induction on the median line:

$$\vec{B}_I = \lambda \left(\frac{\alpha \mu_0}{n \gamma f} \right)^{1/(n-1)} \hat{e} .$$

The field lines f are expected to be roughly circular with the addition of a small fixed distance at the inner end of the gap. Hence $f = bp + c$, where b is a numerical constant which depends on the rectangularity of the yoke. Eliminating λ in Eq. (A2.1) gives

$$\vec{B}_I = \frac{\hat{e} b \phi (n-2) (bp + c)^{-1/(n-1)}}{(n-1) [(bp + c)^{(n-2)/(n-1)} - c^{(n-2)/(n-1)}]} . \quad (A2.4)$$

Thus the field decreases monotonically from the inner edge of the aperture to the outside of the yoke as expected.

The mean loss of magnetomotance M_I can be expressed by the equation

$$M_I = - \frac{1}{P \mu_0} \int_0^P \frac{f \cdot B_I}{\mu} dp \cdot \frac{2P}{P + P_a} ,$$

where f and B_I are always parallel. The equivalent current I_m is identically equal to the loss of magnetomotance M_I . Hence after integration

$$I_m = - \frac{2}{\lambda \mu_0 b \alpha (P + P_a) (\alpha \mu_0 / n \gamma)^{1/n-1}} \left[\frac{(n-1)}{(2n-1)} f^{(2n-1)/(n-1)} + \gamma \lambda^n \left(\frac{\alpha \mu_0}{n \gamma} \right)^{n/(n-1)} f \right]_c^{bp+c} . \quad (A2.5)$$

To a fair degree of approximation the values of α , γ , and n are respectively 2.2×10^4 , 5.5, and 4 for "Armco" soft iron used to make the magnets described in Section 5.