VISUAL AIDS TO RELATIVISTIC KINEMATICS

G.C. Wick

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1. **INTRODUCTION**

Relativistic kinematics is an old game, and it is hardly possible to invent any new tricks. However, I wish to review some tricks that are new only in the sense that they have largely been forgotten. They are based on the complete similarity ("isomorphism") between the homogeneous Lorentz group and the group of movements of non-Euclidean geometry; a mathematical fact already known in the days of Felix Klein\(^*\). This fact can easily be explained by means of the well-known "Cayley map" of non-Euclidean space into ordinary space; it goes without saying that previous knowledge of these things is not assumed\(^**\)), and only those aspects of the analogy more directly related to the interests of the experimental physicist will be mentioned. The analogy between non-Euclidean space and the set of all possible "physical" velocities (\(v < c\)) permits a convenient visualization of Lorentz frames and of complicated kinematic situations ("velocity diagrams")\(^***\)). As a further possible addition to the usefulness of this picture, I propose here a kind of "geometric calculus" which leads directly from the visual pictures to invariant expressions for the quantities of interest. This calculus is, of course, closely related to the customary tensor methods, and is far more natural, in relativistic problems, than the ordinary (three-dimensional) vector notation, which is still used occasionally because of its simple intuitive appeal. Formally, covariant tensor calculus is, of course, far more appropriate\(^†\)).

We aim at combining the intuitive appeal of the one with the formal precision of the other.

2. **THE MINKOWSKI VELOCITY MANIFOLD**

We start with the naive purpose of simplifying the drawing of velocity diagrams. Minkowski uses a four-dimensional picture: the momentum four-vector \(p^\mu = (p^0; \mu = 0,1,2,3)\) or, more conveniently, the velocity four-vector \(x^\mu\), which is normalized \((x = p\) divided by the rest mass\)) by the condition

\[
g_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1 ,
\]

(1)

is drawn from a fixed origin. The components \(x^\mu\) are then regarded as the Cartesian coordinates of a point in a four-dimensional space (the end point of the four-vector).

Equation (1) together with the condition

\[
x^0 > 0
\]

(1')

then states that \(x\) lies on the "positive sheet" of a hyperboloid (Fig. 1). Our first aim is to learn to "see" points and (draw pictures of) configurations of points on this hyperboloid, a three-dimensional manifold, while ignoring the embedding four-dimensional space. It is, of course, easier to draw a two-dimensional picture than a three-dimensional one, and much easier to draw a three-dimensional picture than a four-dimensional one. This

\(^*)\) See, for example, A. Sommerfeld's discussion of the composition law of relativistic velocities\(^1\)).

\(^**\) The reader endowed with such knowledge can glance through Sections 2 to 5 rather quickly.

\(^***\) I have often used this picture in my own thinking for several years; see, for example, some earlier works\(^2,3\)). Independently, similar ideas have been used by Smorodinski and others. See the references to this work in Smorodinski's review paper\(^5\)).

\(^†\) See, for example, a report by Berman and Jacob\(^6\)).
reduction in the number of dimensions can be achieved (see Fig. 1) by projection (or possibly by other methods of mapping) onto a tangent hyperplane, for example the hyperplane $x^0 = 1$.

The Cayley map (see Appendix B) is such a projection. What it amounts to is viewing the old-fashioned velocity space of classical mechanics as a map of the "true" velocity manifold.

How do we preserve in this picture the metric properties of Minkowski space; or, in other words, how do we describe Lorentz invariance? The simple answer is provided by Gauss’s notion of "intrinsic geometry" (see also Appendix A). Just as a surface in ordinary space has a $ds^2$ defined by the metric of the embedding space, in the same way one can derive from the (indefinite) Minkowski metric of the space, a Lorentz-invariant positive definite $ds^2$ "on" the velocity manifold. Lorentz transformations $^*$ of the four-dimensional space transform the manifold (1) (1') into itself. By construction, the $ds^2$ is invariant under these transformations, which one calls, therefore, the "group of movements" of the intrinsic geometry of the manifold. It turns out that this geometry, and the corresponding group of movements, are those of non-Euclidean (hyperbolic) space. The properties of this space of interest to us will, however, be derived directly from familiar mathematics.

For visual requirements the positive definiteness of the intrinsic $ds^2$ is very important; it means that a sufficiently small piece of our manifold can be mapped isometrically (i.e. without deforming distances or angles) into a piece of ordinary space. There is absolutely no difficulty in visualizing a point of such a space and its neighbourhood. As to larger pieces, we should not ask too much; we cannot easily "see" the curvature of a three-dimensional space. But is it really necessary? Qualitatively, I find no difficulty in visualizing the space, while recognizing that what the mind's eye sees is a (distorted)

$^*$ We shall only refer to transformations which do not reverse the sign of $x^0$, and which are "proper" (i.e. have determinant +1).
map of the manifold into ordinary space\footnote{Minkowski space, being "flat", is allegedly "easy to visualize". This is true to a certain extent for two-dimensional kinematics, requiring only a three-dimensional Minkowski space. Even then what the eye sees is, of course, a map into ordinary space, i.e. into a space with the wrong metric.}}. These distortions are easily described (see Appendix B).

3. VARIOUS KINDS OF COORDINATES

In our geometric picture we have replaced four-vectors (more precisely: time-like four-vectors, representing "physical" velocities, $v < c$) by \textit{points} of a three-dimensional space or manifold. In calculations we can still use our beloved four-components $x^\mu$, with their simple linear behaviour under Lorentz transformations. If we wish, we may regard them as \textit{homogeneous coordinates} of the corresponding point\footnote{That is to say, a point only determines the ratios $x^0 : x^1 : x^2 : x^3$; or: it determines the vector $x$ only up to an arbitrary proportionality constant $k$. Since the vector is time-like, we may restrict $k$ to positive values only, by adding the (Lorentz invariant) condition (I').}}, and treat them as independent variables. Of course, if furthermore we add the normalization condition (1), then only three of the variables are independent, and the four-vector corresponding to a point is determined completely.

In relativistic kinematics, however, we also need space-like four-vectors. For example, a Lorentz frame is often defined by a \textit{tetrad} of mutually orthogonal four-vectors in Minkowski space, and three of these, as is well known, are necessarily space-like. When they are normalized they obey an equation like Eq. (1), but with a -1 on the right-hand side. We have two ways of representing such a vector in our geometric picture. i) We may think of it as corresponding to a pure imaginary point of the manifold, later we shall see that this is a useful notion. ii) Even more useful, however, is the alternative interpretation of a space-like vector as a \textit{set of (homogeneous) coordinates defining a real plane}. A plane (see also Section 4) may be defined as the "locus" of all points $\{x^\mu\}$ satisfying a homogeneous linear equation

\[ x^\mu \xi_\mu = x^0 \xi_0 + x^1 \xi_1 + x^2 \xi_2 + x^3 \xi_3 = 0 , \]  

(2)

where $\xi_0$, etc., are constant coefficients. As is well known (and easy to see), such an equation possesses a continuum of time-like solutions $\{x^\mu\}$ if, and only if, $\xi_\mu$ is space-like:

\[ g^{\mu\nu} \xi_\mu \xi_\nu = - \xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 > 0 . \]  

(3)

The notion of "plane" is obviously invariant against linear transformations of the coordinates $x^\mu$, and in fact a \textit{fixed} plane, i.e. one which is well defined regardless of such transformations, must have $\xi_\mu$ coefficients which transform (simultaneously with the $x^\mu$) in such a way that $x^\mu \xi_\mu$ is an \textit{invariant linear form}: $x$ and $\xi$ "belong to dual vector spaces\footnote{The distinction is convenient (the corresponding geometric entities, points, and planes, being different) although a bit academic, since the only coordinate transformations we envisage are those of the Lorentz group, so that one can switch from one kind of vector to the other by the familiar transformation $\xi_\mu = g_{\mu\nu} x^\nu$.}". 

\[ \xi_0 \neq 0 \]
Just as the coordinates of a point may be (arbitrarily) normalized by Eq. (1), we can fix the coordinates \( \xi \) of a plane by normalization: \( g^{\mu \nu} \xi_\mu \xi_\nu = +1 \). The remaining sign ambiguity, however, must be removed in a different way; for a space-like vector there is no Lorentz invariant inequality similar to condition (1'). On the other hand, just as in ordinary space, a plane [Eq. (2)] separates the velocity manifold into two parts where the form \( x^\nu \xi_\nu \) has opposite signs; choosing the part where the form is positive clearly fixes the sign of the vector \( \xi \). We say that the plane has been "oriented".

Equation (2), which in ordinary language is described as an orthogonality condition, between vectors \( x \) and \( \xi \), is now an incidence condition: point \( x \) belongs to the plane \( \xi \). The geometric picture of a tetrad is therefore a point \( 0 \) -- corresponding to the time-like element of the tetrad -- and three planes \( \xi, \eta, \zeta \), say, through the point \( \xi \). The system is very similar to a reference frame in ordinary space, and it will be easy to learn to use it in a quantitative way.

4. GEOMETRY OF PLANES AND LINES

The planes, defined by Eqs. (2) and (3), are two-dimensional submanifolds of our space\(^{**}\), which enjoy, in common with the planes of ordinary geometry, the property that the shortest path (geodesic) between two points of a plane lies entirely in the plane (see also Appendix A). While the notion of the geodesic line is common to all Riemann spaces, the existence of "planes" endowed with the above property is a special feature of non-Euclidean space, which brings us much closer to ordinary geometry than is generally the case.

It is easy to see, for example, that given two points \( A \) and \( B \), it is possible, in infinitely many ways, to find two distinct planes \( \xi \) and \( \eta \), say, containing both points. The geodesic line joining \( A \) to \( B \) may then be defined (obviously) as the locus of points satisfying the two independent linear equations

\[
x^\nu \xi_\mu = 0 ; \quad x^\nu \eta_\mu = 0 ,
\]

in other words as the intersection of two planes \( \xi \) and \( \eta \). Owing to this analogy with ordinary geometry, the locus defined by Eqs. (4) will be called a straight line or briefly line\(^{***}\).

In the Cayley map (Appendix B), lines and planes are mapped into (straight) lines and planes of ordinary space; more precisely into lines and planes which intersect the unit sphere. Imaginary lines and planes can also be easily visualized!

The notions of plane and line, and the resulting relations of coplanarity and collinearity, allow one to express in a concise geometric language the existence of linear dependence relations between momentum four-vectors. For example, it is easy to see that

\*\) See further conditions in Sections 4 and 5.

\**) When condition (3) is not satisfied, we shall say: Eq. (2) defines an "imaginary plane", which contains no "real" points.

\***\) The above reasoning assumes implicitly that \( \xi \) and \( \eta \) have two real points in common (one is sufficient). Two real planes may have no real point in common; they define, then, an imaginary line. See Appendix B.
three velocity points A, B, and C are *collinear* [they belong to a line, i.e. they satisfy two equations such as Eqs. (4)], if the corresponding coordinate four-vectors \((A^\mu)\), \((B^\mu)\), \((C^\mu)\) satisfy a linear relation

\[
c_1 A^\mu + c_2 B^\mu + c_3 C^\mu = 0
\]

(4')

with non-zero coefficients \(c_1\), \(c_2\), \(c_3\).

Similarly, four points A, B, C, D are *coplanar* (belong to a plane) if they satisfy a relation

\[
c_1 A^\mu + c_2 B^\mu + c_3 C^\mu + c_4 D^\mu = 0
\]

(5)

Let us notice at once that the conservation laws of energy and momentum in a two-body reaction \(A + B + C + D\) are precisely of this form. The velocity points corresponding to the four particles are therefore *coplanar*. (In ordinary language, one states that the velocity vectors are coplanar, but this is only true in special frames of reference, e.g. the centre-of-mass frame). In a similar way the velocity points of two particles and of their centre of mass are *collinear*, i.e. the corresponding momentum four-vectors are linearly dependent [Eq. (4')]. Again the corresponding statement about velocities, in ordinary three-dimensional terminology, holds only in certain selected frames of reference.

The role of these concepts in drawing velocity diagrams and in defining *frames of reference* (or equivalently, Lorentz transformations) should now be obvious. In practice a frame of reference is invariably defined as a rest-system for some particle, or for the centre of mass of certain particles. This defines the time-element of a tetrad, or -- as we have stressed already -- the origin \(O\) of a reference system in the velocity manifold (Fig. 2). The remaining three elements of the tetrad -- in other words the three coordinate planes \(\xi, \eta, \zeta\) through \(O\) -- are also usually defined in terms of existing particles. The figure corresponds to the following assumptions: the \(1\)-axis is chosen parallel to the velocity of particle \(A\) in the rest-frame for particle \(O\). \(A\) lies therefore on the intersection of the planes \(\eta\) and \(\zeta\), on the positive side of the \(\xi\)-plane (i.e. \(A^{\mu\xi} > 0\)). The assumption

![Fig. 2](image-url)  
*Definition of a right-handed reference system in the velocity manifold by means of three non-collinear velocity points \(O, A, B\).*
that $B$ lies in the $\zeta$-plane and $B^\mu \eta_{\mu} > 0$, completes the definition of the $\zeta$-plane; furthermore, it tells us how $n$ is to be oriented. The tetrad will now be completely defined if we add the Minkowski orthogonality conditions between the four-vectors $(\xi_{\mu})$ $(\eta_{\mu})$ $(\zeta_{\mu})$, for instance

$$g^{\mu\nu} \xi_\mu n_\nu = 0,$$

and the condition that the frame be right-handed.

5. ORTHOGONALITY*)

It behaves us, now, to give an intrinsic geometric interpretation of Eq. (6). This is quite simple. If we define the angle $\phi$ between two planes as explained in Appendix B, or by formula (23), Section 7, we see that Eq. (6) means: the planes $\xi$ and $n$ are orthogonal. An orthogonal tetrad in Minkowski space corresponds, therefore, in non-Euclidean space to three mutually orthogonal planes through a real velocity point $O$. This completes the description of a reference frame in our velocity manifold.

There is an alternative interpretation of Eq. (6). Let us write it $\xi^\mu \eta_{\mu} = 0$, with:

$$\xi^\mu = g^{\mu\nu} \xi_\nu.$$

Then let us regard Eq. (7) as a correspondence ("polarity") between different entities **); namely, a plane $\xi$ with coordinates $(\xi_0, \ldots, \xi_3)$ and a point $\xi$ with coordinates $(\xi^0, \ldots, \xi^3)$. We may then interpret Eq. (6), just as Eq. (2), as an incidence condition. Thus all problems demanding the construction of planes $\eta$ orthogonal to a given plane $\xi$ are reduced to a different, and easily solved, class of problems involving the construction of planes through a given point $\xi = (\xi^\mu)$, the "pole" of the plane $\xi$.

The only trouble is that the reality conditions for points and planes are just opposite: when $(\xi_{\mu})$ defines a real plane [see condition (3)] we cannot interpret $\xi^\mu$ as a real point of the velocity manifold. We referred to it earlier as an "imaginary point". This may be better understood as follows. In essence, the use of "homogeneous coordinates" implies that a point on the velocity manifold, Eq. (1), is defined as the intersection of the manifold with a certain line through the origin of Minkowski space, corresponding to certain values of the ratios $x^0:x^1:x^2:x^3$. When these ratios correspond to a space-like vector, the line has no real intersection with the manifold, but it has, of course, imaginary intersections ***)}: if $\xi$ is normalized to satisfy $g^{\mu\nu} \xi_\mu \nu = 1$, then the (pure) imaginary point $(i\xi^0, i\xi^1, i\xi^2, i\xi^3)$ in Minkowski space will satisfy Eq. (1). We may say that it is this point which is represented by the homogeneous coordinates $\xi^\mu$. By analogy, the components $x^\mu$ of a real point, when transformed to lower indices in a similar way, are said to represent an imaginary plane. In the Cayley map, these imaginary points and planes have a very simple representation (see Appendix B).

*) This section contains additional material, which may be left out in a first reading.

**) rather than between different coordinates for the same entity; this departs from current usage, but follows a certain logic: see our conventions in Section 3.

***) As usual, there are two intersections: changing $\xi^\mu$ to $-\xi^\mu$ does not affect Eq. (1).
We may now turn briefly to other questions of orthogonality. Let us first notice the
symmetry of the relation (6); the two incidence relations "$\zeta" contains the pole of $\eta$" and
"$\eta$ contains the pole of $\zeta$" are therefore distinct, but equivalent, geometric statements.
Next, let us consider a plane $\zeta$, and a point 0 in it. There must exist a line $n$ through 0
normal to the plane $\zeta$, i.e. a line such that any plane $\xi$ containing it is orthogonal to $\zeta$.
One sees at once that one obtains this line by joining 0 to the pole ($\zeta^0$) of the plane $\zeta$.
This construction, which is also valid when 0 does not belong to $\zeta$, yields the "coordinates"
$n^0$ of the line $n$ (see Appendix D.5).

Conversely, consider a real point ($0^\mu$) and a line $n$ through it. We construct the plane
normal to $n$ through 0 as follows. Consider the (imaginary) plane $\bar{0}$, with coordinates
$0^\mu = g^\mu_{\nu}0^\nu$, the "polar plane" corresponding to 0. The (imaginary) intersection ($\zeta^0$)
of this plane with $n$ is easily calculated: $\zeta^\mu = 0^\mu n^\nu$, see Eq. (18') later. It is the pole $\hat{z}$
of a real plane $\zeta$. Because of the above-mentioned "symmetry", $0^\mu \zeta^\mu = 0$ implies "$\zeta$ contains $0^\mu";
that is, $\zeta$ is a plane through 0 and $n$ is normal to it.

Finally, we can describe orthogonality between two lines $k$ and $n$, as the relation:
"$k$ is contained in a plane $\zeta$ normal to $n$" or equivalently: "$n$ is contained in a plane $\xi$
normal to $k$". The equivalence may be proved, using the notion of "polar line" (Appendix B),
or from Eq. (29) (orthogonality means $\cos \phi = 0$).

6. AN EXAMPLE

We now wish to describe a relatively simple way of analysing a reaction. More sophis-
ticated ways of calculating are described in the next section; in the long run, I believe
they might represent a saving of time, in spite of the time lost in learning the somewhat
elaborate vector notation.

Suppose we have a reaction $A + B \rightarrow O + \text{anything}$, then $O$ decays (strongly or weakly)
into $C + \text{anything}$, then ... (one may go on). The velocity points $OABC$ define a tetrahedron
(Fig. 3). In order to analyse the decay of $O$, if it has a spin, one wants to transform to
a "rest-system" for $O$, then define the direction of emission of $C$ (for example) in this
system. This is equivalent to saying: we use $O$ as the origin of a Cayley map (see also
Appendix B) of velocity space. Then all angles between velocities in the above-mentioned
rest-system can be interpreted in two ways: a) as true angles, in the intrinsic geometry
of the manifold, between lines through $O$; or b) as ordinary angles on the (Euclidean)
mapping space (the map is "conformal" near the origin). Because of (a), certain angles,
such as $\alpha$, $\beta$, $\gamma$ in the figure, may be calculated in terms of measured four-momenta (no
Lorentz transformation necessary), from the simple formulae for triangles [see Section 7,
Eqs. (21) and (22)].

Any other desired angle "at the origin", such as the angle between planes $AOC$ and $BOC$,
may be calculated, because of (b), by ordinary formulae of spherical trigonometry or ordi-
nary vector calculus. For example, if $\hat{a}$, $\hat{b}$, $\hat{c}$ are unit vectors in the direction of the
velocities of $A$, $B$, $C$ in the rest-system, one can write $\hat{c} \cdot \hat{b} = \cos \gamma$, etc., and go on from
there. As an example, suppose we want to use $\hat{a}$ as the direction of the $x$-axis in the rest-
frame, and $\hat{a} \times \hat{b}$ as the direction of the $z$-axis (the customary "normal to the reaction
plane"). Then the angle $\theta$ (one of the polar angles one uses to define $\hat{c}$, the direction of
emission of particle $C$) is obviously given by

$$\cos \theta = \sin \phi \sin \beta = \frac{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}{\sin \gamma}.$$ 

The azimuth $\psi$ of the direction $OC$, relative to the $zx$-plane for example, can obviously be
evaluated similarly, e.g. from:

$$\cos \beta = \sin \theta \cos \psi.$$ 

As stated, the angles $\alpha$, $\beta$, $\gamma$ are obtained from Eqs. (21) and (22) for triangles. The more
elaborate methods described later arrive, of course, at equivalent results.

We may now want to move on to a "rest-system for $C"$, in order to analyse the polariza-
tion of particle $C$, and its effect on a subsequent decay or scattering experiment. We do
not have to perform any Lorentz transformation, for again its effect may be described as
drawing a Cayley map in which $C$ is the origin. We can describe what happens without drawing
a new picture; we simply use $C$ as we have used $O$ before. A system of axes may be chosen in
many ways. The line $OC$ could be the new $z$-axis (a "helicity" convention for the spin of
particle $C$). Then the plane $AOC$ could be used, as we used $OAB$ before, to define the new
$zx$-plane, with $C$ as an origin. If $C$ emits a particle $D$, then $CAOD$ would be a tetrahedron,
and we could start the same calculation again. There is no point in describing specific
choices of axes, as the reader may have his own reasons for choosing one or the other. The
technique is sufficiently explained.

As we stated, it may pay to use the more sophisticated vector calculus of Section 7.
Using measured four-momenta, one could then begin by calculating coordinates $\xi^{ab}$ or $\lambda_{ab}$ for
all the lines that one needs, and $\xi_\alpha$ for all the planes that one needs. Angles between
planes, between lines, or between planes and lines, are then given by a few general formulae.
7. A VECTOR CALCULUS FOR KINEMATICS*)

The natural tool here is the exterior product of Grassman Algebra **), defined in Appendix C, Eq. (C.1); see also, as simple examples, Eq. (8) below and Eq. (C.2).

Using $A$, $B$, $C$, ... (sometimes also $x$) to denote velocity points as well as the corresponding four-vectors $(A^1)$, $(B^1)$, ..., etc., we shall encounter exterior products of two, three, or occasionally four vectors: $A \wedge B$, $A \wedge B \wedge C$, $A \wedge B \wedge C \wedge D$. Let us state at once how these are used (the why will soon be clear).

1) $A \wedge B$ corresponds to the line segment from $A$ to $B$ ***). It is used as a set of coordinates, defining the line $AB$.

2) $A \wedge B \wedge C$ corresponds to the oriented triangle $ABC$. It is used as a set of coordinates [easily related to the $\xi$'s of Eq. (2)] defining the plane through $A$, $B$, and $C$, and an orientation for it.

3) $A \wedge B \wedge C \wedge D$ corresponds to an oriented tetrahedron.

In order to avoid cluttering up the main ideas with details, most statements concerning orientations will be postponed to Appendix D, where they are summarized by figures.

Let us continue. We shall use $\lambda$, $\mu$, $\nu$, ... to denote "bivectors", i.e. antisymmetric tensors with two indices. The exterior product $A \wedge B$ is such a tensor [compare Eq. (C.1)]; $\lambda = A \wedge B$ stands for

$$\lambda^{ij} = A^i B^j - A^j B^i . \quad (8)$$

A bivector of this form is called "decomposable"; its six components ($\lambda^{12}$, $\lambda^{13}$, $\lambda^{23}$, $\lambda^{31}$, $\lambda^{12}$) satisfy an identity (Plücker) [see Eq. (C.18)]. Not every bivector, therefore, is decomposable. The products $A \wedge B \wedge C$ and $A \wedge B \wedge C \wedge D$ are likewise tensors $\tau^{\alpha\beta}$ and $\rho^{\alpha\beta\gamma\delta}$ with three and four indices, respectively. An exterior product of vectors is antisymmetric in its factors, e.g. $A \wedge B = -B \wedge A$, $A \wedge B \wedge C = -B \wedge A \wedge C = B \wedge C \wedge A = ...$, etc.

In Grassman Algebra one extends the $\wedge$ symbol to products of these tensors. The product of two antisymmetric tensors with $p$ and $q$ indices, respectively [as defined in Eq. (C.1)], obeys the $(-1)^{pq}$ symmetry rule:

$$\begin{align*}
(p, q) &= (1, 2) & A \wedge \lambda &= \lambda \wedge A \\
&= (1, 3) & A \wedge \tau &= -\tau \wedge A \\
&= (2, 2) & \lambda \wedge \mu &= \mu \wedge \lambda .
\end{align*} \quad (9)$$

*) The more abstract techniques described in this section may interest only some readers. In essence we use nothing more arcane than antisymmetric tensors with upper or lower indices. By picking judiciously (we hope) from the vast literature on the subject, what seemed a convenient and practical formalism, we have aimed at a relatively simple and intuitive form of calculus, not unlike the ordinary vector calculus.

**) A simple introduction is given elsewhere (*). A more systematic but less accessible approach can also be found in the literature (**).

***) Two line segments correspond to the same value of $A \wedge B$ [with a suitable normalization: see Eq. (C.12)] if, and only if, they are a) on the same line, b) equally oriented, and c) of equal length.
We increase our calculational power if we develop in parallel an exterior algebra of tensors with lower indices, defining for example $\Lambda = \xi \wedge \eta$ by

$$\Lambda_{ab} = \xi_a \eta_b - \xi_b \eta_a.$$  \hspace{1cm} (10)

We use $\Lambda, M, \ldots$ to denote tensors with two lower indices. We shall see that $\Lambda = \xi \wedge \eta$ can be used (just like $A \wedge B$) to describe a line, \textit{if it is defined as the intersection of two planes}, $\xi$ and $\eta$ [see Eqs. (4)].

Finally, let us mention that the exterior product is associative:

$$\left[ A \wedge B \right] \wedge C = A \wedge \left[ B \wedge C \right] = A \wedge B \wedge C$$ \hspace{1cm} (11)

$$\left( A \wedge B \wedge C \right) \wedge D = \left( A \wedge B \right) \wedge \left( C \wedge D \right) = A \wedge B \wedge C \wedge D$$ \hspace{1cm} (12)

and that one must be careful not to mix tensors with upper and lower indices in exterior products. The proper way to handle mixed products is, of course, by "saturation of indices", which leads to the \textit{interior} products of Appendix C. The one important fact that remains to be mentioned is the "canonical" relationship* between antisymmetric tensors with $k$ upper indices, and tensors with $4 - k$ lower indices. It involves, of course, the "universal" antisymmetric Kronecker tensors $\varepsilon_{\alpha \beta \gamma \delta}$ and $\varepsilon_{\alpha \beta \gamma \delta}$ (see conventions in Appendix C). For instance, $q = A \wedge B \wedge C \wedge D$ is transformed into a scalar** quantity $e q$ or

$$[ABCD] = A^\alpha B^\beta C^\gamma D^\delta \varepsilon_{\alpha \beta \gamma \delta},$$ \hspace{1cm} (13)

and similarly $t = A \wedge B \wedge C$ is transformed into a dual vector $\xi$ [compare Appendix C, Eq. (C.14)]:

$$\varepsilon_{\alpha \beta} = \frac{1}{3!} \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{\alpha \beta \gamma \delta} = A^\alpha B^\beta C^\gamma \varepsilon_{\alpha \beta \gamma \delta},$$ \hspace{1cm} (14)

which we also abbreviate to $ct$ or

$$\xi = e [A \wedge B \wedge C] = [ABC].$$ \hspace{1cm} (14')

Finally, an $\varepsilon_{\alpha \beta}$ is transformed "canonically" into $\Lambda = e \xi$:

$$\Lambda_{\alpha \beta} = \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{\alpha \beta \gamma \delta}. $$ \hspace{1cm} (15)

Notice that the components of $\Lambda$ are simply a re-labelling of the components of $\xi$:

$$\Lambda_{\alpha \beta} = \xi_{jk}, \quad \varepsilon^{\alpha \beta} = \Lambda^{jk},$$ \hspace{1cm} (16)

where $i,j,k$ is an even permutation of $1,2,3$.

* Invariant with respect to linear transformations with determinant $+$1.

** Or rather pseudoscalar; the distinction is washed out, since we do not use "improper" transformations.
This algebraic scheme, which is a bit more complicated than ordinary vector calculus, but not much, can handle concisely all incidence relations (and hence also orthogonality questions) with the greatest ease. The following propositions are trivially proved:

i) "A coincides with B" is equivalent to \( A \wedge B = 0 \). Similarly, "planes \( \xi \) and \( \eta \) coincide" is equivalent to \( \xi \wedge \eta = 0 \);

ii) "A, B, C are collinear" is equivalent to \( A \wedge B \wedge C = 0 \) *

iii) "A, B, C, D are coplanar" is equivalent to \( A \wedge B \wedge C \wedge D = 0 \) or also to \([A B C D] = 0\).

The reasons for statements (1) and (2) earlier in this Section are now becoming clear. We see that:

iv) the equations of the line through two (distinct) points A and B may be written \( A \wedge B \wedge x = 0 \), i.e. [see Eq. (11)] with \( \ell = A \wedge B \):

\[
\ell \wedge x = 0.
\] (17)

Using the canonical transformation Eq. (14'), and the definition Eq. (15), Eq. (17) takes the form **

\[
x^\alpha \Lambda_{\alpha B} = 0.
\] (18)

It is now obvious why the six components \( x^{\alpha B} \) of the bivector \( \ell \) (or equivalently those of \( \Lambda \)) may be used as (homogeneous) coordinates of the line.

v) The equation of the plane defined by non-collinear points A, B, C is \( A \wedge B \wedge C \wedge x = 0 \), i.e.

\[
t \wedge x = 0,
\] (19)

where \( t = A \wedge B \wedge C \). Multiplying by \( \varepsilon \), as in Eq. (13) and transforming \( t \) according to Eq. (14), the equation also takes the equivalent forms

\[
[ABCx] = 0; \quad \text{or} \quad x \cdot \xi \equiv x^\alpha \xi_\alpha = 0.
\] (20)

Thus Eq. (14') yields the dual vector \( \xi \) defining the plane.

vi) It is easy to see that if Eq. (17) is not satisfied, then \( \xi_\alpha = \Lambda_{\alpha B} x^\alpha \) defines the plane through line \( \ell \) and point \( x \), if \( \Lambda = \varepsilon \ell \). In a similar way

\[
x^\nu = \xi_\nu \ell^{\nu} ,
\] (18')

if it is not zero, defines the point of intersection of the plane \( \xi \) with the line \( \ell \). If it is zero, the plane contains the line.

* Again there is a similar statement about three planes \( \xi, \eta, \zeta \) having a line in common; but we shall not have occasion to use it.

** At first sight, \( B = 0, 1, 2, 3 \) in (18) yields four equations (too many!) for the line. The rank of the system is, however, only 2, as explained in Appendix C after Eq. (C.18).
Without giving detailed proofs, we shall now state various "metric" results, which are obtained by examining, in the light of the discussion of Appendices A and B, the geometric meaning of various absolute invariants depending on the coordinates of 2, 3, or 4 points, or of two planes, or two lines, etc.

11. The one and only invariant depending on two points determines the distance (or "rapidity") between them [see Eqs. (C.12) and (C.13)].

12. Next we consider a triangle ABC. Let \( \hat{\alpha} \) be the internal angle at vertex A. The only independent invariants are \( \rho_{AB}, \rho_{AC}, \rho_{BC} \) (or alternatively \( \rho_{AB}, \rho_{AC} \) and \( \cos \hat{\alpha} \)), and they determine all the others. We consider most simply the symmetric invariant \( |A \wedge B \wedge C| \equiv |\xi| \), where \( \xi \) is the vector of Eq. (14'). When divided by \( |A||B||C| \), this becomes an "absolute" invariant (i.e. homogeneous of order zero in each of the vectors A, B, C) which in a sense measures the "size" of the triangle (size being not quite the same thing as area). We also have (see Appendix C)

\[
\frac{|A \wedge B \wedge C|}{|A||B||C|} = \frac{\rho_{AB}}{AB} \frac{\rho_{AC}}{AC} = \frac{|A \wedge B| |A \wedge C|}{|A|^2 |B||C|} \sin \hat{\alpha},
\]

which is analogous to the formula which expresses the area of an ordinary triangle as \( \frac{1}{2} \rho_{AB} \rho_{AC} \sin \hat{\alpha} \). It may be used to calculate \( \hat{\alpha} \). In order to distinguish between \( \hat{\alpha} \) and \( \pi - \hat{\alpha} \), however, it is preferable to use another absolute invariant

\[
\cos \hat{\alpha} = \frac{(\overline{C} \cdot A)(\overline{A} \cdot B) - (\overline{A} \cdot A)(\overline{C} \cdot B)}{|A \wedge B||A \wedge C|} = \frac{\left(\begin{array}{c} A \\ B \\ C \end{array}\right)}{|A|^2 \left(\begin{array}{c} A \\ B \\ C \end{array}\right)}.
\]

Finally, we may use bivectors \( \lambda = A \wedge B, m = A \wedge C \) to describe the sides of the triangle, and use Eq. (29), and \( \rho = 0 \) (see later) to calculate \( \cos \hat{\alpha} \).

13. Two dual vectors \( \xi \) and \( \eta \) defining planes, give rise to the invariant:

\[
\xi \cdot \eta \equiv \eta, \xi = |\xi||\eta| \cos \phi,
\]

where \( \phi \) is the angle between the two planes.*

One can also calculate \( \sin \phi \) as follows. Equation (10) gives the bivector \( \Lambda \) associated to the intersection of the two planes (see also Appendix C.3). If we normalize \( \Lambda \) by \( |\Lambda| = 1 \), then

\[
\frac{\xi \wedge \eta}{|\xi||\eta|} = \sin \phi \Lambda.
\]

*) This is the easiest one to remember! Actually \( \phi \) is an angle only if

\[-|\xi||\eta| \leq \xi \cdot \eta \leq |\xi||\eta|.

One can show that if this is not the case, the two planes do not intersect, and \( \cos \phi = \cosh \rho_{PQ}, \rho_{PQ} \) being the distance between the two planes, or the minimum value of the distance between a point on \( \xi \) and a point on \( \eta \). The minimal line segment \( P \wedge Q \) then is orthogonal to both planes.
I4. The four vertices $O, A, B, C$ of a tetrahedron define six independent invariants (the six distances $\rho_{OA}, \ldots, \rho_{BC}$ may be varied independently). These include, of course, the invariants of the triangles $OAB, \ldots,$ etc., and of the pairs of planes, such as the angle $\phi$ between the planes $\xi = [O A C], \eta = [O B C]$. These have, of course, a real intersection $O \wedge C$, so that Eq. (23), which may now be written

$$
\cos \phi = \frac{[O \wedge A \wedge C] \cdot [O \wedge B \wedge C]}{|O \wedge A \wedge C||O \wedge B \wedge C|} = \frac{[O A C]}{|O A C|^2},
$$

must yield a real value for $\phi$.

A new invariant is the determinant $[O A B C]$ of the components $O^u, A^u, \ldots,$ etc., which obeys the relation

$$
\frac{[O A B C]}{|O A B||B C|} = \pm \frac{1}{2} [O A B C]^{\frac{1}{2}}.
$$

The $\pm$ ambiguity is, as usual, due to the existence of a "mirror image" (see Appendix D.2). Other invariants involving all four points in an essential way are the Gram determinants of the type $[O C]$. All these invariants may be interpreted geometrically in a variety of ways. In particular we have

$$
\sin \phi = \frac{|O \wedge C|}{|O \wedge A \wedge C||O \wedge B \wedge C|} [O A B C] = \frac{k}{|\xi / \zeta|} [O A B C],
$$

where $k = O \wedge C$. All norms are defined as in Appendix C. The sign of $\phi$ as determined by this equation corresponds to the conventions of Appendix D.3, if $\xi = O \wedge C \wedge A, \eta = O \wedge C \wedge B$.

Other interpretations arise if we introduce orthogonal elements, for instance the normal $n = O \wedge \xi$ to the plane $\xi = [O A B]$, see Fig. 3. Then $\theta$ being the angle between $n$ and the line $k = O \wedge C$, we have

$$
\cos \theta = \frac{|O [O A B C]|}{|O A \wedge B||O A \wedge C|}.
$$

This equation, which is equivalent to Eqs. (2.2) or (2.4) in the paper by Berman and Jacob*, could of course be derived from Eq. (26) ($\phi$ being the angle denoted by $\Phi_C$ in Fig. 3) by spherical trigonometry (see Section 6). Similarly Eq. (2.3) (loa. cit.) is of a type which may be derived from Eq. (22) by the well-known relation $\cos a = \cos b \cos c$ between the hypotenuse and the legs of a spherical right triangle. If, for example, $\psi$ in Fig. 3 is the azimuthal angle between the two planes through the normal $n$, containing point A and point C, respectively, it may also be described as one of the legs of a spherical triangle,

*) See Ref. 5. One must bear in mind the different definition of polar angles in the said paper.
the other leg being \((\pi/2) - \theta\). The hypotenuse is the angle \(\theta\) at 0 in the triangle \(AOC\). Therefore

\[
\sin \theta \cos \psi = \frac{0 A}{O A} \frac{A C}{O C}.
\]  

(28)

I5. Finally, for the sake of completeness, we give the interpretation of the invariants associated with two lines, defined by bivectors \(\ell\) and \(m\). Assume \(|\ell| = |m| = 1\). In general the two lines are not coplanar; it is easy to see that the distance \(\rho_{PQ}\) between a point \(P\) on \(\ell\), and a point \(Q\) on \(m\), must have a minimum. This minimum corresponds to a line \(n = P \wedge Q\) normal to \(\ell\) and \(m\), and may be called the distance \(\rho\) between the lines. The angle \(\phi\) between the planes containing \(\ell\) and \(n\), respectively \(m\) and \(n\), may be called the angle between the lines. One has the relations

\[
-\vec{X} \cdot \vec{m} = -\frac{1}{2} \epsilon_{\mu \nu} m^{\mu \nu} = \cosh \rho \cos \phi
\]

(29)

\[
-\epsilon m \cdot \ell = -\frac{1}{2} \epsilon_{\rho \theta \mu \nu} m^{\rho \theta} m^{\mu \nu} = \sinh \rho \sin \phi.
\]

(30)

These two equations are elegantly summarized in one:

\[
-\cosh (\rho + i\phi) = (\vec{X} + i\vec{\ell}) \cdot \vec{m}.
\]

(31)
APPENDIX A

INTRINSIC GEOMETRY

We sketch here some of the differential-geometric background\(^*\) to the considerations of Sections 2 and 4. The "intrinsic" geometry, according to Gauss, of an ordinary surface is defined by measurements performed entirely on the surface, or more formally by the ds\(^2\), a quadratic form in the differentials du, dv of the curvilinear coordinates, which are used to define a point on the surface. In a similar way we can calculate the infinitesimal distance ds between two points (x\(^{i2}\)) and (x\(^{i3}\) + dx\(^{i3}\)) on the velocity hyperboloid, Eq. (1). It is defined by the metric of the embedding space as:

\[
ds^2 = g_{ij} \, dx^i \, dx^j.\tag{A.1}
\]

The indefinite character of this metric disappears, when we take into account the fact that the four differentials dx\(^2\), ..., dx\(^3\) are not independent; since both points obey Eq. (1), one has \(g_{ij} x^i x^j = 0\). As one easily sees, this implies that the expression (A.1) is \textit{positive}. If, for example, we use the ordinary velocity components \(v_i = x^i / x^2\) (i = 1, 2, 3) as independent variables (curvilinear coordinates) to define a point in the manifold, and set \(v^2 = \sum v_i v_i^2\) and \(\gamma = \gamma(v) = (1 - v^2)^{-\frac{1}{2}}\) as usual, we find that Eq. (A.1) reduces to

\[
ds^2 = \gamma^2 \sum v_i v_i + \gamma^4 \left( \sum v_i \, dv_i \right)^2,\tag{A.2}
\]

which is obviously positive definite. If we split the vector \(\vec{v}\) into a longitudinal component \(dv = v^{-1} \sum v_i \, dv_i\) and a transverse component \(dv_\perp\), we can also write

\[
ds^2 = \gamma^2 \, dv^2 + \gamma^4 \, dv_\perp^2,\tag{A.3}
\]

which will be used later. The differentials dv\(_i\) may be regarded as components (along three different directions) of a tangent vector. It is possible, by means of a linear transformation \(\omega_i = \sum A_{ij} \, dv_j\), to reduce the form (A.1) to a sum of squares: \(ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2\). The coefficients \(A_{ij}\) are functions of \(v_1, v_2, v_3\); in a sufficiently small neighbourhood of a point \(x\) of the manifold, however, they may be treated as constants. If we now consider \(\omega_1, \omega_2, \omega_3\) as orthogonal components of a vector \(\vec{v}\) in ordinary space, we see that ds is the length of this vector. In short, a little thought will show\(^*\) that this construction implies that a sufficiently small neighbourhood of a point \(x\) on the manifold can be mapped "isometrically" onto a piece of ordinary space\(^\dagger\). This implies, in particular, that the geometry of directions (or of tangent vectors) \textit{from a given point} \(x\) is identical with that of ordinary space. For example, the angle \(\theta\) between two displacements \((dv_1)\) and \((dv')\) is naturally defined, after transforming to vectors \(\vec{u}\) and \(\vec{u}'\), respectively, by\(^\ddagger\)

\[
ds \, ds' \cos \theta = \vec{u} \cdot \vec{u}' = \sum \omega_i \omega_i'.\tag{A.4}
\]

\(^*\) This conclusion applies to any three-dimensional Riemann space with a positive definite ds\(^2\).

\(^\dagger\) Infinitesimal displacements or tangent vectors \textit{at different points}, are not so easily compared!
The right-hand side is easily recognized as the bilinear form \((dv_1^2, dv_1')\) associated to the quadratic form \((A.2)\). More generally, all problems relative to angles between lines through a point are subject to the rules of ordinary spherical trigonometry.

The length of a line (on the manifold) from A to B being defined as the integral of ds along the line, calculating the shortest path (geodesic) between two points becomes an easy variational problem. We shall only give the result. First of all the geodesic distance \(\rho_{AB}\) between two points A and B is given by the well-known "rapidity" relation

\[
v_{AB} = \tanh \rho_{AB},
\]

where \(v_{AB}\) is the velocity of B in the rest system of A (or vice versa). The most direct formulae for calculation are Eqs. (C.12) and (C.13) later. Next, the geodesic lines themselves are obtained in the Minkowski picture (Fig. 1) as intersections of the velocity hyperboloid with (two-dimensional) planes\(^*) through the centre of the hyperboloid (the "origin" \(\Omega\)); in other words, they are the straight lines of Section 4. The result is quite analogous to the well-known fact that shortest paths on a spherical surface are arcs of great circles, i.e. of intersections of the sphere with planes through the centre of the sphere. After this result the definitions of planes and lines given earlier become quite natural! We notice, in particular, the analogy between the incidence axioms of ordinary geometry and the following propositions (which are easily proved):

a) there is one, and only one, line through two given points A and B;

b) there is one, and only one, plane through three given non-collinear points A, B, C, or through a line and a point, not on the line;

c) two planes with a common point have a line in common.

Here we refer specifically to real elements (points, planes, lines). For imaginary points, etc., see Appendix B.

Besides incidence axioms, of course, specifically metric properties are most important. Amongst these is the "mobility axiom". Movements are one-to-one transformations of a space into itself, which leave all metric relations (or more simply: the \(ds^2\)) invariant\(^**\). They obviously constitute a group. Movements which leave a certain point O unchanged may be called "rotations about O"; they form a subgroup of the group of movements. The mobility axiom states that non-Euclidean space possesses full mobility, i.e. any point A can be moved to any other point B in the space, and furthermore the group of rotations about any point is the same as ("isomorphic to") the rotation group of ordinary space.

That this is so follows from the Lorentz invariance of Eq. (1) and of the quadratic forms \((A.1)\) and \((A.2)\). One easily recognizes that the transformations of the homogeneous Lorentz group, applied to the coordinates \(x^\mu\) of a point of the velocity manifold possess all the characteristics stated above; they are therefore the movements we have been speaking of.

An obvious consequence of full mobility is that the neighbourhoods of two different points look exactly alike; in particular, the curvature K of the space must be the same,

\(^*)\) A plane in Minkowski space is the intersection of two hyperplanes, i.e. it consists of points satisfying two linear equations such as \((4)\).

\(^**\) We only consider "true" movements, which do not reverse the orientation of the space (i.e. do not interchange right-hand with left-hand).
i.e. constant over the space. Calculation shows that it is negative ($K = -1$). This means, for example, the following: if we join three non-collinear points $A$, $B$, $C$ by straight line segments, we obtain a triangle, the internal angles of which may be called $\hat{A}$, $\hat{B}$, $\hat{C}$. The generalization of Cavalieri's theorem states that the "spherical excess", $\hat{A} + \hat{B} + \hat{C} - \pi$, is equal to the area of the triangle multiplied by $K$ (hence it is negative). This fact is strictly connected with Wigner's "rotation of the spin" in a Lorentz transformation (see Fig. 4).

Fig. 4

The "Wigner" rotation

Two observers, using rest-frames for particles $A$ and $B$, respectively, have agreed to use the direction $AB$ as the $z$-axis. Both define the spin direction of particle $C$ relying on measurements made in a "rest-frame for $C". In order to specify this rest-frame, however, "A" uses a "boost" from his own rest-frame. If the spin is as indicated in the figure, "A" decides that the spin lies in the direction $AC$, i.e. forms an angle $\hat{A}$ with the $z$-axis. Reasoning in a similar manner, hence using a boost from his own rest-frame, "B" concludes that the spin forms an angle $\hat{B}$ with the direction $BC$, hence an angle $\pi - \hat{B} - \hat{C}$ with the $z$-axis. The angles $\hat{A}$, $\hat{B}$, $\hat{C}$ are measured in the respective rest-frames. Hence they are, also, the angles of the non-Euclidean triangle $ABC$, so that the angle of the spin with the $z$-axis, measured by "B", i.e. $\pi - \hat{B} - \hat{C}$ is equal to $\hat{A}$ plus the area of the triangle. (Compare Ref. 2.)
APPENDIX B

CAYLEY MAPS

Cayley's famous Model of Non-Euclidean Space may be described as a map of this space onto the interior of a sphere in ordinary space. In this map, the lines and planes of NES are represented by straight lines and planes intersecting the sphere (or more precisely, if we think of a line or plane as a locus of real points, we must consider only that part of the said lines and planes in ordinary space which lies inside the sphere). "Movements" are real projective transformations of ordinary space which transform the surface of the sphere into itself\(^*\)). Finally, the metric properties of the space are neatly contained in Cayley's formula, which expresses the non-Euclidean distance between two points as the logarithm of a cross ratio (not shown here). We mention these things as background, without going into details.

In connection with our kinematic interpretation of the space, the Cayley map is a triviality, since it amounts to the following. First choose an arbitrary inertial system of reference; then treat the components \(v_1^n\) (i = 1, 2, 3) of any velocity, measured relative to the above system, as Cartesian coordinates of a point in ordinary space, so that in particular the origin \(O\) of the coordinate system represents zero velocity. The "map" is, therefore, just the old-fashioned velocity space of prerelativistic days\(^*\)). Points corresponding to particles moving with the velocity of light \(c\) will lie on the surface of a sphere of radius \(c\) with the centre in the origin. Setting \(c = 1\), it will be called the unit sphere (or the light sphere). As the non-Euclidean distance of these points on the surface from points inside the sphere is infinite, they are, strictly speaking, not proper elements of the space ("points at infinity").

Since this picture is useful, let us also explain it in another way, with reference to Fig. 1. Let us assume that the "chosen reference system" is just the same as the one used in setting up the axes in Minkowski space. Then it is clear that the "Cayley map" is nothing but a projection of the manifold into the tangent hyperplane (in the figure: a plane, since we have suppressed the \(x^3\) coordinate) \(x^0 = +1\). The projection is a "central projection" from the origin of Minkowski space, which is also the centre of the hyperboloid. The analogy with the central projection of a sphere onto a tangent plane gives an intuitive explanation of many of our results (e.g. the fact that geodesics project into straight lines!).

The region near the origin (non-relativistic velocities, i.e. \(|v| < c\), i = 1, 2, 3) has the important property that there the \(ds^2\) [Eq. (A.2)] may be approximated by \(ds^2 = \Sigma v^2\), which is the \(ds^2\) of the mapping (Euclidean) space. This is the same as saying that the map is undistorted near the origin, which is more or less the same as the

\(^*\) They must necessarily transform the interior of the sphere into itself, since interior points may be defined as those points from which no (real) tangent to the sphere may be drawn.

\(^*\) The picture is often discarded, because it may create the wrong impression that the relative velocity of two moving systems is represented by the vector joining the corresponding velocity points. We just have to get rid of that habit, by remembering that we are dealing with a map, and that maps generally involve distortions.
fact that small velocities are also conveniently handled by ordinary vector calculus. An important consequence is the following simple connection between angles defined by non-Euclidean geometry, and the angles one usually talks about.

Consider in Fig. 3 the angle between two lines intersecting at a point 0, or two planes intersecting along the line OC. We can always choose a reference system such that the velocity represented by 0 is zero. Then in a Cayley map, adapted to this system, 0 is the origin and the vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$, respectively, represent the velocities $\vec{v}_A$, $\vec{v}_B$, $\vec{v}_C$, measured in such a system (a "rest-system for 0"). The (non-Euclidean) angle $\gamma$ is then simply the angle between $\vec{v}_A$ and $\vec{v}_B$. Similarly $\phi$ [Eq. (24)] corresponds to the formula:

$$\cos \phi = \frac{V_A \times V_C \cdot (V_B \times V_C)}{|V_A \times V_C| |V_B \times V_C|}.$$  \hspace{1cm} \text{(B.1)}$$

Formula (24) could be obtained, as is often done, by guessing the covariant expression which reduces to Eq. (B.1) in the said rest system. It is then obvious that all the angles commonly employed in the analysis of complicated events can be most simply described as angles defined by the intrinsic geometry of the velocity manifold.

Cayley maps also provide a convenient representation of imaginary elements. Since for a space-like $x^0$ the velocity is non-physical, the corresponding point in the map is a real point outside the unit sphere. Similarly a time-like $\xi^0$ (an imaginary non-Euclidean plane) corresponds to a real plane (or a real plane) which does not intersect the sphere, and an imaginary line is mapped into a line which does not intersect the sphere. One has then a very simple picture of polar elements. The pole $\xi$ of a real plane, as defined in Appendix C, Section 3, is a point outside the sphere; it is easy to show, by ordinary methods of analytic geometry, that the tangent cone, whose vertex is $\xi$, touches the sphere along the intersection of the sphere with the plane $\xi$ (see Fig. 5). The polar line corresponding to a given line may be similarly described; one of the two lines intersects the sphere; the two planes tangent to the sphere at the intersection points P and Q, intersect in the other line (see Fig. 6).

Let us now describe briefly the distortions of the Cayley map. Equation (A.3) shows clearly how true distances are more and more contracted in the map as one approaches "infinity". This is how points at infinity can be mapped into points at unit Euclidean distances from the centre, in the map. Furthermore, infinitesimal distances in the radial direction are contracted in the ratio $1: \gamma^2$, more than those in a transverse direction (1:$\gamma$). The result is a distortion of angles; the non-Euclidean angle $\Theta$ in Fig. 7 is not the same as the angle $\Theta_{\text{Eucl}}$ which one can "read off the map". From Eq. (A.3) one sees that the relation between the two is:

$$\tan \Theta_{\text{Eucl}} = \gamma \tan \Theta.$$  \hspace{1cm} \text{(B.2)}$$

Such angles are "smaller than they look". If Eq. (B.2) looks familiar, it is not surprising, since $\Theta$ is also obtainable by Lorentz-transforming point P to the origin.

*) The reader should carefully note that the angles indicated in all diagrams are never defined as the angle that the eye sees (except, of course, near the origin, when there is no difference).
Fig. 5
Every line through the pole $\xi$ is orthogonal to the plane $\xi$.

Fig. 6
The planes are tangent to the sphere at points $P$ and $Q$. The "real" line $\lambda$ and the "imaginary" line $m$ are conjugate (one is the "polar line" of the other). As explained in Section 7, the line $\lambda$ is defined by coordinates $R^B_A$, or equivalently by the canonically transformed tensor $\Lambda_{QQ}$, Eq. (15). Let $\Lambda^B_A$ and $\Lambda_{QQ}$ be the corresponding tensors for the polar line $m$. The relation between the two lines is then expressed by $M = \mathbb{I}$, or equivalently $\Lambda = -B$. This means, in the notation of Eq. (16):

$m^B_A = \xi^i \xi^j$, $m_{jk} = -\xi^i \xi^j$.

Fig. 7
Distortion of angles in the Cayley map for velocities close to the velocity of light. $\theta$ is the "true" (non-Euclidean) angle. It is smaller than the angle $\theta_{\text{Eucl}}$ on the map, because the "true" sides of the infinitesimal triangle undergo a different contraction in the map.
APPENDIX C

ADDITIONAL CONVENTIONS AND RULES OF CALCULATION

1. The exterior product $s \wedge t$ of two antisymmetric tensors, $s = (s^{a \cdot \cdot \cdot})$ and $t = (t^{\lambda \cdot \cdot \cdot})$, where $a \cdots$ and $\lambda \cdots$ are $p$ and $q$ indices, respectively, is an antisymmetric tensor with $p + q$ upper indices defined by

$$
(s \wedge t)^{a_1 \cdots a_p \lambda_1 \cdots \lambda_q} = \frac{1}{p! q!} (s^{a_1 \cdots a_p} t^{\lambda_1 \cdots \lambda_q} \varepsilon) ,
$$

(C.1)

where $\varepsilon$ denotes antisymmetrization *). The product is obviously zero if $p + q > n$, the dimensionality of the vector space ($n = 4$ in our case). It is associative, as one easily shows: $(s \wedge t) \wedge u = s \wedge (t \wedge u)$. As an example of Eq. (C.1), if $p = 1$, $q = 2$, one has:

$$
(s \wedge t)^{a \lambda u} = s^a t^{\lambda u} + s^a t^{\mu a} + s^\mu t^{a \lambda} .
$$

(C.2)

Exterior products of tensors with lower indices are defined in exactly the same way.

2. With "tensors" we always mean antisymmetric tensors. We use, of course, the "symmetric tensors" $g^{\mu \nu}$ and $g_{\mu \nu}$ to raise and lower indices. They are defined with the signature ($-, +, +, +$). The "universal" antisymmetric tensor $\varepsilon$ is normalized by

$$
\varepsilon^{0123} = +1; \text{ hence: } \varepsilon^{0123} = -1 .
$$

(C.3)

3. Notations: $\xi, \bar{\xi}, \xi, \bar{\xi}$. We restate the convention: vectors with components $A^\mu, B^\mu, \ldots, x^\mu$ with upper indices represent points, dual vectors with components $\xi_\mu, \eta_\mu, \ldots$ represent planes.

$\xi$ is a point with coordinates $\xi^\mu = g^{\mu \nu} \xi_\nu$. Similarly $\bar{\xi}, \check{\xi}$ are planes with coordinates $A_\mu = g_{\mu \nu} A^\nu$ or $x_\mu = g_{\mu \nu} x^\nu$; $\bar{\xi}$ is similarly defined by "lowering the indices" of $\xi$:

$$
\bar{\xi} = (\xi_{ab}); \check{\xi}_{ab} = g_{ab} g_{cd} \xi^{cd} ,
$$

(C.4)

$\bar{\xi}$ by raising the indices of $\Lambda$.

4. Scalar products and norms. The invariant bilinear form

$$
\xi \cdot x = \xi^\mu x_\mu
$$

(C.5)

allows one to regard a dual vector $\xi$ as a linear form in $x$, or conversely $x$ as a linear form in $\xi$. We may abbreviate $g_{\mu \nu} A^\mu B^\nu$ as $\bar{\Lambda} \cdot B$ ($= \bar{B} \cdot \Lambda$) and likewise $g^{\mu \nu} \xi_\mu \eta_\nu$ as $\xi \cdot \eta$ ($= \eta \cdot \xi$). More generally we may define a scalar product of tensors with an equal number ($p$) of indices:

$$
\tau \cdot t = \frac{1}{p!} \tau_{a_1 \cdots a_p} t^{a_1 \cdots a_p}
$$

(C.6)

*) This is a sum over all $(p + q)!$ permutations of the indices, with the usual sign factor.
and abbreviate
\[ \frac{1}{2} \varepsilon_{\mu \nu} m^{\mu \nu} = \bar{\Lambda} \cdot m. \]  

**Norms** for vectors \( \Lambda \), bivectors \( \varepsilon \), and the corresponding dual elements \( \xi \), \( \Lambda \) are then defined by
\[ |\Lambda| = (\bar{\Lambda} \cdot \Lambda)^{\frac{1}{2}} = (-g_{\mu \nu} \Lambda^{\mu} \Lambda^{\nu})^{\frac{1}{2}} \]  
\[ |\varepsilon| = (-\bar{\varepsilon} \cdot \varepsilon)^{\frac{1}{2}} = (-\frac{1}{2} \varepsilon_{\mu \nu} \varepsilon^{\mu \nu})^{\frac{1}{2}} \]  
\[ |\xi| = (\xi \cdot \xi)^{\frac{1}{2}}, \quad |\Lambda| = (\Lambda \cdot \Lambda)^{\frac{1}{2}}. \]

The minus signs are dictated by the reality conditions for the corresponding elements [Eqs. (1) and (3)], and similarly for \( \varepsilon = \Lambda \wedge B \), \( \Lambda = \xi \wedge r \).

5. **Gram determinants** are a convenient notation for certain scalar products. If \( A_1, \ldots, A_p, B_1, \ldots, B_p \) are 2p points with components \( A_i^\mu, \ldots \), etc., we write:
\[ s_{ij} = g_{\mu \nu} A_i^\mu B_j^\nu = A_i^\mu B_j^\nu = \bar{B}_j \cdot A_i. \]

One easily proves
\[ (\bar{B}_1 \wedge \bar{B}_2 \wedge \ldots \wedge \bar{B}_p) \cdot (A_1 \wedge A_2 \wedge \ldots \wedge A_p) = \text{Det} (s_{ij}). \]

We simplify many formulae [see Section 7] if we introduce a minus sign and a normalization, defining:
\[ \left( \begin{array}{c} A_1 & A_2 & \cdots & A_p \\ B_1 & B_2 & \cdots & B_p \end{array} \right) = -\text{Det} (s_{ij}) \left/ \prod_k |A_k||B_k| \right.. \]

In this notation the Cayley formula (not derived) for the "distance" \( \rho_{AB} \) is expressed by:
\[ \cosh \rho_{AB} = -\frac{g_{\mu \nu} A^\mu B^\nu}{|A||B|} = \left( \begin{array}{c} A \\ B \end{array} \right) \]
\[ \sinh \rho_{AB} = +\frac{|A \wedge B|}{|A||B|} = \left( \begin{array}{c} A \\ B \end{array} \right)^{\frac{1}{2}} \]
and
\[ \rho_{AB} = \log \left\{ \left( \begin{array}{c} A \\ B \end{array} \right) + \left( \begin{array}{c} A \\ B \end{array} \right)^{\frac{1}{2}} \right\}. \]

6. The "canonical" transformation of a tensor \( t^{\alpha \cdots} \) with \( p \) upper indices, into a tensor \( \tau^\lambda \cdots \) with \( 4 - p \) lower indices is defined by
\[ \tau = \epsilon t; \quad \tau^\lambda_{\ldots} = \frac{1}{p!} \epsilon_{\alpha \cdots}^{\lambda \ldots} t^{\alpha \cdots}, \]
with saturation of the $p$ indices $\alpha \ldots$. The transformation is easily inverted by:

$$
e^{\alpha \ldots \beta \ldots} = \frac{1}{(4-p)!} \epsilon^{\alpha \ldots \beta \ldots \gamma \ldots} \tau_{\gamma \ldots},$$

(C.15)

which we write symbolically $t = \phi \tau$; hence also

$$\phi = \epsilon^{-1}; \quad \phi \epsilon = 1, \quad \epsilon \phi = 1.$$

(C.16)

These equations are also valid when $p = 0$ or $4$. In the main text we have also used $[A B]$ for $\epsilon(A \wedge B)$, $[A B C]$ for $\epsilon(A \wedge B \wedge C)$, $[A B C D]$ for $\epsilon(A \wedge B \wedge C \wedge D)$ [see Eqs. (13) and (14')].

Notice that

$$\frac{1}{2} \epsilon(\ell \wedge \ell) = \epsilon^{\alpha \beta \gamma} \ell_{\alpha} \ell_{\beta} \ell_{\gamma},$$

(C.17)

is the so-called Pfaffian of the antisymmetric matrix $(\epsilon^{\alpha \beta})$. Its square is equal to the determinant of $(\epsilon^{\alpha \beta})$. For a decomposable $\ell$ ($= A \wedge B$) one has Plücker's relation:

$$\epsilon(\ell \wedge \ell) = \epsilon(A \wedge B \wedge A \wedge B) = 0.$$

(C.18)

The rank of $(\epsilon^{\alpha \beta})$ is then $< 4$, and is therefore equal to 2 (it can only be even; it is zero only if $\ell = 0$).

More general rules *)

Consider the tensor product of a dual $p$-vector $\tau$ and a $q$-vector $t$. If we saturate over $p$ or $q$ indices (whichever is smaller), we obtain a generalization of the scalar product $\tau \cdot t$ [Eq. (C.6)]. We distinguish two "interior" products: if we denote the $|p - q|$ surviving indices by $\lambda \ldots$, and the $p$ (or $q$) dummy indices by $\alpha \ldots$, we write

$$\tau \vee t = \frac{1}{p!} \tau_{\alpha \ldots} t^{\alpha \ldots \lambda \ldots}$$

(C.19)

$$\tau \wedge t = \frac{1}{q!} \tau_{\alpha \ldots} t^{\alpha \ldots \lambda \ldots}.$$

(C.20)

Combining the canonical operations $\epsilon$ and $\phi$ with the exterior product (C.1), we can also define the two interior products as follows:

$$\tau \vee t = \phi(\tau \wedge t)$$

(C.21)

$$\tau \wedge t = \epsilon(\phi \tau \wedge t).$$

(C.22)

A combinational calculation shows that Eq. (C.21) is equal to Eq. (C.19) when $p \leq q$. It is equal to zero if $p > q$ (since $\tau \vee t$ is then $= 0$). Likewise Eq. (C.22) is equivalent to Eq. (C.20) when $p \geq q$, and equal to zero if $p < q$. When $p = q$, all of the above expressions reduce to the scalar product $\tau \cdot t$.

*) The remark made at the beginning of Section 7 applies a fortiori to this paragraph. The notations here are from Ref. 7.
For mnemonic reasons, we read the "right-hand" interior product \( \tau \cdot t \) as: "\( \tau \) absorbs \( t \)", the left-hand product \( \tau \wedge t \) as \( \tau \) is absorbed by \( t \). We also note the rules
\[
\sigma \cdot (\tau \wedge t) = (\sigma \wedge \tau) \cdot t \tag{C.23}
\]
\[
(\tau \wedge t) \cdot u = \tau \cdot (\tau \wedge u), \tag{C.24}
\]
where \( \sigma \) is a dual \((q-p)\)-vector and \( u \) a \((p-q)\)-vector, and the more general relations:
\[
\tau \wedge (s \wedge t) = (\tau \wedge s) \wedge t \tag{C.25}
\]
\[
(\sigma \wedge t) \wedge t = \sigma \wedge (\tau \wedge t),
\]
Factors in a vector product "may be absorbed one at a time".

There is no corresponding rule if \( \tau \), in \( \tau \wedge t \), is an exterior product. If, however, \( \tau \) is decomposed into a product of dual vectors, or correspondingly if \( t \) in \( \tau \wedge t \) is decomposed, there are useful rules of calculation, in particular -- for our purposes -- the following:

\[
p = 1, \ q = 2 \quad \xi \wedge (A \wedge B) = (\xi \cdot B)A - (\xi \cdot A)B
\]

\[
p = 1, \ q = 3 \quad \xi \wedge (A \wedge B \wedge C) = (\xi \cdot C)A \wedge B + (\xi \cdot B)C \wedge A + (\xi \cdot A)B \wedge C
\]

\[
p = 2, \ q = 3 \quad M \wedge (A \wedge B \wedge C) = (M \cdot (B \wedge C))A + (M(C \wedge A))B + (M \cdot (A \wedge B))C
\]
and similarly:

\[
p = 2, \ q = 1 \quad (\xi \wedge \eta) \wedge A = (\xi \wedge A)\eta - (\eta \wedge A)\xi, \text{ etc.}
\]

These rules may be used to "calculate", for example, the intersection of a line with a plane. The identity for \( p = 1, \ q = 2 \) gives the intersection as a linear combination of two points on the line, while \( p = 2, \ q = 3 \) gives the result as a linear combination of three points in the plane. If in the two expressions \( \xi = \varepsilon(A \wedge B \wedge C) \) and \( M = \varepsilon(A \wedge B) \), the expressions must give an identity, which is easily verified.
APPENDIX D

CONVENTIONS FOR ORIENTATIONS OF LINES AND PLANES

1. $\ell = A \wedge B$ defines a line, oriented from A to B.

2. $t = A \wedge B \wedge C$ orients the plane A B C by defining a sense of rotation in the plane. Looking at the positive face one sees the rotation as counter-clockwise.

The positive side is also defined by $x \cdot \xi > 0$, where $\xi = \epsilon t = [A B C]$.

The tetrahedron A B C D is positively oriented (D is on the positive side) if:

$$[A B C D] > 0$$

3. Two planes being oriented as shown by dual vectors $\xi$ and $\eta$, the wedge-shaped region:

$$\xi \cdot x > 0; \quad \eta \cdot x < 0$$

is bordered by the two half-planes. The rotation from the $\xi$ half-plane to the $\eta$ half-plane, and the corresponding orientation of the intersection, are defined as positive by the choice:

$$\Lambda = \xi \wedge \eta$$

4. Assume that the line $\ell$ and the plane containing $\ell$ and C are oriented as shown. If the orientation of the line is defined by the bivector $\ell$ (or $\Lambda = \epsilon \ell$), the orientation of the plane is given by $\xi$ where

$$\xi^\mu = \Lambda_{\alpha \mu} C^\alpha$$

5. The normal $n$ to an oriented plane $\zeta$, through a point 0, is given by

$$n = 0 \wedge \zeta, \quad \text{i.e.} \quad n^\mu = \delta^\mu_\zeta - 0^\mu \zeta^\lambda$$

The relative orientations of line and plane are as shown in the figure.
REFERENCES

1) A. Sommerfeld, Phys. Z. 10, 826 (1909).
8) E. Cartan, Leçons sur la géométrie des espaces de Riemann (Gauthier-Villars, Paris, 1951), especially p. 139, Eq. (15).