CERN-TH.4944/88

TOPOLOGICAL STABILITY IN HIGHER-DIMENSIONAL THEORIES

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A B S T R A C T

In theories whose premise is that there are more than four dimensions to space-time it is often difficult to analyze the stability of the compact extra space against quantum fluctuations. We proffer some elementary criteria for a qualitative discussion of this problem, and illustrate their use with a simple model. The influence of background gauge topology is found to play a key rôle in determining whether a manifold is stable or not.

CERN-TH.4944/88

January 1988

1. - INTRODUCTION

The experimental verification, five years ago, of the existence of massive vector particles mediating the weak force, and the clear indications of vectors mediating strong interactions between quarks has led us to now regard gauge invariance as a fundamental principle operating in nature. Such internal gauge symmetries amongst the particles and their interactions are a beneficial aid to the construction of other viable theories and indeed govern the mathematics of renormalization in such a way as to yield results which have a physical meaning. In an attempt to find an origin for the gauge principle many have been inspired by the observation of Kaluza and Klein that under certain circumstances an internal symmetry arises from extra dimensions to our Universe, being expressed as translation invariance on the extra space therein. Thus an internal symmetry, usually concerning itself with unmeasurable quantities such as phases of wave functions, is traded for something we have a more classical picture of, namely movement along a certain direction. Such postulated extra dimensions must be compact and small to have escaped our attention.

The dynamical nature of these dimensions is governed by the relevant components of the Einstein equations for this new general co-ordinate invariance and so the compactification length scale Λ_c is intimately linked to the Planck length Λ_p . However, gauge symmetries are quantum symmetries and we have to immediately consider how quantum fluctuations influence the dynamics of the extra compact space. If one believes the quantum principle to be the prime motivation of physical processes and modes arising from compactification have to be regarded as quantum fields, this is inescapable. To treat the problem fully one would have to enter into the shady realm of quantized gravity. Nevertheless, a half-way measure can be adopted for a qualitative investigation of some of the physics.

If $\Lambda_c \gg \Lambda_p$ the size of the extra space is much larger than the length scale at which there will be severe fluctuations in the gravitational metric. We could surreptitiously dissociate the matter fields from the gravity and its inherent problems. We would be left with an effective field theory defined on some manifold with a certain background metric. Higher-order interactions in this field theory will be ordered in powers of Λ_p . If, on the other hand, $\Lambda_c \lesssim \Lambda_p$ there is no well-defined field theory limit.

One should expect this effective field theory to respect the choice gravity has made of the background topology of the manifold. Even if we attempt to investigate the quantum corrections solely amongst the matter fields a nonsensical result would signify an inadmissible topology. Higher-order interactions to the effective theory from the corrections will yield terms in higher powers of Λ_c as

gravity did before. However, there is a field independent quantity which is generated, and which does not follow this decoupling rule, namely the vacuum energy or the minimum of the effective potential. It is this, we believe, that should give the first indications of whether some topology is stable or unstable.

An analogy can be drawn here with the Casimir effect [1]. Introducing conducting surfaces into space imposes boundary conditions on the allowable field modes. The spectrum of modes is thus changed and one can calculate the energy difference due to such a change. One has to perform work to alter the geometric configuration of the surfaces and hence there is a pressure on them. It is not the total energy of the system that is the physically relevant object but rather its variation. Compact spaces also impose boundary conditions on field modes and changing the geometrical properties of the space influences the effective potential.

If the form of the vacuum energy suggests that it is energetically favourable for Λ_c to decrease it means that the compact space remains compact and the topology is stable. It cannot decrease too far as it would soon approach Λ_p invalidating our assumptions. However, if the whole theory is well-behaved the gravitational vacuum energy at this level ought to cancel the matter vacuum energy. Whatever happens, we believe that if the matter vacuum energy suggests a smaller Λ_c the topology is stable.

Alternatively, if Λ_c increases there are no new interactions which can prevent it from running away to infinity. We consider this a sign of unstable topology. Concentrating on vacuum energies, though, does have its shortcomings. It is this quantity which usually ascribes a value to the classical cosmological constant, which is found to be extremely small on the scale of ordinary physics. Why this is so is still a mystery, most likely some deep property of a complete theory of gravity. As we are focusing our attention on fields other than gravity we are unable to say anything about it. To illustrate the application of such criteria and to gain some feeling for them in practice we have developed a simple model. It is essentially massless QED on the six-dimensional space $M^4 \times T^2$, with a few added ingredients. M^4 is the usual Minkowski space and T^2 is a toroidal space. The torus is the simplest Ricci-flat complex manifold. The effective action in this case is,

$$S = \int d^6 z \left\{ -\frac{1}{4} F_{MN} F^{MN} + \frac{i}{2} (\bar{\Psi} \Gamma^M D_M \Psi - D_M \bar{\Psi} \Gamma^M \Psi) \right\} \quad (1.1)$$

$F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength of the $U(1)$ gauge field A , Ψ is a Dirac fermion and the gauge covariant derivative is $\underline{D} = \underline{d} - ie\underline{A}$. Γ are some representation of the Dirac matrices in six dimensions. We assume that A is a fundamental field, not arising from components of the metric. The manifold admits a monopole on the

two-space and gives rise to zero mode chiral fermions in four dimensions. As the torus has no intrinsic scale at the tree level, the two radii of the torus are input parameters to the theory and play the rôle of Λ_c . Also, since there are no classical vacuum expectation values to be determined by the dynamics, the vacuum energy density will be simply a constant times some power of $1/\Lambda_c$. Having specified our model let us proceed to a few calculations.

2. - THE MONOPOLE ON THE TORUS

It is very straightforward to construct a monopole on the extra two-space of the torus. Start with a sphere S^2 for which we know the two polar co-ordinates θ, φ have the range $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \pi$. Of course [3] we need at least two patches on this manifold, and they can be chosen to be just the upper and lower hemispheres with intersection at the equator $\theta = \pi/2$,

$$H_+ : 0 \leq \theta \leq \pi/2, \quad 0 \leq \varphi < 2\pi$$

$$H_- : \pi/2 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi$$

For a matter field with a $U(1)$ charge residing on this base manifold, its phases on the two co-ordinate patches must be related by,

$$\phi_+(\theta, \varphi) = e^{in\varphi} \phi_-(\theta, \varphi)$$

to be single-valued on the equator. Thus n is an integer. This is a large gauge transformation and n is the winding number. For a gauge covariant derivative of the form $\underline{D} = \underline{d} - i\underline{A}$ this tells us that the gauge potentials on the two patches are related by the gauge transformation,

$$\underline{A}_+ = \underline{A}_- + n \underline{d}\varphi$$

Indeed the Dirac monopole has potentials,

$$\underline{A}_\pm = \frac{n}{2} (\cos\theta \mp 1) \underline{d}\varphi$$

each of which is regular on their own patches and thus avoids the problems of string singularities at the north and south poles. The field strength is then $\underline{F} = \underline{d}\underline{A}_\pm = n/2 \sin\theta \underline{d}\theta \wedge \underline{d}\varphi$ which is proportional to the surface element of S^2 . In other words it is a geometric invariant.

The torus T^2 can be constructed from S^2 by restricting the θ co-ordinate of the sphere to the range $\alpha < \theta < \pi - \alpha$ and then equating $\theta = \alpha$ with $\theta = \pi - \alpha$ as depicted in Fig. 1. Instead of angles for co-ordinates, we would prefer to use the actual dimensionful co-ordinates $y_{1,2}$ where $0 < y_{1,2} < a_{1,2}$ and the $a_{1,2}$ are the circumferences of the two circles $S^1 \otimes S^1$ generating the surface of the torus. We choose $y_{1,2}$ to be analogous to the θ, φ directions respectively. The relations between the potentials and fields on the patches $H_{\pm}'$ are then,

$$\begin{aligned} \mathcal{A}'_{+}(y_1, y_2) &= \mathcal{A}'_{-}(y_1, y_2) + \frac{2\pi n}{a_2} dy_2 \\ \phi'_{+}(y_1, y_2) &= \exp\left(\frac{2\pi i n}{a_2} y_2\right) \phi'_{-}(y_1, y_2) \end{aligned} \quad (2.1)$$

We associate $\theta = \alpha$ with $y_1 = 0$ and $\theta = \pi - \alpha$ with $y_1 = a_1$. As we are equating these on the manifold, cf. Fig. 1, we impose the boundary conditions,

$$\begin{aligned} \mathcal{A}'_{+}(0, y_2) &= \mathcal{A}'_{-}(a_1, y_2) \\ \phi'_{+}(0, y_2) &= \phi'_{-}(a_1, y_2) \end{aligned} \quad (2.2)$$

Expressed as a co-ordinate condition on the functions on the "+" patch, these are,

$$\begin{aligned} \mathcal{A}'_{+}(a_1, y_2) &= \mathcal{A}'_{+}(0, y_2) + \frac{2\pi n}{a_2} dy_2 \\ \phi'_{+}(a_1, y_2) &= \exp\left(\frac{2\pi i n}{a_2} y_2\right) \phi'_{+}(0, y_2) \end{aligned} \quad (2.3)$$

and we can dispense with the "-" patch altogether if so desired. Hence a choice of potential,

$$\mathcal{A}'_{+}(y_1, y_2) = \frac{2\pi n}{a_1 a_2} y_1 dy_2 \quad (2.4)$$

satisfies these conditions and constitute a monopole on the space T^2 with winding number n . The field strength is,

$$F = d\mathcal{A}'_{+} = \frac{2\pi n}{a_1 a_2} dy_1 \wedge dy_2 \quad (2.5)$$

proportional to the surface element.

We could choose any other $\mathcal{A}'_{\pm}$ satisfying the boundary conditions (2.3). It would differ from (2.4) only by an exact form, whether it be a large gauge transformation or a small one. This can be absorbed into the $\phi'_{\pm}$ via a gauge transformation.

3. - THE FERMION MODES

On the manifold $M^4 \otimes T^2$ it is convenient to label the co-ordinates $z^M = (x^\mu; x^5, x^6) = (x^\mu; y_\alpha)$ where $\alpha = 1, 2$. We choose to work in a timelike metric with signature $(+---; --)$. The manifold is flat. It is useful to decompose the Dirac gamma matrices for this six-dimensional space as a tensor product,

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes 1_2 \\ \Gamma^5 &= i\gamma^5 \otimes \tau^1 \\ \Gamma^6 &= i\gamma^5 \otimes \tau^2\end{aligned}\tag{3.1}$$

where γ^μ are some representation of the usual four-dimensional matrices and the $\tau^{1,2}$ are 2×2 Pauli matrices. We could write the last two of these as $\Gamma^\alpha = i\gamma^5 \otimes \tau^\alpha$, it being understood that an early Greek letter denotes a toroidal co-ordinate. Then the six-dimensional chirality operator is,

$$\overline{\Gamma} = -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 \Gamma^6 = \gamma^5 \otimes \tau^3\tag{3.2}$$

the product of the four-dimensional chirality operator and a diagonal matrix.

We assume that the fermion field is separable in its space-time and toroidal co-ordinates. With the decomposition (3.1) of the Dirac matrices we are led to the ansatz,

$$\Psi(z) = \begin{bmatrix} \psi_+(x) \\ \psi_-(x) \end{bmatrix} \phi(y)\tag{3.3}$$

where the $\psi_\pm(x)$ are four-dimensional Dirac spinor fields, having both left and right four-dimensional chiralities, and the $\pm$ refer to the eigenvalue of τ^3 viz.,

$$\tau^3 \Psi(z) = \begin{bmatrix} \psi_+(x) \\ -\psi_-(x) \end{bmatrix} \phi(y)$$

Recalling our results from the last section, we can construct a gauge background with components,

$$e\phi_M(z) = \left(0; 0, \frac{2\pi n}{a_1 a_2} y_1\right) \quad (3.4)$$

which is clearly Poincaré invariant on the four-space but has non-trivial topology on the torus. The field strength has non-zero components $eF_{56} = (2\pi n/a_1 a_2)$. We now wish to find the eigenmodes of the Dirac operator in this background, i.e., solve,

$$\begin{aligned} i\not{D}\Psi_j(z) &= i(\partial_M - ie\phi_M(z))\Gamma^M\Psi_j(z) \\ &= \lambda_j\Psi_j(z) \end{aligned}$$

For our purposes it is simpler to consider the square of the operator,

$$(i\not{D})^2\Psi = \left[-\partial_\mu\partial^\mu + \frac{\partial^2}{\partial y_1^2} + \left(\frac{\partial}{\partial y_2} - \frac{2\pi in}{a_1 a_2} y_1\right)^2 + \frac{2\pi n}{a_1 a_2} \tau^3 \right] \Psi \quad (3.5)$$

which is separable and diagonal. It was shown in Ref. [2] that the $\phi(y)$ are harmonic oscillator eigenfunctions,

$$\left[\frac{\partial^2}{\partial y_1^2} + \left(\frac{\partial}{\partial y_2} - \frac{2\pi in}{a_1 a_2} y_1\right)^2 \right] \phi_{Nm}(y) = -E_N \phi_{Nm} \quad (3.6)$$

where the eigenvalues are,

$$E_N = \frac{4\pi|n|}{a_1 a_2} (N + 1/2) \quad (3.7)$$

with N and m integers. This follows from the simple observation that the operator (3.6) is, up to a factor, the Hamiltonian for the quantum mechanical problem of a charged particle moving in an external magnetic field whose direction is perpendicular to the plane of motion. This is similar to Landau diamagnetism and results in a harmonic oscillator spectrum for the particle's energy levels.

The eigenfunctions have boundary conditions imposed on them from (2.3), and the complete orthonormal set is,

$$\phi_{Nm}(y) = \left(a_2 \sqrt{\frac{a_1 a_2}{2|n|}} 2^N N! \right)^{-1/2} \sum_{k=-\infty}^{\infty} f_{N, m+kn}(y) \quad (3.8)$$

where the f_{Nm} are the more familiar oscillator solutions,

$$f_{N,m}(y_1, y_2) = \exp\left(\frac{2\pi i m}{a_2} y_2\right) \exp\left(-\frac{\pi |n|}{a_1 a_2} \left(y_1 - \frac{m}{n} a_1\right)^2\right) \\ \times H_N\left(\sqrt{\frac{2\pi |n|}{a_1 a_2}} \left(y_1 - \frac{m}{n} a_1\right)\right)$$

Clearly $\phi_{N, m+n} = \phi_{N, m}$, a cyclic condition implying an $|n|$ -fold degeneracy for each value of N . Translating by a multiple of $a_1/|n|$ in the y_1 direction, followed by a gauge transformation, turns one ϕ_{Nm} into another.

Hence expanding in those modes we can write,

$$\Psi(z) = \sum_{N=0}^{\infty} \sum_{m=0}^{|n|-1} \begin{bmatrix} \psi_{+Nm}(x) \\ \psi_{-Nm}(x) \end{bmatrix} \phi_{Nm}(y) \quad (3.9)$$

and we find that,

$$(i\mathcal{D})^2 \Psi(z) = \sum_{N,m} \begin{bmatrix} (-\partial_\mu \partial^\mu - M_+^2(N)) \psi_{+Nm}(x) \\ (-\partial_\mu \partial^\mu - M_-^2(N)) \psi_{-Nm}(x) \end{bmatrix} \phi_{Nm}(y) \quad (3.10)$$

where the mass squared eigenvalues are,

$$M_{\pm}^2(N) = \frac{4\pi |n|}{a_1 a_2} \left(N + \frac{1}{2}(1 \mp \sigma)\right) \quad (3.11)$$

with $\sigma = \text{sign}(n)$. The eigenmodes of $-\partial_\mu \partial^\mu$ can be taken to be plane waves as usual.

Immediately it is clear from (3.11) that $M_\sigma^2(0) = 0$ and so the modes $\phi_{\sigma 0m}$ are massless, e.g., $M_-^2(0) = 0$ if $n < 0$. As there is an $|n|$ -fold degeneracy we have an explicit demonstration of the index theorem relating the number of zero modes of the Dirac operator to the winding number (first Chern class) of the gauge field. Also apparent is that $M_{-\sigma}^2(N) = M_\sigma^2(N+1)$ and hence the Dirac particles $\phi_{\sigma, N+1, m}$ and $\phi_{-\sigma, N, m}$ are expected to represent two massive Dirac particles. Integrating $i\bar{\Psi}\Psi$ over the torus shows this to be, in fact, the case.

Maintaining four-dimensional Poincaré invariance for the gauge field background demanded that its non-zero components resided solely on the extra two dimensions. With the ansatz (3.9) it is clear that the background fermion fields must vanish, i.e., $\langle 0 | \psi_{\pm Nm}(x) | 0 \rangle = 0$. This is all we need to calculate the quantum corrections for the system considered herein. This does not preclude the question of whether the fermions could dynamically condense due to quantum effects yielding a non-zero vacuum expectation value for some fermionic bilinear. The present system is inadequate for the purposes of such an investigation and it is addressed elsewhere [4].

4. - QUANTUM CORRECTIONS

Having determined the mass spectrum of the fermion modes in the presence of a background monopole we now proceed to estimate the quantum corrections to the vacuum energy due to the excitation of these modes, or more correctly their zero-point energies. It is also an easy procedure to include the corrections due to excitations of the gauge field from its own background. It will be shown that the fermion corrections are stable on the topology of the torus only if there is no monopole present to produce the spectrum of the last section, whereas the gauge corrections imply an unstable manifold unless certain conditions are satisfied. Both those results are already known [5,6] for the case of no gauge topology, stability coming simply from the sign of fermion versus boson loops. However, with non-trivial gauge topology the situation is qualitatively very different.

Begin with the action (1.1) for our system and extremise it to find the equations of motion for the classical (background) fields,

$$\begin{aligned} i \not{D}(\mathcal{A}) \Psi &= 0 \\ \partial_M F^{MN} &= -e \bar{\Psi} \Gamma^N \Psi \end{aligned} \quad (4.1)$$

For Poincaré invariance on M^4 we must have $\langle 0 | \psi | 0 \rangle = 0$. The equations (4.1) are satisfied for a vanishing classical gauge field (winding number $n = 0$) or indeed with the topologically non-trivial situation of (3.4). Denote the background gauge field by $\bar{A}_M$.

Now expand the gauge field around its vacuum expectation value to obtain,

$$\mathcal{A}_M(z) = \bar{A}_M(z) + V_M(z) \quad (4.2)$$

where V_M represents the quantum fluctuations. The specific form of $\bar{A}$ implies a choice of gauge which permits no further large gauge transformations. The field V_M is regular everywhere; it cannot have any discontinuities since it is a small excitation and all the topological information is contained in $\bar{A}$. The action (1.1) would still be invariant under the small gauge transformations,

$$\begin{aligned} \Psi(z) &\rightarrow e^{i\omega(z)} \Psi(z) \\ e V_M(z) &\rightarrow e V_M(z) + \partial_M \omega(z) \end{aligned} \quad (4.3)$$

where the parameter $\omega(z)$ is continuous everywhere. This is a variant on the method of Ref. [7].

Thus expanding the action around the background,

$$S(\bar{\psi}, V, \Psi) = \int d^6 z \left(-\frac{1}{4} \bar{F}_{MN} \bar{F}^{MN} - \frac{1}{4} V_{MN} V^{MN} + \frac{i}{2} \bar{\Psi} \overleftrightarrow{D}(\bar{\psi}) \Psi + e \bar{\Psi} V_M \Gamma^M \Psi \right) \quad (4.4)$$

where $V_{MN} = \partial_M V_N - \partial_N V_M$. As advertised, this is still invariant under (4.3) so we follow the standard procedure of introducing a gauge fixing term and a pair of ghost fields. Whence, order by order in $\hbar$,

$$S(\bar{\psi}, V, \Psi) = S_0 + S_2(\bar{\psi}, V, \Psi, \omega, \eta) + S_3 + \dots$$

where the lowest order term is purely classical,

$$\begin{aligned} S_0 &= \int_{M^4 \otimes T^2} d^6 z \left(-\frac{1}{4} \bar{F}_{MN} \bar{F}^{MN} \right) \\ &= a_1 a_2 \int_{M^4} d^4 z \left(-\frac{2\pi^2 n^2}{(ea_1 a_2)^2} \right) \end{aligned} \quad (4.5)$$

attributable to the energy density of the background gauge configuration. The first-order term vanishes by virtue of (4.1) and the second-order term, in the Feynman gauge for V_M ,

$$S_2 = \int d^6 z \left(-\frac{1}{4} V_{MN} V^{MN} - \frac{1}{2} (\partial_M V^M)^2 - \eta \partial^2 \omega + i \bar{\Psi} \overleftrightarrow{D}(\bar{\psi}) \Psi \right) \quad (4.6)$$

where η, ω are the ghost fields.

If we are only interested in the first, one-loop, quantum effects we may neglect S_3 and upwards as they contribute at the two-loop level and above. As we are interested in the qualitative properties of topological stability this should suffice for our discussion. Thus our generating functional,

$$Z(\bar{\psi}, J) = \int \mathcal{D}V_M \mathcal{D}\eta \mathcal{D}\omega \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp(iS + iS_J)$$

where S_J is a source term for the fluctuations, can be approximated by,

$$Z_1(\bar{\psi}, J) = e^{iS_0} \int \mathcal{D}V_M \mathcal{D}\eta \mathcal{D}\omega \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp(iS_2 + iS_J) \quad (4.7)$$

The effective potential at this order can be shown to be simply,

$$\exp(-i \int d^6 z V_{\text{eff}}^{(1)}) = e^{i \dot{S}_0} \int \mathcal{D}U_M \mathcal{D}\eta \mathcal{D}\omega \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{i \dot{S}_2} \quad (4.8)$$

the sources being unnecessary for the zero-point function which is just what the vacuum energy is.

From the expression (4.6) we see that the path integrals are Gaussian and can be readily performed,

$$\exp(-i \int d^6 z V_{\text{eff}}^{(1)}) = e^{i \dot{S}_0} \det(g_{MN} \partial^2)^{-1/2}_v \det(-\partial^2)_{gh} \times \det(i \not{D}(\omega))_{\psi} \quad (4.9)$$

where the subscripts refer to the vector, ghost and fermion contributions, respectively. g_{MN} is our flat space metric tensor. The determinants are most readily evaluated using the techniques of zeta function regularization [8]. The details of this are presented in the Appendix and the results discussed in the next section.

5. - THE FERMIONIC CONTRIBUTION

Let us concentrate on the fermionic determinant in (4.9) ignoring the gauge and ghost terms for the moment. We may write,

$$\det(i \not{D}(\omega)) = \det([i \not{D}(\omega)]^2)^{1/2} \quad (5.1)$$

which facilitates calculation since the latter operator is diagonal. It is also useful to perform a Wick rotation to Euclidean space to render the operator's eigenvalues non-negative. Taking the logarithm of (4.9),

$$\begin{aligned} - \int d^6 z_E V_{\text{eff}}^{(1)} &= S_{0E} + \frac{1}{2} \ln [\det(i \not{D})_E^2] \\ &= \int d^4 z_E \left(-\frac{2\pi^2 n^2}{e^2 a_1 a_2} \right) - \frac{1}{2} \zeta_1'(0) \end{aligned} \quad (5.2)$$

if a monopole is present, where $\zeta_1'(0)$ is the zeta function evaluation of the determinant of (A.9). Whence, cancelling the Euclidean four volume we obtain,

$$V_{\text{eff}}^{(1)}(n) = \frac{2\pi^2 n^2}{(e a_1 a_2)^2} - \frac{|n|^3}{2\pi^2 (a_1 a_2)^3} \zeta_R(3) \quad (5.3)$$

valid for non-zero winding number, $n \neq 0$.

For the situation where there is no background gauge topology we use the fact that the Dirac operator squared is just the d'Alembertian times the unit matrix of the spinor representation,

$$\begin{aligned} \det(i\cancel{D})_E &= \det(-\cancel{\partial}_M \cancel{\partial}_M 1_4 \otimes 1_2)_E^{1/2} \\ &= \det(-\cancel{\partial}_M \cancel{\partial}_M)_E^4 \end{aligned} \quad (5.4)$$

The effective potential for the fermion loops becomes,

$$-\int d^6 z_E V_{\text{eff}}^{(1)} = -4 \zeta_2'(0)$$

and from (A.16) yield the expression,

$$V_{\text{eff}}^{(1)}(0) = \frac{16}{(\pi a_1 a_2)^3} \left\{ \zeta_R(6) (\rho^3 + \rho^{-3}) + 2 \sum_{k_{1,2}=1}^{\infty} (k_1^2 \rho + k_2^2 \rho^{-1})^{-3} \right\} \quad (5.6)$$

where $\rho = \frac{a_1^2}{a_2^2}$. This ratio parametrizes the complex structure [3] of the torus, an intrinsic topological property of the manifold.

Let us now discuss the implications of (5.3) and (5.6). First, there is no mechanism for tunnelling between vacua of differing values of n . Whether one vacuum energy is greater or less than another is irrelevant as far as this system is concerned. This is in keeping with our treating $\bar{A}$ as an external field, and n as an additional external parameter in the path integral. It is also interpreted as the arbitrariness of the zero-point of energy in flat space quantum mechanics. Secondly, for $n \neq 0$ the potential only depends on the area of the torus whilst for $n = 0$ it also depends on the ratio of the two radii. This is a consequence of the fact that the monopole is a geometric invariant on the torus.

When coupled to a gravitational field the parameters $a_{1,2}$ are regarded as the vacuum expectation values of scalar modes arising from the (5,5) and (6,6) components of the sechsbein. Indeed the product $a_1 a_2$ is akin to the dilaton and the ratio a_1/a_2 is akin to the scalar, which results in deformations of the complex structure of T^2 . (Actually it is the logarithms of these parameters that are normally associated with the fields.)

We can expect the deformation scalar to take on any value for its vacuum expectation value for $n \neq 0$ from (5.3). Hence it will remain massless. However, as the form of (5.6) attests, for $n = 0$ one would expect its vacuum to relax to zero, $\rho = 1$, minimizing the potential. The scalar should also acquire a mass from the

positivity of $V_{\text{eff}}^{(1)}(0)$. In either case the fermionic loops do not lead to an instability of the complex structure of the manifold.

What of the area of the torus? Certainly if we desire our extra-dimensional space to be small we should expect our theoretical expressions to reflect this. Starting from some full-blown model we could satisfy this criterion by expanding order by order in the area and ensuring that each successive term has diminishing effect. In our situation we look at $V_{\text{eff}}^{(1)}$ as a function of $a_1 a_2$ and see if $a_1 a_2 \rightarrow 0$ is natural in the sense of the discussion in the Introduction.

Due to the positivity of $V_{\text{eff}}^{(1)}(0)$ lower energies are those with larger areas, and suggest an instability of the torus. For $V_{\text{eff}}^{(1)}(n)$ we write the dimensionful coupling $e = g\sqrt{a_1 a_2}$ to find,

$$V_{\text{eff}}^{(1)}(n) = c (a_1 a_2)^{-3}$$

where,

$$c = \frac{2\pi^2 n^2}{g^2} - \frac{|n|^3}{2\pi^2} \zeta_R(3) \quad (5.7)$$

As before, if $c > 0$ then the torus is unstable; if $c \leq 0$ then there is indication that it is stable. This translates into a lower bound on the coupling,

$$\frac{g^2}{4\pi} \geq \frac{\pi^3}{|n| \zeta_R(3)} \quad (5.8)$$

The physical significance of this can be seen if we had originally used the dimensionful parameters of the torus to rescale the fields in (1.1),

$$\phi_M \rightarrow \frac{1}{\sqrt{a_1 a_2}} \phi_M$$

the Ψ field not requiring scaling because of the normalization of the $\phi(y)$ when integrated over the torus and the decomposition (3.9). Then the g would be related to dimensionless gauge couplings of the vector to the massive fermion modes in four dimensions. Thus, via the renormalization group equations for this parameter, we should expect g to increase with higher values of momentum subtraction point. Taking the subtraction point as the reciprocal area $Q^2 \sim (a_1 a_2)^{-1}$ this implies that smaller areas enforce the condition (5.8) if g is regarded as a running coupling. Hence the fermion spectrum resulting from background gauge topology is such as to stabilise the torus, provided the gauge coupling is above a certain value.

6. - THE GAUGE CONTRIBUTION

Returning to the gauge and ghost determinants in (4.9) we find the contribution to $V_{\text{eff}}^{(1)}$ is of the form, in Euclidean space,

$$-\ln \det (-\partial_M \partial_M)_E + \frac{1}{2} \ln \det (-\delta_{MN} \partial_P \partial_P)_E = -2 \zeta'_2(0) \quad (6.1)$$

This is the same for the cases $n = 0$ and $n \neq 0$ and is a negative energy.

Nevertheless, for $n = 0$ the fermionic contribution, which is of the same form, still overwhelms this. By the arguments of the last section we would expect a massive deformation scalar, $a_1 = a_2$, and the area of the torus to increase signifying an instability of the geometry. Of course, we could have imposed a Weyl constraint on the fermions at the outset, $\bar{\Gamma}\Psi = \Psi$, which would result in a cancellation of gauge and fermion contributions since their number of degrees of freedom would be identical.

For $n \neq 0$ and at a fixed value of the area we see that $\rho = 1$ is an unstable maximum. Hence, ρ tends to zero or infinity indicating that the complex structure of the torus is unstable. However, if we had forced $\rho = 1$ in some way, we could still balance off the monopole, fermion and vector vacuum energies leading to a constraint similar to (5.8).

The conclusion is that the bosonic modes have a destabilizing effect on the topology. However, the vectors are difficult to interpret purely in the context of the effective theory. When we expand the fermion kinetic term in (1.1) by integrating over T^2 we find kinetic terms and a tower of masses in four dimensions which are perfectly well understood. Not so for the vector. There is a tower of massive spin-one particles and two towers of scalars (V_5, V_6), but no apparent Higgs mechanism for the former, making a conventional field theory calculation for these particles rather dubious. We will comment further on this in the final section.

7. - DISCUSSION

The example we have just presented highlights some fundamental problems in higher dimensional physics. If it is possible to isolate the matter physics from the gravitational physics below the Planck energy our criteria can be applied to almost any resulting effective theory to gain a qualitative feel for the stability

of the geometry. Of course it is no substitution for a complete calculation, but in many cases this is extremely difficult to do. At present theories based upon superstrings [11] purport to be consistent finite theories of gravity. Whether they are particle theories remains to be seen but we could imagine applying our criteria to any such model.

The model we examined showed that a gauge topology on the compact space radically altered the properties of the vacuum. An intimate link between the matter and metric backgrounds seems to have a profound effect on the quantum dynamics. There is no tunnelling mechanism allowing transitions from one n -vacuum to another. $\bar{A}$ is just treated as an external field. If tunnelling were to be allowed somehow we would have found that $|n| \rightarrow \infty$, indicating an instability of the gauge topology also. It is claimed [12] that a lower vacuum energy for a monopole compared to the monopole-free case in an $SU(2)$ theory enforces gauge symmetry breakdown. At zero temperature there is no tunnelling and so this cannot happen. However, it can happen at finite temperature, and similarly in our case the vacuum and gauge topology are unstable at finite temperature. Our criteria seem only to be applicable at $T = 0$.

Our other main difficulty with the approach is the interpretation of the massive spin-one states and the subsequent sum over all one-loop vacuum graphs incorporating them. The effective theory in four dimensions is not well defined in the absence of a Higgs mechanism which could invalidate the calculations in the spin-one sector.

ACKNOWLEDGEMENTS

MJD and GS were supported in part by DOE3071T. They also wish to thank the Department of Theoretical Physics, Oxford, for its hospitality during the course of this work.

APPENDIX

There are two types of operators whose determinants we require. One is the d'Alembertian $\partial_M \partial^M$ and the other is the Dirac operator $i\cancel{D}(\bar{A})$. For an operator Λ with eigenvalues λ_k the generalized zeta function is defined as,

$$\zeta(s; \Lambda) = \sum_k \lambda_k^{-s} \quad (\text{A.1})$$

and the determinant of the operator is formally derived from,

$$\ln(\det \Lambda) = \text{tr} \ln \Lambda = -\frac{d}{ds} \zeta(s; \Lambda) \Big|_{s=0} = -\zeta'(0, \Lambda) \quad (\text{A.2})$$

It can be shown that this function is analytic in s , and therefore it can be used to continue from values of s where the sum is defined to $s = 0$, the point at which we require its value.

The zeta function for any operator of the form $-\partial_\mu \partial_\mu + M^2$ in Euclidean space is evaluated by introducing a plane wave expansion allowing us to sum over eigenvalues of $\partial_\mu \partial_\mu$,

$$\zeta(s; -\partial_\mu \partial_\mu + M^2) \Big|_E = \int d^4 z_E \sum_M \int \frac{d^4 p_E}{(2\pi)^4} \left(\frac{p_E^2 + M^2}{\mu^2} \right)^{-s} \quad (\text{A.3})$$

where the integral is over four-dimensional Euclidean momentum (the subscript E refers to Euclidean space) and we also sum over the mass spectrum. μ is an arbitrary (renormalization) parameter for obvious dimensional reasons. The angular integration may be immediately performed. Denoting the Euclidean space-time volume by vol_4 and substituting $x = p_E^2/\mu^2$ we obtain,

$$\zeta(s; -\partial_\mu \partial_\mu + M^2) \Big|_E = \frac{\text{vol}_4 \mu^4}{16\pi^2} \sum_M \int_0^\infty dx \, x \left(x + \frac{M^2}{\mu^2} \right)^{-s} \quad (\text{A.4})$$

which is ultra-violet divergent for $s \leq 2$. We calculate this for our two mass spectra in the region $s > 2$ and then analytically continue back to $s = 0$.

The Dirac operator squared in Euclidean space is, from (3.10),

$$[i\cancel{D}(\bar{A})]_E^2 = 1_A \otimes \begin{bmatrix} -\partial_\mu \partial_\mu + M_+^2(N) & 0 \\ 0 & -\partial_\mu \partial_\mu + M_-^2(N) \end{bmatrix}$$

in the basis (3.9). There is an $|n|$ fold degeneracy for each value of N . In all there are $|n|$ modes with $M^2 = 0$ and $2|n|$ modes with $M^2 = (4\pi|n|/a_1 a_2)N$ for each value of $N \neq 0$. Hence, up to an irrelevant factor, the zeta function for $(i\cancel{p})^2$ is,

$$\zeta_1(s) = \frac{\text{vol}_4 \mu^4 |n|}{4\pi^2} \int_0^\infty x dx \left\{ x^{-s} + 2 \sum_{N=1}^\infty \left(x + \frac{4\pi|n|}{a_1 a_2 \mu^2} N \right)^{-s} \right\} \quad (\text{A.5})$$

the extra factor of four coming from the trace of l_4 . Clearly, the first term in the integrand is divergent for $s > 2$ and, following Hawking [8], we therefore cut off the lower integration range at $x = \varepsilon$; we will ultimately take the limit $\varepsilon \rightarrow 0$. Then,

$$\zeta_1(s) = \frac{\text{vol}_4 \mu^4 |n|}{4\pi^2} \left\{ \frac{\varepsilon^{2-s}}{s-2} + \left(\frac{4\pi|n|}{a_1 a_2 \mu^2} \right)^{2-s} \frac{2}{(s-1)(s-2)} \sum_{N=1}^\infty N^{2-s} \right\} \quad (\text{A.6})$$

and the remaining sum is just $\zeta_R(s-2)$ the familiar Riemann zeta function.

Continuing to $s = 0$ we find, upon differentiating with respect to s , that the previously infra-red divergent piece vanishes in the limit $\varepsilon \rightarrow 0$. The Riemann zeta function of a negative even integer vanishes and so,

$$\begin{aligned} \zeta_1'(0) &= \frac{\text{vol}_4 \mu^4 |n|}{4\pi^2} \left(\frac{4\pi|n|}{a_1 a_2 \mu^2} \right)^2 \zeta_R'(-2) \\ &= \frac{4 \text{vol}_4 |n|^3}{(a_1 a_2)^3} \zeta_R'(-2) \end{aligned} \quad (\text{A.7})$$

From the reflection formula for ζ_R [9],

$$2^t \Gamma(1-t) \zeta_R(1-t) \sin\left(\frac{\pi t}{2}\right) = \pi^{1-t} \zeta_R(t) \quad (\text{A.8})$$

we deduce that $\zeta_R'(-2) = -\zeta_R(3)/4\pi^2$ and $\zeta_R(3)$ is a calculable number. Whence

$$\zeta_1'(0) = - \frac{\text{vol}_4 |n|^3}{\pi^2 (a_1 a_2)^2} \zeta_R(3) \quad (\text{A.9})$$

the zeta function of $[i\cancel{p}(\bar{A})]^2$ evaluated in Euclidean space.

Similarly we can construct the zeta function for the d'Alembertian. The mass spectrum in this case is,

$$M^2(k_1, k_2) = 4\pi^2 \left(\frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} \right) \quad (\text{A.10})$$

where $k_{1,2}$ are integers, this arising from solutions to,

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \phi(y) = -M^2 \phi(y)$$

and the ϕ has periodic boundary conditions on the torus. We separate out the case $k_1 = k_2 = 0$,

$$\begin{aligned} S_2(s) &= \zeta(s; -\partial_M \partial_M)_E \\ &= \frac{\text{vol}_4 \mu^4}{16\pi^2} \int_0^\infty x dx \left\{ x^{-s} + \sum'_{k_{1,2}=-\infty}^\infty \left(x + \frac{4\pi^2}{\mu^2} \left(\frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} \right) \right)^{-s} \right\} \quad (\text{A.11}) \end{aligned}$$

The first term is the infra-red divergence and can be discarded in the final result as before. We drop it here for convenience. The second term integrates to,

$$S_2(s) = \frac{\text{vol}_4 \mu^4}{16\pi^2} \frac{1}{(s-1)(s-2)} \sum'_{k_{1,2}=-\infty}^\infty \left(\frac{4\pi^2}{a_1^2 \mu^2} k_1^2 + \frac{4\pi^2}{a_2^2 \mu^2} k_2^2 \right)^{2-s} \quad (\text{A.12})$$

This sum is an Epstein zeta function [9],

$$\sum_{s=2}^{(2)} \left(\frac{4\pi^2}{a_1^2 \mu^2}, \frac{4\pi^2}{a_2^2 \mu^2} \right)$$

For an Epstein zeta function such as this there is a reflection formula similar to (A.8),

$$\begin{aligned} \pi^{-t} \Gamma(t) \sum_t^{(2)}(\alpha_1, \alpha_2) &= (\alpha_1 \alpha_2)^{-1/2} \pi^{t-1} \Gamma(1-t) \\ &\times \sum_{1-t}^{(2)} \left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2} \right) \quad (\text{A.13}) \end{aligned}$$

Thus (A.12) can be reflected to the form,

$$\begin{aligned} S_2(s) &= \frac{\text{vol}_4 \mu^4}{16\pi^2} \frac{1}{(s-1)(s-2)} \left(\frac{4\pi^2}{\mu^2 a_1 a_2} \right)^{-1} \pi^{2s-5} \\ &\times \frac{\Gamma(3-s)}{\Gamma(s-2)} \sum_{3-s}^{(2)} \left(\frac{a_1^2 \mu^2}{4\pi^2}, \frac{a_2^2 \mu^2}{4\pi^2} \right) \quad (\text{A.14}) \end{aligned}$$

and we can absorb the factor $(s-1)(s-2)$ into $\Gamma(s-2)$ to yield $\Gamma(s)$. Continuing to $s = 0$ the only non-zero contribution to $\zeta'_2(0)$ comes from differentiating the $1/\Gamma(s)$ and so,

$$\begin{aligned} \zeta'_2(0) &= \frac{2 \text{vol}_4 a_1 a_2}{\pi^3} \left(\frac{\mu^2}{4\pi^2} \right)^3 \sum_3^{(2)} \left(\frac{a_1^2 \mu^2}{4\pi^2}, \frac{a_2^2 \mu^2}{4\pi^2} \right) \\ &= \frac{2 a_1 a_2}{\pi^3} \text{vol}_4 \sum_{k_{1,2}=-\infty}^{\infty} \left(k_1^2 a_1^2 + k_2^2 a_2^2 \right)^{-3} \end{aligned} \quad (\text{A.15})$$

Introducing the ratio $\rho = a_1^2/a_2^2$ and separating out the $k_1 = 0$ and $k_2 = 0$ parts we arrive at,

$$\zeta'_2(0) = \frac{4 \text{vol}_4}{\pi^3 (a_1 a_2)^2} \left\{ \zeta_R(6) (\rho^3 + \rho^{-3}) + 2 \sum_{k_{1,2}=1}^{\infty} \left(k_1^2 \rho + k_2^2 \rho^{-1} \right)^{-3} \right\} \quad (\text{A.16})$$

We make full use of the properties of this and (A.9) in Sections 5 and 6.

It can also be deduced from (A.6) and (A.14) that $\zeta(0) = 0$, reflected in our final results as the disappearance of the arbitrary parameter μ . This is a consequence of there being no continuous order parameters, and the effective potential does not require renormalization. For a further discussion on this, see Ref. [10].

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Fig. 1