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CURVATURES AND GAUSS-BONNET THEOREM IN NEW MINIMAL SUPERGRAVITY

S. Ferrara

CERN - Geneva

and

University of California, Los Angeles, CA 90024

S. Sabharwal and M. Villasante

University of California, Los Angeles, CA 90024

ABSTRACT

We derive the component forms of the gravitational multiplets of new minimal Poincaré supergravity at the full non-linear level. As a by-product, a complete proof of the super-Gauss-Bonnet theorem is obtained.

Recently, a renewed interest has emerged for the new minimal formulation of $N = 1$ 4D-supergravity.¹ This revival has been motivated by the study of four-dimensional heterotic superstrings.² An explicit construction in these theories of supervertex operators for the gravitational multiplet has, in fact, revealed that the underlying superspace geometry corresponds to the new-minimal formulation.³ These results have been confirmed by direct string calculations.⁴

A necessary ingredient to study the effective interactions of the massless modes of superstrings requires a complete understanding of the supergravity couplings. In the present note we give the basic facts on the geometry of new minimal supergravity. We will give all curvature multiplets which, at the linearized level, were derived in Ref. 5. A simple proof of the super--Gauss-Bonnet theorem is obtained. These results allow us to construct general couplings in new minimal supergravity including higher derivative interactions.

The Graviton Multiplet. Let's recall the basic gravitational gauge multiplet.¹ It contains four fields, the physical graviton $e_{a\mu}$, the gravitino $\psi_{\mu\alpha}$ and two auxiliary gauge fields A_μ and $b_{\mu\nu}$ with associated gauge invariances $\delta A_\mu = \partial_\mu \phi$, $\delta b_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. This formulation differs from the standard old minimal form⁶ in the "auxiliary field" sector for which the latter contains a non-gauge vector A_μ and two scalars S and P . The two formulations are different descendants of conformal supergravity by choosing different compensator multiplets.⁷ However, the new minimal formulation looks much simpler than the old minimal one since all fields ($e_{a\mu}$, ψ_μ , $b_{\mu\nu}$, A_μ) are gauge fields. A particular role is played by the $b_{\mu\nu}$ supercovariant field strength three-form:

$$H = dB + \frac{1}{8} \bar{\psi} \gamma \psi \quad (1)$$

which will play the role of a supplementary term both for spin (torsion) and U(1) chiral connections.

More specifically, let us define

$$\omega_{a\lambda b}^{\pm} = \omega_{a\lambda b}(e, \psi) \pm H_{a\lambda b}, \quad \omega_{a\lambda b} = -\omega_{b\lambda a} \quad (2)$$

$$A_{\mu}^{+} = A_{\mu} - H_{\mu}, \quad A_{\mu}^{-} = A_{\mu} - 3H_{\mu}, \quad (3)$$

where $H_a = -\frac{1}{3!} \epsilon_{abcd} H_{bcd}$ and $\omega_{a\lambda b}(e, \psi)$ is the usual supercovariant spin-connection. Then the local (Poincaré) supergravity transformations of the supergravity gauge fields* read

$$\begin{aligned} \delta e^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi \\ \delta \psi &= -D^+ \epsilon \\ \delta B &= \frac{1}{4} \bar{\epsilon} \gamma \psi \\ \delta A^{-} &= \frac{1}{4} \bar{\epsilon} \gamma \gamma_5 \sigma_{ab} \psi_{ab} \end{aligned} \quad (4)$$

where

The γ -matrix conventions are as follows: $g_{ab} = (-, +++)$, $\epsilon_{0123} = +1$, $\{\gamma^a, \gamma^b\} = -2g^{ab}$, $\sigma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b]$, $\sigma^{ab} = \frac{1}{2} \epsilon_{abcd} \sigma^{cd} = i\gamma_5 \sigma^{ab}$, $\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\bar{\psi} = \psi^T C$.

$$D^\pm = d + \delta_L(\omega_{ab}^\pm) + \delta_A(A^\pm), \quad \delta_A(\phi) = iT_A\phi, \quad \delta_L(\omega_{ab}) = \frac{i}{2} \omega_{ab} S^{ab},$$

$$\psi_{ab} = e_a^\mu e_b^\nu \psi_{\mu\nu}, \quad \psi_{\mu\nu} = D_\mu^+ \psi_\nu - D_\nu^+ \psi_\mu.$$

The commutator algebra gives^{1,7}

$$\begin{aligned} [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = & \delta_{GCT}(\xi^\lambda) + \delta_\lambda(\xi^\lambda \omega_{a\lambda b}^-) + \delta_A(\xi^\lambda A_\lambda^-) + \delta_Q(\xi^\lambda \psi_\lambda) \\ & + \delta_B(\xi^\lambda (B_{\lambda\mu} + \frac{1}{2} g_{\lambda\mu})) \end{aligned} \quad (5)$$

when $\xi^\lambda = \frac{i}{2} \bar{\epsilon}_1 \gamma^\lambda \epsilon_2$. Note that the commutator depends only on ξ^λ in contrast to the old minimal formulation⁶ when also the $\bar{\epsilon}_1 \sigma_{\mu\nu} \epsilon_2$ bilinear appears. On "covariant" quantities the $\{Q, Q\}$ commutator gives a supercovariant derivative

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] \phi = \xi \cdot \hat{D}^- \phi, \quad \hat{D}^- = D^- + \delta_Q(\psi). \quad (6)$$

General Multiplets. We now consider general multiplets. They are completely specified by the Lorentz and U(1) quantum numbers of their first component, and by the fact that they are b-inert.⁷

For a general (complex) vector multiplet $V = (C, \chi, H, K, V_a, \lambda, D)$ with chiral weight n (i.e. $T_A = n$) and Lorentz spin matrix S_{ab} , the local supersymmetry transformation rules are:

$$\delta C = -\frac{1}{2} \bar{\epsilon} \chi$$

$$\delta \chi = \frac{1}{2} \{i \hat{D}^- C + \gamma_5 V + H - i \gamma_5 K\} \epsilon(-)^A$$

$$\begin{aligned}
 \delta(H \pm iK) &= \frac{1}{2} \bar{\epsilon} (1 \pm \gamma_5) \{ \gamma_5 \lambda - \hat{D}^- \chi + 2i\gamma_5 M \chi + i\zeta C \} \\
 \delta V_a &= \frac{1}{2} \bar{\epsilon} \{ \gamma_a \lambda - \gamma_5 \hat{D}_a^- \chi + i\gamma_a M \chi \} \\
 \delta \lambda &= \left\{ \frac{1}{4} \sigma^{ab} G_{ab} + \frac{i\gamma_5}{2} D \right\} \epsilon (-)^A + \frac{1}{2} \zeta (\bar{\epsilon} \gamma_5 \chi) \\
 \delta D &= \frac{1}{2} \bar{\epsilon} \gamma_5 \left\{ \hat{D}^- \lambda + \gamma^a \zeta V_a - \left(\frac{n}{2} \hat{R}^+ + \frac{1}{2} \hat{F}_{ab}^+ S_{ab} \right) \chi \right\} \quad (7)
 \end{aligned}$$

where

$$\begin{aligned}
 \zeta &= -\frac{1}{2} n \gamma_5 \sigma^{ab} \psi_{ab} + \frac{1}{2} \psi_{ab} S_{ab} \\
 G_{ab} &= \hat{D}_a^- V_b - \hat{D}_b^- V_a - 2H_{abd} V_d + \frac{1}{2} \bar{\psi}_{ab} \gamma_5 \chi .
 \end{aligned}$$

The multiplication rule for arbitrary vector multiplets is

$$f(V^1) = (C_f, \chi_f, H_f, K_f, V_{af}, \lambda_f, D_f) \quad (8)$$

where

$$\begin{aligned}
 C_f &= f(C^1) \\
 \chi_f &= \chi^1 f_i \\
 H_f &= H^1 f_i + \frac{1}{4} \bar{\chi}^1 \chi^j f_{ji} \\
 K_f &= K^1 f_i + \frac{1}{4} \bar{\chi}^1 \gamma_5 \chi^j f_{ji}
 \end{aligned}$$

$$V_a = V_a^i f_i + \frac{1}{4} \bar{\chi}^i \gamma_5 \gamma_a \chi^j f_{ji}$$

$$\begin{aligned} \lambda_f = \lambda^i f_i - \frac{i\gamma_5}{2} \{ i\hat{D}^- C^i + \gamma_5 \mathcal{Y}^i - H^i - i\gamma_5 K^i \} \chi^j f_{ji} + \frac{i}{8} \{ \gamma_5 \chi^j (\bar{\chi}^k \chi^i) \\ - \chi^j (\bar{\chi}^k \gamma_5 \chi^i) \} f_{ikj} \end{aligned}$$

$$\begin{aligned} D_f = D^i f_i + \frac{1}{2} \{ -V^i V^j - \hat{D}^- C^i \hat{D}^- C^j + H^i H^j + K^i K^j \} f_{ji} \\ - i\bar{\chi}^i \left\{ \gamma_5 \lambda^j - \frac{1}{2} \hat{D}^- \chi^j + \frac{3i}{2} \gamma_5 M \chi^j + i\zeta^j C^i \right\} f_{ji} \\ + \frac{1}{4} \bar{\chi}^j \{ H^k + i\gamma_5 K^k + \gamma_5 \mathcal{Y}^k \} \chi^i f_{ikj} \\ - \frac{1}{4} \bar{\chi}^j \frac{1}{2} (1+\gamma_5) \chi^k \bar{\chi}^i \frac{1}{2} (1-\gamma_5) \chi^l f_{likj} . \end{aligned}$$

For a real linear multiplet $L = [C, \chi, V_a]$ the transformation rules are

$$\delta C = -\frac{1}{2} \bar{\epsilon} \chi$$

$$(-)^A \delta \chi = \frac{1}{2} (i\hat{D}^- C + \gamma_5 \mathcal{Y}) \epsilon$$

$$\delta V_a = -\frac{1}{2} \bar{\epsilon} \gamma_5 \sigma_{ab} \hat{D}^- \chi + \frac{1}{2} \bar{\epsilon} \gamma_a M \chi - \frac{1}{4} \bar{\epsilon} \gamma_5 \gamma_a \psi_{cd} S_{cd}^C \quad (9)$$

with the constraint

$$\hat{D}^- \cdot (V + 2CH) + \frac{i}{2} \hat{F}_{ab}^+ S^{ab} C + \frac{1}{2} \bar{\psi}_{ab} \gamma_5 \left(S_{ab} + \frac{\sigma_{ab}}{2} \right) \chi = 0 .$$

The embedding of a linear multiplet onto a vector one is:

$$H = K = 0 ,$$

$$\lambda = \gamma_5 \hat{\bar{D}} \chi - 2iH\chi + \frac{1}{2} \gamma_5 \psi_{ab} S_{ab}^C ,$$

$$D = -\hat{\bar{D}} C + 2H \cdot V + \frac{1}{2} \bar{\psi}_{ab} \left(S_{ab} + \frac{\sigma_{ab}}{2} \right) \chi . \quad (10)$$

For a chiral multiplet $\Sigma = (z, \chi, h)$, the transformation rules are

$$\delta z = \frac{1}{2} \epsilon \chi$$

$$(-)^A \delta \chi = i \hat{\bar{D}} z \epsilon + h \epsilon$$

$$\delta h = \frac{1}{2} \bar{\epsilon} (i \hat{\bar{D}} + 2H) \chi - i \frac{n}{2} \bar{\epsilon} \sigma_{ab} \bar{\psi}_{ab} z + \frac{1}{2} \bar{\epsilon} \bar{\psi}_{ab} S_{ab} z . \quad (11)$$

The vector embedding is given by

$$\left(z, \chi, h, -ih, -i \hat{\bar{D}}_a z, i \zeta z, -\frac{n}{2} \hat{R}^+ z - \frac{1}{2} \hat{F}_{ab}^+ S^{ab} z \right) . \quad (12)$$

The chiral projection of a general vector multiplet $\Pi(V) = -\frac{1}{4} \bar{V}^2 V$ is a chiral multiplet (z_π, χ_π, b_π) with components:

$$z_\pi = \frac{H - iK}{2} .$$

$$\chi_\pi = i \left(\hat{\bar{D}} \chi + \lambda + 2iH\chi + \frac{n}{2} \sigma^{cd} \psi_{cd}^C + \frac{1}{2} \psi_{cd} S_{cd}^C \right)$$

$$b_\pi = \frac{1}{2} \left\{ D - i(\hat{\bar{D}} - 2iH) \cdot (V + i \hat{\bar{D}} C) + \frac{n}{2} \hat{R}^+ C + \frac{1}{2} \hat{F}_{ab}^+ S_{ab}^C + \right.$$

$$+ i \left(n + \frac{1}{2} \right) \bar{\psi}_{cd} \bar{\sigma}^{cd} \bar{\chi} + i \bar{\psi}_{cd} S_{cd} \bar{\chi} \left. \right\} . \quad (13)$$

Curvature Multiplets. In new-minimal supergravity, the basic curvature multiplets are the Einstein multiplet E_a , the Weyl multiplet $W_{ab\alpha}$, and the scalar curvature multiplet W_α .^{1,7,8} These multiplets were given at the linearized level in Ref. 5.

The Einstein multiplet is a linear real multiplet while the Weyl and scalar curvature multiplets are chiral. The Weyl and scalar multiplets are actually obtained by a more fundamental chiral multiplet, the Riemann multiplet $T_{ab\gamma}$, by taking the traceless and trace part:

$$W = \sigma^{ab} T_{ab} , \quad W_{ab} = \frac{1}{24} (3\sigma_{cd}\sigma_{ab} + \sigma_{ab}\sigma_{cd}) T_{cd} . \quad (14)$$

These multiplets are related by the following identities

$$\begin{aligned} \nabla_a E_a &= 0 , \quad \nabla W = -\bar{\nabla} \bar{W} , \quad W = -i\sigma^{a\bar{b}} \nabla E_a \\ \nabla T_{ab} &= -\bar{\nabla} \bar{T}_{ab} = i(\nabla_a E_b - \nabla_b E_a) + \frac{i}{4} \epsilon_{abcd} (\bar{\nabla}_c \bar{\nabla} - \nabla_c \bar{\nabla}) E_d \\ \nabla W_{ab} &= i(\nabla_a E_b - \nabla_b E_a) - *(\nabla_a E_b - \nabla_b E_a) + \frac{1}{6} \nabla_{\sigma ab} W \\ T_{ab} &= W_{ab} - \frac{1}{6} \sigma_{ab} W - \frac{1}{2} \epsilon_{abcd} \sigma^{c\bar{d}} \nabla E^d \end{aligned} \quad (15)$$

where ∇ are appropriate gravitational derivatives.

The component form of the Einstein (linear) multiplet is

$$E_a = [H_a, i\gamma_5 R_a, \frac{1}{2} (\hat{E}_{ab}^+ - {}^*\hat{F}_{ab}^+)] . \quad (16)$$

On the other hand, the components of the Riemann (chiral) multiplet are:

$$T_{ab} = \left[\frac{1}{2} \psi_{ab}, -\frac{1}{4} \sigma_{cd} \hat{R}_{cdab}^+ - \frac{1}{2} \hat{F}_{ab}^+, \frac{1}{2} \hat{D}^- \psi_{ab} \right] . \quad (17)$$

The scalar curvature multiplet is

$$W = \left[\sigma \cdot \bar{R}, -\frac{1}{2} \left(\hat{R}^- + \sigma^{ab} \hat{F}_{ab}^- \right), -i \hat{D}^- \sigma \cdot R \right] \quad (18)$$

where R is the supercovariantized Rarita-Schwinger operator.

Because of the Bianchi-Identities in (15) one actually has

$$W_\alpha = -\frac{1}{4} \bar{\nabla}^2 \nabla_\alpha V_P \quad (19)$$

where V_P is a gravitational vector gauge multiplet¹ with components

$$V_P = \left(A_\mu^-, \gamma_5 \gamma \cdot R, -\frac{1}{2} \hat{R}^- \right) . \quad (20)$$

The standard supergravity action is simply the Fayet-Iliopoulos term for the multiplet V_P .

Gauss-Bonnet Invariants and Chern-Simons Forms. A major simplification occurs in the derivation of the super--Gauss-Bonnet theorem.⁹

In fact, all topological invariants can be constructed through the F-density formula of the product of the T_{ab} multiplet with its dual

$$T_{ab}^* T^{ab} . \quad (21)$$

This chiral multiplet has the following components

$$T_{ab}^* T^{ab} = \left(\frac{1}{4} \psi_{ab}^* \psi_{ab} , \left(\frac{1}{4} \sigma^{cd} \hat{R}_{cdab}^+ - \frac{1}{2} \hat{F}_{ab}^+ \right) \psi_{ab} , \frac{1}{2} \psi_{ab} i \hat{D}^{*-} \bar{\psi}_{ab} \right. \\ \left. - \frac{1}{4} \hat{F}_{ab}^+ \hat{F}_{ab}^+ + \frac{1}{8} \hat{R}_{abcd}^+ \hat{R}_{abcd}^+ + \frac{1}{8} \hat{R}_{abcd}^+ \hat{R}_{abcd}^{**} \right) \quad (22)$$

and multiplication of chiral multiplets is the same as for global supersymmetry.

Note that the last component of T^*T contains the Euler characteristic in the F part and the U(1) Chern-Simons and Hirzebruch densities in the G part.

The chiral density formula for the T^*T multiplet is

$$(dx) \left[\frac{1}{2} T_{ab}^* T_{ab} \right]_F = - \frac{1}{8} R_{ab}^+ R_{ab}^+ - \frac{1}{8} R_{ab}^+ \hat{R}_{ab}^+ + \frac{1}{4} F^{+2} \\ - d \left(\bar{D}^+ \psi \left(\frac{1-\gamma_5}{2} \right) \left(R + \frac{1}{2} \gamma \gamma \cdot R - \gamma_5 \not{H} \psi \right) \right) . \quad (23)$$

Eq. (23) is a total derivative dX where

$$X = \left[\frac{1}{8} \omega_{ab}^+ \left(R^+ - \frac{1}{3} \omega^{+2} \right)_{ba} \right] - \frac{1}{8} \epsilon^{abcd} \omega_{ab}^+ \left(R^+ - \frac{1}{3} \omega^{+2} \right)_{cd} \\ + \frac{1}{4} A^+ dA^+ - \bar{D}^+ \psi \left(\frac{1-\gamma_5}{2} \right) \left(R + \frac{1}{2} \gamma \gamma \cdot R - \gamma_5 \not{H} \psi \right) . \quad (24)$$

Finally, one can derive the real Chern-Simons multiplet¹⁰ Ω which satisfies the Bianchi Identity

$$\Pi(\Omega) = T^*T . \quad (25)$$

The Ω multiplet which solves Eq. (25) is obtained following the method of Ref. 10, and its components are:

$$\Omega = (C^\Omega, \chi^\Omega, 2 \operatorname{Re} z, -2 \operatorname{Im} z, V_a^\Omega, i\gamma_5 \chi + \gamma_5 (\hat{D}^- - 2i\gamma_5 H) \chi^\Omega, \\ 2\operatorname{Re} h - \hat{D}C^\Omega + 2H \cdot V^\Omega + \frac{1}{4} \bar{\psi}_{ab} \sigma_{ab} \chi^\Omega) \quad (26)$$

with the constraint:

$$\hat{D} \cdot (V^\Omega + 2C^\Omega H) + \frac{1}{4} \bar{\psi}_{ab} \gamma_5 \sigma_{ab} \chi^\Omega = -2 \operatorname{Im} h. \quad (27)$$

Since Ω is obtained up to a linear multiplet, we can choose $\Omega-L$ with $C^\Omega = C^L$, $\chi^\Omega = \chi^L$. This allows us to solve the constraint equation (27) for the multiplet $\Omega-L$ for which

$$\partial_\mu (e V^\mu) = -2 \operatorname{Im}[T^* T]_F, \quad V_\mu = V_\mu^\Omega - V_\mu^L. \quad (28)$$

Note that due to the special property of the T^*T multiplet, one could actually obtain another multiplet Ω' whose vector component would be the Euler current rather than the (dual of) Chern-Simons density. However, when Yang-Mills Chern-Simons terms are added, this would be no longer possible since only the imaginary part of the W^2 density is a total derivative. The component expansion of $\Omega-L^*$ is

*Incidentally in Ref. 10, the last two components should have explicit gravitino terms as in Eq. (29).

$$\begin{aligned}
 (\Omega-L) = & (0, 0, 2\text{Re}z, -2\text{Im}z, V_a, i\gamma_5\chi + \frac{1}{2}\gamma_5\gamma^a(\gamma_5\psi + 2\text{Re}z \\
 & + 2i\gamma_5\text{Im}z)\psi_a, 2\text{Re}h + 2H\cdot V + \frac{1}{4}\bar{\psi}_a(\gamma_5\psi + 2\text{Re}z + 2i\gamma_5\text{Im}z)\psi_a) . \quad (29)
 \end{aligned}$$

Conclusions. In this note we have given the essentials of the geometry of new minimal Poincaré supergravity. In particular, we have emphasized the simpler structure of this formulation relative to the old minimal one.

A major simplification occurs due to the only appearance of gauge fields in the graviton multiplet and to the role played by the $b_{\mu\nu}$ field strength as a modification of the spin and $U(1)$ connections. The multiplets reported here are the building blocks for constructing Lagrangians in the new minimal formulation, including higher curvature terms. These multiplets should play an important role in the description of the point field limit aspects of heterotic strings.

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