THE THEORY OF IONIZATION GROWTH IN GASES
UNDER PULSED AND STATIC FIELDS

A.J. Davies and C.J. Evans
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THE THEORY OF IONIZATION GROWTH IN GASES
UNDER PULSED AND STATIC FIELDS

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PREFACE

The phenomenon of electrical breakdown of a gas is encountered in many fields of physics and engineering. Consequently, investigations aimed at understanding the ionization and growth processes which lead to the breakdown have been carried out over many years. Experimental investigations, commencing with those of Townsend (see Ref. 3), have had a great deal of success, and a large number of the basic physical processes involved are well understood. It is only recently, however, that a sound basis has been laid with respect to the theory of the growth of the ionization currents in space and time.

Many people have told the present authors of the difficulty of locating and understanding the literature describing the theory of the ionization growth, and have suggested that a complete account of the theory would be of great value to all those concerned with electrical discharge phenomena.

In the present work, therefore, an account is given of the statistical and non-statistical theory of ionization growth in gases under pulsed and static fields in as far as it has been developed up to the present time.
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LIST OF SYMBOLS

The following table lists the most frequently used symbols with their meanings. Certain of the symbols, however, may appear with different meanings at various points of the text.

Subscripts used to identify particles and processes

- e  electron
- i  ion
- p  positive ion
- n  negative ion
- m  metastable atom
- ph photon
- ex excitation
- recomb recombination

Subscripted symbols

(The subscript k identifies the particle species or process.)

- \( n_k \) charge (or number) density
- \( n_k \) charge (or number of particles) per unit length of a discharge
- \( J_k \) current density
- \( I_k \) current
- \( f_k \) velocity distribution function
- \( W_k \) drift velocity
- \( \mu_k \) mobility
- \( s_k \) mean free path at 1 Torr
- \( s_k \) mean free path
- \( D_k \) diffusion coefficient
- \( \tau_k \) lifetime or delay time
- \( \gamma_k \) Townsend secondary ionization coefficient
- \( \varepsilon \) average energy of electrons
Other symbols

\( \alpha \) Townsend primary ionization coefficient
\( \alpha' \) excitation coefficient
\( \alpha_m \) primary coefficient for production of metastable atoms
\( \alpha_{\text{eff}} \) effective primary ionization coefficient
\( \beta \) ionization coefficient for collisions between positive ions and gas molecules
\( a \) attachment coefficient
\( \mu \) detachment rate (Chapter I) or amplification factor (Chapter IV)
\( R \) recombination coefficient
\( I_0 \) externally maintained electron current at cathode generation period
\( \tau \) generation period
\( e \) electronic charge
\( E \) electric field
\( p \) gas pressure (also the parameter in the Laplace transform)
\( d \) electrode separation
\( V \) potential difference
\( V_i \) ionization potential
\( V_s \) breakdown potential
\( W \) defined by \( 1/W = 1/W_e + 1/W_p \)
\( V(z) \) generating function for probability distribution
\( \lambda \) exponential growth constant
CHAPTER I

THE BASIC IONIZATION AND TRANSPORT PROCESSES

Under normal conditions a gas is an almost perfect insulator. If, however, a sufficiently large potential difference is applied between electrodes in a gas, it undergoes a transition from an electrically insulating state to a conducting state, in which a small ionization current may be amplified almost without limit. The transition to a conducting state is termed the "electrical breakdown" of the gas. A study of electrical breakdown, besides providing information concerning the fundamental atomic collision processes responsible for ionization growth, also contributes to the design of many important devices in modern electronics (for example, thyatars, gas lasers, etc.).

The fundamental collision and ionization processes which can contribute to the growth of ionization have been understood for a long time [see Llewellyn Jones\(^1\) or Grey Morgan\(^2\)], but it is necessary to elucidate which of these processes are important under any given conditions and to determine by what means they interact to produce the observed current growth.

From Townsend (1903)\(^3\) onwards a considerable effort has gone into experimental investigations of the ionization phenomena. It is only recently, however, that a firm basis has been given to the theory of the ionization growth, and it is with these theoretical investigations that we will be concerned in the present work. We shall show how it is now possible to give a complete description of the ionization growth in terms of the fundamental atomic processes both when the electric field is uniform or when it is distorted by the space charge of the electrons and ions. It will also be shown how the theory is modified when the effect of statistical fluctuations in the ionization currents is taken into consideration.

1. TRANSPORT PHENOMENA

Consider an electron swarm in equilibrium with a field \(E\) in a gas at a pressure \(p\), the electrons having a distribution function \(f_e(x, y, z, v_x, v_y, v_z, t)\), where \(f_e\) \(dx\) \(dy\) \(dz\) \(dv_x\) \(dv_y\) \(dv_z\) is the number of electrons with velocity in the range \(v_x \sim (v_x + dv_x), v_y \sim (v_y + dv_y), v_z \sim (v_z + dv_z)\), and in the volume element \(x \sim (x + dx), y \sim (y + dy), z \sim (z + dz)\). The motion of the electrons will be described by the Boltzmann equation

\[
\frac{\partial f_e}{\partial t} + \left( v_x \frac{\partial f_e}{\partial x} + v_y \frac{\partial f_e}{\partial y} + v_z \frac{\partial f_e}{\partial z} \right) + \left( \frac{eE_x}{m} \frac{\partial f_e}{\partial v_x} + \frac{eE_y}{m} \frac{\partial f_e}{\partial v_y} + \frac{eE_z}{m} \frac{\partial f_e}{\partial v_z} \right) = \left( \frac{\partial f_e}{\partial t} \right)_{\text{collisions}}.
\]

It is seldom possible to determine experimentally the distribution function itself, but various moments may be measured, such as
the particle number density \( \rho_e = \int f_e \, dx \, dy \, dz \, dv_x \, dv_y \, dv_z \),

the average energy \( \varepsilon = \frac{m}{2N_e} \int v^2 f_e \, dv_x \, dv_y \, dv_z \),

the drift velocity \( W_e = \frac{1}{N_e} \int v \, f_e \, dv_x \, dv_y \, dv_z \).

In general we consider the magnitude of the drift velocity \( W_e \) in the direction of any applied field, and we assume that the motion of all the electrons can be described by this drift velocity which is assumed to be attained almost instantaneously. It can be shown quite simply that \( W_e \) is, in general, a function only of the parameter \( E/p \). For example, when the gas is weakly ionized so that the number of electrons and ions is small compared to the number of gas molecules, and if the mean free path is the same for all electrons and is independent of \( \gamma \), then

\[
W_e = \frac{2}{3} \frac{e}{m} \frac{\rho^0_e \langle 1 \rangle}{\gamma} \frac{E}{p} = \mu_e E, \tag{I.1}
\]

\( \mu_e \) being the electronic mass, \( \rho^0_e \) the mean free path of electrons at 1 Torr, and \( \mu_e \) the electronic mobility.

Similarly the drift velocity of a positive ion \( W_p \) is also a function of \( E/p \), and to a good approximation

\[
W_p = \frac{e}{m} \frac{E}{p} \frac{\rho^0_p \langle 1 \rangle}{\gamma} = \mu_p E. \tag{I.2}
\]

If \( \rho_e \) and \( \rho_p \) are the electron and positive ion number densities at any point in space, then the corresponding current densities \( J_e \) and \( J_p \) will be given by

\[
J_e = \rho_e W_e \tag{I.3}
\]

and

\[
J_p = \rho_p W_p \tag{I.4}
\]

Furthermore, if the ionization currents are spread uniformly over an area of cross-section \( A \), then the total electron and ion currents flowing through this area will be given by

\[
I_e = \rho_e A W_e = n_e W_e \tag{I.5}
\]

and

\[
I_p = \rho_p A W_p = n_p W_p, \tag{I.6}
\]

where \( n_e \) and \( n_p \) denote the number of electrons and positive ions per unit length of the discharge.

In general, when we consider the motion of charged particles, diffusion effects will be small compared with the effects due to drift. Where they are important, however, we will assume that diffusion can be represented by a macroscopic diffusion coefficient giving rise to additional diffusion current densities:

\[
J_e = -D_e \nabla \rho_e \tag{I.7}
\]

and

\[
J_p = -D_p \nabla \rho_p \tag{I.8}
\]
in the steady state, where \( D_e \) and \( D_p \) are the electron and positive ion diffusion coefficients given approximately by

\[
D_e = \frac{\kappa_e v_e}{3} \tag{1.9}
\]

and

\[
D_p = \frac{\kappa_p v_p}{3}. \tag{1.10}
\]

Besides the motion of charged particles, we shall also be considering the behaviour of excited and metastable atoms and their role in the ionization growth. Here again the motion can be described by macroscopic diffusion coefficients \( D_{ex} \) and \( D_m \), with

\[
J_{ex} = -D_{ex} \nabla \rho_{ex} \tag{1.11}
\]

and

\[
J_m = -D_m \nabla \rho_m. \tag{1.12}
\]

The potential energy associated with excited and metastable atoms may be used to eject secondary particles from metal surfaces, or may cause ionization or excitation of other gas molecules. The lifetime of an excited atom is typically of the order of \( 10^{-8} \) sec or less, and because of their short lifetime they will normally play a limited role in ionization phenomena. Metastable atoms, on the other hand, can only return to the ground state by collisions with another body or by excitation to a higher level followed by reversion to the ground state, and have typical lifetimes of the order of \( 10^{-3} \) sec. Under certain conditions they may make an important contribution to the ionization growth.

A resonance photon travelling in a gas may be absorbed to cause excitation of the levels in a gas molecule. After a mean lifetime \( \tau_{ex} \sim 10^{-8} \) sec a photon may be emitted, and this photon has a high probability of being re-absorbed by adjacent atoms, to be subsequently re-emitted (and scattered as the emission is isotropic), and so on. During the lifetime \( \tau_{ex} \) the photon is effectively trapped in the atom. Consequently the transport of these resonance photons is much slower than that of non-resonance photons, which have a small probability of being absorbed (and thus scattered) as they pass through the gas.

Owing to the high reabsorption probability, we may regard the motion of resonance radiation as a diffusion process with characteristic velocities of the same order as those of metastable atoms and excited atoms. For diffusion distances that are large compared with the mean free path between absorptions, there will be a great number of scatterings and a high probability that one of these causes a perturbation in the frequency of the photon. Subsequently, the motion is similar to that of a non-resonance photon, and a large distance compared with the mean free path may be covered before further scattering takes place. The trapped radiation thus remains near to where it is produced for a certain characteristic delay time, after which it moves like a non-resonance photon. Depending on the relative dimensions of the mean free path to that of the apparatus the motion can either be regarded as a diffusion process or as a simple non-resonance motion with a characteristic delay time.
2. **PRODUCTION AND LOSS MECHANISMS**

**OF VARIOUS PARTICLE SPECIES**

2.1 **Gaseous processes in pure gases**

2.1.1 **Primary ionization**

Consider a pure gas at a pressure $p$ in which has been established a field $E$. Electrons travelling through the gas will come into equilibrium and will travel with the drift velocity, and have a mean energy, corresponding to the particular value of $E/p$. Because of their spread of velocities a certain proportion of the electrons will acquire sufficient energy to make it possible for gas molecules to be ionized in collisions between the electrons and the molecules. We may define the Townsend Primary Ionization Coefficient as the average number of ionizing collisions an electron undergoes in travelling unit distance in the direction of the field. For a given value of $E/p$ the mean electron energy is fixed and we should expect the probability of ionization per collision to be constant. Now $\alpha$ is the mean number of ionizations per unit distance, and the mean free path is inversely proportional to the gas pressure $p$, so that the probability of ionization per collision will be proportional to the ratio $\alpha/p$. Thus $\alpha/p$ will be a function of $E/p$.

2.1.2 **Excitation**

In travelling through the gas, electrons will also undergo excitation collisions with gas molecules. By analogy with the $\alpha$ coefficient we may define an excitation coefficient $\alpha'$ as the average number of excitation collisions undergone by an electron in travelling unit distance in the direction of the field.

2.1.3 **Recombination**

In a partially ionized gas there will be collisions between electrons and positive ions which may result in neutralization to form neutral gas atoms or molecules, thus removing charged particles from the gas. The rate of recombination will be proportional to the product of the electron and ion densities $\rho_e \rho_p$, and we may define a recombination coefficient $R$ such that $R \rho_e \rho_p$ is the probability per unit volume per unit time of an electron and ion recombining, i.e.

\[
\frac{\partial \rho_e}{\partial t}_{\text{recomb}} = \frac{\partial \rho_p}{\partial t}_{\text{recomb}} = -R \rho_e \rho_p. \tag{I.13}
\]

Since the rate of recombination is proportional to the product of $\rho_p$ and $\rho_e$ it will not be of importance at small ionization densities. The probability of recombination will depend on the time an electron is within the interaction range of the ion, thus, $R$ will be a decreasing function of electron energy. If, in addition, negative ions are present in a gas, recombination will also be possible between the negative and positive ions.

2.1.4 **Attachment**

Electrons may be lost from a region of a gas by attachment to neutral gas molecules forming negative ions. We define an attachment coefficient $a$ as the probability of attachment per unit distance travelled by an electron in the direction of the field, so that attachment will have the effect of reducing the number of ionizations per unit distance from $\alpha$ to $\alpha - a$. The coefficient $\alpha - a$ is sometimes called the effective primary ionization coefficient $\alpha_{\text{eff}}$. 

- 4 -
Attachment will be of particular importance in electronegative gases where the potential energy of the negative ion is smaller than that of the atom or molecule in its normal state.

2.1.5 Detachment

As the reverse process to attachment, negative ions will have a probability \( \mu \) of detaching per unit time to give an electron and a neutral atom or molecule. The reciprocal of this probability \( 1/\mu \) is thus the mean lifetime of a negative ion. Since the detachment probability will be a linear function of \( n \), it may, unlike recombination, be an important process at low current densities.

2.1.6 Ionization by positive ions

In his early work Townsend\(^1\) not only considered ionization of gas molecules by electrons but also by positive ions produced in the body of the gas. He defined an ionization coefficient \( \beta \) as the average number of ionizing collisions per unit distance travelled by a positive ion in the direction of \( E \), and (like \( \alpha/p \)) \( \beta/p \) is a function of \( E/p \). Here we shall consider this process because of its historical importance, but it is now accepted that in all fields normally met with in the laboratory its contribution is negligible compared with the contributions of other possible processes\(^1,2\). Positive ion energies \( \geq 300 \text{ eV} \) are necessary before any appreciable ionization can be detected.

2.2 Gaseous processes in gas mixtures

There are a number of additional ionization processes to be considered when the gas is not pure or is a mixture of pure gases.

2.2.1 Photo-ionisation

Consider a gas mixture where the potential energy \( eV^{ex}_{1} \) of an excitation level of one of the components is greater than the ionization energy \( eV^{i}_{1} \) of a second component. When one of the excited atoms of the first species decays, the photon emitted will have sufficient energy to cause photo-ionization of an atom of the second species.

In a pure gas, photo-ionization can only occur if the photon interacts with an atom that is already in an excited state of energy \( eV^{ex}_{i} \), where \( h\nu = eV^{ex}_{i} > eV^{i}_{1} - eV^{ex}_{1} \). It is also possible for an excited ion to return to the ground ionized state with the emission of a photon having an energy greater than the ionization energy \( eV^{i}_{1} \) of the neutral atom. Both these processes are, however, second-order processes, and in general photo-ionization may be neglected in pure gases.

2.2.2 The Penning effect

Metastable atoms of one gas component may possess sufficient potential energy \( eV^{m}_{m} \) to be able to ionize, by collision, the atoms of a second component or impurity. This process is known as the Penning effect. Since the ionization energies of the common molecular gases -- oxygen, nitrogen, and hydrogen -- are about 15 eV, it is clear that the Penning effect can be very important in monatomic gases such as helium (\( eV^{m}_{m} \approx 20 \text{ eV} \)) and neon (\( eV^{m}_{m} \approx 16 \text{ eV} \)) because very slight traces of impurity of \( H_{2} \) in \( He \), for example, can lead to increased ionization. The highest degree of gas purity is thus essential when investigating ionization phenomena in rare gases.
The effect of this process is to modify the primary ionization coefficient. Each electron has a certain probability of producing a metastable atom per unit distance, and each metastable atom has a fixed probability of ionizing an impurity atom during its lifetime. Thus the effect is to produce a delay in the production of some of the electrons and ions which is comparable with the lifetime of the metastable atoms.

2.3 Electrode processes

In order to consider the various ionization processes that can occur at metal surfaces in a gas, it is convenient, for the moment, to consider two plane parallel electrodes, separated by a distance \( d \), immersed in a gas at a pressure \( p \). A potential difference \( V \) is applied between the electrodes, producing a uniform field \( E \) in the gas.

2.3.1 The initiation of the discharge

In any gaseous discharge it is necessary to have some active particles, normally electrons, present in the discharge space in order to initiate the ionization processes. These particles may be produced, for example, by the ionization of gas molecules by cosmic rays or by field, photo, or thermionic emission from the cathode surface. For the purpose of mathematical analysis it is convenient to discuss two cases: a) when a steady current \( I_0 \) of electrons is generated at the cathode, or b) when an instantaneous pulse of \( N_0 \) electrons is produced, also at the cathode. We shall also consider the condition where unit charge is generated at an arbitrary point in the discharge space at the initial instant. The solution for the ionization currents at any subsequent time will obviously be the appropriate Green's function for an arbitrary charge distribution at time zero. A simple generalization will also give the solution when the current \( I_0 \) of electrons leaving the cathode is a function of time.

2.3.2 Secondary emission due to the incidence of positive ions on the cathode

Electrons drifting in the field \( E \) will undergo ionizing collisions with gas molecules producing one further electron and one ion in each such collision. These ions will drift back to the cathode with the positive ion drift velocity \( W_p \) and will eventually collide with the cathode where they may eject secondary electrons by virtue of either their kinetic or potential energies. The details of the ejection mechanisms are discussed fully in the literature and we shall simply assume that each ion incident on the cathode will have a probability \( \gamma_i \) (sometimes written \( \gamma \)) of producing a particular field and gas pressure.

Since each ion is generated in an ionizing collision, \( \gamma_i \) may also be defined as the probability that a secondary electron is produced per ionizing collision in the gas.

2.3.3 Secondary electron emission due to the incidence of photons on the cathode

A fraction of the photons produced (by excitation collisions of the primary electrons) in any region of the gas will sooner or later arrive at the cathode surface where the photons will have a fixed probability of producing a secondary electron by the photoelectric effect. Each photon will have a certain energy \( h\nu \) associated with it and may be a resonance or non-resonance photon. In the latter case it will reach the cathode after a delay time of the order of the lifetime of the excited atom (\( \approx 10^{-8} \) sec), while a resonance photon may have a considerable delay time due to its repeated absorption and scattering.
For photons of a given energy and delay time we may define a secondary emission coefficient $\delta$ as the probability of the production of a secondary electron by this type of photon per unit distance travelled by the primary electron. The coefficient $\delta$ incorporates a factor representing the fraction of the photons, emitted at a given point in space, which reach the cathode. In general, therefore, $\delta$ must be a function of the geometry of the electrodes and of the position of the point under consideration. For infinite plane parallel electrodes, however, there will be no geometric effect.

It is normal practice to represent this secondary process by a coefficient $\gamma_{ph}$ (sometimes written $\delta/a$) which denotes the probability of production of a secondary electron per ionizing collision. If there are a number of different species of photons having different energies and delay times, then there will be a series of coefficients $\gamma_{ph}^1, \gamma_{ph}^2, \ldots$ with associated delay times $\tau_{ph}^1, \tau_{ph}^2, \text{etc.}$

2.3.4 Secondaries from metastable and excited atoms on the cathode

Excited and metastable atoms produced in the body of the gas will diffuse at a rate determined by the neutral gas density, and a fraction of those produced in a given region of the discharge space will eventually reach the cathode, where they may cause a secondary electron to be emitted. Obviously, in general this process will be more probable for metastable atoms because of their greater lifetime, so that there is much less likelihood of their decaying before reaching the electrodes.

As in the previous cases, we may define coefficients $\gamma_m$ (for metastables) and $\gamma_{ex}$ (for excited atoms) as the probability of the production of a secondary electron per ionizing collision of the primary electron. Mathematically there is no difference between the two processes apart from the different lifetimes involved.

The coefficients $\gamma_m$ and $\gamma_{ex}$ will be functions of the distance from the cathode due to geometric effects and to the greater probability of an atom returning to the ground state the further away from the cathode it is created. The distance dependence will be especially important when the lifetimes are of the order of the time the particles take to diffuse across the electrode separation.
CHAPTER II

THE NON-STATISTICAL, UNIFORM FIELD THEORY OF IONIZATION GROWTH

1. THE CONTINUITY EQUATION

Consider a volume $V$ of a gas enclosed by a surface $S$. Let $dV$ be an element of volume at a point $P$, and $dS$ a vector element of the surface: see Fig. II.1. We suppose that the rate of gain of charge, say electrons, by the elemental volume $dV$ is $(\partial \rho_e / \partial t) dV$, where $\rho_e$ is the number density of electrons in the neighbourhood of $P$. Now the elemental volume may be gaining these electrons either by the creation of electrons or by the motion of electrons from other parts of the region. It will be gaining electrons at a rate $\rho_e / \text{unit volume}$, say, due to primary ionization, detachment, or injection, and will be losing electrons by recombination and attachment if these processes are present.

The net rate of gain of electrons will be given by

$$ \int_V \frac{\partial \rho_e}{\partial t} \, dV = \int_V \left( q_e - R \rho_e \rho_p - a \rho_e W_e + \mu_p n \right) \, dV - \int_S \left( \rho_e W_e - D_e \nabla \rho_e \right) \cdot dS, $$

where the second integral on the right-hand side represents the net outward flux of electrons through the surface $S$ due to drift and diffusion. This last relation may be transformed by Gauss's integral theorem to

$$ \int_V \frac{\partial \rho_e}{\partial t} \, dV = \int_V \left( q_e - R \rho_e \rho_p - a \rho_e W_e + \mu_p n \right) \, dV - \int_V \left[ \nabla \cdot \left( \rho_e W_e \right) - D_e \nabla^2 \rho_e \right] \, dV, $$

and for any elemental volume we have

$$ \frac{\partial \rho_e}{\partial t} = q_e - R \rho_e \rho_p - a \rho_e W_e + \mu_p n - \nabla \cdot \left( \rho_e W_e \right) + D_e \nabla^2 \rho_e. \tag{II.1} $$

Similarly, for the rate of gain of positive ions we have

$$ \frac{\partial \rho_p}{\partial t} = q_p - R \rho_p \rho_e - \nabla \cdot \left( \rho_p W_p \right) + D_p \nabla^2 \rho_p, \tag{II.2} $$

while if negative ions are to be considered then

$$ \frac{\partial \rho_n}{\partial t} = q_n + a \rho_n W_e - \mu_n n - \nabla \cdot \left( \rho_n W_n \right) + D_n \nabla^2 \rho_n. \tag{II.3} $$
If natural radioactivity and the action of cosmic rays etc. is negligible, then \( q_e \) and \( q_p \) will be the rate of gain of electrons and positive ions due, in the main, to the primary \( \alpha \) process of ionization.

Consider now a discharge gap consisting of infinite plane parallel electrodes separated by a distance \( d \), with a voltage \( V \) producing a uniform field \( E \) between the electrodes. We suppose that an initiatory current \( I_0 \) of electrons is generated at the cathode by, for example, illumination with ultra-violet light: see Fig. II.2.

We note that by convention \( x \) is measured from the cathode \((x = 0)\) so that the field \( E \) will be positive in the negative \( x \)-direction. We may now take our elemental volume as a lamina of gas of thickness \( dx \) distance \( x \) from the cathode. Equations (II.1), (II.2), and (II.3) become

\[
\frac{\partial p_e}{\partial t} = \alpha \rho_e \rho_a + \beta \rho_p \rho_p - R \rho_e \rho_p - \alpha \rho_e W_e + \mu \rho_n - \frac{3}{\partial x} (\rho_e W_e) + D_e \frac{\partial^2 \rho_e}{\partial x^2}, \tag{II.4}
\]

\[
\frac{\partial p_p}{\partial t} = \alpha \rho_e \rho_e + \beta \rho_p \rho_p - R \rho_e \rho_p + \frac{3}{\partial x} (\rho_p W_p) + D_p \frac{\partial^2 \rho_p}{\partial x^2}, \tag{II.5}
\]

and

\[
\frac{\partial \rho_n}{\partial t} = \alpha W_e \rho_e - \mu \rho_n - \frac{3}{\partial x} (\rho_n W_n) + D_n \frac{\partial^2 \rho_n}{\partial x^2}. \tag{II.6}
\]

The plus sign appears before the term \((\partial/\partial x)(\rho_e W_e)\) since the drift velocity \( W_e \) is always considered positive, and the positive ions are moving in the negative \( x \)-direction. Since the applied field \( E \) is constant at all points in the discharge space, provided that the charged particle densities are not sufficiently high to cause space charge distortion of the field, the ionization coefficients and drift velocities remain constant for all values of \( x \) and the equations are linear.

With the advent of modern, high-speed, digital computers it is possible to solve Eqs. (II.4) to (II.6) numerically. It is still essential, however, to consider formal solutions to these equations as these bring out the physical processes underlying the ionization growth and also can save a large amount of computing time. It is impractical to consider all the ionization processes acting simultaneously, as the solutions then become very cumbersome. Therefore we will only consider a few of the possible processes at any given time.

2. STEADY STATE SOLUTIONS

If the potential difference between the electrodes is below a certain critical value (called \( V_S \)), then it is found that the current flowing in the discharge space settles down to a steady value with a fixed current \( I \) flowing between the electrodes. Under these conditions we wish to solve Eqs. (II.4) to (II.6) with the left-hand sides equated to zero.
2.1 Ionization by electrons and secondary ionization by positive ions in the gas

In his early work Townsend\(^1\) considered the case where the only operative ionization processes were primary ionization by electrons and secondary ionization by positive ions in the gas. Equations (II.4) and (II.5) become

\[
\frac{d}{dx} (\alpha e W_e) = \alpha \rho_e W_e + \beta \rho_p W_p
\]

and

\[
\frac{d}{dx} (\alpha_p W_p) = -\alpha e W_e - \beta \rho_p W_p ,
\]

or

\[
\frac{dI_e(x)}{dx} = \alpha I_e(x) + \beta I_p(x) = (\alpha - \beta) I_e(x) + I
\]

(II.7)

and

\[
\frac{dI_p(x)}{dx} = -\alpha I_e(x) - \beta I_p(x) = -(\alpha - \beta) I_e(x) - \beta I ,
\]

(II.8)

where \(I_e(x)\) and \(I_p(x)\) are the electron and positive ion currents at \(x\), and \(I = I_p(x) + I_e(x)\) is the gap current. In the steady state \(I\) is a constant independent of \(x\), otherwise there would be an accumulation of charge at some point in the gap.

Integrating Eq. (II.7) between 0 and \(x\), and Eq. (II.8) between \(x\) and \(d\) yields

\[
I_e(x) = \frac{\beta I}{\beta - \alpha} + \frac{[(\alpha - \beta) I_e(0) + \beta I]}{\alpha - \beta} \exp \left[(\alpha - \beta)x\right]
\]

and

\[
I_p(x) = I_p(d) + \frac{[(\alpha - \beta) I_e(0) + \beta I]}{\alpha - \beta} \left[e^{(\alpha - \beta)d} - e^{(\alpha - \beta)x}\right] .
\]

We have now to insert expressions for \(I_e(0)\) and \(I_p(d)\). In his work Townsend did not take into consideration any electrode secondary ionization processes so that \(I_e(0)\) is equal to the externally generated current of photo-electrons \(I_\alpha\) at the cathode. Positive ions are only generated in the body of the gas and not at the electrodes; thus, the density of positive ions at the anode must be zero, i.e. \(I_p(d) = 0\). Inserting these values in the above expressions we have

\[
I_e(x) = \frac{\beta I}{\beta - \alpha} + \frac{[(\alpha - \beta) I_e(0) + \beta I]}{\alpha - \beta} e^{(\alpha - \beta)x}
\]

(II.9)

and

\[
I_p(x) = \frac{(\alpha - \beta) I_\alpha + \beta I}{\alpha - \beta} \left[e^{(\alpha - \beta)d} - e^{(\alpha - \beta)x}\right] .
\]

(II.10)

To find the total current \(I\) flowing in the discharge gap we remember that \(I = I_e(x) + I_p(x) = I_e(d)\), since \(I_p(d) = 0\), and find

\[
I_e(d) = I = \frac{I_\alpha(\alpha - \beta)}{\alpha - \beta} e^{(\alpha - \beta)d} ,
\]

(II.11)
or
\[
I_e(d) = I = \frac{I_e e^{(a-s)d}}{1 - \frac{\beta}{\alpha - \beta} \left[ e^{(a-s)d} - 1 \right]}, \quad (II.12)
\]
which is the familiar Townsend equation. We note that once I is known, \(I_e(x)\) and \(I_p(x)\) can be found from Eqs. (II.9) and (II.10).

2.2 Primary ionization by electrons plus electrode secondary effects

As we have mentioned in Section 2.1 of this chapter, under all normal conditions met with in the laboratory the \(\beta\) process is negligible, so that Eqs. (II.7) and (II.8) become
\[
\frac{d}{dx} [I_e(x)] = \alpha I_e(x) \quad (II.13)
\]
and
\[
\frac{d}{dx} [I_p(x)] = -\alpha I_e(x) \quad (II.14)
\]
in the steady state. These have the solutions
\[
I_e(x) = I_e(0) e^{\alpha x} \quad (II.15)
\]
and
\[
I_p(x) = I_e(0) \left( e^{\alpha d} - e^{\alpha x} \right), \quad (II.16)
\]
remembering that \(I_p(d) = 0\).

In this section we include, besides the externally generated \(I_0\), all the other possible secondary electron currents.

2.2.1 The \(\gamma_1\) process

We have defined the secondary ionization coefficient \(\gamma_1\) as the probability per ionizing collision of a secondary electron being liberated at the cathode. This is identical with the probability per positive ion striking the cathode, since each ionizing collision involves the production of one positive ion.

Now the rate at which positive ions reach the cathode is \(I_p(0) = I_e(0)(e^{\alpha d} - 1)\) so that the rate of production of secondary electrons at the cathode is \(\gamma_1 I_e(0)(e^{\alpha d} - 1)\), which is the contribution to \(I_e(0)\) of the \(\gamma_1\) process.

2.2.2 The \(\gamma_{ph}\) process

The coefficient \(\gamma_{ph}\) represents the probability per ionizing collision of the production of a secondary electron by the process of excitation and the subsequent transport of the resulting photons to the cathode, where secondary electrons are produced by the photo-electric effect. For electrodes of finite diameter \(\gamma_{ph}\) will be a function of position, and the secondary electron current will be given by
\[
\int_0^d \gamma_{ph}(x) a I_e(x) \, dx.
\]
In the case where $\gamma_{ph}$ is independent of $x$, however, as will be the case for infinite plane parallel electrodes, this expression becomes

$$\gamma_{ph} I_e(0) \left( e^{ad} - 1 \right) .$$

2.2.3 The $\gamma_m$ and $\gamma_{ex}$ processes

By analogy with the last section, the secondary electron current at the cathode, owing to the incidence of metastable and excited atoms, will be given by

$$\int_0^d \left[ \gamma_m(x) + \gamma_{ex}(x) \right] a I_e(x) \, dx$$

or

$$\left( \gamma_m + \gamma_{ex} \right) I_e(0) \left( e^{ad} - 1 \right) ,$$

depending on whether or not $\gamma_m$ and $\gamma_{ex}$ are functions of $x$.

Combining all these effects the total electron current at the cathode is given by

$$I_e(0) = I_0 + \gamma_1 I_e(0) \left( e^{ad} - 1 \right) + \int_0^d \left( \gamma_{ph} + \gamma_m + \gamma_{ex} \right) a e^{ax} \, dx ,$$

or

$$I_e(0) = I_0 + I_e(0) \left[ \gamma_1 + \gamma_{ph} + \gamma_{ex} + \gamma_m \right] \left( e^{ad} - 1 \right) = I_0 + I_e(0) \left[ \sum_k \gamma_k(e^{ad} - 1) \right] ,$$

so that the total current flowing in the discharge space $I_e(d)$ is given by

$$I_e(d) = I = \frac{I_0 \, e^{ad}}{1 - \gamma_1(e^{ad} - 1) - \int_0^d \left( \gamma_{ph} + \gamma_m + \gamma_{ex} \right) a e^{ax} \, dx} , \quad \text{(II.17)}$$

or

$$I_e(d) = I = \frac{I_0 \, e^{ad}}{1 - \sum_k \gamma_k(e^{ad} - 1)} = \frac{I_0 \, e^{ad}}{1 - \gamma_T(e^{ad} - 1)} . \quad \text{(II.18)}$$

Equation (II.18) is known as the Townsend equation and we notice that it has the same form as Eq. (II.12); $\gamma_T$ (often referred to as $\omega / \alpha$) = $\gamma_1 + \gamma_{ph} + \gamma_m + \gamma_{ex}$ is the generalized Townsend secondary coefficient.

The Townsend equation is valid for infinite plane parallel electrodes and will also be accurate provided that the area of cross-section of the discharge is small compared with the electrode area. Strictly speaking, even for infinite electrodes, the coefficients $\gamma_m$ and $\gamma_{ex}$ may be functions of distance, since metastable atoms and excited atoms that are created a large distance away from the cathode will have a smaller probability of arriving there before decaying, than those produced in the neighbourhood of the cathode. Furthermore, metastable and excited atoms produced near the anode will have a high probability of diffusing to the anode and being absorbed, thus being removed from the discharge.
2.2.4 Experimental determination of $\alpha$ and $\gamma_T$

We see from Eq. (II.18) that the variation of the current flowing in a discharge between parallel plane electrodes with electrode separation $d$ is controlled by $\alpha$ and the action, either singly or together, of a number of secondary ionization processes represented by the general coefficient $\gamma_T$. Since $\alpha/p$ and $\gamma_T$ are found to be functions of $E/p$, the coefficients $\alpha$ and $\gamma_T$ may be found experimentally by measuring the current $I$ as a function of $d$, keeping $E/p$ constant. A typical experimental layout is shown in Fig. II.3 together with the form of a typical measured curve in Fig. II.4:

![Circuit diagram](image)

**Fig. II.3** Circuit used to measure the steady-state current flowing in a discharge gap.

![Graph](image)

**Fig. II.4** Typical variation of the current $I$ flowing in a discharge gap with electrode separation $d$.

Values of $\alpha$ and $\gamma_T$ may be determined by fitting Eq. (II.18) to the measured curve using a least squares technique. Note that for small $d$ the term $\gamma_T(e^{\alpha d} - 1)$ is negligible compared with unity; thus, the $\ln (I/I_0)$ versus $d$ curve is approximately linear with slope $\alpha$. As $d$ is increased, $e^{\alpha d}$ increases rapidly and, despite the fact that $\gamma_T$ is small, $\gamma_T(e^{\alpha d} - 1)$ attains values comparable with unity. A limit is reached when at a certain critical distance $d_S$ (called the sparking distance)

$$\gamma_T(e^{\alpha d_S} - 1) \to 1$$

and

$$I = I_0 e^{\alpha d}/[1 - \gamma_T(e^{\alpha d} - 1)] \to \infty.$$

When $d = d_S$ the denominator of Eq. (II.18) is zero, and a current will flow even in the absence of $I_0$. A replacement condition has been set up by the primary and secondary processes which does away with the need for the external irradiation of the cathode to maintain the current. Each electron leaving the cathode will produce $(e^{\alpha d} - 1)$ ionizing collisions in travelling to the anode, each of which produces a secondary electron with probability $\gamma_T$. The total number of secondary electrons generated is $\gamma_T(e^{\alpha d} - 1)$, so that if the replacement criterion

$$\gamma_T(e^{\alpha d_S} - 1) = 1$$  \hspace{1cm} (II.19)
is satisfied then each electron leaving the cathode will produce on the average one secondary electron. Thus once a current is established between the electrodes it will continue to flow even in the absence of \( I_0 \). The condition (II.19) is known as the Townsend Breakdown Criterion.

We note that only the total secondary coefficient \( \gamma_T \) is determined by the above measurements and it is not possible to evaluate the individual coefficients composing \( \gamma_T \).

2.3 Primary ionization, secondary electron production at the cathode, attachment and detachment

When attachment and detachment processes are important we have to consider the third continuity equation (II.6) for the negative ion density. From Eqs. (II.4) to (II.6) we see that the relevant equations to be satisfied are

\[
\frac{d}{dx} I_e(x) = (\alpha - a) I_e(x) + \frac{\mu}{W_n} I_n(x),
\]

(II.20)

\[
\frac{d}{dx} I_p(x) = -\alpha I_e(x),
\]

(II.21)

and

\[
\frac{d}{dx} I_n(x) = a I_e(x) - \frac{\mu}{W_n} I_p(x),
\]

(II.22)

with the boundary conditions

\[
I_e(0) = I_0 + \gamma_i I_p(0) + \gamma_{ph} \int_0^d a I_e(x) \, dx,
\]

(II.23)

\[
I_p(d) = 0,
\]

(II.24)

and

\[
I_n(0) = 0.
\]

(II.25)

From Eqs. (II.20) and (II.22) we have

\[
\frac{d^2I_e(x)}{dx^2} = \left( \alpha - a - \frac{\mu}{W_n} \right) \frac{dI_e(x)}{dx} - \frac{\alpha \mu}{W_n} I_e(x) = 0,
\]

and solving this in conjunction with condition (II.24) gives

\[
I_e(x) = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x},
\]

(II.26)

\[
I_p(x) = a \left\{ \frac{A_1}{\lambda_1} (e^{\lambda_1 d} - e^{\lambda_1 x}) + \frac{A_2}{\lambda_2} (e^{\lambda_2 d} - e^{\lambda_2 x}) \right\},
\]

(II.27)

and

\[
I_n(x) = \frac{W_n}{\mu} \left\{ (\lambda_1 - a + a) A_1 e^{\lambda_1 x} + (\lambda_2 - a + a) A_2 e^{\lambda_2 x} \right\},
\]

(II.28)

where \( A_1 \) and \( A_2 \) are constants and \( \lambda_1 \) and \( \lambda_2 \) are roots of

\[
\lambda^2 - \left( \alpha - a - \frac{\mu}{W_n} \right) \lambda - \frac{\alpha \mu}{W_n} = 0.
\]

(II.29)
From condition (II.25)
\[
\frac{A_1}{A_2} = -\frac{\lambda_2 - \alpha + a}{\lambda_1 - \alpha + a} = -\frac{\lambda_1 + (\mu/W_n)}{\lambda_2 + (\mu/W_n)} \quad \text{remembering that } \lambda_1 + \lambda_2 = \alpha - a - \frac{\mu}{W_n},
\]
so that condition (II.23) may be written
\[
A_1 + A_2 = I_0 + \gamma_1 \alpha \left\{ \frac{A_1}{\lambda_1} \left( e^{\lambda_1 d} - 1 \right) + \frac{A_2}{\lambda_2} \left( e^{\lambda_2 d} - 1 \right) \right\}
\]
or
\[
A(\lambda_1 - \lambda_2) = I_0 + \alpha \gamma_1 \left\{ \frac{\lambda_1 + \mu/W_n}{\lambda_1} \left( e^{\lambda_1 d} - 1 \right) - \frac{\lambda_2 + \mu/W_n}{\lambda_2} \left( e^{\lambda_2 d} - 1 \right) \right\}, \quad (II.30)
\]
where
\[
A_1 = (\lambda_1 + \mu/W_n)A, \quad A_2 = -(\lambda_2 + \mu/W_n)A.
\]
The total current flowing in the discharge will be given by
\[
I = I_0(d) + I_a(d)
\]
\[
= A_1 e^{\lambda_1 d} + A_2 e^{\lambda_2 d} + \frac{W_n}{\mu} \left\{ (\lambda_1 - \alpha + a)A_1 e^{\lambda_1 d} + (\lambda_2 - \alpha + a)A_2 e^{\lambda_2 d} \right\}
\]
\[
= A \left\{ \lambda_1 \left( 1 + \frac{W_n \lambda_1}{\mu} \right) e^{\lambda_1 d} - \lambda_2 \left( 1 + \frac{W_n \lambda_2}{\mu} \right) e^{\lambda_2 d} \right\}
\]
which, after substituting for A from Eq. (II.30), gives
\[
I = \frac{I_0 \left( e^{\lambda_1 d} - (\alpha - \lambda_2) e^{\lambda_2 d} \right)}{(\lambda_1 - \lambda_2) - \gamma_1 \left( (\alpha - \lambda_2) (e^{\lambda_1 d} - 1) - (\alpha - \lambda_1) (e^{\lambda_2 d} - 1) \right)} \quad \text{remembering } \lambda_1 \lambda_2 = -\frac{\alpha u}{W_n}. \quad (II.31)
\]
If there is no detachment, this equation reduces to
\[
I = \frac{I_0 \left( e^{(\alpha - a)d} - \frac{a}{\alpha} \right)}{\frac{\alpha}{a} - \gamma_1 \left( e^{(\alpha - a)d} - 1 \right)} \quad \text{remembering } \lambda_1 \lambda_2 = -\frac{\alpha u}{W_n}. \quad (II.32)
\]
Equation (II.32) is the basis of most of the experiments which have been used to find the attachment coefficient. As described in the previous section, the experiment consists of measuring I as a function of d, keeping the pressure p, the electric field E, and hence the values of the ionization and attachment coefficients constant. The form of a typical measured curve is shown in Fig. II.5.

In the region BCD, e^{(\alpha - a)d} is much larger than a/\alpha, although not so large as to make the denominator of Eq. (II.32) differ appreciably from (\alpha - a)/\alpha. At D the term

![Fig. II.5 Typical variation of the current I flowing in a discharge gap with electrode separation in the presence of attachment.](image)
In the denominator is beginning to become comparable with \((\alpha - \frac{a}{\alpha})\), and eventually becomes equal to it at the point E, where the value of \(I\) given by Eq. (II.32) tends to infinity. The portion BCDE of the curve is indistinguishable from that obtained in an experiment where attachment is absent, provided that \(a_{eff} = \alpha - a\) is interpreted as the primary ionization coefficient, \(\gamma_{T}a_{eff}\) as the secondary coefficient, and \(a_{1}\) as the effective maintained photo-current.

At small values of \(d\), \(a/a\) is no longer negligible compared with \(e^{(\alpha-a)d}\), and as a result the graph of \(\ln(I)\) against \(d\) shows a "downcurving" region AB. It is from this region of the graph that values of \(\alpha\) and \(a\) have to be deduced. Because \(a/\alpha\) is usually a small fraction, the downcurving is never very marked, and even when the current is measured very accurately, and accurate curve-fitting procedures are used, it is seldom possible to obtain a value of \(a\) to the same degree of accuracy as is possible for the ionization coefficient \(\alpha\) in non-attaching gases.

The analysis of experimental results becomes even more difficult when detachment is present. Let us suppose for the moment that we are working in the region ABC of Fig. II.5, so that the secondary processes can be neglected. We then have

\[
I = \frac{I_{0}}{\lambda_{1} - \lambda_{2}} \left\{ \lambda_{1} - \alpha \right\} e^{\lambda_{1}d} \left\{ \lambda_{2} - \alpha \right\} e^{\lambda_{2}d},
\]

which in the region BC approximates to

\[
\frac{I_{0}}{\lambda_{1} - \lambda_{2}} (\alpha - \lambda_{2}) e^{\lambda_{2}d},
\]

where \(\lambda_{1}\) is the positive root of Eq. (II.29). Since the ranges of currents and distances encountered along BC are fairly large, it is possible to obtain an accurate value of \(\lambda_{1}\), which gives a relation between \(\alpha\), \(a\), and \(\mu/W_{n}\) by virtue of Eq. (II.29). There are therefore sets of values of these three coefficients, each of which will give the same slope for the linear part of the graph, BC.

Figure II.6 shows graphs of \(\ln(I)\) against \(d\), for the two sets of data:

a) \(\alpha = 3.2\); \(a = 0.2\); \(\mu/W_{n} = 0\); \(I_{0} = 1\) (arbitrary units), and

b) \(\alpha = 4\); \(a = 3\); \(\mu/W_{n} = 6\); \(I_{0} = 44/45\).

These sets are chosen so as to give the same value of \(\lambda_{1}\) (= 3), and the same value of \(I\) at large \(d\). Set (a) gives the relation \(I_{a} = \frac{1}{\sqrt{2}} (e^{3d} - \sqrt{4})\), while from set (b), \(I_{b} = \frac{1}{\sqrt{2}} (e^{3d} - \sqrt{4})\). The difference between the two curves is at a maximum at \(d = 0.154\), where \(I_{a} = 1.626\), \(I_{b} = 1.668\) (a difference of 2.5%) and at \(d = 0\), where \(I_{a} = 1\), \(I_{b} = 0.978\), a difference of 2.24.

It should be emphasized that the two curves differ by only a small amount (\(\approx 2.5\%\)), although there have been large changes in \(\alpha\) (from 3.2 to 4), \(a\) (from 0.2 to 3), and \(\mu/W_{n}\) (from 0 to 6). We would therefore not expect to obtain accurate values of the detachment coefficient, even when the error in measuring the current is only 1%, and it is clear that quite large values of the detachment coefficient would go unsuspected in steady-state experiments. It is therefore of great interest to investigate the temporal growth of ionization in attaching gases to see if this would enable accurate values of the attachment and detachment coefficients to be found.
2.4 Other processes

Obviously the above examples do not exhaust the possible combinations of ionization processes. They do, however, illustrate the general methods used to analyse the steady-state distribution of ionization currents in a discharge space, and other processes may be considered in an analogous manner. Note that although we have indicated how the various ionization coefficients may be measured using the relations (II.18) and (II.32), their real importance is in showing how to compute the current distribution when the coefficients are assumed to be known.
3. THE TEMPORAL GROWTH OF IONIZATION IN THE ABSENCE OF DIFFUSION, ATTACHMENT, AND RECOMBINATION

If the potential difference between the electrodes of a discharge gap is increased above the breakdown potential $V_S$, then the ionization currents do not settle down to steady-state values but increase indefinitely with time. In order to analyse the ionization growth we must therefore make use of the full continuity equations (II.4) to (II.6).

As before, it would be too cumbersome to consider all possible ionization processes, and we will here consider those combinations which have been found to be of practical importance.

3.1 Continuity equations and boundary conditions

Since in this section we are not considering attachment, detachment, or diffusion, the appropriate equations to be solved are

$$\frac{\partial \rho_e}{\partial t} = \alpha W_e \rho_e - \frac{\partial}{\partial x} \left( \rho_e W_e \right)$$

and

$$\frac{\partial \rho_p}{\partial t} = \alpha W_e \rho_e + \frac{\partial}{\partial x} \left( \rho_p W_p \right) ,$$

or

$$\frac{1}{W_e} \frac{\partial}{\partial t} I_e(x,t) = \alpha I_e(x,t) - \frac{\partial}{\partial x} I_e(x,t) \quad \text{(II.33)}$$

and

$$\frac{1}{W_p} \frac{\partial}{\partial t} I_p(x,t) = \alpha I_e(x,t) - \frac{\partial}{\partial x} I_p(x,t) . \quad \text{(II.34)}$$

From Eqs. (II.33) and (II.34) and the boundary condition $I_p(d,t) = 0$ we obtain directly

$$I_e(x,t) = e^{\alpha x} I_e(0,t - \frac{x}{W_e}) \quad \text{(II.35)}$$

and

$$I_p(x,t) = \int_0^x \alpha I_e \left( 0, t - \frac{x'}{W_e} - \frac{x'}{W_p} \right) e^{\alpha x'} \, dx' , \quad \text{(II.36)}$$

where $1/W = 1/W_e + 1/W_p$. Thus it is only necessary to obtain $I_e(0,t)$, and the complete solution for $I_e(x,t)$ and $I_p(x,t)$ may be immediately written down.

The form of the boundary condition used at the cathode depends on the number and type of the secondary processes that are assumed to be acting. In general the current of electrons leaving the cathode will be

$$I_e(0,t) = I_0 + \sum_{r=1}^{n} I_r(0,t) , \quad \text{(II.37)}$$

where $I_r(0,t)$ is the current produced at time $t$ at the cathode by the $r^{th}$ secondary process.
If electrons are emitted from the cathode because of the incidence of positive ions, a term $\gamma_1 I_p(0,t)$ appears on the right-hand side of Eq. (II.37). Similarly, if electrons are produced by the incidence of non-resonance photons, a term

$$\int_0^d \gamma_{ph} \alpha I_e(x,t) \, dx$$

must be added.

If the photons have a delay time $\tau_{ph}$ associated with them, then the contribution to the right-hand side of Eq. (II.37) becomes

$$\int_0^\frac{t}{\tau_{ph}} \frac{1}{\tau_{ph}} e^{-(t-t')/\tau_{ph}} \int_0^d \gamma_{ph} \alpha I_e(x,t') \, dx \, dt', $$

where delay times are assumed to have an inverse exponential distribution. Several secondary processes of this type, with different values of $\tau_{ph}$ may, of course, be considered. Since we will be considering diffusion-type processes later, these are the only secondary processes which we will be concerned with at the moment.

Equation (II.37) thus becomes

$$I_e(0,t) = I_0 + \gamma_1 I_p(0,t) + \gamma_{ph} \alpha \int_0^d I_e(x,t) \, dx + \gamma_{ph} \alpha \int_0^\frac{t}{\tau_{ph}} e^{-(t-t')/\tau_{ph}} \int_0^d I_e(x,t') \, dx \, dt'. $$

($II.38$)

3.2 Davidson's exact solution

The exact solution of the continuity equations (II.33) and (II.34), subject to the correct boundary conditions, was first published (without proof) by Davidson\(^1\), originally for the case of the $\gamma_1$ and $\gamma_{ph}$ processes only.

For convenience Davidson divided his solution into two parts: 1) the values which $I_p(x,t)$ and $I_e(x,t)$ would have if there were no charges in the gap at time zero; and, 2) the values which they would have if the constant maintained current density $I_0$ were absent. The general solution is then simply the sum of these two parts.

3.3 Part 1 of the exact solution

We wish to solve Eqs. (II.33) and (II.34) subject to the boundary conditions (II.38) with $\rho_e(x,0) = \rho_p(x,0) = 0$ and $I_p(d,t) = 0$.

Application of the Laplace transformation\(^*) to Eqs. (II.33) and (II.34) gives

$$\frac{P}{W_e} \bar{\tau}_e(x) = \alpha \bar{\tau}_e(x) - \frac{d \bar{\tau}_e(x)}{dx} + \frac{I_e(x,0)}{W_e},$$

(II.39)

and

$$\frac{P}{W_p} \bar{\tau}_p(x) = \alpha \bar{\tau}_e(x) + \frac{d \bar{\tau}_e(x)}{dx} + \frac{I_p(x,0)}{W_p},$$

(II.40)

\(*\) The Laplace transform $\mathcal{F}(x,p)$ of a function $F(x,t)$ is defined by

$$\mathcal{F}(x,p) = \int_0^\infty \bar{F}(x,t) e^{-pt} \, dt.$$
while the boundary condition transforms to

\[ \bar{I}_e(0) = \frac{I_0}{p} + \gamma \bar{I}_p(0) + \left( \gamma \phi + \frac{\gamma \phi}{1 + p \tau} \right) \alpha \int_0^d \bar{I}_e(x) \, dx \]  

(II.41)

and

\[ \bar{I}_p(d) = 0. \]  

(II.42)

Setting \( \bar{I}_e(x,0) = \bar{I}_p(x,0) = 0 \), the solutions of the differential equations (II.39) and (II.40) are

\[ \bar{I}_e(x) = A \, e^{\phi x} \]

and

\[ \bar{I}_p(x) = \frac{\lambda A}{\phi} \, e^{\phi x} + B \, e^{\psi x / \psi p}, \]

where \( A \) and \( B \) are constants, \( \phi = \alpha - \psi / \psi \), and \( \psi = \alpha - \psi / \psi \).

From Eq. (II.42) we have \( B = (\lambda A / \phi) e^{\phi d} \), and substituting for \( \bar{I}_p(0) \) and \( \bar{I}_e(x) \) in Eq. (II.41) gives

\[ A = \frac{I_0}{pf(p)}, \]

where

\[ f(p) = 1 - \frac{\gamma \alpha}{\phi} (e^{\phi d} - 1) - \left[ \frac{\gamma \phi}{\psi} + \frac{\gamma \phi}{\psi (1 + p \tau)} \right] (e^{\phi d} - 1). \]  

(II.45)

Thus

\[ \bar{I}_e(x) = \frac{I_0 \, e^{\phi x}}{pf(p)} \]  

(II.44)

and

\[ \bar{I}_p(x) = \frac{\alpha I_0}{pf(p)} \, e^{\psi x / \psi p} \left( \frac{e^{\phi d} - e^{\phi x}}{\phi} \right). \]  

(II.45)

If two or more secondary processes of the delayed photon type are present, a sum of terms of the form

\[ \frac{\gamma \phi}{\psi (1 + p \tau)} \]

will appear in the expression \( f(p) \) and all the subsequent theory will be unaltered.

The general theorem for the inversion of a Laplace transform \( \tilde{F}(p) \) states that

\[ f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \tilde{F}(p) \, e^{pt} \, dp, \]

where \( \beta \) is chosen so that the line of integration \( L \) lies to the right of any pole of \( \tilde{F}(p) \) in the complex \( p \)-plane. By using Cauchy's residue theorem, this line may be replaced by the arc \( C \) of an infinite semicircle, with its centre at the origin, lying to the right of \( L \) (see Fig. II.7).
Applying the inversion theorem to Eqs. (II.44) and (II.45),

\[ I_e(x, t) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{I_0 e^{\phi x + pt}}{p f(p)} \, dp \quad \text{(II.46)} \]

and

\[ I_p(x, t) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{a I_0 e^{\phi d} - e^{\phi x}}{p f(p)} \, e^{pt + px/\lambda_p} \, dp . \quad \text{(II.47)} \]

For \( t < 0 \), the integrals are zero on the contour \( \mathcal{C} \), giving \( I_e(x, t) = I_p(x, t) = 0 \), which we should expect on physical grounds because of the initial conditions applied. For \( t > 0 \), either integral taken round the arc of the infinite circle to the left of the line \( \mathcal{L} \) is zero and we may then extend the contour \( \mathcal{C} \) to an infinite circle. The value of each integral is then equal to 2\( \pi i \) times the sum of the residues at all the poles of the integrand.

Thus writing down the residues at the poles we have, at all positive times,

\[ I_e(x, t) = I_0 \left\{ \frac{e^{\phi x}}{f(0)} + \sum_{\lambda} \frac{e^{\phi x + pt}}{p \left( \frac{\partial f(p)}{\partial p} \right)_{p=\lambda}} \right\} \quad \text{(II.48)} \]

and

\[ I_p(x, t) = I_0 \left\{ \frac{e^{\phi d} - e^{\phi x}}{f(0)} + \sum_{\lambda} \frac{\partial (e^{\phi d} - e^{\phi x})}{p \left( \frac{\partial f(p)}{\partial p} \right)_{p=\lambda}} \, e^{pt + px/\lambda_p} \, dp \right\} \quad \text{(II.49)} \]

where the summation extends over every value \( \lambda \) of \( p \) which satisfies \( f(\lambda) = 0 \). Writing \( e^{\phi x + pt}/\lambda(\partial f/\partial p)_\lambda = Q \), say, and \( R = a e^{\phi (d-x)} - 1 \), we have

\[ I_e(x, t) = I_0 \left\{ \frac{e^{\phi x}}{1 - \gamma_1(e^{\phi d} - 1)} + \sum_{\lambda} Q \right\} \quad \text{(II.50)} \]

and

\[ I_p(x, t) = I_0 \left\{ \frac{e^{\phi d} - e^{\phi x}}{1 - \gamma_1(e^{\phi d} - 1)} + \sum_{\lambda} R Q \right\} . \quad \text{(II.51)} \]

3.3.1 Flash illumination

These exact expressions for the ionization growth were first given by Davidson\(^*\)), who also described various other alternative forms\(^*\)) of the exact solution, which we shall consider later (Section 3.5 of this chapter).
The theory so far has been derived on the assumption of constant illumination of the cathode giving rise to a maintained photo-electron current $I_0$. A large number of experiments have also been carried out employing "flash illumination" in which there is no distribution of charges in the gap up to the instant $t = 0$, when a brief flash of light is allowed to fall on the cathode. This may be regarded as a special case of Part 2 of Davidson's solution where unit negative charge is present at the cathode $x = 0$ at time zero. However, the expression for the current in this case may be obtained quite simply by differentiating the expressions for the Part 1 solution, formulae (II.46) and (II.47), and replacing $I_0$ by $N_0$, the total charge released from unit area of the cathode at time zero, i.e.

$$I_e(x,t) = \frac{1}{2\pi i} \int \frac{N_0}{f(p)} e^{\phi x + pt} \, dp$$  \hspace{1cm} (II.52)

and

$$I_p(x,t) = \frac{1}{2\pi i} \int \frac{\alpha N_0}{f(p)} \frac{e^{\phi d} - e^{\phi x}}{\phi} e^{pt+px/\hbar} \, dp.$$  \hspace{1cm} (II.53)

These expressions correspond to the "unit pulse" solution described by Aufer\(^3\) which is, therefore, no more than a special case of the more general solution given by Davidson\(^4\).

By a generalization of expressions (II.52) and (II.53) we also see that if the cathode photo-current is a function of time $I_0(t)$, then the resulting $I_e(x,t)$ and $I_p(x,t)$ will be given by

$$I_e(x,t) = \frac{1}{2\pi i} \int_0^t \int \frac{I_0(t')}{f(p)} e^{\phi x + p(t-t')} \, dp \, dt'$$  \hspace{1cm} (II.54)

and

$$I_p(x,t) = \frac{1}{2\pi i} \int_0^t \int \frac{\alpha I_0(t')}{f(p)} \frac{e^{\phi d} - e^{\phi x}}{\phi} e^{p(t-t')+px/\hbar} \, dp \, dt'. \hspace{1cm} (II.55)$$

3.4 Davidson's approximate solution

As has been pointed out previously, in practice it is only necessary to derive an expression for the electron current at the cathode $I_e(0,t)$, since $I_e(x,t)$ and $I_p(x,t)$ may then be computed from formulae (II.55) and (II.56). Now the exact solution (II.50) may be written

$$I_e(0,t) = A + \sum_{\lambda} B \, e^{\lambda t},$$

the summation being extended over all the roots of $f(p) = 0$ [i.e. Eq. (II.43)]. Davidson has shown that $f(p) = 0$ has one real root $\lambda_0$, and an infinite number of complex roots, the real parts of which are smaller than $\lambda_0$. The oscillating terms involving the complex roots are therefore damped relative to the term $B \, e^{\lambda_0 t}$. Provided that $B$ is given the value

$$B_0 = \frac{I_e/\lambda_0}{(\partial I_e/\partial p)|_{p=0}},$$

specified by the exact solution, the expression $A + B \, e^{\lambda_0 t}$ eventually, for large $t$, represents the value of $I_e(0,t)$ with negligible percentage error and is in this sense an asymptotic solution.
In the whole range of \( t > 0 \) the curve giving the exact solution will then perform an oscillation of complex form about the curve given by the approximate solution \( A + B e^{\lambda t} \) (the two curves intersecting repeatedly) and the amplitude of the oscillation will, at large \( t \), become negligible in comparison with \( A + B e^{\lambda t} \).

At small overvoltages \( B \) does not differ much from \(-A\); thus, for small overvoltages,

\[
I_e(0, t) = \frac{I_0}{1 - \gamma_1(e^{\alpha d} - 1)} (1 - e^{\alpha t}) \quad \text{(II.56)}
\]

is an adequate approximation at large times.

3.5 Solutions in particular time ranges\(^{5-7}\)

The expressions (II.50) and (II.51), given by Davidson\(^1\), represent only one form in which the exact solution (II.46) and (II.47) may be expressed. Since the summations involve an infinite number of terms, it is sometimes more convenient to express the results in a rather different form depending on the time range that is of interest.

3.5.1 The \( \gamma_1 \) secondary process only, and times of the order of a few ion transit times

When the \( \gamma_1 \) process is the only operative secondary process we have

\[
\frac{I_e(0, t)}{I_0} = \frac{1}{2\pi i} \int \frac{e^{\alpha t}}{pf(p)} \, dp,
\]

where \( f(p) = 1 - (\gamma_1 \alpha / \phi)(e^{\phi d} - 1) \). We may write the integral as

\[
\frac{1}{2\pi i} \int \frac{e^{\alpha t}}{pR} \left[ 1 - \frac{Z}{R} \right]^{-1},
\]

where

\[
Z = \frac{\gamma_1 \alpha}{\phi} e^{\phi d} \quad \text{and} \quad R = 1 + \frac{\gamma_1 \alpha}{\phi},
\]

and using the binomial expansion

\[
\frac{I_e(0, t)}{I_0} = \int \frac{e^{\alpha t}}{p} \left( \sum_{m=0}^{\infty} \frac{Z^m}{R^{m+1}} \right) \, dp = \int \frac{e^{\alpha t}}{p} \left\{ \sum_{m=0}^{\infty} \frac{\gamma_1^m \alpha^m e^{m \phi d}}{p^{m+1}} \right\} \, dp.
\]

Now if \( t < 0 \) then all the terms are zero on \( C \);

- if \( 0 < t < d/W \) the only non-zero term is the first \((m = 0)\);
- if \( d/W < t < 2d/W \) the only non-zero terms are the first and second \((m = 0, 1)\);
- if \( nd/W < t < [(n + 1)d]/W \) the only non-zero terms are the first to \((n + 1)\)th \((m = 0, \ldots, n)\).

Considering the integral of the \( m \)th term in the summation, we see that there are poles at \( p = 0 \) and \( p = p_0 = \alpha W (1 + \gamma_1) \), with residues

\[
\frac{\gamma_1^m e^{m \phi d}}{(1 + \gamma_1)^{m+1}}.
\]
and
\[
\frac{1}{m!} \left\{ \frac{d^n}{dp^n} \left[ \frac{\phi \Gamma^{\alpha} e^{m \alpha d} e^{pt}}{p^{(1/W)}} \right] \right\}_{p=p_0},
\]
and thus
\[
\frac{I_e(0,t)}{I_o} = \sum_{n=0}^{n} \gamma_1^{m} e^{n \alpha d} \left( \frac{1}{(1 + \gamma_1)^{n+1}} + \left[ -\alpha W \right]^{m} \right) \left\{ \frac{d^n}{dp^n} \left[ \frac{(1 - \alpha W)}{p} e^{p \left( t - \left[ n d/W \right] \right)} \right] \right\}_{p=p_0},
\]
\[
\frac{nd}{W_e} < t < \frac{(n + 1)d}{W_e} \tag{II.57}
\]

We note that this series has only a finite number of terms so that it is more convenient to use than expressions (II.50) and (II.51) provided \( n \) is not too large. In practice, Eq. (II.57) may be easily evaluated up to about five ion transit times, after which time Davidson's approximate solution may be used.

3.5.2 The \( \gamma_{\text{ph}} \) secondary process only, and times of the order of a few electron transit times

The calculation in this case is identical with that in the previous section, only now \( \gamma_{\text{ph}} \) replaces \( \gamma_1 \) and \( W_e \) replaces \( W \). The electron current at the cathode is given by
\[
\frac{I_e(0,t)}{I_o} = \sum_{n=0}^{n} \gamma_{\text{ph}}^{m} e^{n \alpha d} \left( \frac{1}{(1 + \gamma_{\text{ph}})^{n+1}} + \left[ -\alpha W_e \right]^{m} \right) \left\{ \frac{d^n}{dp^n} \left[ \frac{(1 - \alpha W_e)}{p} e^{p \left( t - \left[ n d/W_e \right] \right)} \right] \right\}_{p=p_0},
\]
\[
\frac{nd}{W_e} < t < \frac{(n + 1)d}{W_e} \tag{II.58}
\]

3.5.3 \( \gamma_{\text{ph}} \) and \( \gamma_{1} \) secondary processes and times of the order of a few electron transit times \((t < d/W)\)

Here we may write
\[
\frac{I_e(0,t)}{I_o} = \frac{1}{2\pi i} \int_{c}^{e^{pt}} dp = \frac{1}{2\pi i} \int_{c}^{e^{pt}} \frac{dp}{p R} \left[ 1 - \frac{2}{R} \right]^{-1}
\]
\[
= \frac{1}{2\pi i} \int_{c}^{e^{pt}} \frac{dp}{p R} \left( \sum_{n}^{Z} \frac{\alpha d}{R^{n+1}} \right)
\]
where
\[
f(p) = 1 - \frac{\gamma_1 \alpha}{\phi} (e^{\alpha d} - 1) - \frac{\gamma_{\text{ph}} \alpha}{\psi} (e^{\alpha d} - 1),
\]
\[
R = 1 + \frac{\gamma_{\text{ph}} \alpha}{\psi} + \frac{\gamma_1 \alpha}{\phi}, \quad \text{and} \quad Z = \frac{\gamma_{\text{ph}} \alpha}{\psi} e^{\alpha d} + \frac{\gamma_1 \alpha}{\phi} e^{\alpha d}.
\]
Since we are only considering times \( t < d/W \), all terms in \( e^{\alpha d} \) will be zero when integrated round \( C \); thus we may write
\[
Z = \frac{\gamma_{\text{ph}} \alpha}{\psi} e^{\alpha d}.
\]
All terms involving \(Z^{n+1}\) and higher powers will be zero when integrated round \(C\), so that

\[
\frac{I(0,t)}{I_0} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{e^{pt}Z^n}{R^{m+1}}, \quad \frac{nd}{W_e} < t < (n + 1)\frac{d}{W_e}.
\]

Considering the \(m\)th term in this series we see that the integrand has poles at \(p = 0\) (simple) and at \(p_1\) and \(p_2\) the roots of \(R \equiv 1 + (\gamma_{ph}/\psi) + (\gamma_{ph}/\phi) = 0\) (order \(m + 1\)). The residue at \(p = 0\) is

\[
\frac{\gamma_{ph}^m e^{n\phi d}}{(1 + \gamma_{ph} + \gamma_{ph})^{n+1}},
\]

and that at \(p = p_1\) is

\[
\frac{\gamma_{ph}^m m!}{\int_1 \frac{\phi^{n+1} e^{pt+n\phi d}}{p(p - p_2)^{n+1}}} \bigg|_{p=p_1}
\]

with a similar expression when \(p = p_2\). Thus

\[
\frac{I(0,t)}{I_0} = \sum_{n=0}^{\infty} \left\{ \gamma_{ph}^m e^{n\phi d}(1 + \gamma_{ph} + \gamma_{ph})^{-(n+1)} \right. \]

\[
+ \frac{\gamma_{ph}^m m!}{\int_1 \frac{\phi^{n+1} e^{pt+n\phi d}}{p(p - p_2)^{n+1}}} \bigg|_{p=p_1} \bigg\}, \quad \frac{nd}{W_e} < t < (n + 1)\frac{d}{W_e}
\]

(II.59)

where \(\Sigma_1\) means a summation over an interchange of the suffixes 1 and 2. Thus, when \(t\) is less than \(d/W_e\),

\[
\frac{I(0,t)}{I_0} = \left\{ (1 + \gamma_{ph} + \gamma_{ph})^{-1} + \frac{WW_e}{p_1 - p_2} \left( \frac{\psi_{ph} e^{p_1 t}}{p_1} - \frac{\psi_{ph} e^{p_2 t}}{p_2} \right) \right\},
\]

and so forth.

3.5.4 \(\gamma_{ph}\) and \(\gamma_{ph}\) secondary processes and times of the order of a few ion transit times

We first note that if \(\gamma_{ph}\) is a fairly large fraction of \(\gamma_T\) (greater than about 1/4), and the applied voltage is not more than a few per cent greater than the breakdown potential \(V_S\), then the current at times of order \(d/W_e\) depends only slightly on the value of \(W_e\) and will remain almost unaltered if we let \(W_e \to \infty\). This is because in the equation \(f(p) = 0\), \(p\) appears only in the two quantities \(\phi = \alpha - p/W_e\) and \(\psi = \alpha - p/W_e\). Since \(W_e\) is usually a large multiple of \(W\) (greater than 50), putting \(\psi = \alpha\) will have a very small effect on the real root of \(f(p) = 0\), and on those complex roots with the largest real parts, which are the most important in determining the current at times of order \(d/W_e\).

We may therefore write

\[
\frac{I(0,t)}{I_0} = \frac{1}{2\pi i} \int_C \frac{e^{pt} dp}{p[1 - \gamma_{ph}(e^{\phi d} - 1)] + (\gamma_{ph}/\phi) - (\gamma_{ph}/e^{\phi d} - \phi)}
\]

\[
= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{e^{pt} Z^n}{p R^{m+1}} dp,
\]
where

\[ R = 1 - \gamma_{ph}(e^{\alpha d} - 1) + \frac{\gamma_{\alpha}}{\phi} \quad \text{and} \quad Z = \frac{\gamma_{\alpha}}{\phi} e^{\phi d}. \]

The \( m \)th integrand has two poles at \( p = 0 \) (simple) and \( p_1 = \alpha \mathcal{W}[1 + \gamma_1 (1 - \gamma_{ph}(e^{\alpha d} - 1))] \) (order \( m + 1 \)) with residues

\[ \frac{\gamma^m e^{\alpha d}}{[1 + \gamma_1 - \gamma_{ph}(e^{\alpha d} - 1)]^{m+1}} \]

and

\[ \left[ \frac{\mathcal{W}}{1 - \gamma_{ph}(e^{\alpha d} - 1)} \right]^{m+1} \frac{\alpha^n \gamma^m}{m!} \frac{d^n}{dp^n} \left[ \frac{\phi e^{\phi d + p t}}{p} \right] \bigg|_{p=p_1} \]

so that

\[ \frac{I_{e}(0, t)}{I_0} = \sum_{n=0}^{\infty} \left\{ \frac{\gamma^m e^{\alpha d}}{[1 + \gamma_1 - \gamma_{ph}(e^{\alpha d} - 1)]^{m+1}} + \left[ \frac{\mathcal{W}}{1 - \gamma_{ph}(e^{\alpha d} - 1)} \right]^{m+1} \frac{\alpha^n \gamma^n}{m!} \frac{d^n}{dp^n} \left[ \frac{\phi e^{\phi d + p t}}{p} \right] \right\} \left( \frac{d}{W} \right)^n \left( \frac{d}{W} \right)^n. \]

This solution neglects all oscillations having periods of order \( d/W_0 \). In many experimental applications this is not serious, since a time of order \( d/W_0 \) may elapse before the impulse voltage applied to the discharge gap increases to its final constant value.

3.6 The steady-state solution

In evaluating Davidson's exact solution, we have to determine the real and complex roots \( \lambda \) of the equation \( f(p) = 0 \). If the condition

\[ 1 > \frac{\gamma_{\alpha}}{\phi} (e^{\alpha d} - 1) + \frac{\gamma_{ph} \alpha}{\psi} (e^{\alpha d} - 1) \]

is satisfied, then the real root is negative and we see from expressions (II.50) and (II.51) that \( I_e(x,t) \) and \( I_p(x,t) \) tend asymptotically to the constant values

\[ I_e(x, \infty) = \frac{I_0 e^{\alpha x}}{1 - (\gamma_1 + \gamma_{ph})(e^{\alpha d} - 1)} \]

and

\[ I_p(x, \infty) = \frac{I_0 (e^{\alpha d} - e^{\alpha x})}{1 - (\gamma_1 + \gamma_{ph})(e^{\alpha d} - 1)}. \]

Thus the total current flowing in the discharge gap is

\[ I = I_e(d, \infty) = \frac{I_0 e^{\alpha d}}{1 - (\gamma_1 + \gamma_{ph})(e^{\alpha d} - 1)} \]

which, as we would expect, is the Townsend equation (II.18).
3.7 Part 2 of Davidson's exact solution

In many situations the cathode of a discharge gap may not be illuminated so that the
initiatory photo-current $I_p$ is absent. No current will flow between the electrodes until
some initiatory charged particles are produced in the discharge space, say by cosmic rays
or other particles (an obvious example is a nuclear particle leaving a trail of ionization
products in a spark chamber). We would like, then, to be able to calculate the subsequent
growth of the ionization currents.

In order to do this it is convenient to calculate the required $I_e(x, t)$ and $I_p(x, t)$
from the initial arbitrary charge density distribution $\rho_e(x)$, $\rho_p(x)$ by means of the four
Green's functions:

$$g_e^p(x, t, x_1) - the \ I_e(x, t) due to the presence at time zero of
unit positive charge/unit length in the region of x = x_1;$$
$$g_p^p(x, t, x_1) - the \ I_p(x, t) due to the presence at time zero of
unit positive charge/unit length in the region of x = x_1;$$
$$g_e^e(x, t, x_1) - the \ I_e(x, t) due to the presence at time zero of
unit negative charge/unit length in the region of x = x_1;$$
$$g_p^e(x, t, x_1) - the \ I_p(x, t) due to the presence at time zero of
unit negative charge/unit length in the region of x = x_1. $$

The required $I_e(x, t)$ of Part 2 is then given by

$$\int_0^d \left[ g_e^e(x, t, x_1) n_e(x_1) + g_p^p(x, t, x_1) n_p(x_1) \right] dx_1 \quad (II.61)$$

and the corresponding $I_p(x, t)$ is

$$\int_0^d \left[ g_p^e(x, t, x_1) n_e(x_1) + g_p^p(x, t, x_1) n_p(x_1) \right] dx_1 \quad (II.62)$$

where $n_e(x)$ and $n_p(x)$ are the numbers of electrons and positive ions per unit length of the
discharge at time zero.

3.7.1 To find $g_e^e$ and $g_p^p$

The Laplace transforms of $g_e^p$ and $g_p^p$ are first of all found by solving Eqs. (II.39) and
(II.40) with the boundary conditions

$$\bar{I}_e(0) = \gamma_1 \bar{I}_p(0) + \gamma_{ph} \int_0^d \alpha \bar{I}_e(x) \, dx \quad (II.63)$$
$$\bar{I}_p(d) = 0 \quad (II.64)$$
$$I_e(0, x) = 0 \quad (II.65)$$

and

$$I_p(0, x) = \delta(x, x_0) W_p \quad (II.66)$$

$\delta$ being the Dirac delta function $[\delta(x, x_1) = 0 \ except in the region of x_1, where
\int_{x_1-\Delta x}^{x_1+\Delta x} \delta(x, x_1) \, dx = 1].$
A short calculation gives the results
\[ g_e^{P}(x, x_i) = \gamma_i e^{-px_i/W_p} e^{ix} / f(p) \]  
(II.67)

\[ g_p^{P}(x, x_i) = \frac{\alpha Y_i}{\phi} \frac{e^{-px_i/W_p}}{f(p)} e^{ix} e^{i(x-(d-x))} \left[ e^{i(d-x)}-1 \right] + e^{px/W_p} \int_{-\infty}^{\infty} e^{-px/W_p} \delta(x, x_i) \, dx', \]  
(II.68)

and, applying the inversion theorem,
\[ g_e^{P}(x, t, x_i) = \frac{1}{2\pi i} \int_{C} \frac{\gamma_i e^{-px_i/W_p} e^{ix} e^{ip\tau}}{f(p)} \, dp, \]  
(II.69)

\[ g_p^{P}(x, t, x_i) = \frac{1}{2\pi i} \int_{C} \frac{\alpha Y_i}{\phi} \frac{e^{-px_i/W_p} e^{ix}}{f(p)} \left[ e^{i(d-x)}-1 \right] e^{p\tau} \, dp + W_p \delta(x_i - x, W_p t) H(x, x) \]  
(II.70)

\[ H(\alpha, \beta) \] is the Heaviside unit pulse function, which is unity for \( \alpha \geq \beta \) and zero for \( \alpha < \beta \).

The contour \( C \) may be extended to the infinite circle and the integrals expressed as a series to give
\[ g_e^{P}(x, t, x_i) = \sum_\lambda Y_i \lambda e^{-\lambda x_i/W_p} \]  
(II.71)

\[ = \sum_\lambda G_e^{P}(x, t, x_i) \quad t > 0, \]  
(II.72)

\[ g_p^{P}(x, t, x_i) = \sum_\lambda R G_p^{P}(x, t, x_i) \quad t > (x_i - x)/W_p. \]  
(II.72)

\[ [Q \text{ and } R \text{ are defined in Eqs. (II.50) and (II.51)}]. \]

We see from the character of Eqs. (II.69) and (II.70) on the contour \( C \), that \( g_e^{P} \) and \( g_p^{P} \) take on simple forms at small times. In particular
\[ g_e^{P}(x, t, x_i) = 0 \quad t < \frac{x}{W_e} + \frac{x_i}{W_p}, \]  
(II.73)

and
\[ g_p^{P}(x, t, x_i) = W_p \delta(x_i - x, W_p t) H(x, x_i) \quad t < \frac{x}{W_e} + \frac{x_i}{W_p}, \]  
(II.74)

i.e. \( g_p^{P} \) is zero for all times less than \( (x/W_e) + (x_i/W_p) \), except for the instant when the original unit positive charge density arrives at the point \( x \), when it rises momentarily to the value \( W_p \).

3.7.8 To find \( g_e^{P} \) and \( g_p^{P} \)

The Green's functions \( g_e^{P} \) and \( g_p^{P} \) are found in a similar manner to that described above, with the exception that (II.65) and (II.66) are now replaced by
\[ I_e(0, x) = \delta(x, x_i) W_e \]  
(II.75)

and
\[ I_e(0, x) = 0. \]  
(II.76)
A short calculation gives
\[ g^q(x, x_i) = e^{q(x-x_i)} \frac{f(x_i)}{f(p)} \left[ f(x_i) - f(p) \right] + e^{qX} \int_0^x e^{-qX'} \alpha(x, x_i) \, dx' \]  
(II.77)

[by \( f(x_i) \) we mean \( f(p) \) with \( d \) replaced by \( x_i \)] and
\[ g^q_p(x, x_i) = \frac{\alpha e^{q(x-x_i)}}{\phi f(p)} \left[ f(x_i) - f(p) \right] \left[ e^{q(d-x)} - 1 \right] + \alpha e^{qX} \int_0^x \int_0^{e^{qX'}} e^{-qX'} \alpha(x', x_i) \, dx' \, dx'' \]  
(II.78)

and, using the inversion theorem
\[ g^q_c(x, t, x_i) = \frac{1}{2\pi i} \int_C \frac{e^{q(x-x_i)}}{f(p)} \left[ f(x_i) - f(p) \right] e^{pt} \, dp + W e^{i\phi t} \alpha(x-x_i, \frac{W}{W_e}) H(x, x_i) \]  
(II.79)

and
\[ g^q_p(x, t, x_i) = \frac{1}{2\pi i} \int_C \frac{\alpha e^{q(x-x_i)}}{\phi f(p)} \left[ f(x_i) - f(p) \right] \left[ e^{q(d-x)} - 1 \right] e^{pt} \, dp \]
\[ + \alpha W e^{i\phi [t+(x-x_i)/W_p]} \left[ H \left( t, \frac{x-x_i}{W_e} \right) - H \left( t, \frac{d-x_i}{W_p} + \frac{d-x_i}{W_e} \right) \right] \text{ for } x > x_i, \]
\[ + \alpha W e^{i\phi [t+(x-x_i)/W_p]} \left[ H \left( t, \frac{x-x_i}{W_p} \right) - H \left( t, \frac{d-x_i}{W_p} + \frac{d-x_i}{W_e} \right) \right] \text{ for } x < x_i. \]  
(II.80)

Equation (II.79) may also be expressed in the form
\[ g^q_c(x, t, x_i) = \sum_{\lambda} Q f(x_i) e^{-qX_i} \]
\[ = \sum_{\lambda} C^q_{\lambda}(x, t, x_i), \quad t > (x-x_i)/W_e. \]  
(II.81)

The contour in formula (II.80) may be extended to the infinite circle for all times greater than \((d - x)/W_p + (d - x_i)/W_e\). However, the second term in formula (II.78) can be combined with the first term to give
\[ g^q_p(x, x_i) = \frac{1}{2\pi i} \int_C e^{qX} \frac{\alpha e^{q(x-x_i)}}{\phi} \left[ e^{q(d-x)} - e^{qX} \right] \frac{f(x_i)}{f(p)} + B e^{pt} \, dp, \]  
(II.82)

where \( B = 0 \) if \( x > x_i \) and \( B = e^{i\phi x} - e^{i\phi x_1} \) if \( x < x_i \). In both cases \( C \) may be extended to the infinite circle for all positive times and, since the term involving \( B \) has zero residue at the pole \( \phi = 0 \), we may write
\[ g^q_p(x, t, x_i) = \sum_{\lambda} D^q_{\lambda}, \quad t > 0. \]  
(II.83)

The second terms in Eqs. (II.79) and (II.80) represent the contributions to the current density of the originating unit negative charge density and its resulting avalanche. In Eq. (II.79) it is zero for \( x < x_i \), and is a pulse of amplitude \( W e^{i\phi t} \) occurring at time \( t = (x-x_i)/W_e \), for \( x > x_i \). The second term in Eq. (II.80) is also zero for distances less than \( x_i \). For greater distances it represents a step function which is zero for times less than \((x-x_i)/W_e\) and for times greater than \((d-x)/W_p + (d-x_i)/W_e\). In the intermediate range it has an amplitude given by \( \alpha W \exp \left[ i\phi \left[ t + (x-x_i)/W_p \right] \right] \).
Once again it is possible to put the exact solution of Part 2 in alternative forms by suitable expansions of the integrand in the contour integrals (II.66), (II.70), (II.79), and (II.80). Since the procedure is identical with that described in Part 1, we will not go through the corresponding calculation at this point.

3.8 The breakdown criterion

If the condition

\[ 1 = \gamma_p \left( e^{\alpha d} - 1 \right) \]  

(II.84)

is satisfied, then the real root of the equation \( f(p) = 0 \) is zero. The expressions (II.44) and (II.45) no longer have simple poles at the origin; thus the \( I_e(x,t) \) and \( I_p(x,t) \) given in Eqs. (II.50) and (II.51) are no longer valid.

The residues of the integrands at \( p = 0 \) in Eqs. (II.46) and (II.47) are now

\[ \frac{I_p}{D(0)} \left( e^{\alpha d} - e^{\alpha x} \right) t + \frac{I_e}{D(0)} \left\{ e^{\alpha d} \left[ \frac{1}{\alpha W} + \frac{x}{W_p} - \frac{d}{W} - \frac{F(0)}{2D(0)} \right] - e^{\alpha x} \left[ \frac{1}{\alpha W} - \frac{x}{W_e} - \frac{F(0)}{D(0)} \right] \right\}, \]

so that

\[ I_e(x,t) = \frac{I_p}{D(0)} \left[ t - \frac{x}{W_e} - \frac{F(0)}{2D(0)} \right] e^{\alpha x} + \sum_{\text{complex } \lambda} Q \]  

(II.85)

and

\[ I_p(x,t) = \frac{I_p}{D(0)} \left( e^{\alpha d} - e^{\alpha x} \right) t + \frac{I_e}{D(0)} \left\{ e^{\alpha d} \left[ \frac{1}{\alpha W} + \frac{x}{W_p} - \frac{d}{W} - \frac{F(0)}{2D(0)} \right] 
- e^{\alpha x} \left[ \frac{1}{\alpha W} - \frac{x}{W_e} - \frac{F(0)}{2D(0)} \right] \right\} + \sum_{\text{complex } \lambda} RQ, \]

(II.86)

where

\[ D(0) = \left[ \frac{\partial f(p)}{\partial p} \right]_{p=0} = \left( \frac{\gamma_i}{\alpha W} + \frac{\gamma_{ph}}{\alpha W_0} \right) \left[ 1 - e^{\alpha d(1-\alpha d)} \right] \]

and

\[ F(0) = \left[ \frac{\partial^2 f(p)}{\partial p^2} \right]_{p=0} = \left( \frac{\gamma_i}{\alpha^2 W^2} + \frac{\gamma_{ph}}{\alpha^2 W_0^2} \right) \left[ 2 - e^{\alpha d(\alpha^2 d^2 + 2 - 2\alpha d)} \right]. \]

We see from these expressions that at large times (when the terms involving complex \( \lambda' \)s are negligible), \( I_e(x,t) \) and \( I_p(x,t) \) (and thus the average current \( \bar{I} \)) grow linearly with time.

The expressions for the current do not become of practical interest until \( t \) is large enough to make \( I_e(0,t) \) a large multiple of \( I_0 \). The constant \( F(0)/D(0) \) is of order unity;
thus in Eqs. (II.85) the term containing this factor only contributes a term of order \( I_s \) to \( I_e(0,t) \). Therefore as soon as \( t \) is large enough to make \( I_e(0,t) \) a large multiple of \( I_s \), \( t \) has become large compared with \( F(0)/D(0) \). At such times the oscillatory terms are relatively negligible. Thus at times of practical interest, the right-hand side of Eq. (II.85) is nearly \( I_s \ e^{\alpha \omega t} D(0) \). Similar remarks apply to Eqs. (II.86), the right-hand side of which becomes \( I_s (e^{\alpha d} - e^{\alpha x} t)/D(0) \).

The condition (II.84) is the familiar Townsend "Breakdown Criterion", and the value of the applied voltage for which it is satisfied (at a fixed value of the electrode separation \( d \)) is called the "breakdown potential" \( V_s \).

The Green's functions of Part 2 tend asymptotically to constant values when \( V = V_s \); thus, once a current has started flowing in a discharge gap, it will continue flowing indefinitely even if there is no external illumination of the cathode. [It should be noted that this conclusion has to be modified when the effect of statistical fluctuations in the ionization currents are taken into consideration (see Chapter IV).] The breakdown potential can thus be measured experimentally by finding the minimum voltage which will maintain a small current in a discharge gap without external illumination of the cathode.

The right-hand side, \( \gamma_p (e^{\alpha d} - 1) \), of condition (II.84) represents the average number of secondary electrons produced by an electron leaving the cathode and travelling to the anode. Thus the breakdown criterion may be interpreted physically as a replacement condition in which every electron leaving the cathode produces, on the average, one secondary electron.

We note that the definition of \( V_s \) given above is independent of the value of the current flowing in the gap -- the only restriction being that the current should not be so large as to produce appreciable space-charge distortion of the electric field.

4. THE TEMPORAL GROWTH OF IONIZATION INCLUDING
THE EFFECT OF ATTACHMENT AND DETACHMENT

4.1 Continuous illumination of the cathode

In the presence of attachment and detachment, the relevant continuity equations are

\[
\frac{1}{W_e} \frac{\partial I_e(x,t)}{\partial t} = (\alpha - a) I_e(x,t) + \frac{\mu}{W_n} I_n(x,t) - \frac{\partial}{\partial x} I_e(x,t), \quad (II.87)
\]

\[
\frac{1}{W_p} \frac{\partial I_p(x,t)}{\partial t} = \alpha I_e(x,t) + \frac{\partial}{\partial x} I_p(x,t), \quad (II.88)
\]

and

\[
\frac{1}{W_n} \frac{\partial I_n(x,t)}{\partial t} = \alpha I_e(x,t) - \frac{\mu}{W_n} I_n(x,t) - \frac{\partial}{\partial x} I_n(x,t). \quad (II.89)
\]

Taking the Laplace transforms of Eqs. (II.87) and (II.89) yields

\[
\frac{P}{W_e} \bar{I}_e(x) = (\alpha - a) \bar{I}_e(x) + \frac{\mu}{W_n} \bar{I}_n(x) - \frac{\partial}{\partial x} \bar{I}_e(x); \quad (II.90)
\]
\[
\frac{P}{W_n} I_n(x) = a I_n(x) - \frac{\mu}{W_n} T_n(x) - \frac{\partial}{\partial x} T_n(x), \\
(\text{II.91})
\]

again assuming that there is zero charge distribution in the gap at the initial instant. Equation (II.88) may be solved to give the positive ion current at the cathode

\[
I_p(0, t) = \int_0^d \alpha I_e(x, t - \frac{x}{W_p}) \, dx. \\
(\text{II.92})
\]

Eliminating \( I_e(x) \) from Eqs. (II.90) and (II.91),

\[
\frac{\partial^2 I_e(x)}{\partial x^2} + \left( a + \frac{\mu}{W_n} + p \frac{W_n + W_e}{W_n W_e} - \alpha \right) \frac{\partial I_e(x)}{\partial x} + \left( \frac{a p}{W_n} + \frac{p^2}{W_e W_n} + \frac{\mu p}{W_e W_n} - \alpha \frac{p + \mu}{W_n} \right) I_e(x) = 0,
\]

the solution of which is

\[
I_e(x) = A_1 e^{B_1 x} + A_2 e^{B_2 x},
\]

where \( A_1, A_2 \) are constants, and \( B_1, B_2 \) are the roots of

\[
B^2 + \left( a + \frac{\mu}{W_n} + p \frac{W_n + W_e}{W_n W_e} - \alpha \right) B + \frac{a p}{W_n} + \left( \frac{p + \mu}{W_n} \right) = 0.
\]

Substituting for \( I_e(x) \) in Eq. (II.90) gives

\[
I_n(x) = \frac{W}{\mu} A_1 \left( B_1 - \alpha + a + \frac{p}{W_e} \right) e^{B_1 x} + \frac{W}{\mu} A_2 \left( B_2 - \alpha + a + \frac{p}{W_e} \right) e^{B_2 x}.
\]

Applying the boundary condition \( I_n(0) = 0 \) to this equation gives the values of \( A_1 \) and \( A_2 \) yielding

\[
I_n(x) = \frac{I_e(0)}{B_1 - B_2} \left\{ \left( B_1 + \frac{p + \mu}{W} \right) e^{B_1 x} - \left( B_2 + \frac{p + \mu}{W} \right) e^{B_2 x} \right\} = I_e(0) G(x, p),
\]

where \( G(x, p) \) is the transform of \( g(x, t) \), say, and \( g(x, t) \) is defined by

\[
I_e(x, t) = \int_0^t I_e(0, t') g(x, t - t') \, dt'.
\]

Considering now only the \( \gamma_1 \) and delayed \( \gamma_{ph}^d \) process we have the boundary condition

\[
I_e(0, t) = I_e + \gamma_1 I_p(0, t) + \gamma_{ph}^d \frac{\alpha}{\gamma_{ph}} \int_0^t e^{-(t-t')/\gamma_{ph}} \int_0^d I_e(x, t') \, dx \, dt',
\]

(II.94)

(the undelayed photon secondary process may simply be considered by setting \( \gamma_{ph} = 0 \)).
Equation (II.94) may now be written in the form

\[
I_e(0, t) = I_0 + \alpha \gamma_i \int_0^t \int_0^{\frac{t'}{\tau_{ph}}} \left[ \frac{W_p}{W_p + W_e} \right] g(x, t - t') \, dx \, dt' + \frac{\alpha \gamma_{ph}}{\tau_{ph}} \int_0^t e^{-\left( t - t' \right)/\tau_{ph}} \int_0^{t''} \int_0^{t'''} g(x, t' - t''') \, dx \, dt'' \, dt'''
\]

which becomes on taking the Laplace transform,

\[
I_e(0) = \frac{I_0}{\mathcal{L}} + \alpha \gamma_i \mathcal{I}_e(0) \int_0^{\mathcal{L}} C(x, p) e^{-px/W_p} \, dx + \frac{\alpha \gamma_{ph}}{1 + p \tau_{ph}} \mathcal{I}_e(0) \int_0^{\mathcal{L}} C(x, p) \, dx.
\]

Thus

\[
I_e(0) = \frac{I_0}{\mathcal{L}(p)},
\]

where

\[
\mathcal{L}(p) = 1 - \int_0^{\mathcal{L}} \left( \alpha \gamma_i e^{-px/W_p} + \frac{\alpha \gamma_{ph}}{1 + p \tau_{ph}} \right) C(x, p) \, dx.
\]

\(I_e(x)\) may be immediately written down since

\[
\mathcal{I}_e(x) = \mathcal{I}_e(0) C(x, p) = \frac{I_0 C(x, p)}{\mathcal{L}(p)}.
\]

As in the previous sections, \(I_e(0, t)\) is conveniently expressed as the contour integral

\[
\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{I_0 e^{pt}}{\mathcal{L}(p)} \, dp
\]

taken anticlockwise around an infinite semicircle to the right of the imaginary axis. This in turn may be expressed at \(t > 0\) as the sum of the residues of the integrand, evaluated at \(p = 0\), and at all the points \(p = \lambda\), where \(\mathcal{L}(p) = 0\), giving

\[
I_e(0, t) = I_0 \left\{ \frac{1}{\mathcal{L}(0)} + \sum_{\lambda} \frac{e^{\lambda t}}{\lambda \left( 2\pi \left( \frac{\mathcal{L}'(p)}{\mathcal{L}(p)} \right) \right)} \right\}.
\]  \hspace{1cm} (II.95)

In many of the applications of this result where the current at large times is required, it will be sufficient to use an approximate formula that is analogous to Davidson's approximate solution, Section 3.4 of this chapter. This involves only the largest real root \(\lambda_0\) of \(\mathcal{L}(p) = 0\). However, considerable labour is still required to determine \(\lambda_0\), unless a computer is programmed to do the calculation.

As remarked previously (Section 3.3), the corresponding solution for the case of flash illumination is easily obtained from Eq. (II.95) by differentiating with respect to \(t\). The general solution for the case where a charge \(N_0\) electrons is released from the cathode at time zero is

\[
I_e(0, t) = N_0 \sum_{\lambda} \frac{e^{\lambda t}}{\left( 2\pi \left( \frac{\mathcal{L}'(p)}{\mathcal{L}(p)} \right) \right)}.
\]  \hspace{1cm} (II.96)
4.2 Solution for a single avalanche initiated by pulsed illumination of the cathode

In experiments which observe the growth of single avalanches initiated by, say, \( N_e \) electrons leaving the cathode, it is of interest to know what effect attachment and detachment have on the ionization growth. The results could be obtained from the general solution given in the previous section, but it is more convenient to derive the result from first principles.

Equations (II.87) and (II.89) may be simplified by putting \( y = x - W_n t \); that is, by choosing a system of coordinates which moves with the negative ions. The resulting equations are

\[
\frac{\partial I_e(y,t)}{\partial t} = W_e(\alpha - \alpha) I_e(y,t) + \frac{\mu W_e}{W_n} I_n(y,t) - (W_e - W_n) \frac{\partial I_e(y,t)}{\partial y} \tag{II.97}
\]

and

\[
\frac{\partial I_n(y,t)}{\partial t} = W_n a I_e(y,t) - \mu I_n(y,t), \tag{II.98}
\]

where \( y \) and \( t \) are now considered as the independent variables. Applying the Laplace transformation to Eqs. (II.97) and (II.98), and assuming that there is a distribution \( I_e(x,0) \) of electrons in the gap at time zero,

\[
pT_e(y) - I_e(y,0) = (\alpha - \alpha) W_e T_e(y) + \frac{\mu W_e}{W_n} I_n(y) - (W_e - W_n) \frac{\partial I_e(y)}{\partial y} \tag{II.99}
\]

and

\[
pT_n(y) = a W_n T_n(y) - \mu T_n(y). \tag{II.100}
\]

In the case where the ionization is initiated by a pulse of \( N_e \) electrons at the cathode, \( I_e(y,0) = I_e(x,0) \) will be \( N_e W_e \delta(y) \), where \( \delta(y) \) is the Dirac \( \delta \)-function.

Substituting \( I_n(y) = \left[ a W_n/(p + \mu) \right] T_e(y) \)

\[
\frac{\partial I_e(y)}{\partial y} - \frac{W_e(\alpha - \alpha) - p + a \mu W_n/(p + \mu)}{W_e - W_n} I_e(y) = \frac{W_n N_e}{W_e - W_n} \delta(y)
\]

so that

\[
I_e(y) = \frac{W_n N_e}{W_e - W_n} \exp \left[ \frac{W_e(\alpha - \alpha) - p + a \mu W_n/(p + \mu)}{W_e - W_n} y \right] \tag{II.101}
\]

and

\[
I_n(y) = \frac{a W_n W_e N_e}{(p + \mu) (W_e - W_n)} \exp \left[ \frac{W_e(\alpha - \alpha) - p + a \mu W_n/(p + \mu)}{W_e - W_n} y \right]. \tag{II.102}
\]

To obtain \( I_e(x,t) \) from formula (II.101), we start with the relation

\[
\int_{\delta}^{\xi} e^{-\nu t} \sqrt{\frac{\xi}{t}} I_e(\xi \sqrt{\nu t}) \, dt = e^{\nu t} \frac{\xi^{3/2}}{3}. 
\]
where \( I_1 \) (in the usual notation) is the well-known Bessel function. Hence

\[
\exp \left[ \frac{aW_ey}{(p + w)(W_e - W_n)} \right]
\]

is the transform of

\[
e^{-\mu t} \left\{ \sqrt{\frac{aW_ey}{(W_e - W_n)t}} \cdot I_1 \left( 2 \sqrt{\frac{aW_eyt}{W_e - W_n}} \right) + \delta(t) \right\}
\]

The factor \( \exp \left[ \frac{-py}{(W_e - W_n)} \right] \) in Eq. (II.101) may be taken into account by using the following theorem.

If \( H(p) \) is the transform of \( h(t) \), then \( e^{-\mu t} H(p) \) is the transform of \( f(t) \), where

\[
f(t) = \begin{cases} 
0 & \text{for } 0 < t < A \\
h(t - A) & \text{for } t \geq A.
\end{cases}
\]

Thus \( I_e(y, t) = 0 \) for \( 0 < t < y(W_e - W_n) \), while for \( t \geq y/(W_e - W_n) \) it is

\[
\frac{W_eN_n}{W_e - W_n} \exp \left( \frac{(a - a)W_ey}{W_e - W_n} \right) \cdot \exp \left( -\mu t + \frac{y\mu}{W_e - W_n} \right) \times
\]

\[
\sqrt{\frac{aW_ey}{(W_e - W_n)t - y}} \cdot I_1 \left( 2 \sqrt{\frac{aW_ey}{W_e - W_n}} \left( t - \frac{y}{W_e - W_n} \right) \right) + \delta \left( t - \frac{y}{W_e - W_n} \right)
\]

Finally, we replace \( y \) by \( x - W_n t \), and after a short calculation obtain

\[
I_e(x, t) = N_n \exp \left( \frac{(a - a)W_e(x - W_n t) + \mu(x - W_e t)}{W_e - W_n} \right) \times
\]

\[
\sqrt{\frac{aW_e(x - W_n t)}{W_e - x}} \cdot I_1 \left( 2 \sqrt{\frac{aW_e(x - W_n t)(W_e t - x)}{(W_e - W_n)^2}} \right) + \delta \left( t - \frac{x}{W_n} \right)
\]

Equation (II.103) represents a distribution of electrons, which at time \( t \) are confined to the interval \( W_n t \leq x \leq W_e t \). The \( \delta \)-function represents a pulse of \( N_n \exp (a-a)x \) electrons, which reaches the distance \( x \) at time \( t = x/W_e \). These electrons have never been involved in the attachment process, and would constitute the entire distribution if there were no detachment.

Similarly \( I_e(x, t) \) may be obtained from Eq. (II.102). Starting with the expression \( (1/p) e^{-k/p} \), which is the transform of \( I_s(2\sqrt{k}t) \), and applying the same theorems as were used in obtaining \( I_e(x, t) \) we finally obtain

\[
I_e(x, t) = \frac{aW_eW_nN_n}{W_e - W_n} \exp \left( \frac{(a - a)W_e(x - W_n t) + \mu(x - W_e t)}{W_e - W_n} \right) \times I_0 \left\{ 2 \sqrt{\frac{aW_e(x - W_n t)(W_e t - x)}{(W_e - W_n)^2}} \right\}
\]

(II.104)
(Iₜ is Bessel's function of zero order and should not be confused with the current Iₜ).

Equation (II.103) is particularly important in the experimental determination of the detachment coefficient. The variation of Iₑ(x, t) with time at a fixed distance from the cathode is conveniently measured using a photomultiplier to observe the light emitted by the discharge. When detachment is present, the light output consists of a pulse at time x/Wₑ, followed by a continuous "tail" up to a time x/Wₑ*. Care must be taken in interpreting the experimental results, however, since a similar continuous signal could be obtained if the secondary processes are not negligible.

4.3 The process of delayed ionization

It has been suggested [Hornbeck⁹ and Molnar¹⁰] that, in the inert gases, an electron may be produced by a collision between an unexcited atom having only thermal energy, and an atom which has become excited to an energy level near to the ionization energy. In helium, for example, the two processes required to produce a new electron would be

\[ \text{He} + e = \text{He}^* + e \]
and

\[ \text{He}^* + \text{He} = \text{He}_2^* + e . \]

For this process to be of practical importance, the high-energy level must be either metastable or sufficiently long-lived for there to be a large probability of a suitable collision occurring during its lifetime. The excited atom, being uncharged, will remain, on the average, at the point where the original excitation occurred, and so it is clear that in a steady-state experiment this process will be indistinguishable from the ordinary ionization process, except when the discharge gap is so short that there is appreciable diffusion to the electrodes. However, we expect delayed ionization to make a significant difference to the theory of temporal growth of ionization, both for the case of continuous current growth initiated by constant ultra-violet illumination of the cathode, and for the case of a single avalanche initiated by a pulse of electrons at the cathode. In certain gas mixtures, collisions between metastable and unexcited atoms provide the dominant mechanism of primary ionization. It is then known as the Penning effect (see Section 2.2.2 of Chapter I).

The problem is mathematically very similar to that already treated in Sections 4.1 and 4.2. Let N be the density of the highly excited atoms, and \( \mu' \) be the fraction of those which in unit time change to a different form, with or without producing an electron. Let \( \mu \) be the fraction of these which do produce electrons, the remainder falling spontaneously to lower levels. The following differential equations are obtained, where \( a \) is the excitation coefficient of the highly excited level:

\[
\frac{1}{W_e} \frac{\partial I_e(x, t)}{\partial t} = (a - a') I_e(x, t) + \mu N(x, t) - \frac{\partial I_e(x, t)}{\partial x} \tag{II.105}
\]

\[
\frac{\partial N(x, t)}{\partial t} = a I_e(x, t) - \mu' N(x, t) , \tag{II.106}
\]

together with Eqs. (II.88) and (II.94), and \( I_p(d, t) = 0 \).
Let \( I_0 \) be the externally maintained cathode electron current, and suppose that the discharge gap is empty of electrons and excited atoms at time zero. Taking the Laplace transforms of Eqs. (II.105) and (II.106)

\[
\frac{dI_e(x)}{dx} + \left( \frac{P}{W_e} - \alpha + a \right) I_e(x) = \mu N(x)
\]

and

\[
(p + p') N(x) = aI_e(x),
\]

from which we obtain

\[
I_e(x) = I_e(0) \exp \left( a - a - \frac{p}{W_e} + \frac{a\mu}{p + p'} \right) x,
\]

which corresponds to Eq. (II.93) for the case of negative ion detachment; \( I_0(0,t) \) is still given by Eq. (II.92). Substituting in the cathode boundary condition (II.94), we obtain

\[
I_e(0) = \frac{I_0}{pf(p)},
\]

where

\[
f(p) = 1 - \int_0^x \left( \alpha_0 e^{-px/W_p} + \frac{\alpha_0' p}{1 + p' \phi} \right) e^{\psi x} \, dx.
\]  

This is a more general form of the equation \( f(p) = 0 \) obtained in Section 3.3 of this chapter (for the case of \( a = 0 \)), with \( \phi \) and \( \psi \) now given the values

\[
\phi = a - a - p \left( \frac{1}{W_e} + \frac{1}{W_p} \right) + \frac{a\mu}{p + p'}
\]

and

\[
\psi = a - a - \frac{p}{W_e} + \frac{a\mu}{p + p'}.
\]

Again, the complete solution may be written as a contour integral. If the current growth at large times is all that is required, it is sufficient to find the real root \( \lambda_0 \) of \( f(p) = 0 \) to obtain the asymptotic solution

\[
I_e(0,t) = \frac{I_0}{\lambda(0)} + \frac{I_0 e^{\lambda t}}{\lambda(0) \lambda(p)}.
\]  

The solution for the case of pulse illumination of the cathode, when secondary processes are present, is now easily obtained as remarked in Section 3.3. The exact solution is

\[
I_e(0,t) = \sum_\lambda \frac{e^{\lambda t}}{\lambda(p)}
\]

taken over all the roots \( \lambda \) of \( f(p) = 0 \) [Eq. (II.107)].

The solution for a single avalanche may be found in an identical manner to that described in Section 4.2, and we find that

\[
I_e(x,t) = N_0 e^{\left( a - a \right) x} \frac{W_e x + \mu' \left( x - W_e t \right)}{W_e} \left\{ \frac{aW_e x}{W_e t - x} \right\} I_1 \left\{ 2 \sqrt{\frac{aW_e x}{W_e t - x}} \right\}
\]

\[
\text{(II.110)}
\]
in the range \(0 < x < W_e t\), together with a pulse of \(N_0 \exp (\alpha - a) W_e t\) electrons at the point \(x = W_e t\), and

\[
N(x, t) = a N_0 \exp \left( \frac{(\alpha - a) W_e x + u'(x - W_e t)}{W_e} \right) I_0 \left( 2 \sqrt{\frac{\alpha x}{W_e}} \right), \tag{II.111}
\]
in the range \(0 < x < W_e t\).

As in the case of negative ion detachment, a very sensitive way of detecting the process of delayed ionization is by using a photomultiplier to observe the variation in the light emitted from a small region of the discharge gap. The photomultiplier should work in a region of the spectrum not containing any lines produced (directly or indirectly) by transitions from the high-energy excited state, or from any other long-lived state. The light output is then proportional to the electron current density given by Eq. (II.110).

5. **THE TEMPORAL GROWTH OF IONIZATION WHEN THE SECONDARY IONIZATION PROCESSES INVOLVE DIFFUSION**

5.1 **The continuity equations and boundary conditions**

In the secondary processes so far discussed, the active particles (photons or positive ions) do not move to the cathode by diffusion. The ions move to it with a steady drift velocity due to the electric field, and the photons move to it either instantaneously or after a certain delay time.

As we have discussed in Section 2.2, cathode emission can also be produced by the incidence of excited atoms in metastable states. Since the metastable atoms are uncharged, their motion to the electrodes is entirely due to diffusion, and the resulting current amplification will usually be much slower than that produced by unscattered photons or by positive ions drifting in the electric field.

Let \(\alpha_m\) be the number of metastable atoms which an electron generates in travelling unit distance in the \(x\)-direction, and \(1/\tau_m\) be the fraction of the active particles, in any region, which are destroyed per unit time by their collisions with unexcited atoms.

The continuity equation for the metastable atoms is

\[
\frac{\partial \rho_m(x, t)}{\partial t} = - \frac{\partial J_m(x, t)}{\partial x} + \alpha_m e^{\alpha x} J_e \left( 0, t - \frac{x}{W_e} \right) - \frac{\rho_m(x, t)}{\tau_m},
\]

where \(\rho_m\) is their spatial density and \(J_m\) their current density. This may also be written

\[
\frac{\partial n_m(x, t)}{\partial t} = - \frac{\partial I_m(x, t)}{\partial x} + \alpha_m e^{\alpha x} I_e \left( 0, t - \frac{x}{W_e} \right) - \frac{n_m(x, t)}{\tau_m}, \tag{II.112}
\]

where \(n_m\) is the density/unit length of the discharge, and the I's are the total currents at any cross-section.

Following Davidson\(^{11}\)) we calculate a diffusion coefficient which makes

\[
I_m = -D_m \frac{dn_m}{dx}. \tag{II.113}
\]
a correct expression when applied to the steady state; and if $D_m$ is constant in space we write the term $-\alpha_{m}(x,t) / \alpha$ in the time-varying equation (II.112) as $D_m \partial^2 n_m(x,t) / \partial x^2$, though it is not strictly accurate. Thus Eq. (II.112) becomes

$$\frac{\partial n_m(x,t)}{\partial t} = D_m \frac{\partial^2 n_m(x,t)}{\partial x^2} + \alpha_m e^{\alpha x} I_0 \left( 0, t - \frac{x}{\tau_m} \right) - \frac{n_n(x,t)}{\tau_n}. \quad \text{(II.114)}$$

As we have mentioned previously, the diffusion coefficient $D_m$ may be written approximately as $(1/3) \ell_m \bar{v}_m$, where $\ell_m$ is a suitably defined mean free path of the metastable particles and $\bar{v}_m$ their mean kinetic velocity. We note in passing that we may take $\ell_m / \bar{v}_m \tau_m$ to be a small fraction, since otherwise the metastable atoms generated in the gas will nearly all be destroyed after only making a few collisions, and such cases are of no practical interest.

We consider now the boundary conditions at the electrodes. Suppose that a fraction $g$ of the $N$ metastables which strike the cathode per unit time are destroyed. If $g$ is a small fraction (say < 1/5), the resulting asymmetry in the directional distribution of the metastables near the cathode is not sufficient to cause a serious failure of the equations there. For $g$ small we may write approximately at the cathode $N = (1/4)n_m \bar{v}_m$, that is, at $x = 0$

$$n_m(x,t) = \frac{4D_m}{g \bar{v}_m} \frac{\partial n_m(x,t)}{\partial x} = \frac{4}{g \bar{v}_m} \frac{1}{3} \ell_m \bar{v}_m \frac{\partial n_m(x,t)}{\partial x}$$

or

$$n_m(x,t) = h \frac{\partial n_m(x,t)}{\partial x} \quad \text{at} \ x = 0, \quad \text{(II.115)}$$

where $h = (4/3)(\ell_m / g)$.

Similarly if a fraction $G$ of the metastables which strike the anode are destroyed, then

$$n_m(x,t) = -H \frac{\partial n_m(x,t)}{\partial x} \quad \text{at} \ x = d, \quad \text{(II.116)}$$

$H$ being $(4/3)(\ell_m / G)$.

Of the metastables destroyed at the cathode, let a fraction $g_m$ cause the emission of an electron. Thus if there is an externally maintained current $I_0$ we have

$$I_0(0,t) = I_0 + g_m D_n \left[ \frac{\partial n_m(x,t)}{\partial x} \right]_{x=0}. \quad \text{(II.117)}$$

Thus for small $g$ and $G$, the relevant boundary conditions are (II.115), (II.116), and (II.117). If $g$ and $G$ are larger fractions, Eqs. (II.115) and (II.116) will, owing to the smallness of $\ell$ compared with such distances as $d$, become almost $n = 0$. For large gap $g$ and $G$, however, there is great asymmetry in the directional distribution of the metastables,
and the differential equations (II.113) and (II.114) will fail at the electrodes. However, the following argument [after Piddock\textsuperscript{12}'] shows that for large $g$ and $G$ we may regard the diffusion equation and Eq. (II.113) as holding right up to the electrode and $n$ as becoming zero there.

Consider an element $dS$ of the surface of the cathode at $\nu = 0$. Suppose initially that the electrode is removed so that the number of metastables crossing $dS$ per second from either side would be $\frac{1}{4} \rho_m \vec{v}_m \cdot dS$ from the velocity of agitation. Diffusion alters the equality and produces a flux $\left(\frac{1}{4} \rho_m \vec{v}_m - \vec{0}\right) dS$ from left to right and $\left(\frac{1}{4} \rho_m \vec{v}_m + \vec{0}\right)$ from right to left. The difference of the fluxes must be equal to $D_m \frac{\partial n_m}{\partial \nu}$.

If conditions are such that the flux from left to right vanishes, the space $\nu < 0$ can be replaced by a metal (which is equivalent to replacing the electrode). Hence the boundary condition at a metal surface is

$$\frac{1}{4} \rho_m \vec{v}_m = \vec{0}$$

or

$$\frac{1}{2} \rho_m \vec{v}_m - D_m \frac{\partial n_m}{\partial \nu} = \vec{0}, \quad \text{[II.118]}$$

and the actual number of atoms falling on $dS$ per second is $D_m (\partial n_m / \partial \nu)$. Now Eq. (II.118) becomes

$$\rho_m - \frac{1}{3} \rho_m \frac{\partial n_m}{\partial \nu} = 0,$$

or

$$\rho_m = 0,$$

since the second term is small in comparison with the first. Hence the number of metastable atoms per unit volume near the metal surface is negligible in comparison with that at some distance away. The flux across the boundary is approximately $D_m (\partial n_m / \partial \nu) \cdot dS$.

Thus, as mentioned previously, we may take Eq. (II.113) and the differential equation to hold right up to the electrodes, with $\rho_m$ (and thus $n_m$) becoming zero there.

5.2 Steady-state solution

In the steady state we wish to solve the equation

$$D_m \frac{d^2 n_m(x)}{dx^2} + \alpha_n I_e(0) e^{\alpha x} = \frac{n_m(x)}{\tau_m}$$

with boundary conditions (II.115), (II.116), and (II.117).

Integrating and using the boundary conditions yields

$$\frac{I_e(0)}{I_e} = 1 \left\{ 1 - \frac{\gamma_1}{\alpha^2 - \mu^2} \left[ \frac{(1 + \alpha H) \left( e^{\alpha H} - \cosh \mu d \right) - (\alpha + \mu \nu) \left( \sinh \mu d \right) / \mu}{(H + h) \cosh \mu d + (1 + Hh \nu) \left( \sinh \mu d \right) / \mu} \right] \right\}, \quad \text{[II.119]}$$

where $\gamma_1 = g_m \alpha_m$ and $\mu = 1 / \sqrt{D_m \tau_m}$. 

If there is a high probability of destruction at the electrodes \((g\) and \(G\) large), then \(\alpha H, (H+h)/d,\) and \(\mu (H+h)\) are small fractions, and Eq. (II.119) reduces to

\[
\frac{I_d[0]}{I_0} = 1 \left\{ 1 - \frac{\gamma_1}{\alpha^2 - \mu^2} \left[ \frac{e^{\mu d} - \cosh \mu d - \alpha (\sinh \mu d)/\mu}{(\sinh \mu d)/\mu} \right] \right\},
\]

(II.120)

which is exact if \(h = H = 0\).

Formulae (II.119) and (II.120) correspond to the Townsend equation (II.18) previously considered. As in Section 3.8 of this chapter, a breakdown potential \(V_s\) may be defined as the applied potential which makes the denominator vanish. If the denominator is negative the expressions are physically meaningless, and no steady-state solution exists.

5.3 Solutions for the temporal growth

We wish to solve the differential equation (II.114) subject to the boundary conditions (II.115), (II.116), and (II.117). In formulae (II.115) and (II.116) we shall take \(h = H = 0\), since the calculation is easily modified to take account of the more general boundary conditions. We assume that there are no metastables or charged particles in the discharge space for times less than zero.

Applying the Laplace transformation to Eq. (II.114) yields

\[
p\tilde{\Pi}_m(x) = D_m \frac{d^2\tilde{\Pi}_m(x)}{dx^2} + \alpha_m e^{\alpha x} e^{-px/W_e} \tilde{\gamma}_e(0) - \frac{\tilde{\Pi}_m(x)}{\tau_m}
\]

or

\[
\frac{d^2\tilde{\Pi}_m(x)}{dx^2} - \frac{p + (1/\tau_m)}{D_m} \tilde{\Pi}_m(x) = - \frac{\alpha_m \tilde{\gamma}_e(0)}{D_m} \exp \left[ x(\alpha - p/W_e) \right].
\]

(II.121)

In order to have a solution that is free from square roots, the multiplier of \(\tilde{\Pi}_m(x)\) should be a perfect square. Equating it to \(z^2\) we have

\[
\frac{p}{D_m} + \frac{1}{D_m \tau_m} = z^2
\]

or

\[
p = D_m (z^2 - \mu^2)
\]

and the equation becomes

\[
\frac{d^2\tilde{\Pi}_m(x)}{dx^2} - z^2\tilde{\Pi}_m(x) = - \frac{\alpha_m \tilde{\gamma}_e(0)}{D_m} e^{\alpha x},
\]

where

\[
\psi = \alpha - \frac{D_m}{W_e} (z^2 - \mu^2),
\]

which has the general solution

\[
\tilde{\Pi}_m(x) = A e^{\alpha x} + B e^{-\alpha x} - \frac{\alpha_m \tilde{\gamma}_e(0)}{D_m (\psi^2 - z^2)} e^{\psi x}.
\]
Substituting in the transformed boundary conditions we have

\[ \bar{I}_e(0) = \frac{I_0}{p} + g_n \frac{dN_n(x)}{dx} \bigg|_{x=0}, \]

or

\[ \frac{\bar{I}_e(0)}{g_n D_n} - \frac{I_0}{g_n D_n(z^2 - \mu^2)} = A z - B z = \frac{\psi a_n}{D_n(z^2 - z^2)} \bar{I}_e(0), \]  \hspace{1cm} (II.122)

with

\[ A + B = \frac{\alpha n \bar{I}_e(0)}{D_n(z^2 - \mu^2)} = 0 \]  \hspace{1cm} (II.123)

and

\[ A + B e^{-2zd} - \frac{\alpha_n \bar{I}_e(0)}{D_n(z^2 - \mu^2)} e^{(\psi - 2d)} = 0. \]  \hspace{1cm} (II.124)

Eliminating \( A \) from the last two conditions gives

\[ B = \frac{\bar{I}_e(0) g_n \left[ 1 - e^{(\psi - 2d)} \right]}{D_n(z^2 - \mu^2)} \left( 1 - e^{-2zd} \right) \]

with \( A \) given by condition (II.123). Substituting for \( A \) and \( B \) in condition (II.122) yields

\[ \frac{\bar{I}_e(0)}{I_0} = \frac{[z^2 - \psi^2] \left( 1 - e^{-2zd} \right)}{(z^2 - \mu^2) \theta}, \]  \hspace{1cm} (II.125)

where

\[ \theta = \xi + 2 \gamma z e^{(\psi - 2d)} - (2 \gamma z + \xi) e^{-2zd}, \]

and

\[ \xi = (z + \psi) \left[ (z - \psi) F - \gamma_1 \right], \]

with \( F = 1 \) (the reason for introducing \( F \) will appear later).

Substituting for \( A, B, \) and \( \bar{I}_e(0) \) in formula (II.121) gives

\[ \bar{n}_m(x) = \frac{\alpha_n}{D_n(z^2 - \mu^2) \theta} \left[ e^{(\psi - 2d)} \right] e^{zx} + [1 - e^{(\psi - 2d)}] e^{-zx} - [1 - e^{-2zd}] e^{\psi} \]  \hspace{1cm} (II.126)

Now the inversion theorem states that

\[ n_m(x,t) = \frac{1}{2\pi i} \int_{\beta = i}^{\beta + i} \bar{n}_m(x) e^{\beta t} \, d\beta, \]

where \( \beta \) is a real number such that all the singularities of \( \bar{n}_m(x) \) lie to the left of the line \( \beta - i\omega \) to \( \beta + i\omega \). This line may therefore be replaced by the semicircle with centre at the origin in the right half plane, and it is described clockwise. Since \( p = D_n(z^2 - \mu^2) \), the semicircle transforms into a quadrant C in the \( z \)-plane, the centre of the arc being on the positive real axis.
Thus
\[ n_m(x,t) = \frac{1}{2\pi i} \oint_C N_m(x) e^{D_m(z^2 - \mu^2)t} \cdot 2D_n z \, dz \quad \text{traversed clockwise} \]
\[ = \frac{i}{\pi} \int_C D_n z \, N_m(x) e^{D_m(z^2 - \mu^2)t} \, dz \quad \text{traversed anticlockwise}. \]

Substituting for \( N_m(x) \) we have
\[ \frac{n_m(x,t)}{I_0} = \frac{i a_n}{\pi D_n} \oint_C \frac{z}{(z^2 - \mu^2)^{\theta}} \left\{ (1 - e^{-2z \psi}) e^{z \chi} + \left[ e^{(y-z)\lambda} - 1 \right] e^{-z \chi} + \left[ z e^{-2z \mu} - e^{(y-z)\lambda} \right] e^{z \chi} \right\} \, dz \]
and
\[ \frac{I_m(0,t)}{I_0} = \frac{i}{\pi} \oint_C \frac{z}{(z^2 - \mu^2)^{\theta}} \left( z^2 - \psi^2 \right) \left( 1 - e^{-2z \mu} \right) \, dz, \tag{II.127} \]
where
\[ \psi = \alpha - D_n \left( z^2 - \mu^2 \right) / \kappa_n, \]
\[ \theta = \xi + \left\{ 2Y_1 z \, e^{(y-z)\lambda} - (2Y_1 z + \xi) \, e^{-2z \mu} \right\}, \]
\[ \xi = \left\{ z + y \right\} \left( z - \psi \right) \left( -y \right) - \left\{ z - \psi \right\}, \]
\[ F = 1. \]

Now the integrand in formula (II.128) is an odd function of \( z \). Thus the integral round an infinite circle is twice the integral round an infinite semicircle lying to the right of the imaginary axis. Furthermore, we note that at \( t > 0 \) the integral in formula (II.128) is not altered by expanding the contour \( C \) in both directions to form this semicircle. Thus the integral round an infinite circle is twice the integral round the contour \( C \). Considering the residues at the poles we see that
\[ \frac{I_m(0,t)}{I_0} = A + \sum_{\lambda} \frac{2F(x^2 - \psi^2) \left( 1 - e^{-2z \mu} \right) e^{D_m(z^2 - \mu^2)t}}{\left( z^2 - \mu^2 \right) \left( \frac{dN}{dz} \right)_{\lambda}}. \tag{II.129} \]
The summation extends over all the values \( \lambda \) of \( z \) (other than 0 and \( \bar{z} \)) which satisfy \( \theta(z) = 0 \), and which lie on the positive real or positive imaginary axes or the quadrant bounded by them. The term \( f \) is a factor which is equal to unity for poles on the axis and to 2 for poles inside the quadrant.

The term \( A \) is the contribution of the pole at \( z = \mu \) and is identical with expression (II.120). If the latter expression is positive [i.e. the denominator in formula (II.120) is positive and the applied voltage is less than the breakdown potential] then there will be an infinite number of \( \lambda \)'s situated on the imaginary axis. There may be one \( \lambda \) on the real axis but its value will be less than \( \mu \). Thus at large times, expression (II.129) reduces to the steady-state solution (II.120).

On the other hand if \( A \) is negative \( (V > V_s) \), one of the \( \lambda \)'s is real and greater than \( \mu \) and thus contributes an exponentially increasing term to the current. All the other roots give terms which are damped relative to this term. Thus the exact solution may be reduced to an approximate formula which is valid at large times by including only the principal terms.

In a similar manner it is also possible to evaluate expression (II.127) for the number density of metastables -- this, however, is not usually of much interest. Normally we wish only to know \( I_e(x,t) \) and \( I_p(x,t) \), both of which can be calculated immediately from \( I_e(0,t) \) using the expressions (II.55) and (II.36).

The treatment is readily generalized to include the secondary cathode processes by adding to the right-hand side of expression (II.117) the terms representing these processes. The only effect is to modify the quantity \( F \) which was unity in the above calculations, so that for the \( \gamma_i \) and \( \gamma_p \) processes, for example,

\[
F = 1 - \frac{\alpha \gamma_{ph}}{\psi} \left( e^{\phi d} - 1 \right) \frac{\alpha \gamma_i}{\phi} \left( e^{\phi d} - 1 \right),
\]

where

\[
\phi = \alpha - D_m(x^2 - \mu^2)/W, \quad \psi = \alpha - D_m(x^2 - \mu^2)/W_e \quad \text{and} \quad \frac{1}{W} = \frac{1}{W_e} + \frac{1}{W_p}.
\]

Since the integrand in formula (II.128) is still an odd function of \( z \), expression (II.129) still remains applicable. As mentioned earlier, to obtain expressions for the current produced by a sudden generation of \( N_0 \) electrons at the cathode, we have merely to differentiate the expressions for \( I_e(0,t) \) with respect to time and replace \( I_0 \) by \( N_0 \).
CHAPTER III

THE TEMPORAL GROWTH OF IONIZATION
IN FIELDS DISTORTED BY SPACE CHARGE

1. INTRODUCTION

In the previous chapter we have discussed the temporal growth of ionization currents under conditions where the applied field \( E \) is constant so that the ionization coefficients and drift velocities are normally independent of position. The solutions obtained there are of great interest in investigations of the relative importance of the various ionization processes during the early stages of the current growth.

As the current growth proceeds, however, a stage is reached when the electric field is no longer uniform, and it is necessary to take into account the influence of field distortion, due to the space charge of the ions and electrons, on the collision and ionization processes.

1.1 Continuity equations and boundary conditions

In the present work we shall consider conditions under which the dominant processes are primary ionization (\( \alpha \) process) and secondary ionization by the \( \gamma_i \) and \( \gamma_{ph} \) processes. Since \( \alpha, W_s, W_p, \) etc., are functions of the field they are no longer independent of \( x \) and \( t \), and the continuity equations (II.4) and (II.5) become

\[
\frac{\partial}{\partial t} \left\{ \frac{I_e(x,t)}{W_e(x,t)} \right\} = \alpha(x,t) I_e(x,t) - \frac{\partial}{\partial x} I_e(x,t) \tag{III.1}
\]

and

\[
\frac{\partial}{\partial t} \left\{ \frac{I_p(x,t)}{W_p(x,t)} \right\} = \alpha(x,t) I_e(x,t) + \frac{\partial}{\partial x} I_p(x,t) \tag{III.2}
\]

with the boundary conditions

\[
I_e(0,t) = I_d(t) + \gamma_i I_p(0,t) + \int_0^d \gamma_{ph}(x,t) \cdot \alpha(x,t) I_e(x,t) \, dx \tag{III.3}
\]

and

\[
I_p(d,t) = 0 \tag{III.4}
\]
1.2 Calculation of the field distortion

In order to solve the above system of equations, it is necessary to relate the field distortion in the discharge to the charge densities of ions and electrons. Several workers have attempted to integrate equations (III.1) to (III.4), but in all cases they have computed the field by means of Poisson's equation in one dimension, i.e.

\[
\frac{dE(x,t)}{dx} = -4\pi\left[\rho_p(x,t) - \rho_e(x,t)\right].
\]  

(III.5)

This process is valid, however, only if the radius of the discharge is infinite, or is large compared with the gap separation, and in practice this is not usually the case. When the discharge approximates to a long thin cylinder, a more accurate estimate of the field distortion can be obtained by the following method:

Consider a cylinder of length \(d\), radius \(r\), containing a distribution

\[\rho(x) = \rho_p(x) - \rho_e(x)\]

of electric charge, uniform over any cross-section of the cylinder (Fig. III.1). To find the field distortion at a point \(P\) on the axis a distance \(x\) from the cathode, we divide the cylinder into a number of discs by equidistant planes perpendicular to the axis. The field at \(P\) due to a disc of thickness \(dx'\) at a distance \(x'\) from \(P\) is

\[
2\pi\rho(x + x') \left[1 - x'(x'^2 + r^2)^{-\frac{1}{2}}\right] dx' \quad (x' > 0)
\]

and

\[-2\pi\rho(x + x') \left[1 + x'(x'^2 + r^2)^{-\frac{1}{2}}\right] dx' \quad (x' < 0) \]

The axial field at \(P\) due to all the charges in the cylinder is thus

\[
2\pi \int_{-x}^{0} \rho(x + x') \left[-1 - x'(x'^2 + r^2)^{-\frac{1}{2}}\right] dx' + \int_{0}^{d-x} \rho(x + x') \left[1 - x'(x'^2 + r^2)^{-\frac{1}{2}}\right] dx'.
\]  

(III.6)

If the function \(\rho(x)\) is given, it may be possible to integrate this expression. Usually, however, the values of \(\rho(x)\) are only known numerically at a number of points along the \(x\)-axis, and in this case some form of numerical quadrature must be used to evaluate formula (III.6).

Since the ends of the discharge gap are bounded by metal electrodes, the field should really be calculated by using an infinite series of discs of a charge obtained by repeated reflections in the electrodes. It has been shown, however [Davies and Evans\(^{14}\)], that in general it is only necessary to consider images up to a distance \(2d\) outside both electrodes, and that in many cases it suffices to take into account those images up to a distance \(d\) on either side of the point at which the field is being evaluated.
In the above, only the field on the axis of the cylinder has been calculated. Ideally we wish to know the field at all points within the discharge space. The field distortion decreases towards the outside of the cylinder, and Poisson's one-dimensional equation, besides giving an incorrect result on the axis, will give an even larger error elsewhere. The integral (III.6) also is not valid for points off the axis, but Davies and Evans\textsuperscript{14} have shown that the average error is of the order of 20\%, whereas using Eq. (III.5) may give an average error of several orders of magnitude. It should be noted that in the limit $r \to \infty$ formula (III.6) gives the same result as using Poisson's equation (III.5), and we should therefore expect it to be more accurate for any finite radius.

1.3 Influence of the external circuit

The potential difference between the electrodes is determined by the external electrical circuit. Under most circumstances we may approximate the external circuit by that shown in Fig. III.2. $I_g(t)$ is the total current flowing in the discharge gap and $V(t)$ the potential difference between the electrodes; $R$ is the resistance and $C$ the capacity of the external circuit, $C_0$ being the capacity of the discharge gap. Any charge flowing from one electrode of the discharge gap to the other will tend to be neutralized by the charge flowing to the electrodes from the reservoir condenser $C$, giving rise to a current $I_c(t)$ in the external circuit.

The differential equations governing the flow of current in this circuit are

$$\frac{dV(t)}{dt} = -\frac{I_g(t) - I_c(t)}{C_0} \quad (III.7)$$

$$\frac{dV_c(t)}{dt} = -\frac{I_g(t)}{C} \quad (III.8)$$

and

$$V_c(t) = V(t) + RI_c(t) \quad (III.9)$$

where

$$I_g(t) = \frac{1}{d} \int_0^d \left[ I_p(x,t) + I_c(x,t) \right] \, dx \quad (III.10)$$

At each stage of the computation these equations must be integrated to determine the voltage $V(t)$ between the electrodes which, in turn, determines the boundary condition on the field distribution in the gap through the relation

$$V(t) = \int_0^d E(x,t) \, dx \quad (III.11)$$
2. NUMERICAL INTEGRATION OF THE CONTINUITY EQUATIONS

2.1 The method of characteristics

It is more convenient to write the continuity equations (III.1) and (III.2) in terms of the linear densities \( n_p \) and \( n_e \) when we have

\[
\frac{\partial n_e(x,t)}{\partial t} = \alpha(x,t) W_e(x,t) n_e(x,t) - \frac{\partial}{\partial x} \left[ n_e(x,t) W_e(x,t) \right]
\]

and

\[
\frac{\partial n_p(x,t)}{\partial t} = \alpha(x,t) W_e(x,t) n_e(x,t) + \frac{\partial}{\partial x} \left[ n_p(x,t) W_p(x,t) \right].
\]

or

\[
\frac{\partial n_e}{\partial t} = \alpha W_e n_e - W_e \frac{\partial n_e}{\partial x} - n_e \frac{\partial W_e}{\partial x} \tag{III.12}
\]

and

\[
\frac{\partial n_p}{\partial t} = \alpha W_e n_e + W_p \frac{\partial n_p}{\partial x} - n_p \frac{\partial W_p}{\partial x} \tag{III.13}
\]

The rate of change, say \( Dn_e/|Dt| \), of \( n_e \) at a point moving in the positive \( x \)-direction with velocity \( W_e \) is

\[
\frac{\partial n_e}{\partial t} + W_e \frac{\partial n_e}{\partial x}
\]

and is thus given by

\[
\frac{Dn_e}{Dt} = \alpha W_e n_e - n_e \frac{\partial W_e}{\partial x}.
\]

Similarly the rate of change, say \( Dn_p/|Dt| \), of \( n_p \) at a point moving in the negative \( x \)-direction with velocity \( W_p \) is given by

\[
\frac{Dn_p}{Dt} = \alpha W_e n_e + n_p \frac{\partial W_p}{\partial x}.
\]

Thus if \( n_e \) and \( n_p \) are known at any time \( T \) for all values of \( x \) in the range 0 to \( d \), their values at a point \( P \) (Fig. III.3) at time \( T + \Delta t \) are given [neglecting terms of order \((\Delta t)^2\)] by

\[
n_e = n_e^R + \left( \alpha n_e W_e - n_e \frac{\partial W_e}{\partial x} \right) \Delta t \quad \tag{III.14}
\]

and

\[
n_p = n_p^S + \left( \alpha n_e W_e + n_p \frac{\partial W_p}{\partial x} \right) \Delta t, \quad \tag{III.15}
\]

where

\[
\Delta t = (x^p - x^R)/W_e^C = (x^S - x^p)/W_p^C.
\]

Fig. III.3 Characteristics (which follow the paths of electrons and ions through the gap) passing through a specified point \( P \) at the future time \( T + \Delta t \).
C⁺ and C⁻, the so-called "Characteristic Curves" or simply "Characteristics", effectively follow the paths of the electrons and ions through the discharge gap. In deriving Eqs. (III.14) and (III.15) it is assumed that for small values of Δt, RP and SP approximate to straight lines:

![Characteristics Curves Diagram](image)

Fig. III.4

Usually the values of \( n_p \) and \( n_e \) are known only at a finite number of equidistant points on the line \( T \) (e.g. A, C, B, in Fig. III.4); thus \( n_e^R \) and \( n_p^S \) must be found by interpolating between known values at the mesh points.

The basic procedure for numerical integration of the continuity equations thus becomes the following:

i) The co-ordinates of R and S are found from

\[
\begin{align*}
x_R^p &= x^p - W_e \Delta t \\
x_S^p &= x^p + W_p \Delta t.
\end{align*}
\]  

(III.16) \hspace{1cm} (III.17)

ii) \( n_p^R, n_p^S, n_e^R, \) and \( n_e^S \) are evaluated by interpolating between known values at A, C, and B.

iii) \( n_e^p \) and \( n_p^p \) are found from

\[
\begin{align*}
n_e^p &= n_e^R + \left( an_e W_e - n_e \frac{\partial W_e}{\partial x} \right)^C \frac{(x^p - x_R^p)/w_e^C}{(x^p - x_R^p)/w_e^C} \\
n_p^p &= n_p^S - \left( an_p W_p + n_p \frac{\partial W_p}{\partial x} \right)^C \frac{(x^p - x_S^p)/w_p^C}{(x^p - x_S^p)/w_p^C}.
\end{align*}
\]  

(III.18) \hspace{1cm} (III.19)

Thus, knowing \( n_p \) and \( n_e \) at a given time \( T \), their values at \( T + \Delta t \) can be found for all points across the gap except near the electrodes where the boundary conditions must be applied.

2.2 Points near the boundary

If the point \( P \) is sufficiently near the boundary \( x = d \), the C⁻ characteristic may not cut the line \( t = T \) at all, but intersects the boundary at the point \( N \) (Fig. III.5). Now
Fig. III.5 The $C_-$ characteristic through a point $P$ near the anode.

The value of $n_p$ at $N$ is zero [from the boundary condition $I_p(d,t) = 0$] so that Eq. (III.19) becomes

$$n_p^P = \left( \alpha W_e n_e + n_p \frac{\partial W_e}{\partial x} \right)^C (d-x^C)/W_e^C$$  \hspace{1cm} (III.20)

for points near the anode.

Similarly, if $P$ is near the cathode, Eq. (III.18) becomes (Fig. III.6)

$$n_e^P = n_e^N + \left( \alpha n_e W_e - n_e \frac{\partial W_e}{\partial x} \right)^C \frac{x^P}{W_e^C}.$$  \hspace{1cm} (III.21)

The value of $n_e^M$ is not known directly but may be extrapolated from the known values of $n_e$ at the cathode at times $T$, $T - \Delta t$, $T - 2\Delta t$, etc.

In this manner we may evaluate $n_e$ and $n_p$ at all points at time $T + \Delta t$ except for the value of $n_e^N$ at the cathode. In order to do this we make use of the boundary condition (III.3). This, however, means relating the linear number densities $n_e$ and $n_p$ at each point to the currents $I_e$ and $I_p$, where $I_e = n_e W_e$ and $I_p = n_p W_p$. Now $W_e$ and $W_p$ will be functions of the field at time $T + \Delta t$, and this field distribution may be found (using the method

Fig. III.6 The $C_+$ characteristic through a point $P$ near the cathode.
described in Section 1.2 of this chapter) from the known values of \( n_e \) and \( n_p \), and from either the value of \( n_e^N \) at the time \( T \) or from an extrapolated value of \( n_e^N \) from earlier times. Note that comparatively large errors in \( n_e^N \) will have a very small influence on the field distribution at \( T + \Delta t \). Having determined the fields, \( I_e(0,T + \Delta t) \) and thus \( n_e(0,T + \Delta t) \) may be found from the boundary condition (III.3).

2.3 Improvements to the basic method of characteristics

The above procedure provides a means of computing \( n_e \) and \( n_p \) at a time \( T + \Delta t \) given their values at a time \( T \). Several improvements may, however, be made to the basic procedure.

a) In the method of characteristics we are essentially integrating along the characteristics \( C_+ \) and \( C_- \) using the values of the ionization coefficients, drift velocities, and velocity gradients at the point \( C \) (Fig. III.4) at time \( T \). Accuracy may be improved by giving \( n_e \) and \( n_p \) in the second terms on the right-hand sides of Eqs. (III.18) and (III.19), their values at the foot of the characteristics, i.e. at \( R \) and \( S \), and by replacing the value of \( \alpha n_p W_e \) at \( C \) by the mean of the values at \( C \) and either \( R \) or \( S \).

b) In the first attempts at calculating current growth, linear interpolation was used to evaluate \( n_e \), \( n_p \), and various coefficients at \( R \) and \( S \). This introduced a first-order inaccuracy and thus, subsequently, quadratic interpolation using a four-point formula was employed.

c) The method is essentially of first order since the characteristics \( C_+ \) and \( C_- \) are taken to be straight lines. Here again, second-order accuracy may be attained by assuming \( C_+ \) and \( C_- \) to be second-degree curves and suitably modifying the computation [for details, see Ralston and Wilf\(^{15} \)].

d) In evaluating Eqs. (III.18) and (III.19) it is assumed that the ionization coefficients and drift velocities retain the values they had at time \( T \). In fact a more accurate approximation would be to give them their average values along the characteristics, which involves knowing their values at the future time \( T + \Delta t \). These values will, in general, be different since the electric field is changing with time. The basic method of characteristics will, however, give a first approximation to the densities and field distribution at \( T + \Delta t \). These values may then be taken as first approximations in an iterative procedure which allows for the variation of \( u, W_e, W_p \), etc., along \( C_+ \) and \( C_- \). The iteration is repeated until the cathode electron current at time \( T + \Delta t \) differs by less than an arbitrary amount in successive iterations.

2.4 The single characteristic method for use at large times

In order that the above procedure may be followed directly, both \( W_e \Delta t \) and \( W_p \Delta t \) must be less than or of the order of \( \Delta x \), the step length in the \( x \)-direction. Now in many cases the value of \( W_e \) is about one hundred times greater than that of \( W_p \) at the same \( E/p \), so that if we choose \( W_p \Delta t \approx \Delta x \), the \( C_p \) characteristic is nearly horizontal and intersects the line \( T \) well outside the interval 0 to \( d \) (Fig. III.7).
Fig. III.7 Direction of the characteristics when the electron drift velocity $W_e$ is very much greater than the ion drift velocity $W_p$.

In one interval $\Delta t$, therefore, several generations of electrons (see Chapter IV, Section 3.1) cross the gap and the photon secondary process almost attains equilibrium. By this we mean that the total current leaving the cathode at any time $t$ is related to the current $I_e(t) + \gamma I_p(0,t)$, which varies much more slowly in time, by an equation similar to Townsend's equation

$$I_2 = \frac{I_e(t) + \gamma I_p(0,t)}{1 - \mu}, \quad (III.22)$$

where

$$\mu = \int_0^x \gamma_{ph} a(x,t) \exp \left\{ \int_{x'}^x a(x',t) \, dx' \right\} \, dx,$$

which reduces to $\gamma_{ph} (e^{ad} - 1)$ for the case of uniform fields. If we can assume that equilibrium has been reached, then $I_2$ approximates $I_e(0,t)$.

Normally the current at any point grows by a very small fraction in a time $d/W_e$, and we may put $W_e = \infty$ in Eq. (III.1) and integrate the resulting equation directly to obtain

$$I_e[x,t] = I_e[0,t] \exp \left\{ \int_0^x \alpha(x',t) \, dx' \right\}. \quad (III.23)$$

Once $n_p$ has been obtained at every point at time $T + \Delta t$ using the method of characteristics, we may calculate an approximate value of the field distortion at this time by giving the electron density the value it had at time $T$. Since $n_e$ is only about one-hundredth of $n_p$ over most of the gap at times of the order of an ion transit time, the error due to assuming that it has not altered is very small. Once again, however, the approximate field so obtained may be used in the first stage of an iterative process. From the field the value of $\alpha$ -- and hence of $\mu$ -- may be calculated. Now $I_2$ is the current which the electron current at the cathode will attain after an infinite number of electron avalanches have crossed the gap ($I_p$ remaining constant). Let $I_1$ be the value of $I_e(0,T)$ and $k$ the number of generations of electrons and photons that have crossed the gap in the subsequent time $\Delta t$. 
Then we may write

\[ I_e(0, T + \Delta t) = I_1 \mu^k + I_2 (1 - \mu^k), \]

\[ = I_1 - (I_2 - I_1) \mu^k. \] (III.24)

Now, provided \( \mu \) is not much greater than unity it can be shown that \( \mu^k \) is equal to the \( e^{\lambda t} \) of Davidson's approximate solution (II.56). Equation (III.24) therefore corresponds exactly to the form of Davidson's approximate solution obtained for the case in which there is a current \( I_1 \) flowing at the start of the experiment and a steady current \( I_0 + \gamma_1 I_p(0,T) \) is maintained at the cathode at subsequent times. Although it is only approximate, it is much more accurate to use Eq. (III.24) than to assume that complete equilibrium of the photon process has been reached.

Since the value of \( I_1 \) \( = I_e(0,T) \) is known and \( I_2 \) can be found from Eq. (III.22), we can thus evaluate \( I_e(0,T + \Delta t) \) from Eq. (III.24) and finally \( I_e(x,T + \Delta t) \) from Eq. (III.23). Since we have already computed the field distribution, the drift velocities at \( T + \Delta t \) may be evaluated and hence \( n_e \) at all points across the gap. With \( n_p \) already known from Eq. (III.19) at \( T + \Delta t \) we have thus integrated the continuity equations over one time-step on the assumption of quasi-equilibrium of the electron current at the cathode.

2.5 The initial data

Before the program (described in the previous section) can be started, the values of the various ionization parameters and charged particle densities have to be found. In general the effect of space-charge distortion will be negligible at the initial instant (since the ionization currents are small); thus the initial charge distribution may be found from the various formulae derived in Chapter II, Section 3, for the case of uniform fields.

In the boundary condition (III.3), the term \( I_0(t) \) represents the current generated at the cathode by external means (normally by illumination with ultra-violet light). In the present work we consider only the cases a) where we have continuous illumination of the cathode so that \( I_0(t) \) is constant, or b) when a pulse of light falls on the cathode, the intensity of illumination having a Gaussian distribution in time so that \( I_0(t) \) also has the familiar Gaussian form.

Provision can also easily be made for an arbitrary distribution of charge between the electrodes at the initial instant.

3. Typical results of the numerical computations

3.1 Pulsed illumination of the cathode.

Double characteristic method

It is of interest, in the first instance, to perform the calculations for the case when the cathode is illuminated by a pulse of ultra-violet light and the space-charge distortion of the ions and electrons is neglected. The results then show the growth of a typical electron avalanche and the resulting secondary avalanches.
The experimental conditions were as follows: nitrogen; \( p = 90 \) Torr; \( E/p = 62 \text{ V} \cdot \text{cm}^{-1} \text{ Torr}^{-1} \); overvoltage = 25.5\%; \( \gamma_{ph} = 3.1 \times 10^{-8} \); number of electrons in initial (Gaussian) pulse \( \approx 400 \). The various drift velocities and ionization coefficients were given by

\[
\alpha/p = 5.7 \exp\left(-260p/E\right),
\]

\[
W_o = 2.9 \times 10^5 \frac{E}{p} ,
\]

and

\[
W_p = 2 \times 10^4 \left[1 - 4 \times 10^{-3} \frac{E}{p}\right] \frac{E}{p} \quad E/p \leq 80
\]

\[
= 1.25 \times 10^7 \sqrt{\left[\frac{E}{p}\right]} - Ap/E \quad E/p > 80.
\]

The constant \( A \) was chosen to make the two formulae agree at \( E/p = 80 \); \( E/p \) is in \( \text{V} \cdot \text{cm}^{-1} \text{ Torr}^{-1} \) in all the above equations.

Electron charge density (Coulomb/cm\(^3\))

---

**Fig. III.8** The development of a discharge in a uniform field due to a pulse of electrons leaving the cathode at time zero. The time in nanoseconds is indicated on the curves.
Figure III.8 shows the variation of electron density $n_e(x,t)$ across the gap separation at various instants of time. We see that the primary avalanche, having a Gaussian form with half-width $d/8$, travels at the appropriate drift velocity. The electrons in the second generation have an almost constant density in space, since the primary and secondary electrons are amplified at the same rate. When the primary avalanche reaches the anode, however, fewer secondary electrons are produced; thus the density diminishes towards the cathode.

The results obtained when allowance is made for the distortion of the field are shown in Figs. III.9 and III.10. The radius of the discharge was taken to be 0.025 cm [estimated from measurements made by Wagner 16, 17]. Figure III.9 shows the distribution of electrons in the discharge gap. At about 100 nsec we see the acceleration of the front of the avalanche towards the anode, while the ionization in the central region is very much reduced compared with Fig. III.8. At 125 nsec a step appears at the back of the distribution and travels towards the cathode.

In Fig. III.10 the corresponding field distributions are plotted. At 100 nsec the field distribution has the typical shape due to a dipole. The maximum on the anode side of the avalanche increases rapidly after this time until it reaches the anode at about 115 nsec, when the field there falls quickly to a low value. The second maximum, on the cathode side of the avalanche, then starts to increase so that secondary electrons approaching the avalanche find themselves in an enhanced field. The resulting increase in the ionization rate maintains the step shown in Fig. III.9 and causes it to move towards the cathode. We see that the maximum in the field, which coincides with the steepest part of the step, also propagates towards the cathode.

![Electron charge density (Coulomb/cm³)](image)

Fig. III.9 The development of a discharge allowing for the distortion of the field. The time in nanoseconds is indicated on the curves.
3.2 Continuous illumination.

Single characteristic method

Figure III.11 shows the growth of a) the average electron current $\bar{I}_e(t)$, b) the average ion current $\bar{I}_i(t)$, and c) the total average current $I_g(t)$, flowing in a discharge in hydrogen between copper electrodes under the following conditions: $p = 1.6$ Torr; electrode separation = 1.55 cm; $V_S = 350$ volts; $V = 358$ volts; $I_o = 4.9 \times 10^{-10}$ A; $\gamma_{ph} = 0.75 \gamma_T$. The functional dependence of $\alpha/p$, $\gamma_T$, $W_e$, and $W_p$ were as follows:

$$\alpha/p = 3.5 \exp \left(-88.9p/E\right) \quad E/p > 130$$

$$\alpha/p = 5.05 \exp \left(-146p/E\right) \quad E/p \leq 130$$

$$\gamma_T = 0.00302 \exp \left(0.0259E/p\right) \quad E/p \geq 6C$$

$$W_p = 6.232 \times 10^9 \times E/p \text{ cm sec}^{-1}.$$  

$$W_e = 100W_p.$$
Since the times considered are large, sufficient accuracy can be obtained using the single characteristic method (Section 2.4 of this chapter).

The shape of the $I_g(t)$ - $t$ curve agrees well with the experimental current growth traces of Bandel\cite{10} and Mene\cite{11}. There is a considerable period [up to about $(2/3)t_f$] where the total average current grows exponentially with the same time constant $\lambda$ as is given by the real root of the equation

$$f(p) = 1 - \frac{\gamma_{1\alpha}}{\phi} (e^{\phi d} - 1) - \frac{\gamma_{2\alpha}}{\psi} (e^{\psi d} - 1) = 0$$

[see Chapter II, Section 3.3, Eq. (II.43)]. This means that in this region the effect of space-charge distortion of the field on the current growth is negligible, and it is still valid to use Davidson's exact solution to compute the temporal growth of the ionization currents.

After about eight microseconds, however, the field distortion becomes appreciable and the current growth becomes greater than exponential. Just before the measured time lag, the field distortion becomes very large and the resulting current growth is very rapid indeed. It is at this time that the collapse of voltage occurs.

From the above considerations we see that it is not valid to use Davidson's solution right up to the measured time lag. In fact it is only valid to use it until about $(2/3)t_f$, after which time numerical computation must be employed.

![Graph](image)

**Fig. III.11** Average ionization currents $\bar{I}_e(t)$, $\bar{I}_p(t)$, $I_g(t)$, and the voltage $V(t)$ plotted as functions of time.
Fig. III.12  Distribution of positive ions at various times (indicated on the curves) during the initial stages of current growth. Data as for Fig. III.11.

Fig. III.13  Field distribution in a discharge gap at different times during the current growth. Data as in Fig. III.11.
It should be noted that although the effect of ion avalanches is shown up clearly in the $\bar{I}_e(t)$ - $t$ graph, there is very little effect on the smoothness of the $I_g(t)$ curve. Since $I_g(t)$ is the quantity which is measured experimentally, this means that it is not possible to determine the ion transit time by observing the early part of the current growth.

Figure III.12 shows the distribution of positive ions between the electrodes in the early stages of the current growth. The passage of ions from near the anode to the cathode during the first ion transit time is clearly indicated.

In Fig. III.13 the field distribution between the electrodes is shown. The field distortion does not become large until just before $t_f$, and the curve corresponding to $t = 12.06$ µsec shows clearly the beginning of the formation of the anode fall region (indicated by the upcurving of the graph near the anode).
CHAPTER IV
THE STATISTICAL THEORY OF IONIZATION PROCESSES

1. INTRODUCTION TO PROBABILITY THEORY AND STATISTICS

1.1 Probabilities over discrete and continuous ranges

It would be out of place here to give a rigorous development of probability theory. We can, however, give a summary of the various ideas which are necessary to the understanding of this chapter. First, consider an experiment which assigns an integer value to some variable \( r \). The experiment is statistical in nature, so that repetitions of the experiment can give different values to \( r \). Let the experiment be repeated \( N \) times and let \( N_0 \) be the number of trials which give \( r = 0 \); \( N_1 \) the number which give \( r = 1 \), \( \ldots \); \( N_n \) the number which give \( r = n \), and so on. As \( N \to \infty \), \( N_\infty /N \) tends to a constant, which is defined to be the probability \( P(n) \) of obtaining \( r = n \). This definition is sufficient for our purposes.

It follows from the above definition that

\[
\sum_{n=0}^{\infty} P(n) = 1 \quad \text{and} \quad 0 \leq P(n) \leq 1 \quad \text{for all} \ n . \quad (IV.1)
\]

If \( r \) can take a continuous range of values, the definition must be revised. Out of a total of \( N \) experiments, let the number which gives a result in the range \( x \leq r < x + \delta x \) be \( N(x, \delta x) \). Then in the limit \( N \to \infty \) and \( \delta x \to 0 \),

\[
\frac{N(x, \delta x)}{N \cdot \delta x} \to P(x) .
\]

In other words, \( P(x) \ dx \) is the probability of having a result in the interval \( x \leq r < x + \delta x \). Corresponding to Eqs. (IV.1) for a discrete distribution, we now have

\[
P(x) \geq 0; \quad \int_{-\infty}^{\infty} P(x) \ dx = 1; \quad 0 \leq \int_{a}^{b} P(x) \ dx \quad a \leq b .
\]

The quantity \( P \), considered as a function of \( x \) or \( n \), is called a probability distribution. When a distribution is illustrated graphically, \( P(x) \) generally takes the form of a smooth curve, while \( P(n) \) should be plotted as a set of isolated points, or as a histogram. We may often show \( P(n) \) as a smooth curve drawn through the points, but no significance should be attached to the value of \( P \) at non-integer \( n \).
1.2 Conditional and combined probabilities

Out of a total population of \(N\) experiments, let \(P(r = m)\) represent the fraction which assign the value \(m\) to the variable \(r\), and let \(P(s = n)\) represent the fraction which assign the value \(n\) to a second variable \(s\). (As \(N \to \infty\), the \(P's\) become probabilities.) Let us consider the \(NP(r = m)\) experiments which have \(r = m\), and enquire what fraction of these will have \(s = n\). The answer we get will depend on whether or not an experiment with \(r = m\) has the same probability of giving \(s = n\) as an experiment with \(r \neq m\). If the probabilities are the same, we describe the two measurements (of \(r\) and \(s\)) as being independent, and so the fraction of the \(NP(r = m)\) experiments which yield \(s = n\) is \(P(s = n)\). Therefore, the number giving \(r = m\) and \(s = n\) is \(NP(r = m)P(s = n)\). Allowing \(N \to \infty\), we obtain the probability

\[
P(r = m \text{ and } s = n) = P(r = m) \cdot P(s = n).
\]

If the measurements are not independent, then those with \(r = m\) may be more likely to yield \(s = n\) than those with \(r \neq m\), or vice versa, and so the above equation does not hold -- it cannot even be generalized to an inequality. If out of a total of \(N\) experiments we again consider only the \(NP(r = m)\) which yield \(r = m\), then a smaller number \(NP(r = m \text{ and } s = n)\) will yield both \(r = m\) and \(s = n\), and these represent a fraction \(P(r = m \text{ and } s = n)/P(r = m)\) of the \(NP(r = m)\) experiments we are considering. As \(N \to \infty\), we define \(P(r = m \text{ and } s = n)/P(r = m)\) to be the conditional probability that \(s = n\) given that \(r = m\).

In the work that follows, we shall not use a distinctive notation for conditional probabilities, but we shall state which is intended.

1.3 Some common distributions

1.3.1 The binomial distribution

Each experiment is assumed to consist of \(K\) independent measurements; each measurement has a probability \(p\) of giving a result 1 and a probability \(1-p\) of giving a result zero. The result of the experiment is the sum of all the \(K\) measurements (and we would expect the mean result, from elementary considerations, to be \(kp\)). We are interested in the distribution of probabilities over the possible results 0, 1, 2, ..., \(K\). Now it is possible to choose \(n\) measurements out of \(K\) in \(K! / n!(K - n)!\) ways. If these \(n\) measurements give 1, while the remaining \(K - n\) give zero, the total result will be \(n\). The probability of this happening is \(p^n(1 - p)^{K-n}\). Since each selection of \(n\) measurements out of the total \(K\) may be assumed to be equally likely, the probability of a total result \(n\) is

\[
P(n) = \frac{K!}{n!(K - n)!} \cdot p^n(1 - p)^{K-n}.
\]

This is the binomial distribution. It is so called because the \(P(n)\) may be identified with the coefficients of \(z^n\) in the binomial expansion of

\[
V(z) = (1 - p + pz)^K.
\]

This relation enables the following results to be calculated more easily. Putting \(z = 1\), \(V(z)\) becomes the sum of all the probabilities

\[
\sum_{n=0}^{K} P(n) = (1 - p + p)^K = 1 \quad \text{as it should be.}
\]
The mean value of \( n \) is

\[
\sum_{n=0}^{\infty} nP(n) = \sum_{n=1}^{\infty} \frac{K(K-1)!}{(n-1)! (K-1-n+1)!} p^n (1-p)^{K-1-n+1} = pK(1-p)^{K-1} = pK.
\]

The function \( V(z) \) is called the generating function of the distribution, and we shall examine its properties more thoroughly later on.

1.3.2 The Poisson distribution

This may be derived from the binomial distribution in the limit \( K \to \infty, \ p \to 0, \ Kp \to m \), where \( m \) is a finite constant. The distribution may be written

\[
P(n) = \frac{K(K-1) \ldots (K-n+1)}{n!} \left( \frac{p}{1-p} \right)^n \left( \frac{1-p}{K} \right)^K \]

\[
= \frac{1}{n!} \left( \frac{pK}{1-p} \right)^n \left( \frac{1-p}{K} \right)^K,
\]

which becomes in the limit

\[
P(n) = \frac{1}{n!} m^n e^{-m}.
\]

This is the Poisson distribution.

We note that

\[
\sum_{n=0}^{\infty} nP(n) = e^m \sum_{n=0}^{\infty} \frac{m^n}{n!} = e^{-m} \cdot e^m = 1,
\]

while the mean value of \( n \) is

\[
\sum_{n=0}^{\infty} nP(n) = m e^{-m} \sum_{n=1}^{\infty} \frac{m^{n-1}}{(n-1)!} = m e^{-m} e^m = m.
\]

The generating function for \( P(n) \) is

\[
V(z) = \sum_{n=0}^{\infty} P(n)z^n = e^{-m} e^{zm} = e^{m(z-1)}.
\]

The Poisson and binomial distributions are functions of an integer variable \( n \) -- that is, they are discrete distributions. The Poisson distribution, however, is not bounded, nor is it ever symmetrical, whereas the binomial distribution is symmetrical when \( p = \frac{1}{2} \).

1.3.3 The Gaussian, or normal error, distribution

When \( p = \frac{1}{2} \), the binomial distribution can be written

\[
P(r) = \frac{(2m)!}{(m+r)! (m-r)!} \left( \frac{1}{2} \right)^{2m}.
\]

where \( 2m = K \), the number of measurements; \( m \) is the mean result; and \( r \) is the difference between an actual result and the mean. \( P(r) \) is symmetrical about \( r = 0 \), and its maximum value is

\[
P(0) = \frac{(2m)!}{(2^m m)!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots 2m}{2 \cdot 2 \cdot 4 \cdot 4 \ldots 2m} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \ldots \frac{2m-1}{2m} = \left( \frac{\pi m}{\sqrt{2m}} \right)^m \left( 1 - \frac{1}{8m} + \ldots \right),
\]
using Wallis's product; \( P(r) \) falls to its minimum value \( 2^{-2m} \) at \( r = \pm m \). Since the area under the graph of a distribution is 1, the width of the distribution does not increase linearly with \( m \), but only to the order of \( m^{1/2} \). If we replace \( r \) in the limit of large \( m \) by the continuous variable \( x = m^{-1/2} \), then the curve (drawn as a function of \( x \)) will tend to a constant width, and we may expect to obtain a limiting form for the whole distribution. We therefore put \( dx = m^{-1/2} \), \( dr = m^{-1} \), and \( P(x) = m^{1/2} P(r) \).

Applying Stirling's formula

\[
 n! = (2\pi)^{1/2} n^{n+1/2} e^{-n}
\]

to the factorials in \( P(r) \):

\[
 \ln P(r) = (2m + \frac{1}{2}) \ln 2m - 2m - 2m \ln 2 - (m + r + \frac{1}{2}) \ln (m + r) + (m + r) - (m + r + \frac{1}{2}) \ln (m - r) + (m - r) - \frac{1}{2} \ln 2\pi.
\]

After some simplification, this reduces to

\[
 r^2 \left( \frac{-1}{m} + \frac{1}{2m} r \right) + r^4 \left( \frac{-1}{6m^3} + \frac{1}{4m} \right) + \ldots - \frac{1}{2} \ln \pi m
\]

and so, as \( m \to \infty \),

\[
 P(r) = (\pi m)^{-1/2} e^{-r^2/m}, \quad \text{or} \quad P(x) = \pi^{-1} e^{-x^2}.
\]

This is the Gaussian distribution. The mean value of \( x \) is zero, from the symmetry of the curve, while the area under the curve is 1.

1.4 Occurrence of the common distributions

in physical situations

The binomial distribution occurs when a finite number of measurements are made, and each measurement gives one of two results with a fixed probability. It therefore gives the probability of obtaining \( n \) "heads" on tossing \( K \) coins (in which case \( p = \frac{1}{2} \)), or the probability of obtaining \( n \) sixes on throwing \( K \) dice (in which case \( p = \frac{1}{6} \)), and so on. We shall encounter the binomial distribution in the course of this chapter, but some limiting process is usually applied, and the Poisson or Gaussian distribution obtained instead.

The Poisson distribution occurs when the number of measurements constituting each experiment becomes very large, and the probability of any one measurement giving a positive result tends to zero. Since it contains only one parameter, \( m \) (the mean result), it is often preferable to use it in situations where the binomial distribution would be more appropriate. For example, consider a sample of radioactive isotope. This will contain a finite number of atoms \( K \), while each atom has a finite probability \( p \) of decaying in a given interval of time (of the order of seconds). Strictly, therefore, the number of atoms decaying in successive intervals is given by the binomial distribution. However, since \( K \) is so large and \( p \) so small, the Poisson distribution containing the single parameter \( Kp \) gives a satisfactory result. Another example arises in the case of photo-electric emission. Suppose electrons are being emitted at an average rate \( I \) per unit time. The probability of
one being omitted in an interval $\delta t$ is $I_0 t$, and the number of such intervals in a finite time $T$ is $T/\delta t$. Letting $I_0 t \to 0$ and $T/\delta t \to \infty$, then $(I_0 t)(T/\delta t)$ remains constant, and the Poisson distribution is obtained with $m = I_0 T$.

The Gaussian distribution occurs when the result of an experiment is the sum of a large number of small increments which can be positive or negative at random. It should therefore apply to the position of a particle which has been making a random walk -- that is, a diffusion process. If the error in a physical measurement is assumed to be due to a large number of small independent errors, then one would expect to find a Gaussian distribution of errors. Most physical measurements do in fact fit such a distribution.

1.5 The generating function, mean, and standard deviation

When $P(n)$ is a discrete distribution over non-negative values of $n$, we may form the infinite series

$$V(z) = \sum_{n=0}^{\infty} P(n)z^n$$

which is called the generating function. Since the series is absolutely convergent, the generating function will exist for $z \leq 1$, which is sufficient for our purposes. Conversely, given $V(z)$, $P(n)$ may be found by expansion or by differentiation. Thus,

$$P(0) = V(0), \quad P(1) = \left(\frac{\partial V}{\partial z}\right)_{z=0}, \quad \text{etc}.$$  

The mean value of $n$ is defined as

$$\bar{n} = \sum_{n=0}^{\infty} nP(n).$$

Now

$$\frac{\partial V}{\partial z} = \sum_{n=0}^{\infty} nz^{n-1} P(n),$$

so that

$$\bar{n} = \left(\frac{\partial V}{\partial z}\right)_{z=1}.$$  

Similarly,

$$\left(\frac{\partial^2 V}{\partial z^2}\right)_{z=1} = \sum_{n=0}^{\infty} n(n-1)P(n) = \sum_{n=0}^{\infty} n^2P(n) - \bar{n}^2.$$  

The standard deviation $\sigma$ is given by

$$\sigma^2 = \text{mean value of } (n - \bar{n})^2$$

$$= \sum_{n=0}^{\infty} (n^2 - 2n\bar{n} - \bar{n}^2) P(n) = \sum_{n=0}^{\infty} n^2P(n) - \bar{n}^2 - \left(\frac{\partial V}{\partial z}\right)_{z=1} - \bar{n}(\bar{n} - 1).$$  

Higher-order moments can also be calculated, but we do not often need them.
1.6 The product of generating functions

Let

\[ V_1(z) = \sum_{n=0}^{\infty} P_1(n)z^n, \quad V_2(z) = \sum_{n=0}^{\infty} P_2(n)z^n. \]

Then the coefficient of \( z^N \) in \( V_1(z)V_2(z) \) is

\[ \sum_{n=0}^{N} P_1(n) P_2(N-n). \]

Now \( P_1(n)P_2(N-n) \) is the probability of getting a result \( n \) in the first distribution and a result \( N-n \) in the second one (assuming the measurements to be independent). When the sum is taken over \( 0 \leq n \leq N \), it is clear that we shall take into account all possible combinations which give a total result \( N \). The generating function \( V_1(z) \cdot V_2(z) \) therefore gives the probability distribution of the sum of the two measurements. More generally, the coefficient of \( z^N \) in \( V_1(z)V_2(z) \cdot \ldots \cdot V_k(z) \) is the probability of having a total result \( N \) from \( k \) independent experiments.

1.7 Use of generating functions in solving problems

We start by giving an alternative derivation of the binomial distribution. The result of each of the \( K \) measurements which constitute an experiment can be 0, with probability \( 1 - p \), and 1, with probability \( p \). The generating function for the single measurement is therefore \( 1 - p + pz = V_1(z) \), say. When \( K \) independent measurements are made, the generating function for the total result is obtained by multiplying together \( K \) factors all equal to \( V_1(z) \). That is,

\[ V_K(z) = [V_1(z)]^K = (1 - p + pz)^K, \]

which is the result we obtained in Section 1.3.1 of this chapter.

When \( z = 1 \), \( \partial V_K(z)/\partial z = Kp \), which is therefore the mean value \( \bar{n} \).

Differentiating again, \( \partial^2 V_K(z)/\partial z^2 = K(K - 1)p^2 \) when \( z = 1 \). The standard deviation is given by \( \sigma^2 = K(K - 1)p^2 - Kp(Kp - 1) \) and so

\[ \sigma = \left[ Kp(1 - p) \right]^{1/2}. \]

We may also define the relative width of the distribution to be

\[ \frac{\sigma}{\bar{n}} = \frac{1 - p}{\bar{n}^{1/2}}. \]

We can obtain the Poisson distribution for photo-electric emission in a similar way. Let \( P(n, t) \) be the probability that \( n \) electrons have been emitted during the time interval 0 to \( t \), \( V(z, t) \) the corresponding generating function, and let \( Idt \) be the probability of an electron being emitted during any interval \( dt \). \( P(n, t + dt) \) can be expressed in terms of \( P(n, t) \) as follows.
There will be \( n \) electrons at time \( t + dt \), either
\[ \text{a) if there are } n \text{ at time } t, \text{ and no more are emitted in the subsequent interval } dt, \text{ or} \]
\[ \text{b) if there are } n - 1 \text{ at time } t, \text{ and one is emitted in the subsequent interval } dt. \]

Since the two processes in (a) and the two in (b) are independent,
\[ P(n, t + dt) = P(n, t) (1 - I dt) + P(n - 1, t) I dt \quad (n = 1, 2, ...) \]
\[ P(0, t + dt) = P(0, t) (1 - I dt). \]

Multiplying each term by \( z^n \) and summing from \( n = 0 \) to \( \infty \),
\[ \sum_{n=0}^{\infty} z^n P(n, t + dt) = (1 - I dt) \sum_{n=0}^{\infty} z^n P(n, t) + z I dt \sum_{n=1}^{\infty} z^{n-1} P(n - 1, t) \]
that is,
\[ V(z, t + dt) = (1 - I dt) V(z, t) + z I dt V(z, t). \]

Dividing through by \( dt \),
\[ \frac{dV}{dt} = (z - 1)IV \quad \text{giving} \quad V = A e^{(z-1)I}t \]

where \( A \) is a constant. Note that if we had considered other processes, such as the emission of two electrons in the time \( dt \), the probability would have involved \( (dt)^2 \) and higher powers. On dividing by \( dt \) and taking the limit, these terms would have vanished.

At \( t = 0 \), no electrons have been emitted. Therefore
\[ P(0) = 1, P(1) = P(2) = ... = 0 \quad \text{and so} \quad V(z, 0) = 1. \]

Also, It is the mean number of electrons expected in the time \( t \). Putting \( m = It \),
\[ V = \mu^{m(z-1)} \] which is the expression we obtained in Section 1.5.1. Differentiating we get
\[ \frac{\partial V}{\partial z} = m \mu^m(z-1), \]
which becomes \( m \) when \( z = 1 \). This confirms that the mean value is \( m \). Differentiating again we get
\[ \frac{\partial^2 V}{\partial z^2} = m^2 \mu^m(z-1), \]
which becomes \( m^2 \) when \( z = 1 \). Thus the standard deviation is given by \( \sigma = [m^2 - m(m-1)]^{\frac{1}{2}} = m^{\frac{3}{2}} \).

The relative width \( \sigma/\mu = m^{-\frac{1}{2}} \).

2. STATISTICS OF A SINGLE ELECTRON AVALANCHE

In Chapter I we defined the ionization coefficient \( \alpha \) by saying that \( \alpha dx \) was the probability of an electron ionizing in travelling a distance \( dx \). Since we were then concerned with currents consisting of large numbers of avalanches, it was possible to work with the average size of an avalanche \( e^{\alpha x} \), and to neglect the statistical variation. Now, however, we wish to consider the statistics of a single avalanche.
We may easily show that there is a finite probability that a primary electron will not ionize at all in crossing a discharge gap of length \(d\). If we divide the gap into \(N\) intervals of length \(d/N\), the probability of not having an ionizing collision in any one interval is \((1 - \alpha d/N)^N\). The probability of not ionizing at all is

\[
\left(1 - \frac{\alpha d}{N}\right)^N + e^{-\alpha d}\quad \text{as} \quad N \to \infty.
\]

Since this probability is appreciable for \(\alpha d < 5\), we may expect to find a large statistical spread in avalanche sizes.

Let \(P(n, x)\) be the probability that an avalanche will contain \(n\) electrons by the time it has moved a distance \(x\), \(V(z, x)\) the corresponding generating function, and let \(a\ dx\) be the probability of any one electron ionizing in the subsequent distance \(dx\). We follow the same steps as we did in obtaining the Poisson distribution in Section 1.7 of this chapter; \(P(n, x + dx)\) can be expressed as the sum of two probabilities:

a) there are \(n\) electrons at \(x\) and none ionize in \(dx\); or

b) there are \(n - 1\) electrons at \(x\) and one ionizes in \(dx\).

Therefore

\[
P(n, x + dx) = P(n, x) (1 - a dx) + P(n - 1, x) a(n - 1) dx \quad [n = 2, 3, 4, \ldots]
\]

\[
P(1, x + dx) = P(1, x) (1 - a dx).
\]

Again, terms involving \((dx)^2\) are neglected, since they vanish on taking the limit. Multiplying each term by \(z^n\) and summing from \(n = 1\) to \(n = \infty\),

\[
V(z, x + dx) = V(z, x) - az dx \frac{\partial V(z, x)}{\partial z} + az^2 dx \frac{\partial^2 V(z, x)}{\partial z^2},
\]

which gives

\[
\frac{\partial V}{\partial x} = az(z - 1) \frac{\partial V}{\partial z}.
\]

(IV.2)

Comparing this with the equation

\[
dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy
\]

suggests that we should look for solutions of the auxiliary equations

\[
dV = 0 \quad \text{and} \quad \frac{dx}{1} + \frac{dz}{az(z - 1)} = 0.
\]

Two independent solutions are \(V = \text{constant}\), and \(e^{\alpha x} (z - 1)/z = \text{constant}\). The solution of the original equation (IV.2) will therefore be obtained by putting one constant equal to some function of the other. In fact we may show that

\[
V = f \left( e^{\alpha x} \frac{z - 1}{z} \right)
\]
is the general solution, where \( f \) is an arbitrary function. The form of the function \( f \) cannot be found from the differential equation, so we use the initial condition that at \( x = 0 \) there is only one electron present. The generating function is then \( V(z,0) = z \), so that \( f[z/(z - 1)] \equiv z \). Only one function satisfies this identity, namely \( f(u) = 1/(1 - u) \). Therefore for all \( x \),
\[
V(z,x) = \frac{1}{\left(1 - z - \frac{1}{z}A\right)},
\]
where \( A = e^{\alpha x} \). To find \( P(n,x) \) we must expand \( V \) as a power series in \( z \)
\[
V = \frac{z}{A} \left[1 - \frac{z}{A} (A - 1)\right]^{-1} = \frac{z}{A} \sum_{n=0}^{\infty} \left(1 - \frac{1}{A}\right)^n z^n
\]
so that
\[
P(n,x) = \frac{1}{A} \left(1 - \frac{1}{A}\right)^{n-1} \quad (n = 1, 2, \ldots)
\]
\[
P(0,x) = 0.
\]
To get the mean value \( \bar{n} \) we evaluate
\[
\frac{\partial V}{\partial z} = \frac{A}{(z + A - Az)^2} = A \quad \text{when} \quad z = 1.
\]
The mean value of \( n \) is therefore \( e^{\alpha x} \), which is the value we would have obtained in the absence of statistical fluctuations. The distribution can now be written in terms of \( \bar{n} \):
\[
P(n,x) = \frac{1}{\bar{n}} \left(1 - \frac{1}{\bar{n}}\right)^{n-1},
\]
which becomes
\[
P(n,x) = \frac{1}{\bar{n}} e^{-n/\bar{n}} \quad \text{for large} \quad \bar{n}.
\]
Differentiating we get
\[
\frac{\partial^2 V}{\partial z^2} = \frac{-2A(1-A)}{(z + A - Az)^3} = 2\bar{n}(\bar{n} - 1) \quad \text{when} \quad z = 1.
\]
Therefore
\[
\sigma^2 = 2\bar{n}(\bar{n} - 1) - \bar{n}(\bar{n} - 1),
\]
and so, for large \( \bar{n} \),
\[
\sigma = \sqrt{\bar{n}}.
\]
The above distribution was discovered by Fursey\(^{28}\) for the early stages of development of cosmic-ray showers, but was later found independently by Legler\(^{21}\) and Wijsman\(^{22}\) for electron avalanches. It is often referred to as the Fursey distribution. It has a maximum at \( n = 1 \) and a large width \( \sigma = \bar{n} \) (compare the Poisson distribution which only has \( \sigma = \bar{n}^{3/2} \)).

The distribution will be applicable to actual experimental results, provided that ionization is not the predominant mechanism of energy loss; that is, provided \( E \gg \alpha V_i \), where \( V_i \) is the ionization potential in volts. When \( E \) is only slightly larger than \( \alpha V_i \),
the analysis becomes invalid, since the probability of an electron ionizing depends on the
distance it has travelled since the last ionizing collision.

Legler\textsuperscript{23} has analysed the more general case, where the probability of ionization is
not constant, and his results give better agreement with experiment. The resulting distribution
is narrower than the Furry distribution, and the maximum can be at any value of \( n \),
not only \( n = 1 \). It is obvious that the correct distribution will be bounded, since an
electron must travel a distance of at least \( V_{ij}/E \) between ionizing collisions, and so the
maximum number of electrons at a distance \( x \) is

\[
\exp \left( \frac{E \ln 2}{V_{ij}} x \right).
\]

2.1 Statistics of the electron current during the first
electron transit time, with a photon secondary process

2.1.1 Case of one cathode electron at time zero

If instead of calculating the number of electrons reaching the anode, we calculate the
total number of electrons in the gap at any instant (and suppose that some apparatus may be
constructed to measure this quantity), then the contribution of a secondary photon process
may be easily taken into account up to a time \( d/W_e \).

An electron drifting for a time \( dt \) moves a distance \( dx = W_e dt \), and has a probability
\( \alpha W_e dt \) of producing another electron in the gas, and \( \delta W_e dt \) of producing a new photo-electron
at the cathode (where \( \delta = \alpha_{ph} \)). The above analysis therefore holds provided we replace
\( \alpha \) \( dx \) by \((\alpha + \delta)W_e dt \). Thus the probability of having \( n \) electrons after a time \( t \) is

\[
P(n, t) = \frac{1}{n!} \left( 1 - \frac{1}{n} \right)^{n-1} (n = 1, 2, \ldots)
\]

\[
P(0, t) = 0,
\]

where \( n \), the mean value, is \( \exp (\alpha + \delta)W_e t \).

2.1.2 Case of maintained illumination of the cathode

As in the last paragraph, an electron has probability \( \alpha W_e \) per unit time of undergoing
an ionizing collision in the gas, and a probability \( \delta W_e \) per unit time of producing a photo-
electron at the cathode. In addition, there is a uniform, externally maintained illumination
of the cathode which causes electrons to be emitted with a probability \( I \) per unit time. Let \( P(n,t) \)
be the probability of having \( n \) electrons in the gap at a time \( t \) (< \( d/W_e \)), and
\( V(z,t) \) the corresponding generating function. By an argument which should now be familiar,
we find that

\[
P(n, t + dt) = P(n, t) \left[ 1 - (\alpha n + I) dt \right] + P(n - 1, t) \left[ \alpha (n - 1) + I \right] dt,
\]

where \( A = (\alpha + \delta)W_e \). Multiplying by \( z^n \) and summing,

\[
\frac{\partial V}{\partial t} = I(z - 1)V + \alpha z(z - 1) \frac{\partial V}{\partial z} \quad \text{or} \quad V = \frac{1}{I(z - 1)} \frac{\partial V}{\partial t} - \frac{\alpha z}{I} \frac{\partial V}{\partial z}.
\]
Comparing this with the equation

$$dV = dt \frac{\partial V}{\partial t} + dz \frac{\partial V}{\partial z}$$

suggests that we should look for solutions to

$$\frac{dV}{V} = I(z-1) \ dt = - \frac{I \ dz}{Az} .$$

The second and third terms give $dt = -dz/Az(z - 1)$ so that $e^{At}(z - 1)/z = $ constant. The first and third give $V_z^{I/A} = $ constant. Putting one constant equal to an arbitrary function of the other,

$$V = z^{-m} f \left( \frac{z-1}{z} e^{At} \right) ,$$

where $m = I/A$. Since cathode electrons are being generated by external illumination, we can take as our initial condition $n = 0$ at $t = 0$. That is, $V(z,0) \equiv 1$, so that $f$ must satisfy

$$z^{-m} f \left( \frac{z-1}{z} \right) \equiv 1 .$$

This shows that $f$ must be of the form $f(u) = (1 - u)^{-m}$. The solution for any value of $t$ is therefore

$$V(z,t) = z \left( 1 - \frac{z-1}{z} e^{At} \right)^{-m} = e^{-It} [1 - z(1 - e^{-At})]_{-m} .$$

Expanding this as a power series,

$$P(n,t) = e^{-It} \frac{f(m + n)}{n! f(m)} (1 - e^{-At})^n .$$

We may show as usual that the mean $n$ is $\bar{n} = m(e^{At} - 1)$, while $\sigma^2 = m e^{At}(e^{At} - 1)$. Thus the relative width of the distribution is approximately

$$m^{-1} = \frac{1}{(\alpha + \delta) \bar{n}} .$$

For small $m$, the distribution has a large relative width because many experiments will contain no electrons at all.

In extreme cases (for example, $n$ much larger than $m$) the result is capable of further simplification [Davidson24]. We cannot apply the above formulae for times greater than $d/N_e$, because we have not allowed for the possibility of any electrons leaving the gap.

The exact treatment of the subsequent processes has not yet been given, but only approximate treatments based on a discrete generation period, which we shall derive later.

2.2 Solution for pulse illumination of the cathode

We first obtain the distribution due to a pulse of exactly $m$ electrons leaving the cathode. The initial condition to be satisfied is $V(z,0) \equiv z^m$, where $V(z,x) = f[e^{\alpha x}(z-1)/z]$ as in Section 2.
The only function satisfying the initial condition is \( f(u) = (1 - u)^{-m} \), so that

\[
V(z, x) = \left( \frac{z}{z - e^{\alpha x(z - 1)}} \right)^m.
\]

Alternatively we could have derived this result by observing that the generating function for \( m \) avalanches is the product of \( m \) generating functions, one for each avalanche.

Expanding \( V(z, x) \) as a power series, we obtain the probability

\[
P(n, x) = (1 - e^{-\alpha x})^n (e^{\alpha x} - 1)^{-m} \frac{(n-1)!}{(n-m)! (m-1)!} \quad (n \geq m)
\]

\[= 0 \quad (n < m).\]

We now investigate the experimental problem of observing the distribution of avalanche sizes due to one or \( m \) electrons leaving the cathode. We cannot ensure that exactly \( m \) electrons leave the cathode in each experiment; all we can do is to illuminate the cathode with identical flashes of light. In order to get one initial electron in each experiment, we should use a very weak flash of light, but the best we can do is to get a Poisson distribution with mean value 1. In \( 1/e \) of the experiments no electrons will be emitted at all, two will be emitted in \( 1/2e \) of the experiments, and so on. There would be no cause for concern if the resulting distribution of avalanche sizes was as shown in Fig. IV.1, since we could easily say which avalanches arose from one primary electron, which from two, and so on. Unfortunately, the distribution is completely smooth, and it is impossible to distinguish those experiments which arose from a given number of primary electrons. It is clear, therefore, that we must consider an exact treatment, allowing for the statistics of the light pulse and of the ionization processes simultaneously.

Let \( P(m, n, x) \) be the probability of having an avalanche of \( n \) electrons at the distance \( x \), given that exactly \( m \) electrons left the cathode, and let \( P(m, n, x) \) be the corresponding probability when the only information known about the cathode electrons is that they have a Poisson distribution with mean value \( \bar{m} \). The probability of having \( m \) initial electrons is

\[
Q(m) = \frac{1}{m!} \bar{m}^m e^{-\bar{m}},
\]
and so

\[ P(\bar{m}, n, x) = \sum_{n=0}^{\infty} Q(m) P(m, n, x) \]

\[ = \sum_{n=0}^{\infty} \frac{1}{m!} m^n \left( 1 - e^{-\alpha x} \right)^n \left( e^{\alpha x} - 1 \right)^{-m} \frac{(n-1)!}{(n-m)! (m-1)!} \]

\[ = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{m^n (e^{\alpha x} - 1)^{-m} n! (e^{\alpha x} - 1)^m}{(m-1)!} \frac{1}{e^{\alpha x} - 1} \]

\[ = \frac{1}{e^{\alpha x} - 1} \sum_{m=0}^{\infty} \frac{1}{m! (m-1)!} \]

\[ = \frac{1}{n} \exp \left[ -\left( \bar{m} + n e^{-\alpha x} \right) \right] \left( \bar{m} \right)^{n} \left( n \right)^{-1} \frac{1}{\sqrt{2\pi}} \left( \bar{m} \right)^{1/2} I_1 \left( 2 \bar{m} \right) e^{-\alpha x} \]

where \( I_1 \) is the modified Bessel function of order 1.

Davidson\(^\text{2a}\) obtains this result and also produces two alternative forms: i) if \( \bar{m} \) is small compared with unity,

\[ P(\bar{m}, n, x) = \frac{1}{n} \exp \left[ -\left( \bar{m} + n e^{-\alpha x} \right) \right] \left( \bar{m} \right)^{n} \left( n \right)^{-1} \frac{1}{\sqrt{2\pi}} \left( \bar{m} \right)^{1/2} I_1 \left( 2 \bar{m} \right) e^{-\alpha x} \]

ii) if \( \bar{m} \) is large compared with unity,

\[ P(\bar{m}, n, x) = \frac{1}{2\pi} \frac{1}{\bar{m} \left( \bar{m} \right)^{1/2}} \exp \left[ -\left( n \right) \frac{1}{2} \left( \bar{m} \right)^{1/2} \frac{1}{\sqrt{2\pi}} \right] \frac{1}{\sqrt{2\pi}} \left( \bar{m} \right)^{1/2} I_1 \left( 2 \bar{m} \right) e^{-\alpha x} \]

General properties of the distribution (mean, standard deviation) are best obtained from the generating function. This is easily found by writing

\[ V(\bar{m}, z, x) = \sum_{n=0}^{\infty} z^n P(\bar{m}, n, x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{m^n}{z - e^{\alpha x}} \frac{1}{z - e^{\alpha x}} \left( \bar{m} \right)^{n} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{m!} \frac{m^n}{z - e^{\alpha x}} \left[ \frac{z}{z - e^{\alpha x} (z-1)} \right]^{n} \]

\[ = \exp \left[ \bar{m} \left( \frac{z}{z - e^{\alpha x} (z-1)} - 1 \right) \right] \frac{z - e^{\alpha x} (z-1)}{z - e^{\alpha x} (z-1)} \]

All the above results could be deduced from the generating function, if desired. To find the mean, we differentiate \( V \):

\[ \frac{\partial V}{\partial z} = \frac{\bar{m} e^{\alpha x}}{z - e^{\alpha x} (z-1)} \]

\[ \frac{\partial^2 V}{\partial z^2} = \frac{\bar{m} e^{\alpha x}}{z - e^{\alpha x} (z-1)} + 2 \bar{m} e^{\alpha x} \]

thus \( \bar{m} = \bar{m} e^{\alpha x} \), which was only to be expected, and

\[ \frac{\partial^2 V}{\partial z^2} = \frac{\bar{m} e^{\alpha x} (z-1)}{z - e^{\alpha x} (z-1)} + 2 \bar{m} e^{\alpha x} \]

\[ \frac{\partial^2 V}{\partial z^2} = \frac{\bar{m} e^{\alpha x} (z-1)}{z - e^{\alpha x} (z-1)} + 2 \bar{m} e^{\alpha x} \]
so that

$$\sigma^2 = \bar{n}(2e^{\alpha x} - 1).$$

The relative width of the distribution is

$$\frac{\sigma}{\bar{n}} = \left( \frac{2e^{\alpha x} - 1}{\bar{n}} \right)^{\frac{1}{2}} \approx \left( \frac{2}{\bar{n}} \right)^{\frac{1}{2}}$$

which is a factor of $2^{\frac{1}{2}}$ larger than the corresponding quantity for a Poisson distribution.

This last result indicates how it may be possible to estimate $\bar{n}$ from the experimental results. We know that a fraction $\exp(-\bar{n})$ of the experiments will result in no avalanche at all, but the detecting system is not usually capable of distinguishing these cases from the cases in which a very small avalanche arises from one, two, or more initial electrons. Counting the number of experiments which do not give a measurable signal is likely to lead to an underestimate of $\bar{n}$, whereas measuring $\sigma$ will give the correct value of $\bar{n}$, within the limits of statistical accuracy.

3. THE STATISTICS OF AVALANCHE SERIES

3.1 The generation period $\tau$

In order to investigate the statistics of ionization currents for times longer than the electron transit time, we imagine the time to be split up into a discrete series of generation periods, such that the avalanches leaving the cathode during the $k$th period produce all their secondaries during the $(k + 1)$th period. This is evidently an approximation, since there is a possibility of a secondary electron being emitted immediately after the primary has left the cathode. We shall obtain the best description of reality if we use for the generation period the mean time $\tau$ that elapses between the emission of an electron from the cathode and the appearance there of its secondaries. This time will depend on the active secondary process, and may be expected to be of the order of the appropriate transit time.

When the undelayed photo-electric process is active, the mean rate of production of secondaries at time $t$ is $\delta \exp(\alpha W_0 t)$ ($\delta = \alpha W_0$), assuming that a single electron left the cathode at time $t = 0$ and that the time of flight of a photon back to the cathode may be neglected. The mean delay in the emission of a secondary is therefore

$$\tau = \int_0^{d/W_0} t \exp(\alpha W_0 t) \, dt - \int_0^{d/W_0} \exp(\alpha W_0 t) \, dt$$

$$= \frac{d}{W_0} \left( 1 - \frac{1}{\alpha d} \right) + \text{terms of order } \exp(-\alpha d).$$

Similarly, when the positive ion secondary process is active,

$$\tau = \frac{d}{W_e} + \frac{1}{W_p} \left( 1 - \frac{1}{\alpha d} \right) + \text{terms of order } \exp(-\alpha d).$$

As a somewhat different example, we may consider the diffusion of metastable molecules (with infinite lifetime) back to the cathode. When the primary avalanche has reached $x$, the rate of production of metastables is proportional to $\exp(\alpha x)$. These will reach the cathode
after a further mean time \( x^2/2D_n \), the time \( x/W_e \) being negligible in comparison. We thus obtain

\[
\tau = \frac{1}{2D_n} \int_0^d x^2 \exp(\alpha x) \, dx \int_0^d \exp(\alpha x) \, dx
\]

\[
= \frac{d^2}{2D_n} \left( 1 - \frac{2}{\alpha d} - \frac{2}{\alpha^2 d^2} \right) + \text{terms of order } \exp(-\alpha d).
\]

In the calculations which follow, we shall use \( \tau \) to represent the generation period, and \( \gamma \) the secondary coefficient, irrespective of the type of secondary process, since there is usually no difference in the mathematics.

3.2 The generation amplification factor \( \mu \)

The mean number of secondaries produced by a single avalanche has been shown to be

\[ \mu = \gamma \{ \exp(\alpha d) - 1 \} \]

and it is useful to define \( \varepsilon = 1 - \mu \). The actual number of secondaries produced will be scattered about this mean value because of the statistical nature of both the \( \alpha \) and \( \gamma \) processes. The investigation of this scatter will be our next concern.

3.3 The distribution of secondaries from a single avalanche

We suppose for the moment that the secondary emission is due to the impact of positive ions on the cathode. Then for an avalanche of given size \( N \), a fixed number of positive ions \( N - 1 \) will reach the cathode. Each of these has a fixed probability \( \gamma \) of ejecting a secondary electron, and so the number of secondaries will follow the binomial distribution. Its generating function is

\[ V_N(z) = (\gamma z + 1 - \gamma)^{N-1} = \sum_{n=0}^{\infty} P_n(n)z^n. \]

Now the probability of having \( N \) electrons in the primary avalanche is

\[ Q(N) = e^{-\alpha d}(1 - e^{-\alpha d})^{N-1}, \]

and we have shown that its generating function is

\[ W(z) = \sum_{N=1}^{\infty} Q(N)z^N = \frac{z}{z - (z - 1)e^{\alpha d}}. \]

The total probability of getting \( n \) secondaries for a primary avalanche of any size is

\[ R(n) = \sum_{N=1}^{\infty} Q(N)P_n(n). \]

\( R \) has the generating function

\[
x(z) = \sum_{n=0}^{\infty} R(n)z^n = \sum_{n=0}^{\infty} \left\{ \sum_{N=1}^{\infty} Q(N)P_n(n) \right\} z^n = \sum_{N=1}^{\infty} Q(N) \left\{ \sum_{n=0}^{\infty} P_n(n)z^n \right\}
\]

\[
= \sum_{N=1}^{\infty} Q(N)(\gamma z + 1 - \gamma)^{N-1} = \frac{W(\gamma z + 1 - \gamma)}{\gamma z + 1 - \gamma} = \frac{1}{e^{\alpha d} + (1 - e^{\alpha d})(\gamma z + 1 - \gamma)}.}
\]
Now \( \gamma(\kappa^0 - 1) = \mu \) so that

\[
X(z) = \frac{1}{1 + \mu - \mu z}
\]

which can be expanded to give

\[
R(n) = \mu^n[1 + \mu]^{-[n+1]}.
\]

We shall use these formulae in all subsequent calculations. It is worth noting, however, that a slightly different result would be obtained when the secondary emission is due to photons, since the probability of the production of a photon is independent of the probability of the production of a new electron in the gas. A somewhat larger modification arises if we allow for the change in ionization probability with electron energy. This more realistic model, as we have seen, narrows the distribution of avalanche sizes. As an example, consider the extreme case when the avalanche consists of exactly \( \mathrm{exp} \) (\( \kappa \)) electrons; \( R(n) \) is then simply the binomial distribution written above. Since \( \exp \) (\( \kappa \)) is usually large, this approximates very closely to the Poisson distribution

\[
\frac{e^{-\mu} \mu^n}{n!}.
\]

3.4 The number of avalanches in consecutive generations

Let \( R(m,n) \) be the probability of having \( n \) avalanches (i.e. \( n \) cathode electrons) in the \( k^{th} \) generation, given that the \((k-1)^{th}\) generation had \( m \) avalanches. The result of the last section gives us immediately

\[
R(1,n) = \mu^n[1 + \mu]^{-[n+1]}.
\]

with generating function

\[
V(1,z) = (1 + \mu - \mu z)^{-1}.
\]

Since each of the \( m \) given avalanches produces secondary electrons independently of the others,

\[
V(m,z) = (1 + \mu - \mu z)^{-m}
\]

and so

\[
R(m,n) = \frac{(m + n - 1)!}{(m - 1)! n!} \frac{\mu^n}{(1 + \mu)^{m+n}}.
\]

The mean of this distribution is \( \mu m \), and the standard deviation \((\mu m(1 + \mu))^{1/2}\). The relative width \( \sigma/\mu \) can be quite small when \( \mu m \) exceeds a few hundred, so that non-statistical calculations become sufficiently accurate.

3.5 The number of avalanches in the \( k^{th} \) generation

Let us number the generations 0, 1, 2, ..., \( k \), and suppose that there are \( m \) electrons for \( k = 0 \). Let \( P_m(n,k) \) be the probability of having \( n \) avalanches in the \( k^{th} \) generation and
let $V_m(z,k)$ be the corresponding generating function. Then the result of the last section may be written:

$$P_n(n,1) = \frac{(m+n-1)!}{(m-1)! n!} \frac{\mu^n}{(1+\mu)^{m+n}}, \quad V_n(z,1) = (1 + \mu - \mu z)^{-n}.$$ 

To find the distribution of avalanches at $k = 2$, we must find the probability that $m$ avalanches lead to $r$ in one generation, that $r$ lead to $n$ in the next generation, and sum over all values of the intermediate number $r$. Thus

$$P_n(n,2) = \sum_{r=0}^{m} P_n(r,1) P_r(n,1)$$

and the corresponding generating function is

$$V_m(z,2) = \sum_{n=0}^{\infty} z^n \sum_{r=0}^{m} P_n(r,1) P_r(n,1) = \sum_{r=0}^{m} P_n(r,1) [1 + \mu - \mu z]^{-r}$$

$$= V_n\left(\frac{1}{1+\mu - \mu z}, 1\right) = \left(1 + \mu - \frac{\mu}{1+\mu - \mu z}\right)^{-m}.$$ 

Similarly, $V_m(z,3)$ is obtained by replacing $z$ by $(1 + \mu - \mu z)^{-1}$ in $V_m(z,2)$, and so on. We can show by induction that

$$V_m(z,k) = \{f_k(z)\}^n,$$

where

$$f_k(z) = \frac{(1 - \mu^k) - \mu z(1 - \mu^{k-1})}{(1 - \mu^{k+1}) - \mu z(1 - \mu^k)}.$$

3.6 Application to an experiment

for measuring the sparking voltage

As an example, let us consider the probability $P_m(0,k)$ that a discharge, initiated by $m$ electrons, should have died out in the $k^{th}$ generation; that is, by time $kr$ approximately. Putting $z = 0$ in the generating function

$$P_m(0,k) = \left(\frac{1 - \mu^k}{1 - \mu^{k+1}}\right)^n \quad \text{provided } \mu + 1.$$ (IV.3)

If $\mu = 1$, this expression becomes indeterminate, so we must evaluate the limit as $\mu \to 1$. This gives

$$P_m(0,k) = \left(1 + \frac{1}{k}\right)^{-n} \quad (\mu = 1).$$ (IV.4)

Expressions (IV.3) and (IV.4) show that for $\mu \leq 1$, $P_m(0,k) \to 1$ as $k \to \infty$. Therefore for voltages less than or equal to the breakdown voltage $V_{\delta}$, the discharge will always go out if left long enough. Even when $\mu > 1$, there is a finite probability of the discharge going out at any finite time.

We may contrast this behaviour with that forecast by a non-statistical calculation. According to such a calculation, there will be exactly $m \mu^k$ avalanches flowing in the $k^{th}$ generation. It is only when $\mu = 1$ that the current remains constant. For $\mu > 1$ it will
grow exponentially, so there is no chance of the discharge dying out. For \( \mu < 1 \) it decreases exponentially, and will fall to one avalanche after a time given by

\[
k = \frac{\ln m}{\ln (1/\mu)} = \frac{\ln m}{\epsilon} \quad \text{if } \mu \text{ is near to } 1.
\]

At a slightly later time the current will have died out. Thus if \( k = 10^6 \) and \( m = 10^3 \), the current will die out for all values of \( \epsilon \) greater than \( 10^{-5} \ln (10^3) = 10^{-5} \), that is, for all voltages less than \( V_S = \Delta V \), where \( \Delta V \) is a very small fraction of the voltage applied. These calculations assume, of course, that there is no external supply of electrons and no device for stabilizing the current. It is because of such experiments (usually described from a non-statistical point of view) that the breakdown voltage is considered to be a critical voltage.

When statistical fluctuations are taken into account, we see immediately that the value \( V_S \) is not a critical voltage in the above sense. For \( \mu \) very small (i.e. \( V << V_S \)) there is a very small but finite probability that the discharge is still running, while for large \( \mu \) (\( V >> V_S \)) the discharge is almost certainly still running. We may claim, however, that if the probability of the discharge still running changes from, say, 0.001 to 0.999 over a very small range of voltage, then \( V_S \) is a critical voltage for all practical purposes.

Let us investigate the variation of the probability as a function of voltage for a particular experimental case. We consider the case of hydrogen [Jones and Llewellyn Jones\(^{23}\)] at \( E/p = 140 \) V cm\(^{-1}\) Torr\(^{-1}\), with \( V_S = 350 \) V, \( d = 1.55 \) cm, \( p = 1.6 \) Torr. The dependence of \( \alpha \) on the field is given by

\[
\alpha/p = 3.3 \exp \left( -88.9 \frac{p}{E} \right),
\]

which enables us to show that \( d u/dV = 0.008089 \), assuming that there is no change in the secondary coefficient. The electron current at the cathode at \( t = 0 \) is in the range \( 10^{-7} \) to \( 10^{-11} \) A, and \( \tau \) is of the order of 1 \( \mu \)sec, so that \( m \) lies in the range \( 10^2 \) to \( 10^6 \). We are concerned with measurements of the current made after a few seconds, so that \( k \) is of the order of \( 10^6 \). To illustrate the variation of \( P_m(0,k) \), we shall calculate the values of \( \mu \) which make the probability equal to 0.999 and 0.001, for various values of \( m \). These values of \( \mu \) are tabulated in the second and third columns of Table IV.1. The lower and upper limits of the voltage are given in the fourth and fifth columns.

<table>
<thead>
<tr>
<th>( m )</th>
<th>Value of ( \mu ) required to make ( P_m(0,10^6) = 0.001 )</th>
<th>Value of ( \mu ) required to make ( P_m(0,10^6) = 0.999 )</th>
<th>( V ) (volts) for ( P_m(0,10^6) = 0.001 )</th>
<th>( V ) (volts) for ( P_m(0,10^6) = 0.999 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1000.0</td>
<td>1.001</td>
<td>( \gg 500 )</td>
<td>350.12</td>
</tr>
<tr>
<td>10</td>
<td>1.996</td>
<td>1.0002</td>
<td>( \sim 473 )</td>
<td>350.025</td>
</tr>
<tr>
<td>( 10^2 )</td>
<td>1.072</td>
<td>1.00002</td>
<td>359</td>
<td>350.0025</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>1.007</td>
<td>1.0</td>
<td>350.87</td>
<td>350.0</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>1.0007</td>
<td>0.9999992</td>
<td>350.087</td>
<td>349.9999</td>
</tr>
</tbody>
</table>
We see from the table that, for m less than about 10, a large change in the voltage is required to increase the probability from 0.001 to 0.999. Thus, at small values of I₀, V₅ cannot be measured directly by finding the smallest voltage that will maintain a current for about one second, although its value could be inferred from the statistical distribution of the probability as a function of the applied voltage. If, however, I₀ is increased so that m = 10⁴, then with V = 350.087 volts, only one experiment in a thousand will fail to maintain a current for one second, while with V = 349.9999 volts, only one experiment in a thousand succeeds in maintaining a current for this time. Since the difference in voltage, 0.087 volts, is less than the experimental accuracy in measuring voltages in practice the probability distribution becomes indistinguishable from a step function, the discontinuity being at V₅, to within the experimental error.

We have taken τ to be 1 μsec, since this corresponds to the positive ion secondary process, which is the slower of the two processes occurring in that particular experiment. If we had used the undelayed photo-electric process, the resulting probability distribution would have been even sharper because a time of one second would correspond to nearly 10⁸ generation periods.

4. EXPERIMENTS OF LONG DURATION WITH EXTERNAL ILLUMINATION

When there is an external supply of cathode electrons, the discharge will never go out. It becomes practicable, therefore, to visualize an experiment lasting many seconds (or even hours), during which the current fluctuates about a mean value. One would expect in such an experiment that the choice of origin for the time scale would not affect the probability distribution for long times. If we persist in referring to the "kth generation", however, then we shall introduce an artificial distinction in that the generations near "time zero" are separate while those further from time zero are completely merged. We therefore consider a "generation" of avalanches to be all those emitted from the cathode during the time interval (k-1)τ to kτ, and we call the number of avalanches in the generation the "current number at time kτ". This definition makes the statistics of the current invariant under change of time origin.

Let the external illumination produce an average of h cathode electrons in each generation period. The probability of having a particular number p is given by the Poisson distribution e⁻ʰ²p²/p!, the generating function for which is e⁻²(1-1)²h. On the other hand, the probability that m avalanches in one generation lead to n in the next in the absence of external illumination is given by the generating function (1 + μ - μz)⁻ᵐ. Therefore, the probability that n avalanches are obtained due to both causes is given by the product of the two generating functions. That is,

\[ Vₙ(z,k;m,k-1) = \sum_{n=0}^{∞} z^n pₙ[n,k;m,k-1] = e^{k(z-1)²h} (1 + μ - μz)⁻ⁿ, \]

where \( pₙ[n,k;m,k-1] \) is the probability of having n avalanches in the kth generation given that there were m in the (k-1)th. Since the number of avalanches in the (k-1)th generation is also subject to statistics, we may write

\[ pₙ[n,k] = \sum_{m=0}^{∞} pₙ[n,k;m,k-1] pₙ[m,k-1], \quad (IV.5) \]
where \( P_h(n,k) \) and \( P_h(m,k-1) \) are now the respective probabilities without any restrictive conditions. Multiplying both sides of Eq. (IV.5) by \( z^n \) and summing,

\[
\sum_{n=0}^{\infty} z^n P_h(n, k) = \sum_{n=0}^{\infty} P_h(m, k - 1) \sum_{n=0}^{\infty} z^n P_h(n, k; m, k - 1)
\]

\[
= \sum_{n=0}^{\infty} P_h(m, k - 1) e^{(z-1)h}(1 + \mu - \mu z)^{-m} = e^{(z-1)h} \sum_{n=0}^{\infty} P_h(m, k - 1) \omega^n
\]

where \( \omega = (1 + \mu - \mu z)^{-1} \). Similarly,

\[
\sum_{n=0}^{\infty} P_h(m, k - 1) \omega^n = e^{(\omega-1)h} \sum_{n=0}^{\infty} P_h(r, k - 2) (1 + \mu - \mu \omega)^{-r}
\]

or, using \( V_h(z, k) \) for the generating function,

\[
V_h(z, k) = e^{(z-1)h} V_h\left(\frac{1}{1 + \mu - \mu z}, k - 1\right)
\]

\[
= \exp \left\{ (z-1)h + \left(\frac{1}{1 + \mu - \mu z} - 1\right) h \right\} V_h\left[\frac{1}{1 + \mu - [\mu/(1 + \mu - \mu z)]}, k - 2\right]
\]

\[
= \exp \{ h \cdot g_h(z) \} V_h[f_k(z), 0]
\]

where \( f_k(z) \) can be shown by induction (as in Section 3.5) to be

\[
\frac{(1 - \mu^k) - \mu(1 - \mu^{k-1})}{(1 - \mu^{k+1}) - \mu(1 - \mu^k)},
\]

while \( g_k(z) \) is the sum

\[
\sum_{\sigma=0}^{k-1} \{ f_\sigma(z) - 1 \}
\]

Putting \( a = (1 - \mu z)/(1 - z), \omega = \mu^k \), the sum may be approximated by the corresponding integral expression. Thus

\[
g_k(z) = \sum_{\sigma=0}^{k-1} \frac{(u-1)\mu^k}{a - \mu \cdot u^k} = \int_{1}^{\mu-1} \frac{\mu-1}{\ln \mu - \ln(1-a/\mu)} - \frac{\mu-1}{\ln \mu} \frac{d\omega}{\mu \cdot u^k - 1-a/\mu}
\]

\[
= \frac{1-\mu}{\mu \ln \mu} \ln \left\{ \frac{\mu^{k-1} - a/\mu}{1 - a/\mu} \right\}
\]

\[
= \frac{1-\mu}{\mu \ln \mu} \ln \left\{ n_\sigma - z(n_\sigma - 1) \right\},
\]

where

\[
n_\sigma = \frac{1-\mu^k}{1-\mu}.
\]

If \( n = 0 \) at time zero, then

\[
V_h(z, k) = \exp \{ h g_h(z) \} = \left\{ n_\sigma - (n_\sigma - 1)z \right\}^{-n}, \quad \text{(IV.7)}
\]
where
\[ \eta = \varepsilon h / (\mu \ln \mu) . \]

We should expect this result to be true only for \( k \) large compared with unity, because of the approximation involved in replacing the sum by an integral. We obtain a mean value for \( n \)
\[ \bar{n} = \frac{dV}{dz} \bigg|_{z=1} = \eta n_0 + 1 = \frac{\mu (1 - \mu^{k-1})}{1 - \mu} \cdot \frac{h (1 - \mu)}{\mu \ln \mu} \approx h \frac{1 - \mu^{k-1}}{1 - \mu} , \]
provided that \( \mu \) is near to 1.

Now it is possible to calculate the mean directly from Eq. (IV.6) (that is, without replacing the sum by an integral). Differentiating both sides with respect to \( z \),
\[ \sum_{n=0}^{\infty} nz^{n-1} p_{n}(n, k) = \sum_{n=0}^{\infty} \left( h + \frac{\mu m}{1 + \mu - \mu z} \right) e^{(z-1)h} (1 + \mu - \mu z)^{-n} P_{n}(m, k-1) . \]

Putting \( z = 1 \),
\[ \bar{n}(k) = \sum_{n=0}^{\infty} (h + \mu m) P_{n}(m, k-1) = h + \mu \bar{n}(k-1) . \]

If we imagine a series of experiments, each starting with \( n = 0 \) at \( k = 0 \), then the mean current number after successive generation periods is
\[ \bar{n}(1) = h, \quad \bar{n}(2) = h + \mu h, \quad \bar{n}(3) = h + \mu (h + \mu h) \]
\[ \bar{n}(k) = h(1 + \mu + \ldots + \mu^{k-1}) = h \frac{1 - \mu^k}{1 - \mu} . \]

Thus the distribution (IV.7) will yield the correct mean value if \( n_0 \) is given the slightly different value
\[ n_0 = \frac{1 - \mu^{k+1}}{1 - \mu} . \]

By differentiating Eq. (IV.6) a second time, it is possible to determine the standard deviation of the current fluctuations. It is easier, however, to differentiate Eq. (IV.7) twice (with the corrected value for \( n_0 \)) provided that \( k \) is large compared with unity. Thus
\[ \frac{\partial^2 V}{\partial z^2} = \eta (\eta + 1) [n_0 - 1]^2 \{n_0 - (n_0 - 1)z\}^{-n+2} \]
\[ \sigma^2 = \eta (\eta + 1) [n_0 - 1]^2 - \bar{n}(n_0 - 1) = n_0 \bar{n} = \frac{h}{(1 - \mu)} \frac{1 - \mu^k}{1 - \mu} \frac{(1 - \mu^k) (1 - \mu^{k+1})}{(1 - \mu)^2} \]
\[ \sigma = h \frac{1 - \mu^k}{1 - \mu} = h \frac{1}{2} \bar{n}, \quad \text{for} \ k \gg 1 . \]

This expression gives the standard deviation as a function of time, or in the steady state.

Another result which we shall need later is the probability distribution, the mean and standard deviation of the current number at time \( k \) given that it had a particular value \( n_t \) at time zero.
\( V_h(z, k) \) is then the product of two generating functions: \((n_\theta - (n_\theta - 1)z)^{-n_\theta}\) due to the external illumination; and \((f_k(z))^{n_1}\) due to the avalanches present at \(k = 0\). We note that \(f_k(z)\) may be written in the simpler form

\[
\frac{(n_\theta - 1) - (n_\theta - \mu - 1)z}{\mu n_\theta - \mu(n_\theta - 1)z}
\]

and so the required distribution has the generating function

\[
V_h(z, k) = \mu^{-n_1}[(n_\theta - 1) - (n_\theta - \mu - 1)z]^{n_1} \{n_\theta - (n_\theta - 1)z\}^{-(n_1+n)}.
\]  

(IV.8)

The derivative of this with respect to \(z\) is

\[
\mu^{-n_1}[(n_\theta - 1) - (n_\theta - \mu - 1)z]^{n_1-1} \{n_\theta - (n_\theta - 1)z\}^{-(n_1+n+1)}
\]

\[
[n_\theta(1 + n_\theta(\mu - z)) + n[(n_\theta - 1)(n_\theta - zn_\theta + \mu z) + 1 - z]].
\]

Putting \(z = 1\) we obtain the mean value

\[
\bar{n} = n_\theta \mu^k + h \frac{1 - \mu^k}{1 - \mu}.
\]  

(IV.9)

This is what we should expect, since it shows the mean current due to the \(n_1\) initial avalanches, and the mean current due to the illumination, changing independently. Differentiating again, and putting \(z = 1\)

\[
V_h''(1, k) = n_\theta(n_\theta - 1) \left(1 - \frac{n_\theta - 1}{\mu}\right)^2 \{n_\theta + n\}(n_\theta + n + 1)(n_\theta - 1)^2
\]

\[
+ 2n_\theta(n_\theta + n)(n_\theta - 1) \left(1 - \frac{n_\theta - 1}{\mu}\right).
\]

Subtracting \(\bar{n}(\bar{n} - 1)\) we get the square of the standard deviation

\[
\sigma^2 = n_\theta \frac{n_\theta - 1}{\mu^2} (\mu^2 n_\theta + \mu - n_\theta + 1) + n_\theta(n_\theta - 1)
\]

so that \(\sigma\) is, approximately,

\[
\left[\frac{\mu}{c} (1 - \mu^k) \left\{n_\theta \mu^{k-\nu}(1 + \nu) + \frac{h}{c} (1 + \mu^{k+\nu})\right\}\right]^\frac{1}{2}.
\]  

(IV.10)

4.1 The statistics of breakdown times, with \(\mu > 1\)

If \(\mu > 1\) and \(h \neq 0\), the discharge gap will always break down eventually. We will consider the onset of breakdown as the instant at which the current number reaches some critical value \(N\). The value assigned to \(N\) is in practice the number of avalanches for which the applied field becomes appreciably distorted by space charge. As soon as \(n\) reaches \(N\), amplification proceeds at a greater rate, and breakdown follows with very little further delay. It does not matter if we use a slightly different value for \(N\) -- we shall see that the final result involves \(N\) logarithmically -- because the statistical fluctuations are almost completely smoothed out as soon as \(N\) exceeds a few hundred.
Expanding Eq. (IV.7), we obtain the probability of a current number \( n \) at a time \( k \):

\[
P_B(n, k) = \left( \frac{n - 1}{n_0} \right)^n n_0 e^{-\eta} \frac{n}{n_0} \frac{\Gamma(n + n)}{n! \Gamma(n)} .
\]

Since we are interested in large \( n \), this approximates to

\[
\frac{1}{n_0 \Gamma(n)} e^{-n/n_0} \left( \frac{n}{n_0} \right)^{n-1}.
\]

If we change from the discrete variable \( n \) to the continuous one \( y = n/n_0 \) and put \( dy = dn/n_0 = 1/n_0 \), then the probability per unit range of \( y \) is

\[
\frac{1}{\Gamma(n)} e^{-y} y^{n-1}.
\]

It follows from the definition of the gamma-function that

\[
\int_0^\infty P_B(y, k) \, dy = \frac{1}{\Gamma(n)} \int_0^\infty e^{-y} y^{n-1} \, dy = 1 .
\]

The probability that at any time \( k \) the current number exceeds the critical value \( N \) is

\[
\frac{1}{\Gamma(n)} \int_{N/n_0}^\infty e^{-y} y^{n-1} \, dy = \frac{\Gamma(n, N/n_0)}{\Gamma(n)} ,
\]

where \( \Gamma(n, N/n_0) \) is the incomplete gamma-function, as defined by the integral.

We interpret Eq. (IV.11) as the probability \( P_B(k) \) that the gap has already broken down by the time \( k \). The probability that the discharge has not broken down by the time \( k \) may be written

\[
1 - P_B(k) = \frac{\Gamma(n, N/n_0)}{\Gamma(n)}
\]

where

\[
\gamma(a, x) = \int_0^x e^{-t} t^{a-1} \, dt .
\]

This result is rather difficult to illustrate graphically, since it involves four independent variables \( k, \mu, N, \) and \( n \). We may reduce them to two by writing

\[
x = -\ln \left( \frac{N}{n_0} \right) = -\ln \left( \frac{N(\mu - 1)/[\mu^k - 1]}{\mu - 1} \right) ,
\]

which, for \( \mu^k \) large compared with unity, becomes

\[
K \ln \mu - \ln \{N(\mu - 1)\} .
\]

Therefore with \( x = k \ln \mu - \ln \{N(\mu - 1)\} \) used as a measure of time, the probability that the gap has not broken down becomes

\[
\gamma(n_0 e^{-x}/\Gamma(n)) .
\]
Legler\textsuperscript{26)} has calculated \(1 - P_B(k)\) as a function of \(x\) and \(k\), and his results are illustrated in Fig. IV.2.

It can be seen that the probability drops rapidly when the illumination is strong, while there is a greater spread in breakdown times for the case of weak illumination. In fact, the probability distribution for the case of weak illumination may be deduced by calculating the mean delay before the appearance of a primary electron (and assuming a fixed probability that a single electron can lead to breakdown). The expected exponential behaviour is seen in the graphs.

4.2 The statistics of the Townsend discharge

When \(\mu < 1\) and \(h \neq 0\), we have the case of the Townsend discharge, which has been fully discussed in Chapter II, Section 2. Using the notation of the present chapter, the current number is seen to reach a steady-state mean value of \(h/(1 - \mu)\), corresponding to a total current of

\[
\frac{h e^{\nu d}}{1 - \gamma (e^{\nu d} - 1)}
\]
electrons per generation period. If \(\mu \ll 1\), the secondary processes have negligible effect, and we have simply a succession of primary avalanches. The case when \(\mu\) is near to 1 will therefore be of the greatest interest, and we may often simplify the formulae considerably by assuming that \(\epsilon(\cdot 1 - \mu)\) is small compared with 1.

The results of the last section which are applicable to the steady state are:

Average current number

\[
\bar{\eta} = \frac{h}{\epsilon}
\]

Standard deviation

\[
\sigma = \frac{h^{1/2}}{\epsilon} = \frac{\bar{\eta}}{h^{1/2}}
\]

Probability of current number \(n\)

\[
P_h(n, \infty) = \frac{1}{\epsilon ! ^{\eta}} e^{-\eta (\epsilon n)^{\eta - 1}}
\]

with generating function

\[
V_h(z, \infty) = \left(1 - \frac{\mu z}{\epsilon}\right)^{-\eta}
\]

The result \(\sigma/\bar{\eta} = h^{-1/2}\) indicates that the fluctuations can be quite large. For example, if \(\tau = 1\) usec, and a cathode current of \(10^{-13}\) A is maintained by external illumination, then \(h = 10^{-6} \times 10^{-13} / 1.6 \times 10^{-19} = 0.625\). The standard deviation of the current fluctuations is therefore slightly greater than the mean. If the externally maintained current is increased to \(10^{-12}\) A, then \(h = 6.25\), and the fluctuations amount to 40% of the mean.
4.3 The frequency of the fluctuations

One should not conclude from the above results that measurements on a Townsend discharge will give a wildly fluctuating reading on a meter; whether or not this occurs will depend on the frequency of the fluctuations relative to the pass-band of the amplifying and measuring system.

If the current instantaneously reaches some large value \( n_1 \), then the "fluctuation current" \( n - \bar{n} \) decreases with time. Its average value a time \( k \) later is \( (n_1 - \bar{n}) \mu^k \). One might say that when \( \mu^k = \frac{1}{2} \), then a half of the "memory" of the fluctuation has been lost. The fluctuations therefore have a period of order \( k_f \), where \( k_f \ln \mu = \ln \frac{1}{2} \). If \( \mu \) is near to 1, \( k_f = \frac{0.7}{\epsilon} \), or the frequency is of the order \( \epsilon/\tau \).

Davidson\(^{24}\) has given a more precise explanation of the frequency \( \epsilon/\tau \). It is possible, however, to extend his method so as to obtain the spectrum of the fluctuations [Evans\(^{27}\)]. We start by obtaining the autocorrelation function of the current. This is defined to be

\[
g(k) = \sum_{n_1=0}^{\infty} \sum_{n=0}^{\infty} F(n_1, k, n) (n_1 - \bar{n}) (n - \bar{n})
\]

where \( F(n_1, k, n) \) is the probability of finding current numbers \( n_1 \) and \( n \) separated by an interval \( k \). The probability \( F(n_1, k, n) \) is the product of two probabilities:

\( P_h(n_1, \infty) \) that the current is \( n_1 \) at any large time, and

\( P_h(n, k; n_1) \) that the current is \( n \) at time \( k \) given that it was \( n_1 \) at time zero.

Both these probabilities have been evaluated [Eqs. (IV.12) and (IV.8)].

We may obtain \( F(n_1, k, n) \) from a generating function with two dummy variables, defined as follows:

\[
V_2(z_1, k, z) = \sum_{n_1, n=0}^{\infty} F(n_1, k, n) z_1^{n_1} z^n = \sum_{n_1, n=0}^{\infty} z_1^{n_1} P_h(n_1, \infty) P_h(n, k; n_1) z^n
\]

\[
= \sum_{n_1=0}^{\infty} z_1^{n_1} P_h(n_1, \infty) \left( n_0 - (n_0 - 1)z \right)^{-\eta} \left( \frac{[n_0 - (n_0 - 1)z]}{\mu n_0 - \mu (n_0 - 1)z} \right)^{n_1}
\]

\[
= e^{\eta} \{ n_0 - (n_0 - 1) \left[ z + z_1 \right] + (n_0 - \mu - 1)zz_1 \}^{-\eta}
\]

Thus the autocorrelation function is

\[
g(k) = \sum_{n_1=0}^{\infty} F(n_1, k, n) (n_1 - \bar{n}) (n - \bar{n})
\]

\[
= \left[ \frac{\partial^2 V_2}{\partial z \partial z_1} \right]_{z=z_1=1} - \bar{R}^2 = \frac{h}{e^\epsilon} \mu^k
\]

\[
= h e^{-2} e^{-c k}, \quad \text{if } \epsilon \text{ is small.}
\]

Since \( V_2(z_1, k, z) \) is symmetric in \( z \) and \( z_1 \), we would get the same result if the value \( n_1 \) occurred a time \( k \) after the value \( n \). Therefore, we should write

\[
g(k) = h e^{-2} e^{-c k}.
\]

(IV.13)
The value of \( g(0) \) is \( h/\varepsilon^2 = \sigma^2 \), as it should be. The function \( g(k) \) decreases monotonically as \( k \) increases, a behaviour which lies between that of a pure sine wave [for which \( g(k) \) would oscillate] and that of a completely random signal [for which \( g(k) \) would be a \( \delta \)-function].

Considering a single experiment of long duration, we define \( x(k) \) to be equal to the fluctuation current \( n(k) - \bar{n} \) over the range \(-K < k < K\), and to be zero outside this range. Now \( x(k) \) may be written as the Fourier integral

\[
x(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(\omega) e^{i\omega k} \, d\omega,
\]

where

\[
f(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} x(k) e^{-i\omega k} \, dk.
\]

Now \( f(\omega) \), being a complex number, cannot be determined for a particular value of \( \omega \) by making a single measurement -- for example, the r.m.s. voltage produced by a tuned amplifier -- since some information concerning the phase is also required. A more important quantity for experimental application is the power spectrum, defined as

\[
\rho(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) f^*(\omega) \, d\omega,
\]

which may be interpreted as the mean-square fluctuation current per unit range of \( \omega \). Using the convolution theorem for Fourier integrals, we obtain the pair of equations

\[
g(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} \rho(\omega) e^{i\omega k} \, d\omega, \quad \rho(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} g(k) e^{i\omega k} \, dk.
\]

Thus from Eq. (IV.13)

\[
\rho(\omega) = \frac{2h(2\pi)^{-1}}{c(\varepsilon^2 + \omega^2)}.
\]

This result does not exhibit any particularly striking behaviour, such as a peak for some finite \( \omega \). There is a broad maximum at \( \omega = 0 \), and the spectrum falls to half its peak value at \( \omega = \varepsilon \). The frequency \( \varepsilon/(2\pi\tau) \) may be called the half-width of the distribution, and for typical experimental conditions it will be of the order \( 10^2 \) to \( 10^4 \) Hz.

Our expression for \( \rho(\omega) \) will be expected to be accurate for \( \omega << 1 \), since \( g(k) \) is accurate for \( k >> 1 \) [remembering the approximations used to obtain Eq. (IV.7)]. We should not expect the formula to give us any valid information for \( \omega \geq 1 \).

The power spectrum [or \( x(k) \) itself] is best observed by using a photomultiplier to view the light emitted from the discharge, since there is no complication due to the positive ion current. The power spectrum of the ion current, and the response of a tuned circuit to the signal, have also been considered[27].

4.4 Davidson's continuous-change model

for the Townsend discharge

Consider what happens in a Townsend discharge during an infinitesimal time interval \( dk \). There is a probability \( h \, dk \) that a new avalanche will start as a result of external illumination of the cathode. If we let \( An \, dk \) be the probability that any of the \( n \) avalanches
flowing will produce a secondary electron, then we have a situation exactly analogous to that of Section 2.1. Now, however, we must assume that there is also a probability $B_n \, dk$ that one of the avalanches will disappear from the gap (by reaching the anode). $A$ and $B$ are constants, and we should expect, at first sight, that $A = \mu$ and $B = 1$. If $\mu \neq 1$, however, the avalanches are not distributed uniformly across the gap, and consequently the appropriate value for $A$ is

\[
\text{mean rate of production of secondaries at instant } k = \mu \int_0^k \mu^k \, dk \\
\text{mean rate of production over the interval } k - 1 \text{ to } k = \mu \int_0^1 e^{\ln \mu} \, dk = \frac{\mu \ln \mu}{1 - \mu}.
\]

Similarly, $B = (\ln \mu)/(1 - \mu)$.

It is of course not strictly correct to assume that the current grows at a mean rate $\mu^k$, as this assumption removes any fluctuations on a scale smaller than one generation period. Since our model is not capable of allowing for these fluctuations, their omission should not make any difference.

The above calculation shows that $A$ and $B$ could be evaluated (not very accurately, perhaps) from first principles. Since we have already worked out the statistics of current fluctuations using a discrete-generation model, it will be more satisfactory to leave $A$ and $B$ as undetermined constants, and evaluate them by comparison with the earlier result (IV.7).

By a familiar argument, we obtain the following differential equation for the probability $P_h(n,k)$ of having $n$ avalanches at time $k$

\[
\frac{\partial P_h}{\partial k} = A \left[ (n-1) P_h(n-1,k) - nP_h(n,k) \right] + B \left[ (n+1) P_h(n+1,k) - nP_h(n,k) \right] + h \left[ P_h(n-1,k) - P_h(n,k) \right].
\]

Introducing the generating function

\[
V_h(z,k) = \sum_{n=0}^{\infty} z^n P_h(n,k)
\]

gives

\[
\frac{\partial V_h}{\partial k} = (Az - B) \left[ z - 1 \right] \frac{\partial V_h}{\partial z} + h(z-1)V_h.
\]

Comparing this with the equation

\[
dV_h = dk \frac{\partial V_h}{\partial k} + dz \frac{\partial V_h}{\partial z}
\]

yields the auxiliary equations

\[
\frac{dV_h}{hV_h} = \frac{dz}{B - Az} = (z - 1) \, dk,
\]
two independent solutions of which are

\[ V_h \left( z - \frac{B}{A} \right)^{h/A} = \text{constant} \quad \text{and} \quad \frac{z - B/A}{z - 1} e^{(A-B)k} = \text{constant}. \]

Putting one constant equal to an arbitrary function of the other, we get the general solution for \( V_h \):

\[ V_h = \left( z - \frac{B}{A} \right)^{-h/A} F \left( \frac{z - B/A}{z - 1} e^{(A-B)k} \right). \]

When the current is produced entirely by external illumination,

\[ V_h \equiv 1 \]

at \( k = 0 \), so that

\[ 1 = \left( z - \frac{B}{A} \right)^{-h/A} F \left( \frac{z - B/A}{z - 1} \right). \]

Solving for \( F \),

\[ F(u) = \left( \frac{1 - \frac{B}{A}}{u - 1} \right)^{h/A} \]

and so

\[ V_h = \left\{ \frac{B - A e^{(B-A)k}}{B - A} - \frac{A - A e^{(B-A)k}}{B - A} \right\}^{-h/A}. \]

In order to make this agree with formula (IV.7) we must have

\[ \frac{h}{A} = \eta, \quad \frac{A}{B} = \mu, \quad B - A = \ln \mu. \]

These three equations are consistent, and yield

\[ A = -\mu(\ln \mu)/\epsilon, \quad B = -((\ln \mu)/\epsilon), \]

which are just the values which we tentatively assigned to \( A \) and \( B \) at the start of this section.

With the values of \( A \) and \( B \) derived above, the continuous change model is therefore exactly equivalent to that based on discrete generations. It does not introduce any further inaccuracy, nor of course does it remove any inaccuracy inherent in the discrete model.

4.5 The probability of breakdown, \( \mu < 1 \)

As in Section 4.1, we consider breakdown to follow as soon as the current number reaches a critical value \( N \). The calculation of breakdown probability for \( \mu < 1 \) presents an entirely different mathematical problem from that considered by Legler \(^{21,26}\). When \( \mu > 1 \), the mean current increases exponentially and the fluctuations produce a scatter in the breakdown times about the mean value. We remarked earlier that the effect of the fluctuations is felt for current numbers less than a few hundred, and that for larger currents the rate of growth will be very close to the average rate. Thus, if the current reaches the
value $N$ at any instant, it has almost certainly reached that value for the first time and there is very little chance of it decreasing again (whether or not a "sparking process" is assumed to be acting).

On the other hand, if $\mu < 1$, the mean current is constant and considerably less than $N$. Sparking would not occur at all if statistical fluctuations were absent. The current can only reach the value $N$ during a large fluctuation and (in the absence of a sparking process) will very soon fall again towards the mean value. When a sparking process is assumed to be acting, the situation is radically changed, since the current number cannot now fall once it has reached $N$, and so the whole probability distribution will be altered.

Davidson\(^{2n}\) has considered this problem and has obtained an expression for the mean time to breakdown. He gives an accurate argument, based on the continuous-change model, and an approximate argument which leads to the same result. We shall reproduce here the approximate argument.

Consider a record of the current flowing during a long time, and suppose that there is no sparking process. Then there will be intervals of time of average length $\tau_1$, during which the current exceeds $N$, separated by longer intervals of average length $\tau_2$, during which the current is less than $N$. If we commence the experiment at an arbitrary point, and assume that a sparking process is active, then it is clear that we will have to wait a time of order $\tau_1$ for breakdown to occur. But

$$\frac{\tau_1}{\tau_1 + \tau_2} = \int_0^\infty P_h(n, \infty) \, dn \approx \frac{1}{\Gamma(n)} (cN)^{n-1} e^{-cN}$$

and so a knowledge of $\tau_2$ is equivalent. To estimate $\tau_2$ we use two of our earlier results. Equation (IV.9) shows that if the current is $N$ at a particular instant, its mean value after a time $k$ is

$$\frac{h}{\varepsilon} + \left(N - \frac{h}{\varepsilon}\right)\frac{n}{\mu}_k.$$ 

Equation (IV.10) shows that if the current is $N$ at a particular instant, then the standard deviation after a time $k$ is

$$\left[\frac{n}{\mu} \left(1 - n^k\right) \left\{N \left(1 - n^k\right)(1 + \mu) + \frac{n}{\mu} \left(1 - n^{k+1}\right)\right\}\right]^{1/2}.$$ 

While the standard deviation exceeds $\left[N - (h/\varepsilon)\right](1 - n^k)$, there is an appreciable probability of the current being in the neighbourhood of $N$. The two expressions are equal after a time of order $1/\varepsilon^2$ which is therefore an estimate for $\tau_2$. Thus the time before breakdown is of order

$$\tau_1 = \frac{1}{\varepsilon} \Gamma(n) (cN)^{n-1} e^{cN} \text{ seconds.} \quad (IV.14)$$

Let us evaluate this time for a particular experimental case. As in Section 3.6 we choose the example of hydrogen at low pressure, with $V_s = 350 \, V$, $I_s = 10^{-18} \, A$, $\tau = 1.76 \times 10^{-6} \, \text{sec}$, $\varepsilon = 0.008089 \, \Delta V$, for small undervoltages $\Delta V$. The value of the current at which the effect of space charge becomes appreciable can be deduced from the static $(I,V)$
characteristic, which begins to descend from the Townsend plateau region at a current of about $10^{-5}$ A. We therefore take $N$ to be

$$6.25 \times 10^{18} \times 10^{-5} \times 1.76 \times 10^{-6} e^{-ad} = 1.03 \times 10^6$$

avalanches per generation.

Similarly, $N$ is found to be 0.85 electrons per generation.

If $\varepsilon$ is given the value $h/N = 8.22 \times 10^{-7}$, then the average current flowing in the gap will be equal to the critical current. Breakdown will follow immediately the voltage is applied, the required voltage being only $10^{-3}$ V below $V_s$. If the undervoltage is increased to $10^{-1}$ V, $\varepsilon$ becomes $8.22 \times 10^{-4}$ and Eq. (IV.14) gives the mean time to breakdown = 126 sec.

Now the experimental error in setting the applied voltage is much larger than $10^{-3}$ V -- in practice it is nearer to 0.1 V. For this case, Eq. (IV.14) gives the mean time delay equal to $10^{990}$ sec. Since this is immeasurably long, we conclude that the statistical fluctuations in the above example do not result in an appreciable probability of a spark occurring for voltages that are less than the breakdown voltage.

A similar calculation has been made for a case approximating to a one-centimetre gap in air at atmospheric pressure. It is shown that $\Delta V$ would have to be considerably smaller than 4 V (in 31 kV) before there is any appreciable probability of breakdown within a measurable time.

4.6 The statistics of a stabilized discharge

We found in Section 3.6 that when $\mu \leq 1$, a discharge initiated by a pulse of electrons will always go out eventually. Experiments are usually conducted, however, with a stabilized circuit, in which the discharge gap is supplied from a low-impedance source via a high resistance. If the current starts to increase, the voltage on the gap falls and $\mu$ decreases. Conversely, if the current decreases, $\mu$ will increase. The variation in $\mu$ is therefore such as to stabilize the current and voltage. We shall assume that the fluctuations in voltage are small enough to make the dependence of $\mu$ on voltage approximately linear.

In the non-statistical case, we can produce a gap potential $V_0$ (less than $V_s$) by setting the source potential $E$ equal to $V_0$ and the series resistance $R$ equal to zero. The current number $n$ then has the value $n_0 = h/e_0$, where $e_0 = e(V_0)$, and the current in the external circuit is $n_0$ avalanches per generation, or $n_0I_1A$, where $I_1 = 1.6 \times 10^{-19} e^{ad/\tau}$.

We could achieve the same gap potential and current by giving $R$ any positive value and increasing $E$ from $V_0$ to $V_0 + n_0I_1R$. When fluctuations are taken into account, the circuits are no longer equivalent, so we describe them as the unstabilized circuit and the corresponding stabilized circuit, respectively.

Since a large part of the current in the external circuit is due to positive ions, we may only apply the theory to secondary processes for which $\tau$ is greater than, or of the same order as, the ion transit time. We can then assume that the voltage on the gap at any time is a function of the current number flowing at that instant.

When the current number is $n$, the voltage on the discharge gap is $E - nI_1R = V_0 - mI_1R$, where $m = n - n_0$ is the fluctuation current. Assuming a linear variation of $\mu$ with $V$, we may write

$$\mu = \mu(V_0) + m \left( \frac{d\mu}{dV} \right)_{V_0} \left( \frac{dV}{dn} \right)_{n_0} = \mu_0 - bm = a - bn,$$

(IV.15)
where \( a = u_0 + bn_0, \) \( u_0 = \mu(V_0), \) and \( b = RI_1(d\mu/dV)V_0. \) The variation of \( I_1 \) with \( V \) will be comparatively unimportant.

Let \( P(n) \) be the probability that the current number is \( n \) at a time \( k. \) Then Davidson's continuous-change model leads immediately to the differential equation

\[
\frac{\partial P}{\partial k} = (n - 1) A(n - 1) P(n - 1) - nA[n] P(n) + (n + 1) B(n + 1) P(n + 1) - nB(n) P(n) + hP(n - 1) - hP(n),
\]

where

\[
A(n) = -\mu(\ln \mu)/\epsilon, \quad B(n) = -\left(\ln \mu\right)/\epsilon,
\]

and \( \mu \) is given by Eq. (IV.15).

We shall obtain the steady-state solution for the important case in which \( \epsilon \) is a small fraction. Thus \( A \) may be replaced by \( \mu, \) that is, \( A = a - bn, \) and \( B \) replaced by 1. Putting the left-hand side of Eq. (IV.16) equal to zero, and rearranging, we obtain

\[
(n + 1) P(n + 1) - \{h + a(n - 1) - b(n - 1)^2\} P(n) = nP(n) - \{h + a(n - 1) - b(n - 1)^2\} P(n - 1).
\]

Suppose the left-hand side of Eq. (IV.17) is called \( f(n). \) Then the right-hand side is \( f(n - 1) \) and, since these are equal, \( f(n) \) is a constant which is independent of \( n. \) We must take the constant to be zero since in regions of very small probability each term in \( f(n) \) is very small. Thus, writing \( \phi(n) \) for \( nP(n), \)

\[
\frac{\phi(n + 1)}{\phi(n)} = a - bn + \frac{h}{n}, \quad \text{or} \quad \frac{\phi(n + 1) - \phi(n)}{\phi(n)} = -\left(b + \frac{\epsilon_0}{n}\right) m.
\]

For very large \( n, \) Eq. (IV.18) predicts that \( \phi(n), \) and therefore \( P(n) \) becomes large and alternately positive and negative. Such impossible behaviour arises since \( \mu \) becomes negative in the approximate formula (IV.15) while, in practice, it must obviously be positive. For a sufficiently narrow distribution, however, \( \phi(n) \) will have fallen to a negligible value before equation (IV.15) gives any appreciable error in \( \mu, \) and so the most important part of the distribution will be correct.

From formulae (IV.18) we see that, for both the unstabilized circuit \( (b = 0) \) and the corresponding stabilized circuit, \( \phi(n) \) has a maximum at \( n = n_0 \) (that is, \( m = 0), \) which is the value of \( n \) that both circuits would have at all times, according to a non-statistical treatment. The maximum of \( P(n) \) is at a slightly different position, namely

\[
n = n_0 - \frac{bn_0 + \epsilon_0}{\epsilon_0}.
\]

If we replace \( \phi(n + 1) - \phi(n) \) by the approximation \( d\phi/dn, \) Eq. (IV.18) becomes

\[
\frac{d}{dn} \left\{ \ln \phi(n) \right\} = -\left(b + \frac{\epsilon_0}{n}\right)(n - n_0) = -\left(b + \frac{\epsilon_0}{n}\right) \left(n - \frac{h}{\epsilon_0}\right),
\]
the solution of which is

$$\phi(n) = n \phi(n) = C n^b \exp \left\{-\frac{1}{2} b n^2 - \left( \frac{b}{c_0} \right) n \right\}$$

(IV.19)

in which C has the value which makes $\int \phi(n) \, dn = 1$.

The approximation of writing $d\phi/dn$ for $\phi(n + 1) - \phi(n)$ causes little error in the important region near the maximum of the distribution, and it may be noted incidentally that, for very large n, Eq. (IV.19) has the correct property of being small and positive. Although this does not mean that it is necessarily correct (since it is derived from an approximate equation), it does mean that the integral defining C in Eq. (IV.19) may be evaluated over the limits 0 to $\infty$ -- exactly as it would be in an exact solution.

If the fluctuation current $m$ is small compared with $n_0$, the distribution approximates to a Gaussian

$$\phi(n) = \frac{C}{n_0} \exp \left\{-\frac{1}{2} \left( b + \frac{c_0}{n_0} \right) (n - n_0)^2 \right\},$$

(IV.20)

which has standard deviation $[b + (c_0/n_0)]^{1/2}$.

This will be the true standard deviation of the current fluctuations provided that it is small compared with $n_0$ -- say, equal to or less than $n_0/10$. That is

$$b \geq \frac{100 - h}{n_0}.$$  

4.7 The self-maintained stabilized discharge

The breakdown voltage $V_s$ is commonly measured by finding the voltage at which a continuous current will flow after the external illumination has been cut off. We have discussed such an experiment (Section 3.6), but only for an unstabilized circuit. We now turn to the stabilized case.

We first note that the problem cannot strictly be solved for the steady-state distribution when $h = 0$, since Eq. (IV.16) then only has the trivial solution $\phi(0) = 1; \phi(n) = 0$ for $n \neq 0$. In other words, even a stabilized discharge is bound to die out eventually. There is nothing to stop us making h as small as we like in the solution, however, provided we also reduce $c_0$ (by slightly raising the supply voltage) so as to keep $n_0 = h/c_0$ constant. The distribution (IV.20) then tends smoothly to the form

$$\phi(n) = \left( \frac{b}{2\pi} \right)^{1/2} \exp \left\{-\frac{1}{2} b (n - n_0)^2 \right\}.$$

If b is large enough (greater than about $100/n$) then there is a very small probability,

$$\frac{1}{2} \text{erfc} \left\{ \left( \frac{b}{2} \right)^{1/3} n_0 \right\},$$

that $n$ will be less than some small quantity (a small fraction of $n_0$, for example) at any instant.

Now $n$ must be smaller than this small value for at least one generation period before we are certain that the current has died out. Therefore, the lifetime of the discharge is
at least

$$\left(2nb\right)^{1/2} n_0 \exp\left(\frac{1}{2} b n_0^2\right)$$

generation periods.

If the series resistor is chosen \( \text{from Eq. (IV.15)} \) so as to make the standard deviation of the fluctuations less than \( n_0/10 \), then \( b = 100/n_0^2 \). Substituting in the last result shows that the lifetime of the discharge is at least \( 10^{20} \) generation periods. This is such a long time that we may conclude that the discharge is completely stabilized against dying out.
CHAPTER V

SOME APPLICATIONS OF INTEREST AT CERN

In this final chapter we discuss some applications of the theory of ionization growth to experimental situations of particular interest at CERN.

1. SPARK CHAMBERS

In the development of the narrow-gap spark chamber, the optimum pressure, electrode spacing, gas composition, overvoltage, and pulse-length were discovered largely empirically. With the present interest in streamer chambers, however, the requirements on the power supply and pulse circuitry become much more stringent, and much unnecessary effort and expense can be avoided by a proper consideration of the theory involved, supported by numerical computations.

1.1 The high-voltage spark

First we consider the phenomenon of rapid (~ 100 nsec) electrical breakdown between parallel plates, initiated by a pulse of electrons at the cathode, paying particular attention to the light emitted by the discharge. The development of the discharge has been studied extensively by Wagner, using an electronic image intensifier and streak camera\textsuperscript{16,17}. He investigated nitrogen (with a slight addition of methane) at 91 Torr and a gap spacing of 3 cm. Although the gas used, the pressure, and the gap spacing are not directly comparable with the parameters of spark chambers, we have chosen these experiments to test the theory because they are among the most detailed studies ever carried out. The photographs obtained by Wagner are a record of the light output from the discharge as a function of position and time, and their most prominent features are illustrated schematically in Fig. V.1.

At time zero (not normally recorded on the film), a flat-topped voltage impulse is applied to the electrodes of the discharge gap, and a group of electrons, generated earlier at the cathode by a light flash, starts to move towards the anode. When the resulting avalanche has travelled about half way across the gap, sufficient light is emitted to be recorded on the film. Since at that stage the number of electrons and ions is not sufficient to distort the field appreciably, the centre of the avalanche moves with the electron drift velocity appropriate to the uniform applied field. The avalanche, however, has a finite spread in the longitudinal direction owing to diffusion which occurs before and after the application of the electric field; and so, as time proceeds, electrons which lie some distance away from the centre of the avalanche begin to produce visible traces on the film as a result of excitation collisions in the gas. It is important to realize that the edges of the darkened area do not represent the motion of electrons -- whether the motion is
considered as a diffusion process or a "modified" drift process [Ward\textsuperscript{29,30}]. Thus even at this early stage, there is a possibility of misinterpreting the results.

At a later time, when the avalanche size has reached about $10^7$ electrons, the effects of field distortion can be seen. Not only does the upper edge of the exposed area show an increased velocity, but the region of maximum luminosity or brightness is also accelerated. This stage is sometimes called the "anode-directed streamer". In some cases the formation of the anode-directed streamer is accompanied by a change in the motion of the edge of the luminous region at the rear of the avalanche. This exhibits a deceleration of approximately the same magnitude as the acceleration observed at the front of the avalanche, and the resulting velocity may consequently be positive, zero, or negative. In the latter case it is often said to form a "cathode-directed streamer". The velocity of the cathode-directed streamer appears to increase when secondary electrons generated at the cathode arrive in the region of the primary avalanche.

It is clear that almost any behaviour could be explained by postulating a suitable spatial distribution of electrons in the avalanche. If the avalanche has a long "tail" -- due, for example, to a long time-constant in the decay of the light-flash which initiated the discharge -- then the density gradient in the tail will determine the velocity of the cathode-directed streamer. It is not possible, however, to explain the acceleration of the densest part of the avalanche in this way, and it is necessary to take into account the distortion of the field due to the space charge of the electrons and ions.

The method of calculation is fully described in Chapter III, and more specifically by Davies et al.\textsuperscript{31}). The equations of ionization growth are integrated by the method of characteristics and the electric fields are found by the cylindrical method (Chapter III, Section 1.2). The functional dependence of the ionization coefficients on the electric field was assumed to be the same as in pure nitrogen (Chapter III, Section 3.1)\textsuperscript{*)}.

\textsuperscript{*) Recent work carried out at the Department of Electrical Engineering, University of Strathclyde, Glasgow, has demonstrated almost identical results with pure nitrogen, so that the addition of methane is not vital to the experiment.
Fig. V.2 The results of Fig. III.8 re-plotted to show contour lines of equal light output (in arbitrary units) as a function of $x$ and $t$.

If space charge distortion is neglected, the distribution of light output as the avalanche propagates from the cathode to the anode will be identical in form with Fig. III.8. We may replot Fig. III.8 to show the isodensity contours in the $x$-$t$ plane for the light given out by the discharge (Fig. V.2). The maximum of the light intensity in the primary avalanche is propagated with velocity $W_e$ from the cathode to the anode, reaching the anode at about 153 nsec. Owing to the finite half-width of the Gaussian distribution, however, the contour lines represent points of equal intensity moving with a velocity slightly greater than $W_e$ at the front of the avalanche and slightly less than $W_e$ at the rear.

Since, as mentioned above, the electrons in the second generation have a practically constant density over a large part of the gap, the corresponding contour lines in Fig. V.2 will be parallel to the distance axis. A front of given luminosity will then appear to be propagated towards the cathode with an infinite velocity even though all the electrons and ions are still moving with their uniform field drift velocities.

Figures V.3 and V.4 show the distribution of light intensity between the electrodes and the corresponding isodensity contours when the effects of space-charge distortion are incorporated in the calculation. We note that the contour map, although similar to Fig. V.2, is modified in certain important respects. Until about 90 nsec the maximum of the light intensity in the primary avalanche travels with a velocity $W_e$, but at later times it is accelerated towards the anode where it attains a velocity of about $2W_e$. The time at which the maximum reaches the anode is thereby reduced from 153 nsec ($d/W_e$) to 115 nsec. This acceleration may be taken to represent the so-called "anode-directed streamer", and the calculated velocity at the anode is in good agreement with that measured by Wagner.

When the field distortion has become appreciable, the electric field and the secondary electron density are not constant in space; thus the contour lines are no longer parallel to the $x$-axis but have slopes of the order of a few $W_e$. A much more obvious feature,
Fig. V.3 Light output from the discharge for the data of Fig. III.9. The time in nanoseconds is indicated on the curves.

Fig. V.4 The results of Fig. V.3 re-plotted to show the contour lines of light output as a function of x and t.
however, is the movement towards the cathode of the maximum in the light output, and it is clear that we may identify this with the so-called "cathode-directed streamer".

Owing to the experimental difficulties involved, it is not possible to deduce reliable isodensity curves from the experimental streak photographs, so that a very subjective judgement is involved in comparing the latter with the present results. It is preferable, therefore, to use the converse approach -- that is, to simulate theoretical streak photographs on microfilm, and to compare these with the photographs published by Wagner.

The computer program which computes the ionization growth also produces values of the light output per unit length of the discharge $\Phi$ for each mesh point and each value of the time considered. Omitting an arbitrary scaling factor (involving the size of the optical components and the amplification of the electron image intensifier, etc.) we equate $\Phi$ to the excitation rate $a''(x,t)I_e(x,t)$. For our present purposes we may take the response of the photographic emulsion to be logarithmic, so that the probability $P$ of a given grain of emulsion being exposed will be a linear function of $\log_{10} \Phi$. For most emulsions $P$ goes from zero to unity as $\log_{10} \Phi$ increases by two.

In order to simulate the streak photographs, the computer scans a matrix of equally spaced points in the $x$-$t$ plane and, for each point, generates a random number in the range 0 to 1. If the random number is less than $P$ then a dot is plotted on microfilm at this point. Except for the tails at each end of the response curve, this procedure simulates exactly the process that occurs when a film is exposed to light. The resulting "photograph" will consist of a large number of dots, the number of dots per unit area being a linear function of $\log_{10} \Phi$. The apparent grain size may be adjusted by suitable choice of the matrix spacing and dot diameter. For the sake of greater clarity, we have chosen a slightly finer grain size than that achieved by Wagner.

Figures V.5 and V.6 show the streak photographs corresponding to the contour maps of Figs. V.2 and V.4. In the latter we see clearly the formation of the anode- and cathode-directed streamers. Wagner cut off the applied voltage at 120 nsec, so that his results only show the first part of Fig. V.6, the simulated and experimental photographs being almost identical.

The close agreement obtained indicates that the formation of the anode- and cathode-directed streamers can be satisfactorily explained in terms of the Townsend primary and

![Fig. V.5 Simulated streak photograph corresponding to the contour map of Fig. V.2.](image-url)
photon secondary processes, provided correct allowance is made for the space-charge distortion of the field.

The simulation procedure described above is obviously ideal for the analysis of spark chambers operating in the streamer mode\textsuperscript{12}, and at present the calculations are being extended to cases of more practical interest such as hydrogen at high pressures.

1.2 The track-following spark chamber

A wide-gap spark chamber can also operate in the track-following mode\textsuperscript{33}. The arrangement may be regarded as intermediate between the narrow-gap and the streamer chamber. It has the high light-output that one expects from a narrow-gap chamber, but it is more isotropic, allowing the observation of tracks inclined at up to 45° to the applied field. The lack of complete isotropy has stimulated interest in the streamer chamber, but the track-following mechanism still poses a large number of interesting theoretical problems.

The mode of action of a wide-gap spark chamber is determined mainly by the value of the primary ionization coefficient $\alpha$. Let $N$ be the number of electrons which an avalanche must contain in order to be visible. This number will vary for different gases but will be almost constant for a given gas, not varying appreciably with the pressure or the electric field. The track of a high-energy particle will develop into a series of avalanches, the spacing between them being determined mainly by the pressure. Let a typical spacing be $s$. Now an avalanche may be approximated by a dipole with charge $\pm e$ and length $1/\alpha$. The field at a distance $s$ is therefore proportional to $Ne/(\alpha s^3)$. If this quantity is comparable to the applied field, then there will be interaction between the avalanches and track-following (for sufficiently small angles of inclination) can occur. Otherwise, the avalanches will become visible as separate points or streaks of light. Therefore, track-following tends to occur for small $s$ (high pressure), small $\alpha$ (comparatively low field), and consequently a long voltage pulse is required to produce sufficient amplification. The streamer mode, on the other hand, requires either a lower pressure or a higher field (and therefore a shorter pulse). Since a low pressure produces a more diffuse track, it is usual to turn to a high field.

The problem of calculating the growth of an isolated avalanche, and of a series of avalanches arranged along a track, has been discussed in Ref. 33. A one-dimensional model
cannot be used for computing the track-following behaviour, and the program had to be
generalized to two or three dimensions. Using two-dimensional (cylindrical) geometry, the
behaviour of an isolated avalanche was calculated. The results showed the acceleration of
the front of the avalanche, as in Fig. III.9. The formation of a cathode-directed front,
however, was not observed because no provision was made in the program for the production
of secondary electrons. In addition, a comparatively larger mesh must be used in two
dimensions, and so the errors will reach unacceptable proportions at an earlier stage of
the calculation. Further investigations need to be made in this field, with a view to
speeding up the convergence of the program and using a finer mesh (or higher-order diffe-
rence formulae). Results obtained so far are, however, in good agreement with the one-
dimensional calculations, and renew our confidence in the applicability of the simpler
method.

The computation of the track-following spark requires a full three-dimensional mesh to
be set up, with a consequent increase in the mesh spacing and a further loss of accuracy.
The storage used can be reduced to a half by observing that the problem possesses reflection
symmetry about a plane containing the field direction and the track direction. The effect
of neighbouring avalanches can be included by making the charge density and electric field
components periodic functions of position in the track direction.

Fig. V.7 Field lines and electron-density contours during the development of
a track-following spark. Neon gas, p = 690 Torr, E = 6.73 kV cm⁻¹,
track angle = 30°, avalanche separation = 2 mm, time = 82.8 nsec.
Figure V.7 shows the electron density in the plane of symmetry for a short length of an infinite track at a time 82.8 nsec. The avalanches have already started to turn into the direction of the track, and this tendency is almost complete at 97.9 nsec (Fig. V.8).

The track-following behaviour is predicted for all track angles except $\phi = 90^\circ$ (i.e. when the track is parallel with the plates). However, if we take into account the effect of the electrodes, or of ions produced outside the track, then we see that there must be some limiting angle. When the track has become a continuous tube of current between the electrodes -- as it does in our calculations -- then the electric field in the track direction is $E \cos \phi$, where $E$ is the applied field. Therefore, avalanches which start on the cathode, or in the gas outside the track, will grow faster than the track itself, and will eventually obscure it if $\phi$ is large enough. We cannot estimate the limiting angle unless we take into account higher-order ionization process. For example, if there is a large density of metastable atoms along the track, then the effective ionization coefficient there may well be greater than in the larger uniform field outside it. Consequently, we should not expect track-following to occur as readily in molecular gases as it does in helium and neon. In addition, the time delay involved in ionization by the Penning effect might improve the stability of the spark.

Provided that no spurious sparking occurs outside the track, the final stage of current growth will be equivalent to that in a parallel-plate discharge with a uniform applied field $E \cos \phi$. By analogy with the work described earlier, we would expect to find waves of ionization travelling up and down the track, but such large-scale effects are precluded by seeking a periodic solution.
2. Breakdown Probabilities under d.c. and Pulsed Voltages

Electrical breakdown is a recurring problem in many fields of high-voltage technology, particularly, for example, in the design of high-gradient accelerating tubes.

Recently there has been a design study into the possible upgrading of the present 600 keV CERN pre-injector to approximately 1.5 MeV \( V_s \). This introduced a number of breakdown problems, one of which was that of breakdown between the equipotential spacers on the outside of the accelerating column. At the moment it is possible to apply a d.c. voltage of approximately 640 kV to the column before breakdown between the rings starts to occur.

The suggestion has been made that it might be possible to use a pulsed voltage source (of duration about 100 usec) and to hold off voltages that are considerably in excess of the d.c. breakdown potential \( V_s \). In this section we will investigate how far, in fact, we can exceed \( V_s \) with a negligible probability of breakdown occurring during the application of the pulse voltage.

2.1 The Meaning of "Breakdown"

Before we can proceed any further we have to be quite clear what we mean by the term "breakdown". If one looks at the temporal growth of the ionization currents in a parallel plate gap, then, after the initial transients have died out, the current growth is exponential and only becomes greater than exponential when the space-charge distortion of the field due to the ions and electrons is no longer negligible. The subsequent current growth is very rapid, and eventually the rate of increase of current is so great that the potential difference between the electrodes collapses.

For our present purpose we shall define breakdown as having occurred when this last, catastrophic region of current growth has been reached. The actual value of current flowing will depend upon the geometry of the electrodes and the electrical components in the external circuit. In general we may say, however, that once a current of about \( 10^{-5} \) A has been reached, the subsequent growth is so rapid that collapse of the voltage between the electrodes occurs almost immediately. We shall thus define "breakdown" as the attainment of a current of the order of \( 10^{-5} \) A.

2.2 Statistical and Formative Time Lags

With applied voltages in excess of \( V_s \), the time that elapses between the application of the applied voltage and the occurrence of breakdown is termed the "time lag", and may be divided into two parts called the "statistical" and "formative" time lags, respectively. Before any ionization currents can flow between two electrodes there must be at least one electron present in the discharge space. The statistical time lag is defined as the time delay between the application of the voltage and the appearance of the initial electron which starts the discharge. The formative time lag is the subsequent time which elapses before the occurrence of breakdown, i.e.

\[
\text{total time lag} = \text{statistical time lag} + \text{formative time lag}.
\]

We see, therefore, that if the pulse voltage \( V \) is applied to the electrodes for an indefinite period, then for \( V > V_s \), breakdown will always occur sooner or later. If, however, the pulse is of finite duration, then breakdown will occur if the time lag is less
than the pulse length and not otherwise. We are interested in calculating the probability of the occurrence of breakdown as a function of the magnitude and duration of the pulse voltage. In particular, if the pulse length is fixed (say 100 μsec) we wish to know how high \( V \) may be above \( V_s \) before the probability of breakdown becomes appreciable.

### 2.3 Statistics of the formative time lag

We consider a parallel-plate discharge gap, of separation \( d \), to which is applied a voltage \( V \). If an electron is emitted from the cathode then it will travel to the anode and will undergo ionizing and excitation collisions, thus forming an electron avalanche. In Chapter IV we have examined in detail the statistics of these avalanches and of the secondary electrons produced at the cathode by the \( \gamma_1 \) or \( \gamma_{ph} \) process.

We have defined the generation period \( \tau \) as the mean time that elapses between the emission of an electron from the cathode and the production there of the resulting secondaries. As before, we number successive generations \( 0, 1, 2, \ldots, k, \) and suppose that there is only one electron for \( k = 0 \), i.e. the discharge is initiated by a single electron. The mean number \( \mu \) of secondary electrons produced by one avalanche is given by

\[
\mu = \gamma (e^{ad} - 1),
\]

where \( \gamma \) can represent either the \( \gamma_1 \) or \( \gamma_{ph} \) process.

We will take as a measure of the current flowing in the gap the number \( n \) of electrons that leave the cathode during a generation period, and will denote the critical current defining breakdown by \( N_t \).

Let \( P(n,k) \) be the probability that \( n \) avalanches flow in the \( k^{th} \) generation, i.e. that the current number is \( n \) at time \( k \tau \). Then in Chapter IV, Section 3.5, we showed that the generating function for \( P(n,k) \) is given by

\[
V(z,k) = \sum_{n=0}^{\infty} P(n,k)z^n = \frac{1 - \mu^k - \mu(z - 1 - \mu^{k-1})}{1 - \mu^{k+1} - \mu(z - 1 - \mu^k)},
\]

from which we find

\[
P(n,k) = \frac{(n_0 - 1)^n}{\mu n_0^n} \left[ \frac{n_0 - 1}{n_0} - \frac{(n_0 - \mu - 1)}{n_0 - 1} \right], \tag{V.1}
\]

where

\[
n_0 = \frac{1 - \mu^{k+1}}{1 - \mu}.
\]

The mean electron current \( \bar{n} \), as is to be expected, is

\[
\bar{n} = \mu^k, \tag{V.2}
\]

and the standard deviation \( \sigma \) is given by

\[
\sigma^2 = \frac{\mu^k (\mu - 1 - \mu^{k+1})}{1 - \mu},
\]
which, for $\mu^k \gg 1$, $k \gg 1$, reduces to

$$\sigma = \frac{N}{\sqrt{\mu - 1}}.$$ (V.3)

In Eq. (V.1) all values of $n$ are possible, including $n > N_1$; that is, the breakdown process has not been taken into consideration. Now if $N_1$ is large, once the avalanche number $n$ has exceeded $N_1$ there is a negligible probability that it will fall below $N_1$ at subsequent times provided $\mu \gg 1$. Thus, when we take into account the breakdown process, the probability that breakdown occurs at or previous to time $k\tau$, $W(k)$, is given to a high degree of accuracy simply by

$$W(k) = \sum_{n=0}^{\infty} P(n,k) = \frac{\mu - 1}{\mu} \cdot \frac{1}{1 - (1/\mu^k)} \left[ \frac{1 - (1/\mu^k)}{1 - (1/\mu^{k+1})} \right]^{N_1}.$$ (V.4)

We note that

$$W(\infty) = 1 - \frac{1}{\mu},$$

so that there is a finite probability $1/\mu$ that breakdown does not occur at all, i.e. that the series of avalanches dies out.

Now Eq. (V.4) is only substantially different from zero if $\mu^k \gg N_1$, in which case it may be written (see Fig. V.9):

$$W(k) = \frac{\mu - 1}{\mu} \cdot \exp \left[ -\mu^{-(k-k_0)} \right],$$ (V.5)

Fig. V.9 Probability of breakdown previous to time $k$ plotted as a function of $(k - k_0) \ln \mu$. 

---
where

\[
k_0 = \frac{\ln \left[ N_0 (\mu - 1) \right]}{\ln (\mu)} - 1.
\]

The corresponding frequency distribution is (Fig. V.10):

\[
w(k) = \frac{d}{dk} w(k) = \frac{\mu - 1}{\mu} \cdot \ln (\mu) \cdot \mu^{-(k-k_0)} \cdot \exp \left[ -\mu^{-(k-k_0)} \right].
\] (V.6)

In calculating the mean growth time we note that since there is a finite probability 1/\mu that the avalanche series dies out, then, in a large number of experiments there will be a finite fraction which have an infinite breakdown time. In practice, however, we observe only those series in which breakdown does occur (if the series dies out, unless there are very sensitive current-measuring devices we would not ever be aware that the original electron had been produced!) and we wish to know the mean breakdown time \( \bar{k} \) in these particular cases [a fraction 1 - 1/\mu of the total]. Thus

\[
\bar{k} = \frac{1}{1 - (1/\mu)} \int_0^\infty k w(k) \, dk
\]

\[
= \int_0^\infty k \ln \mu e^{-(k-k_0)\ln \mu} \exp \left[ -e^{-(k-k_0)\ln \mu} \right] \, dk.
\]

By a simple substitution this may be written

\[
\bar{k} = k_0 - \int_{-\infty}^{k_0 \ln \mu} \frac{x e^x e^{-e^x}}{\ln \mu} \, dx.
\]

---

![Graph](image.png)

Fig. V.10  Probability of breakdown per unit time as function of \((k - k_0) \ln \mu\).
Now the integral in this last expression is zero everywhere except in the region of $x = 0$ so that we may extend the upper limit to $\infty$ provided $k_0 \ln(\mu) \gg 0$.

Substituting $t = e^x$ we have

$$\bar{k} = k_0 - \int_0^\infty \frac{e^t \ln t}{\ln \mu} = -\frac{\psi(1)}{\ln \mu} + k_0$$

and

$$\bar{k} = k_0 + \frac{0.5772}{\ln \mu}, \quad (V.7)$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) \quad \text{and} \quad \psi(1) = -\epsilon \quad (Euler's constant = 0.5772).$$

The standard deviation $\sigma'$ is given by

$$\sigma'^2 = \frac{1}{1 - (1/\mu)} \int_0^\infty (k - \bar{k})^2 w(k) \, dk$$

$$= k_0 - \bar{k}^2 + \frac{2 k_0 \epsilon}{\ln \mu} + \int_{-\infty}^\infty x^2 e^x e^{-\epsilon x} \, dx.$$ 

Now this last integral is

$$\frac{1}{\ln^2 \mu} \left[ \frac{d^2 \ln \Gamma(z)}{dz^2} \right]_{z=1} = \frac{[\psi(1) + \epsilon^2]}{\ln^2 \mu},$$

where $\psi^n(z)$ is the polygamma function

$$\psi^n(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z)$$

and since

$$\psi(1) = \frac{\pi^2}{6},$$

$$\sigma'^2 = \frac{1}{\ln^2 \mu} \cdot \frac{\pi^2}{6}$$

and

$$\sigma' = \frac{1.283}{\ln \mu}. \quad (V.8)$$

2.4 Application to air near atmospheric pressure

We will apply the above formulae to the case of air at 700.2 Torr between aluminium electrodes placed 1.17 cm apart, for which case accurate experimental data is available\textsuperscript{35}). The sparking potential $V_s = 31,160$ volts and the positive ion drift velocity
\( W_p = 8.56 \times 10^{-6} \text{ cm sec}^{-1} \). The functional dependence of the effective primary ionization coefficient \( \alpha \) is

\[
\alpha = 6.863 \times 10^{-n} \exp \left( -\frac{2.752 \times 10^5}{V} \right) \text{ cm}^{-1}.
\]

If we take the critical current defining breakdown to be \( 10^{-5} \) A, then \( N_1 = 5.9 \times 10^3 \) electrons per generation period \( \tau (\approx 1.37 \times 10^{-5} \text{ sec for the } \gamma_1 \text{ secondary process}) \). For a pulse length of 100 \( \mu \text{sec} \), \( k = 7.32 \) so that seven generations of avalanches will flow in 95.7 \( \mu \text{sec} \).

We first of all note that if statistical fluctuations are neglected, the value of \( \mu \) for which the mean current \( \bar{n} \) is equal to \( N_1 \) after seven generations (95.7 \( \mu \text{sec} \)) is given by Eq. (V.2) and yields \( \mu = 2.14 \). The corresponding percentage overvoltage \( \Delta V\% \) is 0.28\%, so that even with this small overvoltage the mean current in the discharge will have grown to the critical value in less than 100 \( \mu \text{sec} \).

In order to make the probability of breakdown reasonably small, \( N_1 \) should exceed \( \bar{n} \) at 100 \( \mu \text{sec} \) by at least 5 or 4 times the standard deviation. If we choose \( \bar{n} = (1/10) N_1 \), then \( \mu = 1.66 \), corresponding to an overvoltage of only 0.04\%. Even if the pulse length of the applied voltage was reduced to 40 \( \mu \text{sec} \) (\( k = 3 \)), the overvoltage could only be increased to 0.6% if the probability of breakdown is to remain negligible.

The situation is made even worse when we realize that the above calculations have been performed on the assumption of a 100\% \( \gamma_1 \) secondary process. In practice, a large proportion of the \( \gamma_{ph} \) (photonic) secondary process will be active in air at atmospheric pressure. In this case \( \tau \approx 1.37 \times 10^{-7} \text{ sec} \), \( N_1 \approx 59 \), and with an overvoltage of only 0.04\% the pulse length would have to be less than 0.4 \( \mu \text{sec} \) in order to have a negligible breakdown probability.

A rather more accurate procedure than the one described above is to consider the probability distribution \( W(k) \) of the breakdown time \( k \) which is shown in Fig. V.9. If we require the probability of the formative time lag being less than \( k \), to be 1 in \( 10^3 \), say, then from the graph we have

\[
(k - k_0) \ln (\mu) = -2.
\]

In our case for the \( \gamma_1 \) process \( k = 7 \), and we find from this last relation

\[
\mu = 2.405, \quad k_0 = 127 \mu \text{sec}, \quad k = 136 \mu \text{sec}, \quad \sigma' = 20.5 \mu \text{sec}.
\]

This demonstrates how sharply the \( W(k) \) curve descends at small values of \( k \). Even though \( k \) is only \( k = 2\sigma' \), we have only a 1 in \( 10^3 \) probability of breakdown in times less than 95.7 \( \mu \text{sec} \). The overvoltage corresponding to \( \mu = 2.405 \) is 0.4\% so, once again, a negligible improvement is obtained [this compares with \( \Delta V\% = 0.28\% \) obtained previously for \( \bar{n} = (1/10) N_1 \); when the sharp cut-off of the \( W(k) \) curve to the left of \( k_0 \) was not taken into consideration].

In the case of a 100\% \( \gamma_{ph} \) process for \( \mu = 1.66 \) (\( \Delta V\% = 0.04\% \))

\[
k_0 \equiv 8.6 \times 10^{-7} \text{ sec} \quad \text{and} \quad k \equiv 3.15 \times 10^{-7} \text{sec}.
\]

Thus for only 0.04\% overvoltage we still have a 1 in \( 10^3 \) chance of breakdown in less than 0.32 \( \mu \text{sec} \).
In practice, of course, both the $\gamma_1$ and $\gamma_{PH}$ processes contribute to the ionization growth, and the results will then be intermediate between those given above.

It is difficult to repeat the calculations for the exact conditions existing between two consecutive corona rings on the exterior of the pre-injector. Here the nearest points on the outer annuli are approximately a distance 4.4 cm apart so that we would expect $\tau$ to be about four times greater. The field distribution, however, is extremely non-uniform and, bearing in mind how rapidly $\alpha$ varies with $E/p$ in the region under consideration, it is not possible to repeat the calculations taking into account the non-uniformity of the field. We would nevertheless expect that the order of the percentage overvoltage corresponding to a given breakdown probability will be the same, and since in the above $\Delta V\%$ is always far less than 1% for a 100 usec pulse we may conclude that we gain hardly anything in going from d.c. to pulsed conditions as far as the formative time lag is concerned.

2.5 Statistics of the statistical time lag

In this section we examine the statistics of the time that elapses between the application of a pulse voltage to a discharge gap and the appearance of an electron which will initiate a discharge.

We assume a priori that once the voltage is applied to a discharge gap the probability $I$ of an electron appearing in any time interval $t - t + dt$ ($t$ arbitrary) is constant. Let $V(t)$ be the probability that an electron has appeared previous to time $t$. Thus the probability that an electron appears for the first time in the interval $t$ to $t + dt$ is

$$V(t + dt) - V(t) = \frac{dV(t)}{dt} \cdot dt = \left[1 - V(t)\right] I dt .$$

Thus

$$\frac{dV(t)}{dt} = \left[1 - V(t)\right] I ,$$

and remembering that $V(0) = 0$ we have (see Fig. V.11):

$$V(t) = 1 - e^{-\lambda t} ,$$

and the corresponding frequency distribution is

$$v(t) = \lambda e^{-\lambda t} .$$

The above distribution has been verified experimentally by a large number of workers [see, for example, Grey Morgan and Harrcombe 15]; $\lambda$ represents the rate of electron emission from a given cathode for a given applied voltage.

The mean of the distribution is given by

$$\bar{t} = \frac{1}{\lambda} ,$$

and the standard deviation is also $1/\lambda$ showing that there is a large scatter about the mean.

From Fig. V.11 we see that if we want the probability of the appearance of an electron in less than 100 usec to be of the order of $10^{-3}$, so that breakdown will occur, on the average, in one pulse in a thousand, then

$$\lambda t = 10^{-3} ,$$
Fig. V.11  Probability that the statistical time lag is less than $t$ as function of $\log_{10}(It)$.

and since $t = 10^{-4}$ sec,

$$I \sim 10 \text{ electrons/sec}.$$  

Thus the rate of electron emission must be at most 10 per second if the probability of breakdown is to be less than $10^{-3}$ in 100 µsec.

It is very difficult to find experimental measurements of $I$, but Llewellyn Jones\textsuperscript{1)} quotes values ranging from $5 \times 10^3$ to $10^6$ electrons/sec for stainless steel with various surface finishes for fields of the order of $10^5$ V cm$^{-1}$. In the pre-injector the mean field is $10^4$ V cm$^{-1}$, but the non-uniform geometry will increase the cathode field to much nearer $10^5$ V cm$^{-1}$.

Llewellyn Jones\textsuperscript{1)} has noted that dust has a pronounced effect on the emission, which may be reduced by a factor of 100 by the removal of the dust. The above figures were obtained for electrodes of small area and, although there does not appear to be a pronounced increase of emission with area, when we consider the very large surface of the electrodes on the pre-injector the total emission must be considerably increased -- indeed cosmic radiation alone must account for the production of far more than 10 electrons/sec.

The most promising approach to the improvement of the performance of the pre-injector would seem to be to attempt to increase its d.c. breakdown potential by enclosing the accelerating column in a pressure vessel containing SF$_6$ or by using a suitable insulating coating on the external electrodes\textsuperscript{37}).
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For general reading the readers are referred to the excellent monographs by Professor F. Llewellyn Jones\textsuperscript{1)} and Professor C. Grey Morgan\textsuperscript{2)}. The results obtained by Professors P.M. Davidson and W. Legler have been extensively drawn upon in Chapters II and IV, and more detailed discussions of various points may be found in the original papers (see Refs. 4-7, 11, 20, 22, 25, 25).
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