PION-DEUTERON ELASTIC SCATTERING
AT INTERMEDIATE ENERGIES

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GENEVA
1973
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K. Gabathuler**)
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1. **INTRODUCTION**

The most favourable system for studying multiple scattering processes is clearly the deuteron, as it is composed of only two scatterers, and realistic wave-functions are available. The input quantities in a multiple scattering theory are the scattering amplitudes of the projectiles on the individual scatterers. The more accurately these scattering amplitudes are known, the more seriously the different theoretical formulations can be tested. Consequently, there is a considerable advantage in using pions in the medium energy range (\(\sim 100-300\) MeV), where accurate (on-shell) pion-nucleon amplitudes are available for both isospin states\(^1\). Moreover, pion-deuteron scattering should be easier to understand than scattering of nucleons by deuterons, because of the much simpler spin structure. The marked energy variation of the pion-nucleon amplitudes due to the 3-3 resonance can place strong constraints on theoretical multiple scattering models, providing the Fermi motion of the deuteron is taken properly into account.

Above 900 MeV/c there is a series of \(\pi d\) elastic cross-sections\(^2\), which is successfully reproduced by Glauber theory\(^3\), if the small admixture of the D-state of the deuteron is taken into account.

Glauber's theory is based, among other things, on the assumption that the wave distortion by the individual scatterers is accurately described, over the dimensions of the scattering system, by geometrical optics. As the energy is lowered, this assumption becomes doubtful.

The natural variable for describing the longitudinal development of a diffracted wave is

\[ x = \frac{z}{ka^2}, \]

where \(z\) is the distance from the scatterer along the incident direction, \(k\) the wave-number, and \(a\) the radius of the scatterer. Geometrical optics are then characterized by

\[ x \ll 1. \]

Thus in deuterium the double scattering amplitude will only be accurately described by geometrical optics, when

\[ ka^2 \gg \bar{r}, \]

where \(\bar{r}\) is the r.m.s. separation between neutron and proton. This inequality already fails at momenta in the neighbourhood of 1 GeV/c, and Fresnel diffraction effects should be taken into account. The good results obtained with Glauber theory around 1 GeV/c indicate that Fresnel effects are not important, but they are expected to grow quite rapidly as \(k\) falls far below 1 GeV/c\(^5\), and the Glauber approach should not be used any more.

Another multiple scattering approach becomes very attractive at energies below 300 MeV, where only the lowest partial waves are important\(^6\),\(^7\). The deuteron can be described as two point-like scatterers, if we suppose that the "potentials" of the two nucleons do not overlap.
This is a fair assumption, as the nucleons are widely spaced and weakly bound. The wave equation outside the scatterers is solved with the boundary conditions at the scattering points given by the pion-nucleon amplitudes. This approach is known as the Brueckner model.

Below 300 MeV there are a number of bubble chamber measurements\(^{1-10}\), but they are generally of a rather poor quality. It is the aim of this experiment to provide better data which might yield a test of different multiple scattering approaches. As the multiple scattering effects show up mostly at backward angles, the variation of the fixed angle cross-section (\(\theta_{\text{lab}} = 160^\circ\)) is determined for incident pion energies between 141 MeV and 256 MeV, in addition to a full angular curve at 256 MeV. The data are compared to a Brueckner model calculation including s- and p-wave single and double scattering.

2. EXPERIMENTAL DESIGN

2.1 General principles of the experiment

The main difficulty in scattering experiments involving deuteron targets is to distinguish between elastic and inelastic scattering. The deuteron, having a binding energy of only 2.2 MeV, tends to break up in a scattering process, in particular for large momentum transfers.

In counter experiments the separation of elastic and inelastic events can be obtained in two ways:

i) by detecting the scattered pions by a high resolution system which is able to resolve the elastic scattering peak from the continuum of the break-up reactions,

ii) by ascertaining that the deuteron is still intact after the scattering process, i.e. by identifying the recoiling deuterons.

The first possibility requires a high quality beam and a resolution for the pion detecting device better than 1 MeV. Such experiments have not yet been carried out but are proposed for the near future at Los Alamos\(^{11}\). They have the advantage of not being limited in energy and momentum transfer.

The second method, which requires a much simpler apparatus, has been used several times for momenta above 900 MeV/c\(^{2}\), where the recoiling deuterons could be identified outside the target. At energies in the 3-3-resonance region, however, the momentum transfers at forward angles are so small that the deuterons cannot escape from the target (a gaseous target cannot be used with the present beam intensities). It is therefore necessary to detect the recoiling deuterons inside the target. This can be done if a scintillator containing deuterons is used as target material. Scintillators having a deuterated liquid solvent base are commercially available. The criterion for choosing between them is the relative amount of deuterium to the other constituents; this is highest for deuterated cyclohexane \(\text{[C}_6\text{D}_{12}]^*\). Some of the main characteristics for such a scintillator are compared with a commonly used plastic scintillator in Table 1.

A scintillating deuteron target has, apart from the capability of detecting recoil deuterons inside the target, other advantages compared with a cryogenic deuterium target. Firstly, it is much simpler and cheaper to build and to operate. Furthermore, it gives a

\(*)\) NE 252 supplied by Nuclear Enterprises, Edinburgh.
signal for each pion incident on the scintillator, thus defining its useful volume; this is a considerable advantage in understanding the geometry of the scattering processes.

On the other hand, such a scintillating target has serious disadvantages, for example, the carbon nuclei in the scintillator which give rise to a background that must be subtracted. Moreover, by measuring only the energy deposited in the scintillator by the recoiling particle, we cannot unambiguously specify its rest mass. It is possible that a recoil proton gives the same target signal as a deuteron; however, the problem is not too serious as the recoil protons will produce a continuous flat target signal spectrum which can readily be subtracted from the strongly peaked spectrum produced by the recoil deuterons.

It is advantageous to use this scintillating target technique even for the larger scattering angles where the recoil deuterons escape from the target, since in conjunction with an external (dE/dx, E) telescope it allows a much clearer identification of the recoiling deuterons.

Two different measuring methods are required according to whether the recoiling deuteron can escape from the target:

i) For the small angles of the angular distribution measurement at 256 MeV the deuterons stop in the target. It is necessary to obtain a narrow clearly identifiable peak due to the recoiling deuterons in the spectrum of the target pulses, in order to be able to separate this peak from a possible proton background. In addition, a good angular definition is required, since the cross-section varies quite rapidly with angle (forward peak). These requirements are fulfilled if the scattered pions are analysed by a small angular acceptance magnetic spectrometer, the resolution of which need not necessarily be very high.

ii) For the large angles of the angular distribution measurement at 256 MeV, as well as for the energy variation of the cross-section at \(160^\circ\), the deuterons are unambiguously identified outside the target by measuring their specific energy loss and their energy. Consequently, the pions need not be momentum-selected; however, a large angular acceptance is required for the pion detector, as the scattering cross-section at large angles is extremely small.
Fig. 1 Range of recoiling deuterons in cyclohexane versus pion scattering angle $\theta_\pi$ in the reaction $\pi^+ + d \to \pi^+ + d$ for incident pions of 256 MeV.

Figure 1 shows the range of the recoiling deuterons in cyclohexane versus the pion scattering angle in the reaction $\pi^+ + d \to \pi^+ + d$ for an incident pion energy of 256 MeV, as calculated from a computer program described elsewhere. The change-over from the small- to large-angle measuring system is made at 80°; at this angle scattering data are taken via both systems to provide a consistency check. This angle is chosen as a compromise between extending the large-angle method as far forward as possible and retaining a sufficiently thick target to obtain a reasonable counting rate at the backward angles.

For the scattering angles $0 \leq 80^\circ$ we need a target that has a large dimension along the expected direction of the recoiling deuterons, in order to stop them inside the liquid scintillator volume. For $\theta \geq 80^\circ$ the target dimension along the expected direction of the recoiling deuteron, including the target wall, should not exceed 12 mm; this allows the recoiling deuterons to escape from the target. As the recoil direction varies with scattering angle the targets are mounted on a turn-table and positioned appropriate to each scattering angle.

The background arising from carbon nuclei in the scintillator can be determined by using an identical target cell filled with an identical scintillator liquid in which the deuterated cyclohexane is replaced by ordinary cyclohexane ($C_6H_{12}$). Providing we are able to separate the hydrogen scattering from the carbon scattering in the $C_6H_{12}$ we get a true measurement of the carbon background in $C_6D_{12}$.

Absolute cross-sections are obtained by measuring the reaction $\pi^+ + p \to \pi^+ + p$ with the $C_6H_{12}$ targets and comparing it with the cross-section reconstituted from the pion-nucleon phase shifts:

$$\frac{d\sigma}{d\Omega}(\pi^+ d) = \frac{d\sigma}{d\Omega}(\pi^+ p) \frac{N_{nd}}{N_{np}} \frac{\delta_d}{\delta_p},$$
where $l_d$ and $l_p$ are the length of the $^{2}$H and $^{12}$C targets, respectively, and $N_d$ and $N_p$ the number of elastic $\pi^+$ and $\pi^+$ scatterings for the same number of incident pions. With this method, questions of beam contamination and solid angle are avoided.

2.2 The pion beam

The beam extracted from the CERN Synchro-cyclotron (SC) supplied about $2 \times 10^{11}$ protons per second at a kinetic energy of 600 MeV. These protons were focused onto the pion production target PT (Fig. 2) by the quadrupoles Q1Q2, which produced a focus of dimensions $1 \text{ mm (hor.)} \times 6 \text{ mm (vert.)}$ FWHM as measured with miniature ionization chambers. Pions produced near $0^\circ$ were horizontally collimated into a parallel beam by the quadrupole pair Q3Q4, deflected through $27.5^\circ$ by the magnet M1 to provide momentum dispersion and refocused by the quadrupole pair Q5Q6 onto a single momentum-defining scintillation counter S1 of 15 mm width. Vertically the beam was designed with an intermediate focus between the quadrupole pairs. The momentum dispersion transverse to the optical axis at the focal plane was 15 mm per 1%;

![Fig. 2](image-url)
counter S1 thus defined a momentum bite of 1% (FWHM). Vacuum pipes were utilized to avoid multiple scattering of the beam.

For the angular distribution measurement pions were produced on a 10 cm polyethylene target via the reaction $p + p \rightarrow d + \pi^+$. This reaction yielded the highest, externally-produced pion flux available at the SC. Since this experiment depended crucially on the beam intensity, the pion energy for the angular distribution measurement could not be chosen arbitrarily, but was determined by the kinematics of the reaction $p + p \rightarrow d + \pi^+$.

A 5 cm polyethylene absorber A was placed in the beam immediately in front of S1, in order to protect the scintillating target from the large number of degraded protons accompanying the pions; this was imperative for obtaining clear target pulse spectra. At the scattering target position, the pion beam defined by S1 had the dimensions of 18 mm (hor.) $\times$ 11 mm (vert.) FWHM, a flux of about $10^5$ pions per second and a central momentum of $370 \pm 3$ MeV/c (corresponding to an energy of $256 \pm 3$ MeV). The energy uncertainty is due to the uncertainty in the proton energy of the SC.

For pions of lower energies, a pion production target of 10 cm beryllium was used and the proton absorber was reduced to 2.5 cm polyethylene. The beam intensity was of the order of $10^6$ pions per second. The pion energy was determined by measuring (with the same beam layout) the three pion production reactions $p + p \rightarrow d + \pi^+$, $p + d \rightarrow t + \pi^+$ and $p + ^3He \rightarrow ^3He + \pi^+$, all at $0^\circ$ \cite{13}. For a given proton energy, the pion momenta in these reactions were uniquely determined by kinematics and proportional to the corresponding field values in M1 which were corrected for the measured SC fringe field of $125 \pm 10\%$. The three momentum-to-field ratios agreed within the measuring accuracy of $5 \times 10^{-4}$. This linearity was not sensitive to a 3 MeV change in the proton energy. Hence it was not possible to get a more accurate value for the proton energy and the energy scale for the pions is not known to better than 1%.

2.3 Small-angle scattering apparatus

2.3.1 The targets

Two identical rectangular cells with 2 mm thick perspex walls were mounted on perspex lightguides, and one filled with scintillating C$_6$H$_{12}$ and the other with C$_4$D$_{12}$. These target-lightguide systems could be alternately mounted on a photomultiplier assembly, which was mounted on the centre of a turn-table. The target orientation was determined by the angular scale of the turn-table. The cells with the internal dimensions of 30 mm height, 20 mm thickness and 60 mm width were positioned with their greatest dimension along the expected direction of the recoiling particles, in order to prevent their escaping from the target (Fig. 3a).

The liquid was fed into the target cell through a teflon tube placed in one of two small filling ports by applying a slight nitrogen overpressure on the scintillator reservoir. It was important to keep the scintillator protected from oxygen which would reduce its scintillation efficiency. By bubbling dried nitrogen through the full target cell the oxygen dissolved in the scintillator during the filling procedure was replaced by nitrogen. After filling the ports were sealed; the scintillation properties did not change noticeably over a period of several months.
Fig. 3  Experimental set-up for small angle scattering ($\theta \leq 80^\circ$).  a) Pions traversing the proton absorber A and the beam-defining counter S1 are scattered in the scintillating target T.  b) The scattered pions are momentum-selected by the magnetic spectrometer Q7Q8M2Q9Q10 and detected by the counter telescope (S2,S3).

2.3.2 The pion spectrometer

The scattered pions were selected by a magnetic spectrometer designed for previous experiments$^{13-15}$. It had the same general characteristics as the pion beam, i.e., a horizontally parallel beam between Q8 and Q9 (Fig. 3b) and an intermediate vertical focus. The solid angle accepted by the spectrometer was 4 msr. The bending angle of 45° gave a momentum dispersion of 25 mm per 1° and the focal plane was inclined at 10.9° to the beam axis. The field in M2 was measured by a temperature-stabilized Hall plate. A scintillation counter telescope (S2, S3) of 12 and 10 cm wide counters was placed behind the spectrometer and used to detect the scattered pions. The counter S3 was placed perpendicular to the beam axis at the intercept of the axis and the spectrometer focal plane; it selected the momentum of the scattered pions with a FWHM of 4% $\Delta p/p$. Figure 4 shows the situation for pions of energy 256 MeV scattered elastically on protons, deuterons and carbon nuclei. It can be seen that S3 was able to separate the three scatterings for $\theta \geq 30^\circ$. Smaller angles would have been accessible with a narrower counter, but here pions scattered in the proton absorber would have entered the spectrometer. The counter S2 was used to suppress accidental events caused by particles striking S3 without passing through the spectrometer.
2.4 Large-angle scattering apparatus

2.4.1 The targets

For this case the dimensions of the scintillating target cells were 55 mm × 55 mm in section and 10 mm thick with 0.1 mm perspex windows. They were wrapped in a 6 μ aluminized mylar foil to provide good internal light reflection, and in a 40 μ aluminium foil to protect them from external light. The thin cell windows and reflecting foils were necessary in order to minimize the non-scintillating material through which the recoiling particle must pass. The perspex windows were bowed by the liquid in the cells giving rise to a non-uniformity in target thickness. In order to measure this effect, one layer of aluminized mylar foil was glued on the windows; the target was then placed between two metal pins which were moved inwards until an electrical contact was established between the aluminized foil and the two pins. The distance between the two pins was then measured with a dial gauge. The aluminium acted as a mirror and allowed us to avoid an undesired change in the window curvature due to the pins. The complete target area was mapped in this manner, and the result weighted by the beam distribution showed an average thickness of 10.8 mm and 11.5 mm for the C₄H₁₂ and C₆D₁₂ targets, respectively.

The targets were positioned with their windows perpendicular to the expected direction of the recoil particles in order to minimize their path in the target (Fig. 5).
2.4.2 The pion telescope

A scintillating counter telescope (S2, S3) detected the scattered pions (Fig. 5). The circular counter S3 was placed at a distance of 66 cm from the target and defined a solid angle of 15 msr for $\theta = 80^\circ$, and 30 msr for $\theta \geq 90^\circ$. The counter S2 suppressed events caused by pions scattered by the proton absorber into S3.

2.4.3 The recoil counters

The recoil particles were uniquely determined by measuring their total energy and specific energy loss. The large size of the recoil counters was dictated by the accepted pion solid angle. This immediately excluded the use of solid-state counters and, despite the poor energy resolution, plastic scintillation counters were used.

The counter measuring the specific energy loss ($\Delta E$) consisted of a perspex ring into which a thin accurately-machined circular sheet of scintillator could be inserted. Good optical contact was obtained by the use of optical grease. This system allowed us to change the thickness of the scintillator sheet. The thickness chosen was 1 mm for $\theta = 80^\circ$ and 90$^\circ$ at 256 MeV and for $\theta = 160^\circ$ at 141 and 163 MeV, and 2 mm for $\theta \geq 95^\circ$ at 256 MeV and for $\theta = 160^\circ$ at 163, 185 and 208 MeV. The total energy counter E was 20 mm thick when a 1 mm $\Delta E$ counter was used and 40 mm otherwise. The wrapping of the recoil counters was done in the same manner as described for the targets in Section 2.4.1.
Fig. 6 Scatter plots expected for (a) thick recoil counters, and (b) thin recoil counters, assuming infinitely good energy resolution and equal specific light output of the liquid and plastic scintillators. The crosses indicate the regions where the deuterons are expected.

Assuming infinitely good energy resolution and equal specific light output for the liquid and plastic scintillators for both protons and deuterons we calculated the expected scatter plots for the two counter sizes. The result is shown in Fig. 6, where the abscissa represents the sum of the energy losses of the recoil particles in the E counter and in the target T. The ordinate represents the specific energy loss $\Delta E$. The fact that the deuterons are not concentrated in one dot, for a given incident pion energy and scattering angle, is due to the finite target thickness and angular acceptance of the recoil counters. For 160° scattering at 256 MeV a 6 mm polyethylene absorber was placed between $\Delta E$ and E.

3. MEASUREMENT

3.1 Data-recording system

All scintillation counters except S1 were mounted on Philips 56 DVP photomultipliers utilizing a high current Zener diode distribution chain for the lower dynodes. The counter S1 was viewed by an XP 1021 Philips tube; it had a 50 $\Omega$ coaxial output and gave very clear output pulses with essentially no after-ringing. This tube was used on S1 because of the very high instantaneous rates at this location ($\sim 5 \times 10^6$ particles per second).
The data-recording system for backward angles is shown in Fig. 7. The fast electronic logic consisted of standard NIM units, and the on-line computer was a PDP-8 having a 4K memory. The small-angle system was identical, except the counters $\Delta E$ and $E$ were not utilized.

Large proton pulses in $S_1$, $S_2$ and $S_3$ generated 10 nsec long veto pulses; this effectively suppressed random coincidences. Owing to the proton absorber the number of protons in $S_1$, approximately $3 \times 10^4$ per second, was not large enough to cause serious dead-time problems. An attenuator was placed in the target signal system (A1) to compensate for slightly different light outputs of the two scintillating liquids.

An incoming pion was defined by an appropriate pulse in $S_1$ in coincidence with a pulse in the target $T$ (AND2). The master trigger, indicating a scattering event, was built up by suitable pulses in $S_1$, $S_2$ and $S_3$, a signal in $T$ and, for backward scattering, a signal in $\Delta E$ (AND3). The data-taking was started and stopped by the main start/stop switch (Fig. 7). A scattering event was signalled via AND4 to the computer by an interruption request. Simultaneously, all gates were opened and a fast inhibit level applied on AND4 and AND5 until a computer inhibit signal had arrived from the data transfer unit. Thus all scalers were blocked during the processing of the event. For forward scattering, the target pulse was added to a histogram, whereas for backward scattering, the analog-to-digital converter (ADC) outputs for the $T$, $\Delta E$ and $E$ signals were punched onto paper tape. After the computer had finished recording the event it reset the ADC's and the computer inhibit was removed.

Fig. 7 Data-recording system.
Two random coincidence rates to the master trigger were recorded by delaying $S_2 \cdot S_3$ (AND1) and $\Delta E$, respectively, by 58 nsec corresponding to the time delay between two beam pulses in the microstructure of the SC. As a control many count rates were recorded on 100 MHz scalers, for example, proton and pion numbers in $S_1$, $S_2$ and $S_3$, and total rates in the recoil counters.

3.2 Setting up

The setting up procedure was carried out with the C$_6$H$_{12}$ target and was the same for the two measuring methods. The counter $S_1$ was used as reference for timing in the other counters, since it had the best time resolution. The delay and discriminator settings of $S_2$ and $S_3$ were determined at 0° and corrected from calculations for each scattering angle. The upper edge of the pion discriminator windows was set just below the proton pulse band, in order to include most of the pion Landau tail. The lengths of the output pulses from the discriminators were fixed at 3 nsec for $S_1$ and 5 nsec for $S_2$, $S_3$, and $T$. The $\Delta E$ discriminator threshold was set to reject pions. The recoil counters were timed in by recording a delay curve for $^\pi p$ scattering and calculating from this curve the delay and length of the logic $\Delta E$ pulse required. Due to the finite target thickness, the deuterons had a wide time-of-flight spectrum, and pulse lengths of up to 20 nsec were used in order to be sure of accepting all of them.

The high voltages of counters $T$, $\Delta E$ and $E$ were kept as low as possible in order to cover the wide range of pulse heights without driving the photomultipliers into the saturation region. The scales of the ADC's were adjusted by attenuators ($A_2$ in Fig. 7). The gating pulse into the linear gate was 40 nsec wide to avoid clipping the tail of the very large deuteron pulses.

3.3 Small-angle scattering measurements

3.3.1 Spectrometer calibration

Owing to the low counting rate of $n$d scattering, it was impossible to scan the peak of the elastically-scattered pions by varying the field in the spectrometer. Consequently, the field was kept at a central position and wide scintillation counters were used behind the spectrometer in order to cover most of the peak.

The momentum scale of the spectrometer was calibrated by carefully measuring $^\pi p$ scattering at 0°, 30° and 60°. For this purpose, $S_3$ was replaced by narrow counters of 2.5 or 5 cm width. The observed Hall voltages are shown in column 3 of Table 2, where we compare

<table>
<thead>
<tr>
<th>$\theta$ [deg]</th>
<th>Pion momentum [MeV/c]</th>
<th>Observed Hall voltage [mV]</th>
<th>Calculated Hall voltage [mV]</th>
<th>Difference [mV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>365.4</td>
<td>115.1</td>
<td>reference</td>
<td>-</td>
</tr>
<tr>
<td>30°</td>
<td>345.2</td>
<td>108.5</td>
<td>108.7</td>
<td>-0.2</td>
</tr>
<tr>
<td>60°</td>
<td>298.6</td>
<td>94.0</td>
<td>94.1</td>
<td>-0.1</td>
</tr>
</tbody>
</table>
them with calculated values at 30° and 60° assuming that the momentum-to-Hall voltage ratio measured at 0° was also valid at the other angles. When calculating the Hall probe settings for πd scattering, the momentum-to-Hall voltage ratio of 0° was used and the calculated Hall voltage was roughly corrected using the last column in Table 2.

3.3.2 Data recording

Data were taken between 30° and 80° in steps of 10° for both C6D12 and C6H12; the spectrometer was set to the calculated magnetic field and the targets adjusted so as to align the long dimension of the target parallel to the calculated deuteron recoil angle.

The cross-section calibration was carried out at 30° using the known π+p scattering cross-section, and the background from the carbon nuclei in the C6H12 was measured by tuning the spectrometer off the pion peak due to scattering on protons.

Because of the long decay time of the liquid scintillator, pile-up effects were to be expected in the T counter spectra at high incident pion rates. In order to investigate this effect the T spectra were recorded with a delay of 58 nsec inserted into the T ADC line. The results of this test are discussed later in Section 4.1.4.

3.4 Large-angle scattering measurements

3.4.1 Calibration of the recoil counters

The response of the counters T, ΔE and E to protons of energy between 16 and 82 MeV was measured. For this purpose the proton absorber was removed and the Si electronics was adjusted to respond only to the protons present in the beam. The protons were momentum-analysed by the spectrometer which was positioned at 0°. The counters were equipped with long cables, so that they could be placed at the end of the spectrometer, and spectra were recorded via the counter's computer read-out system (linear gate, shaping amplifier, ADC). In this manner a calibration of the complete pulse-height analysis system of each counter was carried out.

In the data analysis the pulses in T and E were added by the computer (by adding their ADC outputs); therefore it was desirable that the two counters gave identical outputs for equal energy losses. Such a matching of T and E could not be obtained for all energy losses for two main reasons; firstly, the liquid and plastic scintillators have different energy loss versus light output characteristics; secondly, the energy loss per unit path length was much higher in the total absorption counter E than in T and organic scintillators are known to have a non-linear behaviour for high energy loss densities. Nevertheless, it could be shown that a clear separation of deuteron and proton events could be obtained in the scatter plots, even with a factor of two mismatching between E and T.

The outputs of counters T and E were therefore simply adjusted to be proportional to the calculated energy losses for a pion beam of 256 MeV. Figure 8 shows some of the calibration curves; the energy loss is calculated from Ref. 12 and the ADC channel numbers represent the average energy loss.

3.4.2 Data recording

Data were taken at 80°, 90°, 100°, 120°, 140° and 160° for pions of energy 256 MeV, and at 160° for pions of energy 141, 163, 185 and 208 MeV; the machine time remaining was used to measure the critical case of 75° at 256 MeV, where the deuterons were just able to escape from the target. For each point, data were accumulated for both the C6D12 and C6H12 target
Fig. 8 Examples of counter calibration curves as measured with protons of energy between 16 and 82 MeV.

(carbon background). Mechanically it was impossible to utilize a target-to-recoil counter separation smaller than 34 cm; for large scattering angles the separation was kept large enough to prevent the incident beam from entering the recoil counters.

The cross-section calibration was carried out for each data point. In this case, different separations between T and ΔE were utilized to study the scattering geometry and the background arising from carbon in C₆H₁₂ (see Section 4.2.1). This background was also measured by counting recoil protons at angles larger and smaller than that for protons arising from elastic π⁺p scattering; these measurements were then interpolated to the appropriate recoil angles.
In some cases the contents of the counters T, ΔE and E were measured 58 nsec after a master trigger to check the smearing out of the scatter plots due to other particles in the counters. The result of this test is given in Section 4.2.4.

4. DATA ANALYSIS

4.1 Small-angle scattering

4.1.1 Presentation of raw data

The two target pulse spectra measured with C₆D₁₂ and C₆H₁₂ were normalized to the same number of incident pions and then subtracted from each other. Thus the background due to the carbon (typically 15% under the peak) was eliminated. Figure 9 shows for example the net recoil spectrum at θ = 60°. At all angles a significant number of counts were observed in the channels lower than the deuteron peak. These events were accompanied by a pion having the appropriate momentum for a good event; hence they were presumably protons from a break-up reaction with a final state interaction of the two nucleons during the scattering process. Since only the proton effectively loses energy in the scintillator, the energy loss of these events would be smaller than that of deuterons; on the other hand, all scintillators have a larger light efficiency for protons than for deuterons. By extrapolating this background under the peak in two extreme manners (Fig. 9), it was subtracted from the peak. The difference between the two extrapolations defined the error connected with this procedure.

No significant carbon background was observed in the cross-section calibration run at 30°.

![Fig. 9 Pulse-height distribution in the scintillating target (C₆D₁₂) as measured at θ = 60°, after a subtraction of the background measured with C₆H₁₂. The remaining background is extrapolated under the deuteron peak by taking the mean value of the two extreme possibilities shown in the figure.](image)
4.1.2 Spectrometer acceptance correction

The momentum band of ±2% defined by S3 was the largest one which could be utilized, as the solid angle of the spectrometer dropped rapidly outside this band.\(^1\)\(^8\) Pions having larger or smaller momenta were lost. In order to estimate this loss, the particle distribution near S3 was calculated by simply assuming that all relevant quantities have symmetric distributions whose widths are added quadratically.

These widths (HMMP), all in the scattering plane, were taken as follows:

- \(\theta_b\): beam divergence (20 mrad),
- \(\theta_m\): multiple scattering angle (17 mrad) mostly due to the proton absorber,
- \(\theta_s\): angular acceptance of the spectrometer (20 mrad),
- \(\delta p\): momentum spread of the beam of momentum \(p\), including a contribution due to the energy loss in the proton absorber and the target (2.2 MeV/c).

The momentum distribution around the central value \(p'\) entering the spectrometer positioned at an angle \(\theta_\pi\) had the relative half width

\[
\left\{ \left( \frac{\theta_b^2 + \theta_m^2 + \theta_s^2}{3} \right)^2 \left( \frac{1}{p'} \frac{dp'}{d\theta_\pi} \right)^2 + \left( \frac{1}{p'} \frac{dp'}{dp} \delta p \right)^2 \right\}^{1/2},
\]

which multiplied with the spectrometer dispersion (see Section 2.3.2) gave the width \(W_1\) of the pion distribution at S3 for a point source.

The horizontal dimension (HMMP) of the spectrometer source was

\[
S = \left\{ \frac{1}{\sin^2 \theta_R} \left[ b^2 \sin^2 (\theta_R + \theta_\pi) + \frac{t^2}{4} \sin^2 \theta_\pi \right] \right\}^{1/2},
\]

where \(b\) (9 mm) was the horizontal beam width (HMMP) at the target, \(t\) (20 mm) the target thickness, and \(\theta_\pi\) and \(\theta_R\) the scattering and recoil angles, respectively. The terms in \(b\) and \(t\) were added quadratically.

The source image was

\[
W_2 = M \cdot S,
\]

where \(M = 1\) was the magnification of the spectrometer. We finally obtained the width of the pion distribution at S3 as

\[
W = (W_1^2 + W_2^2)^{1/2}.
\]

Assuming a Gaussian distribution the loss of pions could be found by looking up error function tables. In order to check the calculations, the obtained widths are compared in Table 5 with curves recorded for the spectrometer calibration (Section 3.3.1). The agreement is marginal but sufficient in view of the small value of the calculated correction factors for the \(\pi d\) cross-sections (between 0.984 at 30° and 1.023 at 80°).
Table 3

<table>
<thead>
<tr>
<th>Width of S3</th>
<th>Unfolded measured peak width [%]</th>
<th>Calculated peak width [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+ p$ at 30°</td>
<td>25 mm</td>
<td>1.07 ± 0.05</td>
</tr>
<tr>
<td>$\pi^+ p$ at 60°</td>
<td>50 mm</td>
<td>1.25 ± 0.07</td>
</tr>
</tbody>
</table>

In practice this correction factor was combined with the pion decay correction, where we assumed that all pions decaying in the spectrometer were lost, but muons produced in the last metre of flight path were counted. Within this assumption the decay correction to the $\pi d$ cross-sections did not exceed 2%.

4.1.3 Target wall correction

Because of the finite angular acceptance of the spectrometer and the divergence of the incident pion beam, scattering events taking place near the edges of the target cell were likely to produce recoil particles which entered the cell wall. In the case of the cross-section calibration ($\pi^+ p$ scattering) this effect was largely compensated by scattering events in the plexiglas, where the recoil protons entered the scintillator. For $\pi d$ scattering, however, a correction had to be applied.

Using the definitions in Section 4.1.2 the width (HWHM) of the recoil angle distribution was

$$\theta = (\theta_d^2 + \theta_m^2 + \theta_s^2)^{1/2} \frac{d\sigma}{d\theta} = 18 \text{ mrad for all angles}.$$

Assuming a Gaussian distribution for $\theta$, the average thickness of target layer was calculated from which all deuterons entered the wall. Including both walls, the thickness obtained amounted to 2.5% of the deuteron range. As the deuterons lose most of their energy at the end of their range, it was fair to assume that all deuterons reaching the wall were lost. A correction of at most 2% (80°) was obtained.

4.1.4 Random

The number of accidental master triggers was of the order of 5%. For most of these there should be no recoiling particle in the target and they were neglected. The spectrum of fractions of additional pulses in the T-counter during the gating time (see Section 3.3.2) was concentrated to 95% at the lowest channels, whereas the remaining 5% were distributed uniformly up to higher channels giving rise to a slight smearing of the spectra. This effect was taken into account by an error contribution to the final result.

4.2 Large-angle scattering

4.2.1 Presentation of raw data

The paper tapes containing the digitized T, $\Delta E$, and E response for each recoil particle were read into the computer which presented the distributions of the events as scatter plots, i.e. the energy loss in $\Delta E$ versus the summed energy loss in T and E (Fig. 10). The energy
Fig. 10 Energy loss scatter plots measured at 120°/256 MeV (above, thick recoil counters) and 160°/141 MeV (below, thin recoil counters) with (a) C6D12 and (b) C6H12 (background). For comparison the number of counts in (b) has to be scaled by a factor of two. The inner curves indicate the expected half-height contour of the deuteron region, the outer curves are assumed to contain all the deuterons. The energy loss scales refer to protons.
scales, as well as the separation line between protons and deuterons, were calculated using the counter calibration described in Section 3.4.1; in this calculation the energy loss of the pions in the target was also taken into account. The energy loss in the target wall and the air between T and ΔE was negligible. Large accumulations of points showed up in the scatter plots for both C4D2 and C6H2; these accumulations were interpreted as protons from quasi-elastic scattering on deuterons, and as protons from ?p scattering, respectively. Protons from quasi-elastic scattering on carbon might also have contributed to a lesser extent. When selecting the deuterons, all dots on the right-hand side of the proton-deuteron separation line could safely be taken, but with the aid of the counter calibration it was possible to predict a rough half-height contour for the deuteron region (inner curves in Fig. 10). Here the difference in light response for protons and for deuterons was taken into account. This was measured for plastic scintillator\(^{16}\), and for the liquid scintillator it was assumed to be the same. These predicted half-height contours agreed in shape and position with the visible accumulations of points. The deuteron regions selected for the data analysis were chosen to have a simple form (outer curves in Fig. 10), so that their boundaries could easily be fed into the computer which then counted the events contained within them. The carbon background measured with C6H2 was interpreted as deuterons or heavier fractions of broken-up carbon nuclei from the target. It was subtracted after having been normalized to the same number of incident pions and to the same target thickness. Changing the boundaries of the chosen deuteron regions by ±10% (the proton-deuteron separation line was always strictly respected) gave an estimate of the error in this method of analysis. It was only a fraction of the statistical uncertainty.

In the case of the cross-section calibrations performed at each angle, the number of master triggers was taken directly as a measure of the ?p scattering. The carbon background measured at larger and smaller recoil angles gave an interpolated value of about 5%. This was checked at 120° by varying the target-to-recoil counter separation between 34 and 42 cm, where the number of protons from ?p scattering was constant, but the number of protons or heavier particles from an inelastic reaction on carbon would be expected to follow the solid angle accepted by the ΔE counter. The two methods gave consistent answers, which indicated that there was no appreciable structure in the background due to quasi-elastic scattering.

4.2.2 Geometry correction

All calibration cross-section data, i.e. the number of ?p scattering events measured at different angles and target-to-recoil counter distances, were normalized to the same target thickness and to the same number of incident pions, and divided by the corresponding cross-sections which were reconstituted from the ?p phase shifts\(^{17}\). Plotting the result, it was observed that not all points followed the expected horizontal line (Fig. 11a). The differences between measurements made at different recoil counter positions were interpreted as due to recoil particles missing counter ΔE.

The scattering geometry was studied with the aid of a Monte Carlo program using the following variables:

- horizontal and vertical beam divergences given by a beam tracking program (20 and 45 mrad H\(\uparrow\)M, respectively, with Gaussian distributions),
- beam multiple scattering angles due to the proton absorber of 17 mrad horizontally and vertically (Gaussian distributions),
- 20 -

Fig. 1) Cross-section calibration data for different scattering angles and different target-to-recoil counter separations (a) before and (b) after the geometry correction was applied. The data were divided by the corresponding $\pi^p$ cross-sections and should follow a horizontal line.

- an interaction point in the target taking into account the measured beam distribution and the target geometry,
- the scattering angle $\theta$ appropriate to the angular dependence of the cross-section,
- the orientation of the scattering plane.

If the scattered pion struck S2 placed at a given angle, it was subjected to multiple scattering according to its momentum and followed up to S3. In the case where it hit S3, the direction of the recoiling particle was calculated after it had undergone multiple scattering on its way out of the target.

The program gave the following quantities:
- the pion solid angle (very close to the values of 15 and 30 msr given in Section 2.4.2),
- the angular width of the scattered pions accepted by the pion telescope ($2.4^\circ$ HWHM for 15 msr and $4.5^\circ$ for 30 msr),
- the loss of recoil particles for a given target-to-recoil counter separation.

When correcting this loss, the situation improved considerably (Fig. 1b). But there still remained a slight trend toward lower count rates for the larger separations indicating that the Monte Carlo program did not simulate the situation completely. However, since the $\pi^p$ cross-sections were sensitive only to the relative corrections between scatterings on protons
and deuterons, the residual separation dependence should have only a very small effect. The largest correction was applied to the \( ^{1}d \) cross-section at 80° measured with a 42 cm distance, where it amounted to 3%.

The \( ^{1}d \) cross-sections obtained were of course the integral values appropriate to the detector angular acceptance. At 80°, where a rapid variation of the cross-section was observed, the unfolding procedure resulted in a lowering of 5 ± 2%, whereas it could be neglected at other angles.

4.2.3 Break-up of deuterons in the counters

Deuterons slowing down in the counters are to some extent broken up by the electromagnetic and nuclear interactions. Although these processes probably depend on the deuteron energy in a rather complex way, it has been shown experimentally that a simple attenuation model describes this effect quite well\(^{17}\), where the cross-section is assumed to be approximately equal to the geometrical cross-section. The correction factor obtained by this method is shown in Fig. 12, where it is assumed that all broken-up deuterons are lost.

4.2.4 Randoms

The amount of the accidental master triggers was \( \leq 3\% \), except for 160° at 256 MeV, where there was a high particle flux, and where the counters were placed in the beam halo (~25% randoms). In this case the quality of the scatter plots was deteriorated. Since the randoms were roughly the same and had similar distributions for \( C_{6}D_{12} \) and \( C_{6}H_{12} \), they cancelled approximately in the subtraction. For the calibration runs, they were subtracted from the master trigger rate.

The scatter plots of fractions of additional pulses in the counters during the gating time (see Section 3.4.2) showed that 99% of the dots were accumulated close to the origin. Thus the probability of a pile-up effect shifting a proton event into the deuteron region was extremely small.

![Correction factor for break-up of the deuterons in the counters.](image_url)

**Fig. 12** Correction factor for break-up of the deuterons in the counters.
5. RESULTS AND DISCUSSION

5.1 Presentation of data

The measured differential cross-sections are listed in Tables 4 and 5 (where all corrections mentioned above have been applied), together with the values used for the calibration cross-sections. It can be seen that the two different measuring methods gave consistent answers in the overlap region. The uncertainties were obtained by adding the errors of different sources quadratically.

5.2 Errors of small-angle scattering data

The main sources of error were the statistical uncertainty (from $\pm 2.5\%$ at $30^\circ$ to $\pm 13\%$ at $80^\circ$) and the extrapolation procedure described in Section 4.1.1 and Fig. 9 (between $\pm 4\%$ at $30^\circ$ and $\pm 10.5\%$ at $80^\circ$). The uncertainty of the cross-section calibration performed at $30^\circ$ was $\pm 5\%$ including statistics and the target pulse spectrum cuts. To all the cross-sections, an additional error of $\pm 3\%$ was added for the smearing of the spectra by random pulses as mentioned in Section 4.1.4.

Uncertainties in the spectrometer acceptance correction, the pion decay correction, and the target wall correction were, in view of the smallness of the corrections, neglected. The target thicknesses were known to better than 1%.

5.3 Errors of large-angle scattering data

Here the most important error source was of statistical nature: between $\pm 5\%$ and $\pm 10\%$ at 256 MeV and from $\pm 10\%$ to $\pm 30\%$ at the other energies. The choice of the deuteron region resulted in a further error contribution which was never larger than half the statistical one. The cross-section calibration error amounted to typically $\pm 5\%$. If we believe that the deuteron stripping correction is accurate to 30%, we obtain a further error contribution of at most $\pm 3\%$ in the worst case ($160^\circ$ at 256 MeV). The maximum error in the geometric corrections would be below 1% due to cancellations and was neglected. The target thicknesses are known to $\pm 1\%$.

5.4 Comparison with other data

The backward-angle cross-sections at 141 MeV and 185 MeV agree with results from bubble-chamber measurements performed at 142 MeV $^9$) and 183 MeV $^{18}$), respectively. In the momentum range between 496 MeV/c and 1050 MeV/c there are a series of large-angle measurements $^{18}$). In view of the rapid energy variation of the cross-section and the gap of 130 MeV between the two experiments, it is impossible to prove a consistency of the two measurements; however, a smooth line can be drawn through the points of the two experiments. An emulsion measurement $^{19}$), performed at 300 MeV, gives a value of 0.2 mb which is a factor of three too large compared to the value expected from the other experiments.

5.5 Discussion of the experiment

With respect to the high intensity meson factories coming into operation in the near future, our measuring method loses attractiveness. High pion fluxes allow one to utilize gas targets $^{20}$) where the recoil deuterons can always be identified outside the target. Moreover, the scintillating targets would be useless in a high intensity beam, since the long decay time of the liquid scintillator would make a pulse-height analysis impossible. Hence
Table 4

$\pi^+d$ elastic differential cross-section at 256 MeV as measured with the small-angle technique ($30^\circ$-$80^\circ$) and the large angle technique ($75^\circ$-$160^\circ$).

The errors do not contain the uncertainty of the calibration cross-section.

<table>
<thead>
<tr>
<th>$\theta_\pi$ (lab) [deg]</th>
<th>Lab. differential cross-section $(d\sigma/d\Omega)_{\text{lab}}$ [ub/sr]</th>
<th>$\theta_\pi$ (c.m.) [deg]</th>
<th>c.m. differential cross-section $(d\sigma/d\Omega)_{\text{cm}}$ [ub/sr]</th>
<th>$t$ [(GeV/c)$^2$]</th>
<th>$\pi^+ + p \rightarrow \pi^+ + p$ calibration cross-section $(d\sigma/d\Omega)_{\text{lab}}$ [mb/sr]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$(12.37 \pm 0.93) \times 10^3$</td>
<td>35</td>
<td>$(9.16 \pm 0.69) \times 10^3$</td>
<td>0.0556</td>
<td>25.96</td>
</tr>
<tr>
<td>40</td>
<td>$(5.29 \pm 0.38) \times 10^3$</td>
<td>47</td>
<td>$(4.08 \pm 0.51) \times 10^3$</td>
<td>0.0610</td>
<td>-</td>
</tr>
<tr>
<td>50</td>
<td>$(1.87 \pm 0.15) \times 10^3$</td>
<td>58</td>
<td>$(1.52 \pm 0.13) \times 10^3$</td>
<td>0.0910</td>
<td>-</td>
</tr>
<tr>
<td>60</td>
<td>529 $\pm$ 91</td>
<td>69</td>
<td>452 $\pm$ 77</td>
<td>0.1237</td>
<td>-</td>
</tr>
<tr>
<td>70</td>
<td>144 $\pm$ 25</td>
<td>80</td>
<td>133 $\pm$ 23</td>
<td>0.1582</td>
<td>-</td>
</tr>
<tr>
<td>80</td>
<td>65 $\pm$ 12</td>
<td>90</td>
<td>61 $\pm$ 12</td>
<td>0.1925</td>
<td>-</td>
</tr>
<tr>
<td>75</td>
<td>79.8 $\pm$ 7.4</td>
<td>85</td>
<td>74.8 $\pm$ 6.9</td>
<td>0.1754</td>
<td>3.58</td>
</tr>
<tr>
<td>80</td>
<td>51.6 $\pm$ 4.9</td>
<td>90</td>
<td>49.9 $\pm$ 4.7</td>
<td>0.1925</td>
<td>3.17</td>
</tr>
<tr>
<td>90</td>
<td>45.6 $\pm$ 2.9</td>
<td>100</td>
<td>46.8 $\pm$ 3.0</td>
<td>0.2261</td>
<td>2.98</td>
</tr>
<tr>
<td>100</td>
<td>57.7 $\pm$ 3.3</td>
<td>110</td>
<td>41.2 $\pm$ 3.6</td>
<td>0.2574</td>
<td>3.50</td>
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<tr>
<td>120</td>
<td>42.0 $\pm$ 3.0</td>
<td>129</td>
<td>51.4 $\pm$ 3.6</td>
<td>0.3115</td>
<td>4.57</td>
</tr>
<tr>
<td>140</td>
<td>59.6 $\pm$ 5.2</td>
<td>146</td>
<td>79.7 $\pm$ 7.0</td>
<td>0.3512</td>
<td>5.22</td>
</tr>
<tr>
<td>160</td>
<td>58.4 $\pm$ 7.4</td>
<td>163</td>
<td>82.5 $\pm$ 10.5</td>
<td>0.3755</td>
<td>5.67</td>
</tr>
</tbody>
</table>

Table 5

Energy dependence of the $\pi^+d$ elastic differential cross-section at 160°$^{\text{lab}}$.

The errors do not contain the uncertainty of the calibration cross-section.

<table>
<thead>
<tr>
<th>Lab. pion energy [MeV]</th>
<th>Lab. differential cross-section $(d\sigma/d\Omega)_{\text{lab}}$ [ub/sr]</th>
<th>c.m. pion energy [MeV]</th>
<th>c.m. differential cross-section $(d\sigma/d\Omega)_{\text{cm}}$ [ub/sr]</th>
<th>$t$ [(GeV/c)$^2$]</th>
<th>$\pi^+ + p \rightarrow \pi^+ + p$ calibration cross-section $(d\sigma/d\Omega)_{\text{lab}}$ [mb/sr]</th>
</tr>
</thead>
<tbody>
<tr>
<td>141</td>
<td>816 $\pm$ 91</td>
<td>115</td>
<td>1060 $\pm$ 118</td>
<td>0.1769</td>
<td>15.43</td>
</tr>
<tr>
<td>163</td>
<td>398 $\pm$ 44</td>
<td>132</td>
<td>526 $\pm$ 59</td>
<td>0.2115</td>
<td>18.16</td>
</tr>
<tr>
<td>185</td>
<td>193 $\pm$ 36</td>
<td>149</td>
<td>259 $\pm$ 48</td>
<td>0.2483</td>
<td>16.20</td>
</tr>
<tr>
<td>208</td>
<td>141 $\pm$ 53</td>
<td>165</td>
<td>192 $\pm$ 72</td>
<td>0.2874</td>
<td>12.19</td>
</tr>
<tr>
<td>256</td>
<td>58.4 $\pm$ 7.4</td>
<td>200</td>
<td>82.5 $\pm$ 10.5</td>
<td>0.3755</td>
<td>5.67</td>
</tr>
</tbody>
</table>
we conclude that the use of a scintillating target for elastic $\pi d$ scattering is appropriate for the old generation of machines and a good alternative to the bubble-chamber method used until now.

The accuracy of the order of 10% is due mainly to statistical errors. These can of course be reduced, in principle, by investing more running time; however, at a certain stage the other errors become dominant, i.e. the uncertainties in the selection of the deuteron peak in small-angle scattering and the deuteron region in large-angle scattering. The cross-section calibration induces an additional error, but releases us from the problems of the beam contamination and the solid angle. In addition, the weights of some corrections, which are applied to both $\pi p$ and $\pi d$ scatterings, are reduced.

6. BRUECKNER MODEL CALCULATION

6.1 Introduction

It is important in our energy range to incorporate into any calculation of $\pi d$ scattering the nucleon spin-flip, the double charge exchange, the deuteron D-state and a reliable wavefunction. All these features can easily be built into a Glauber model, but, as already mentioned in Section 1, this would be unreliable at our energies, especially at the large angles upon which our effort has been concentrated. We shall therefore examine the consequences of the Brueckner model. As in the Glauber picture, the dynamics of the target are completely neglected. The justification of this is the light mass of the pion, which makes it difficult to transfer energy to the deuteron. Thus we calculate the $\pi d$ scattering amplitude for two fixed scattering centres and average it over all possible deuteron positions by means of the deuteron wave-function.

The other important assumption incorporated in the Brueckner model is that there is no spatial overlap of the two scatterers. This implies that the on-shell amplitude for the scattering from the combined system is a function only of the on-shell individual pion-nucleon amplitudes$^{21)}$. For the deuteron this non-overlapping of the nucleons is a plausible hypothesis.

For simplicity, we restrict ourselves just to s- and p-wave pion-nucleon scattering. However, as we shall see later, the neglect of the deuteron dynamics leads to unfortunate kinematical problems. The calculations are performed strictly in the Breit frame. A consequence of this is that the s- and p-wave $\pi N$ amplitudes of the c.m. frame, being transformed into the Breit frame, show a higher order angular dependence. Therefore, in the Breit frame the s- and p-wave amplitudes must be chosen in a judicious way.

6.2 General procedure

The task is to calculate the $\pi d$ amplitude $F_{d}$ for the scattering from two fixed "potentials", which represent the proton and neutron of the deuteron, and to average the result over all possible positions $\vec{r}_{p}$ and $\vec{r}_{n}$ of the nucleons. This averaging procedure is to be carried out with the help of the deuteron wave-function $\phi_{d}(\vec{r})$:

$$T_{M'M}^{(k,q)} = \int d^{3}r \phi_{M'}^{*}(\vec{r})F_{d}\left(k, q; \frac{\vec{r}_{p}}{2}, \frac{\vec{r}_{p}}{2} = -\frac{\vec{r}}{2}, \frac{\vec{r}}{2}\right)\phi_{M}(\vec{r}).$$

(1)
\( T_{M'M}(\vec{k}, \vec{q}) \) is the transition amplitude from the initial deuteron spin projection \( M \) to the final spin projection \( M' \) for an incident pion of wave vector \( \vec{k} \) and for a momentum transfer \( \vec{q} \). For convenience the origin is chosen at the centre of mass of the deuteron:

\[
\vec{R}_{\text{cm}} = \frac{\vec{r}_p + \vec{r}_n}{2} = 0,
\]

and therefore

\[
\vec{r}_p = -\vec{r}_n = \frac{\vec{r}}{2},
\]

where \( |\vec{r}| \) is the proton-neutron separation.

The deuteron wave-function of spin projection \( M \) and proton-neutron separation \( r \) is:

\[
\phi_M(\vec{r}) = \sum_{m_1m_2} \left[ \frac{1}{2} \right] \begin{pmatrix} \begin{array}{c} m_1m_2 \\ 1/2 \end{array} \end{pmatrix} |M\rangle \begin{pmatrix} 1/2 \end{pmatrix} Y_{00}(\vec{r}) \chi_{m_1}^{p} \chi_{m_2}^{n} + \nonumber \\
\times \begin{pmatrix} 2, 1, M - m_1 - m_2, m_1 + m_2 \end{pmatrix} \begin{pmatrix} 1/2 \end{pmatrix} Y_{2\,M-m_1-m_2}(\vec{r}) \chi_{m_1}^{p} \chi_{m_2}^{n},
\]

where \( \chi_{m_1}^{p} \) and \( \chi_{m_2}^{n} \) are the proton and neutron Pauli spinors of projection \( m_1 \) and \( m_2 \). The functions \( S(r) \) and \( D(r) \) are the S and D radial wave-functions, respectively. They can both be chosen to be real and are normalized so that

\[
\int_0^\infty \left[ S(r)^2 + D(r)^2 \right] \, dr = 1.
\]

Since we need, in our model of non-overlapping scatterers, a radial wave-function with a hard core, we choose one which is derived by Hamberston from the Hamada-Johnston two-nucleon potential\(^{22}\). Its parameters are listed in Ref. 3. In particular, it predicts a D-state probability of 7%.

The \( \pi d \) scattering amplitude \( F_d \) is obtained by solving the wave equation outside the scatterers and by looking at the asymptotic behaviour of the wave-function. Outside the pion-nucleon "potential" the wave-function is given simply by the on-shell pion-nucleon amplitudes, but does not depend on the details of the "potentials" \(^{21}\). In order to calculate the wave-function outside the two non-overlapping scatterers, the two "potentials" can therefore be reduced to two points provided we keep the scattering amplitudes the same.
6.3 *s*-wave scattering

We solve the wave equation outside the two point-like scatterers (and this is valid everywhere but at \( \mathbf{r}_p \) and \( \mathbf{r}_n \)):

\[
\psi^S(\mathbf{x}) \sim e^{i \mathbf{k} \cdot \mathbf{x}} + A e^{i k |\mathbf{x} - \mathbf{r}_p|} + B e^{i k |\mathbf{x} - \mathbf{r}_n|},
\]

Equation (6) contains an incident plane wave plus spherical waves (s-waves) outgoing from \( \mathbf{r}_p \) and \( \mathbf{r}_n \). The amplitude \( \lambda \) of the outgoing wave from \( \mathbf{r}_p \) is the \( \tau \) s-wave elastic scattering amplitude \( f_p(\lambda) \) multiplied by the wave arriving at \( \mathbf{r}_p \):

\[
\lambda = f_p \left( e^{i k \mathbf{r}_p} + B e^{i k |\mathbf{r}_p - \mathbf{r}_n|} \right),
\]

and similarly

\[
\lambda = f_n \left( e^{i k \mathbf{r}_n} + A e^{i k |\mathbf{r}_n - \mathbf{r}_p|} \right).
\]

Solving Eqs. (7a) and (7b) we obtain, using Eq. (5):

\[
\lambda = \left[ f_p \exp \left( i \mathbf{k} \cdot \mathbf{r} \right) + f_p f_n h(\lambda) \exp \left( -i \mathbf{k} \cdot \mathbf{r} \right) \right] \left[ 1 - f_p f_n h(\lambda)^2 \right]^{-1},
\]

with \( h(\lambda) \equiv ke^{i \lambda}/x \).

The \( \pi d \) scattering amplitude is then obtained by looking at the asymptotic behaviour of the wave-function (6)

\[
F_{d}^S(\mathbf{k}, \mathbf{k}') = \lambda \exp \left( -i \mathbf{k} \cdot \mathbf{r} \right) + B \exp \left( i \mathbf{k} \cdot \mathbf{r} \right),
\]

where \( \mathbf{k} \) is the wave vector of the incident pion. Consequently we obtain for a fixed deuteron position

\[
F_{d}^S(\mathbf{k}, \mathbf{k}') = \left[ f_p \exp \left( i \frac{\mathbf{q} \cdot \mathbf{r}}{2} \right) + f_n \exp \left( -i \frac{\mathbf{q} \cdot \mathbf{r}}{2} \right) + 2f_p f_n h(\lambda) \cos \left( \mathbf{k} \cdot \mathbf{r} \right) \right] \times
\]

\[
\left[ 1 - f_p f_n h(\lambda)^2 \right]^{-1},
\]
with
\[
\mathbf{k} = \frac{\mathbf{k} + \mathbf{k}'}{2}, \quad \mathbf{q} = \mathbf{k} - \mathbf{k}', \quad \mathbf{q} \cdot \mathbf{k} = 0.
\]

(11)

In Eq. (10) single and double scattering appears explicitly, and higher order scatterings manifest themselves when we expand the denominator
\[
\left[1 - \frac{f_p f_n h(kr)}{p n h(kr)}\right] = 1 + \frac{f_p f_n h(kr)}{p n h(kr)} + \ldots.
\]

(12)

In the present calculation only single and double scattering will be considered.

6.4 p-wave scattering

For s-wave scattering a nucleon spin-flip is impossible; in p-wave scattering, however, the spin-flip of one or both nucleons must be considered.

The procedure of Section 6.3 for s-wave scattering can, in principle, be applied to p-wave scattering\(^e\), but the complication of spin-flip makes it rather difficult to solve the wave equation. Restricting ourselves to single and double scattering only, the \(\pi n\) scattering amplitude is given by\(^f\)

\[
F_p(\mathbf{k}, \mathbf{k}') = F_p(\mathbf{k}, \mathbf{k}') \exp \left( -i \frac{\mathbf{k} - \mathbf{k}'}{2} \right) + F_n(\mathbf{k}, \mathbf{k}') \exp \left( +i \frac{\mathbf{k} - \mathbf{k}'}{2} \right) -
\]

\[
+ \frac{4\pi}{(2\pi)^3} \int \frac{d^3k''}{k''^2 - k^2} \left\{ e^{i\mathbf{k}'' \cdot \mathbf{r}} e^{-i\mathbf{k}'' \cdot \mathbf{r}} F_p(\mathbf{k}, \mathbf{k}'') F_n(\mathbf{k}, \mathbf{k}') + e^{-i\mathbf{k}'' \cdot \mathbf{r}} e^{i\mathbf{k}'' \cdot \mathbf{r}} F_p(\mathbf{k}, \mathbf{k}'') F_n(\mathbf{k}, \mathbf{k}') \right\} \bigg|_{\epsilon = 0},
\]

(13)

where \(F_N(\mathbf{k}_1, \mathbf{k}_2)\) is the full \(\pi N\) scattering amplitude.

We have not yet incorporated the assumption of non-overlapping scattering centres. This implies that only on-shell effects are relevant and that Eq. (13) indeed is equivalent to the outcome of a calculation using the method described for s-wave scattering, providing that only single and double scattering are taken into account. Thus the s-wave scattering result can also be obtained using Eq. (13), but this method does not show how higher order scattering terms appear in the calculation.

The pion-nucleon p-wave scattering amplitude \(F^p_N\) has a non-spin-flip and a spin-flip part:

\[
F^p_N(\mathbf{k}, \mathbf{k}') = \frac{\alpha^p_N(k)}{k^2} \mathbf{k} \times (\mathbf{k} \times \mathbf{k}') + \frac{\beta^p_N(k)}{k^2} i\mathbf{r} \cdot (\mathbf{k} \times \mathbf{k}').
\]

(14)

The quantities \(\alpha\) and \(\beta\) only depend on the incident wave number \(k\), since we just need to consider on-shell contributions, and can therefore be taken out of the integral in Eq. (13).
Using this p-wave amplitude, we obtain the following amplitudes for a fixed deuteron position:

i) Single scattering, non-spin-flip

\[ F^d_d(\vec{k}, \vec{k}') = \frac{\alpha_p}{k^2} (\vec{k} \cdot \vec{k}') \exp\left(i \frac{\hat{q} \cdot \hat{r}}{2}\right) + \frac{\alpha_n}{k^2} (\vec{k} \cdot \vec{k}') \exp\left(-i \frac{\hat{q} \cdot \hat{r}}{2}\right). \]  \hspace{1cm} (15)

ii) Single scattering, spin-flip

\[ F^d_d(\vec{k}, \vec{k}') = \frac{\beta_p}{k^2} i \hat{r} \cdot (\vec{k} \times \vec{k}') \exp\left(i \frac{\hat{q} \cdot \hat{r}}{2}\right) + \frac{\beta_n}{k^2} i \hat{r} \cdot (\vec{k} \times \vec{k}') \exp\left(-i \frac{\hat{q} \cdot \hat{r}}{2}\right). \]  \hspace{1cm} (16)

iii) Double scattering, non-spin-flip

\[ F^d_d(\vec{k}, \vec{k}') = -2 \frac{\alpha_p \alpha_n}{k^3} \cos (\vec{k} \cdot \hat{r}) \left[ f(\vec{k}) \cos (\vec{k} \cdot \hat{r}) + g(\vec{k}) \frac{(\vec{k} \cdot \hat{r})(\vec{k}' \cdot \hat{r})}{r^2} \right]. \]  \hspace{1cm} (17)

iv) Double scattering, single spin-flip

\[ F^d_d(\vec{k}, \vec{k}') = -2i \frac{\alpha_p \beta_n}{k^3} \cos (\vec{k} \cdot \hat{r}) \left\{ f(\vec{k}) \hat{r} \cdot (\vec{k} \times \vec{k}') + \right. \]

\[ + \frac{1}{2r^2} g(\vec{k}) \left[ \hat{r} \cdot (\vec{k} \times \vec{k}') (\vec{k} \cdot \hat{r}) - \hat{r} \cdot (\vec{k} \times \vec{k}')(\vec{k}' \cdot \hat{r}) \right] \]

\[ - 2i \frac{\beta_p \alpha_n}{k^3} \cos (\vec{k} \cdot \hat{r}) \left\{ f(\vec{k}) \hat{k} \cdot (\vec{k} \times \vec{k}') + \right. \]

\[ + \frac{1}{2r^2} g(\vec{k}) \left[ \hat{k} \cdot (\vec{k} \times \vec{k}') (\vec{k} \cdot \hat{r}) - \hat{k} \cdot (\vec{k} \times \vec{k}')(\vec{k}' \cdot \hat{r}) \right] \}. \]  \hspace{1cm} (18)

v) Double scattering, double spin-flip

\[ F^d_d(\vec{k}, \vec{k}') = -\frac{\beta_p \beta_n}{k^3} \cos (\vec{k} \cdot \hat{r}) \left\{ f(\vec{k}) \left[ \left( \hat{r} \cdot (\vec{k} \times \vec{k}') \right) + \left( \hat{k} \cdot (\vec{k} \times \vec{k}') \right) \cdot (\vec{k}_n \times \vec{k}) \right] + \right. \]

\[ + \frac{g(\vec{k})}{r^2} \left[ \hat{r} \cdot (\vec{k} \times \vec{k}) \hat{k}_n \cdot (\vec{k} \times \vec{k}') + \hat{k} \cdot (\vec{k} \times \vec{k}') \hat{k}_n \cdot (\vec{k} \times \vec{k}) \right] \}. \]  \hspace{1cm} (19)

Here we use the definitions

\[ f(x) = k \frac{1}{x} \frac{d}{dx} \frac{e^{ix}}{x}, \]

\[ g(x) = x \frac{df}{dx}. \]  \hspace{1cm} (20)
In the cases (iv) and (v) terms odd in $\hat{F}$ have been dropped, since they will not contribute when integrated according to Eq. (3).

6.5 Mixed double scattering

In this section we consider the cases of double scattering, where we have s-wave scattering by one and p-wave scattering by the other nucleon. The result is obtained from the integral of Eq. (13), with one p-wave amplitude and one s-wave amplitude $f_{\pi}$. Terms odd in $\hat{F}$ have been dropped.

i) Non-spin-flip

$$p_{d}^{\text{ps}} = -2 \frac{\alpha_{p} f_{\pi} + \alpha_{n} f_{p}}{k^{2}} f(kr)(\hat{R} \cdot \hat{r}) \sin (\hat{R} \cdot \hat{r}) .$$  \hspace{1cm} (21)

ii) Spin-flip

$$F_{d}^{\text{ps}} = i \frac{\delta_{p} f_{\pi}}{k^{2}} f(kr) \hat{q} \cdot (\hat{r} \times \hat{r}) \sin (\hat{R} \cdot \hat{r}) + i \frac{\delta_{n} f_{p}}{k^{2}} f(kr) \hat{q} \cdot (\hat{r} \times \hat{r}) \sin (\hat{R} \cdot \hat{r}) .$$  \hspace{1cm} (22)

6.6 Integration over the deuteron orientation

We choose $\hat{q}$ to lie in the direction of the z-axis (quantization axis) and $\hat{R}$ to coincide with the x-axis. The angular part of the integration in Eq. (1) is performed by first expanding all the $\hat{F}$-dependent quantities in terms of spherical harmonics $Y_{l_{m}}$ and spherical Bessel functions $j_{l}$. The integrand of Eq. (1) contains products of between three and five spherical harmonics which are reduced, where necessary, to products of three spherical harmonics by

$$Y_{l_{1}m_{1}}(\hat{F})Y_{l_{2}m_{2}}(\hat{F}) = \sum_{l_{3}m_{3}} \left[ \frac{(2l_{1} + 1)(2l_{2} + 1)(2l_{3} + 1)}{4\pi} \right]^{1/2} \times$$

$$\times \left( \begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ m_{1} & m_{2} & m_{3} \end{array} \right) \left( \begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array} \right) Y_{l_{3}m_{3}}^{*}(\hat{F}) .$$  \hspace{1cm} (23)

These are then integrated via

$$\int_{\Omega} d\hat{F} Y_{l_{1}m_{1}}(\hat{F})Y_{l_{2}m_{2}}(\hat{F})Y_{l_{3}m_{3}}(\hat{F})$$

$$= \left[ \frac{(2l_{1} + 1)(2l_{2} + 1)(2l_{3} + 1)}{4\pi} \right]^{1/2} \left( \begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ m_{1} & m_{2} & m_{3} \end{array} \right) \left( \begin{array}{ccc} l_{1} & l_{2} & l_{3} \\ 0 & 0 & 0 \end{array} \right) .$$  \hspace{1cm} (24)

The Pauli spin matrix acts on the appropriate spinor of $\Psi_{M}(\hat{r})$ in Eq. (1), and we are left with products of $\chi_{m_{1}}^{N_{1}}\chi_{m_{2}}^{N_{2}}$, which is 1 for $m_{1} = m_{2}$ and 0 otherwise.
6.7 The transition amplitudes

The integration over \( r \) cannot be performed analytically. As the formulae are very lengthy, we shall give here only the deuteron S- and D-wave single scattering amplitudes and the deuteron S-wave double-scattering amplitudes. The deuteron D-wave double-scattering amplitudes are given in the Appendix. For the double-spin-flip case all terms with the deuteron both initially and finally in the D-state are neglected. The following results agree, in the absence of the D-state admixture, with the formula given in Ref. 7. The D-state amplitudes have been checked by comparing them in the limit \( kr \gg 1 \) near the forward direction to the Glauber result.\(^3\)

Using the Dirac notation for the \( r \)-integration we find

\[
T_{11} = T_{-1-1} = \left[ \mathbf{f}_p + \mathbf{f}_n + \left( \alpha_p + \alpha_n \right) \cos \theta \right] \left[ \langle S | j_0 \left( \frac{\mathbf{r}}{2} \right) | S \rangle - \right.
\]

\[
\left. - \sqrt{2} \left( \langle S | j_2 \left( \frac{\mathbf{r}}{2} \right) | d \rangle + \langle d | j_0 \left( \frac{\mathbf{r}}{2} \right) \rangle + \frac{1}{2} \langle j_2 \left( \frac{\mathbf{r}}{2} \right) | d \rangle \right) \right] +
\]

\[
+ 2f_p f_n \langle S | h(kr)j_0(Kr) | S \rangle - 2 \frac{\alpha_p \alpha_n \mathbf{K}^2}{k^2} \cos \theta \langle \mathbf{S} | f(kr)j_0(Kr) | S \rangle -
\]

\[
- \frac{2}{3} \alpha_p \alpha_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) - 2j_2(Kr) \right] | S \rangle +
\]

\[
+ \frac{1}{6} \alpha_p \alpha_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) + j_2(Kr) \right] | S \rangle -
\]

\[
- 2 \beta_p \beta_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | f(kr)j_0(Kr) | S \rangle - \frac{2}{3} \beta_p \beta_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) + j_2(Kr) \right] | S \rangle -
\]

\[
- 2 \left( \alpha_p f_n + \alpha_n f_p \right) \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | f(kr) \cdot r \cdot j_1(Kr) | S \rangle
\]

\[(25)\]

\[
T_{00} = \left[ \mathbf{f}_p + \mathbf{f}_n + \left( \alpha_p + \alpha_n \right) \cos \theta \right] \left[ \langle S | j_0 \left( \frac{\mathbf{r}}{2} \right) | S \rangle +
\]

\[
+ \sqrt{2} \left( \langle S | j_2 \left( \frac{\mathbf{r}}{2} \right) | d \rangle + \langle d | j_0 \left( \frac{\mathbf{r}}{2} \right) \rangle + \frac{1}{2} \langle j_2 \left( \frac{\mathbf{r}}{2} \right) | d \rangle \right) \right] +
\]

\[
+ 2f_p f_n \langle S | h(kr)j_0(Kr) | S \rangle - 2 \frac{\alpha_p \alpha_n \mathbf{K}^2}{k^2} \cos \theta \langle \mathbf{S} | f(kr)j_0(Kr) | S \rangle -
\]

\[
- \frac{2}{3} \alpha_p \alpha_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) - 2j_2(Kr) \right] | S \rangle +
\]

\[
+ \frac{1}{6} \alpha_p \alpha_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) + j_2(Kr) \right] | S \rangle + \beta_p \beta_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | f(kr)j_0(Kr) | S \rangle +
\]

\[
+ \frac{1}{2} \beta_p \beta_n \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | g(kr) \left[ j_0(Kr) - \frac{1}{2} j_2(Kr) \right] | S \rangle - 2 \left( \alpha_p f_n + \alpha_n f_p \right) \frac{\mathbf{K}^2}{k^2} \langle \mathbf{S} | f(kr) \cdot r \cdot j_1(Kr) | S \rangle
\]

\[(26)\]
\[ T_{1-1} = T_{-11} = 2\beta_p \beta_n \frac{K^2}{k^4} \langle S | f(kr) j_0(kr) | S \rangle + \]
\[ + \frac{2}{3} \beta_p \beta_n \frac{K^2}{k^4} \langle S | g(kr) [j_0(kr) + j_2(kr)] | S \rangle + \]
\[ + \frac{1}{2} \beta_p \beta_n \frac{Q^2}{k^4} \langle S | g(kr) j_2(kr) | S \rangle \]  (27)

\[ T_{10} = -T_{01} = -T_{-10} = T_{0-1} \]
\[ = \frac{1}{\sqrt{2}} \left( \beta_p + \beta_n \right) \sin \theta \left[ \langle S | j_0 \left( \frac{q}{2} \right) | S \rangle + \frac{1}{\sqrt{2}} \langle S | j_2 \left( \frac{q}{2} \right) | D \rangle - \frac{1}{\sqrt{2}} \langle D | j_0 \left( \frac{q}{2} \right) - j_2 \left( \frac{q}{2} \right) | D \rangle \right] - \]
\[ - \sqrt{2} \frac{\beta_p \beta_n + \alpha \beta n}{k^2} \sin \theta \langle S | f(kr) j_0(kr) | S \rangle - \]
\[ - \frac{\sqrt{2}}{3} \frac{\beta_p \beta_n + \alpha \beta n}{k^2} \sin \theta \langle S | g(kr) \left[ j_0(kr) - \frac{1}{2} j_2(kr) \right] | S \rangle - \]
\[ - \frac{1}{\sqrt{2}} \left( f_p \beta_n + f_n \beta_p \right) \frac{Q}{k^2} \langle S | f(kr) \cdot r \cdot j_1(kr) | S \rangle . \]  (28)

The unpolarized cross-section is
\[ \frac{d\sigma}{d\Omega} = \frac{1}{3} \sum_{M M'} |T_{M M'}|^2 . \]  (29)

6.8 Pion-nucleon scattering amplitudes and kinematics

6.8.1 General

Since the nucleons are not infinitely heavy, the most convenient frame for our calculations is the Breit or brick-wall frame, the kinematics of which is illustrated in Fig. 13.

![Fig. 13 The Breit frame.](image-url)
It treats the initial and final states symmetrically. The pions are scattered off an infinitely heavy target, losing no energy, as we assumed in our model.

The pion-nucleon scattering amplitudes are obtained in the c.m. frame from phase shifts and then must be transformed into the Breit frame.

6.8.2 Single scattering

Given the laboratory momentum $k_{\text{lab}}$ of the incident pions and the laboratory scattering angle $\theta_{\text{lab}}$, we proceed in the following steps:

i) Calculate for the $\pi$d system the Lorentz invariants $s$ and $t$.

ii) Compute from these values $s$ and $t$ the quantities $k_{\text{CM}}(\pi N)$ and $\theta_{\text{CM}}(\pi N)$ of the $\pi$N system.

iii) Reconstitute the c.m. frame pion-nucleon scattering amplitudes at $k_{\text{CM}}(\pi N)$ and $\theta_{\text{CM}}(\pi N)$ and transform them into the Breit frame via the invariant amplitudes $s^{2}$.

The consequence of this procedure is that the pion-nucleon amplitudes are required, depending on the momentum transfer, at a higher c.m. energy than that of the $\pi$d system. Thus an off-shell extrapolation of the single-scattering amplitudes is avoided.

6.8.3 Double scattering

The integrand of Eq. (13), when averaged over the deuteron orientation, is largest if the two scatterings take place through half the over-all scattering angle. Hence we assume that for a given $\pi$d scattering angle $\theta$, the pion scatters on both nucleons through an angle close to $\theta/2$.

In contrast to single scattering, for double scattering we must know the Breit frame s- and p-wave pion-nucleon amplitudes separately. They are obtained as follows:

i) Given $k_{\text{lab}}$ and $\theta_{\text{lab}}$, calculate for the $\pi$d system the Lorentz invariant $s$ and the Breit frame scattering angle $\theta_B$.

ii) Given $s$ and $\theta_B/2$, compute the quantities $k_{\text{CM}}(\pi N)$ and $\theta_{\text{CM}}(\pi N)$ of the $\pi$N system.

iii) Reconstitute the c.m. frame pion-nucleon scattering amplitudes at $k_{\text{CM}}(\pi N)$ and $\theta_{\text{CM}}(\pi N)$ and transform them into the Breit frame via the invariant amplitudes $s^{2}$.

iv) Separate the obtained Breit frame amplitudes into s- and p-wave amplitudes using

$$ F_N = f_N + \alpha_N \cos \frac{\theta_B}{2} \quad \text{non-spin-flip} \tag{30} $$

$$ G_N = \beta_N \sin \frac{\theta_B}{2} \quad \text{spin-flip}. $$

In practice this ansatz does not reproduce the amplitudes over the whole angular range, since $F_N$ and $G_N$ are strongly distorted in the Breit frame, due to the different c.m. energies at which the amplitudes are required. A typical curve is shown in Fig. 14 for the non-spin-flip case. At a given angle, the curve is approximated by its tangent whose slope represents $\alpha_N$ and whose intercept with the ordinate indicates $f_N$. The quantity $\beta_N$ is obtained by

$$ \beta_N = \frac{G_N}{\sin \frac{\theta_B}{2}}. \tag{31} $$
This method of choosing the double-scattering amplitudes implies that we do not require Eq. (30) to be valid for the full angular range, but only in the neighbourhood of half the over-all scattering angle.

We must include contributions where the pion gives its charge to the neutron and gets it back from the proton (double charge exchange). This is accomplished by replacing

\[ a_p b_n \rightarrow a_p b_n - \frac{1}{4} (a_p - a_n)(b_p - b_n), \tag{32} \]

where \( a \) and \( b \) refer to \( f, a, \) or \( \beta \).

6.8.4 Fermi motion

The Fermi motion of the nucleons is roughly taken into account by smearing out the pion-nucleon amplitudes. This is accomplished by expanding the kinetic energy \( T \) of the pion about the static value \( T_0 \) and keeping only the lowest order correction term. The Breit frame amplitude \( A(T_0) \) has to be replaced by

\[ A(T_0) \rightarrow A(T_0) + \frac{k_B^2(p^2)}{6m_N^2} \left. \frac{d^2A(T)}{dT^2} \right|_{T_0}, \tag{33} \]

where \( k_B \) is the Breit frame pion momentum and \( (p^2) \) the averaged Fermi momentum squared. Verde\textsuperscript{25} gives \( (p^2) = 0.011 \) (GeV/c)\(^2\).

The effect is small, giving rise to a correction of at most 8\%. 
6.9 Results and discussion

In Figs. 15 and 16 the Brueckner model calculation is compared to the experimental data. In order to show the effect of double scattering, the single-scattering approximation is also given in the figures.

In the case of the 256 MeV angular distribution of Fig. 15, the single-scattering curve is expected to have a minimum around 90°, due to the fact that the dominant pion-nucleon p-wave amplitude vanishes near this angle. However, the large spin-flip amplitude fills in the minimum completely. For small angles the single scattering agrees marginally with the experimental points. In our model the double scattering is constructive for angles smaller

![Graph showing differential cross-section for elastic π⁺d scattering at 256 MeV versus laboratory scattering angle. The data are compared to a Brueckner model calculation including single and double scattering and performed in the Breit frame, where only s- and p-wave pion-nucleon scattering is considered.](image-url)
than 60° with the effect that the calculated cross-section is consistently higher than the experimental data. This might be due to the fact that, near the forward direction, our double-scattering kinematics give large s- and p-wave pion-nucleon amplitudes of opposite sign (Fig. 14) and that these contributions do not cancel in an appropriate way. Moreover, higher order scattering is neglected and we do not know what its contribution will be with these artificially large amplitudes. Around 90° the cross-section is very sensitive to the double-scattering amplitude $T_{10}$ (one nucleon spin-flip) involving the deuteron D-state due to destructive interference between single and double scattering. Removing the D-state gives rise to a constructive interference at this angle and increases the cross-section by a factor of

Fig. 16  Energy dependence of elastic $\pi^+d$ scattering at fixed laboratory scattering angle of 150°. The data are compared to a Brueckner model calculation including single and double scattering and performed in the Breit frame, where only s- and p-wave pion-nucleon scattering is considered.
four. At backward angles the result is sensitive to the D-state through single scattering. Removing the D-state reduces the cross-section at 160° by a factor of two.

The theory reproduces the energy variation of the 160° cross-section quite well (Fig. 16). As already mentioned, the D-state acts through single scattering and increases the cross-section; its influence decreases with decreasing energy (decreasing momentum transfer).

In summary, the Brueckner model fits the data at large angles remarkably well, whereas the angular distribution is reproduced only qualitatively.

* * *

Acknowledgements

I am very grateful to Professor J.P. Blaser, Director of SIN, and to Dr. E.G. Michaelis, Leader of the CERN MSC Division, for giving me the opportunity of working at CERN.

I am deeply indebted to Dr. J.J. Domingo and Dr. N.W. Tanner for supervision during the experimental stage of this work. The theoretical part of this report would not have become possible without the helpful guidance of Dr. C. Wilkin, who also thoroughly checked the results.

Many thanks are due to the CERN MSC operating staff. I express my gratitude to the CERN Typing Service for the careful typing of the manuscript.
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APPENDIX

We list here the deuteron D-state double-scattering transition amplitudes. For the double spin-flip case those terms are neglected where the deuteron is in the D-state both initially and finally.

The argument of the functions h, f and g is kr; that of the spherical Bessel functions j_k is Kr.

\[ T_{11} = T_{-1-1} \]

\[ = 2f_p f_n \left[ \frac{1}{\sqrt{2}} \langle S | h \cdot j_2 | D \rangle + \langle D | h \left( j_0 - \frac{1}{4} j_2 \right) | D \rangle \right] - \]

\[ - 2 \frac{a_y n}{k^2} \cos \theta \left[ \frac{1}{\sqrt{2}} \langle S | f \cdot j_2 | D \rangle + \langle D | f \left( j_0 - \frac{1}{4} j_2 \right) | D \rangle \right] - \]

\[ - \frac{2}{3} \alpha a_n \frac{K^2}{k^6} \left[ \sqrt{2} \langle S | g \left( -\frac{1}{5} j_0 + \frac{11}{14} j_2 - \frac{18}{35} j_* \right) | D \rangle + \langle D | g \left( \frac{11}{10} j_0 - \frac{67}{28} j_2 + \frac{9}{35} j_* \right) | D \rangle \right] + \]

\[ + \frac{1}{6} \alpha a_n \frac{q^2}{k^6} \left[ \sqrt{2} \langle S | g \left( \frac{2}{5} j_0 + \frac{11}{14} j_2 + \frac{27}{70} j_* \right) | D \rangle + \langle D | g \left( \frac{4}{5} j_0 + \frac{17}{28} j_2 - \frac{27}{140} j_* \right) | D \rangle \right] + \]

\[ + 4\sqrt{2} \beta \frac{K^2}{k^6} \langle S | f \cdot j_2 | D \rangle - 2\sqrt{2} \beta \frac{K^2}{k^6} \langle S | g \left( -\frac{2}{15} j_0 - \frac{7}{21} j_2 + \frac{6}{35} j_* \right) | D \rangle \]

\[ + \frac{1}{\sqrt{2}} \beta \frac{K^2}{k^6} \langle S | g \left( -\frac{4}{5} j_0 + \frac{4}{7} j_2 - \frac{9}{70} j_* \right) | D \rangle - \]

\[ - 2 \left( \alpha f_n + \alpha_r f_p \right) \frac{K}{k^3} \left[ \sqrt{2} \langle S | f \cdot r \left( -\frac{1}{5} j_0 + \frac{3}{10} j_1 \right) | D \rangle + \langle D | f \cdot r \left( \frac{11}{10} j_1 - \frac{3}{20} j_3 \right) | D \rangle \right] \]

\[ T_{00} = 2f_p f_n \left[ -\sqrt{2} \langle S | h \cdot j_2 | D \rangle + \langle D | h \left( j_0 + \frac{1}{2} j_2 \right) | D \rangle \right] - \]

\[ - 2 \frac{a_y n}{k^2} \cos \theta \left[ -\sqrt{2} \langle S | f \cdot j_2 | D \rangle + \langle D | f \left( j_0 + \frac{1}{2} j_2 \right) | D \rangle \right] - \]

\[ - \frac{2}{3} \alpha a_n \frac{K^2}{k^6} \left[ -2\sqrt{2} \langle S | g \left( -\frac{1}{5} j_0 + \frac{11}{14} j_2 - \frac{18}{35} j_* \right) | D \rangle + \langle D | g \left( \frac{4}{5} j_0 - \frac{17}{14} j_2 - \frac{18}{35} j_* \right) | D \rangle \right] + \]

\[ + \frac{1}{6} \alpha a_n \frac{q^2}{k^6} \left[ -2\sqrt{2} \langle S | g \left( \frac{2}{5} j_0 + \frac{11}{14} j_2 + \frac{27}{70} j_* \right) | D \rangle + \langle D | g \left( \frac{7}{5} j_0 - \frac{25}{14} j_2 + \frac{27}{70} j_* \right) | D \rangle \right] - \]
\[- \sqrt{2} \beta_p \beta_n \frac{q^2}{k^2} \left\langle S \left| f \cdot j \right| D \right\rangle - 2 \sqrt{2} \beta_p \beta_n \frac{K^2}{k^2} \left\langle S \left| g\left(- \frac{4}{5} j_0 - \frac{8}{5} j_2 - \frac{12}{35} j_4\right) \right| D \right\rangle + \]
\[+ \frac{1}{\sqrt{2}} \beta_p \beta_n \frac{q^2}{k^2} \left\langle S \left| g\left(\frac{1}{15} j_0 - \frac{10}{21} j_2 + \frac{9}{35} j_4\right) \right| D \right\rangle - \]
\[- 2 (\alpha_p f_n + \alpha_n f_p) \frac{K}{k^2} \left\{- \sqrt{2} \left\langle S \left| f \cdot r\left(- \frac{2}{5} j_1 + \frac{3}{5} j_3\right) \right| D \right\rangle + \left\langle D \left| f \cdot r\left(\frac{1}{5} j_1 + \frac{3}{10} j_3\right) \right| D \right\rangle \right\} \]

\[T_{1-1} = T_{-11} = 2 f_p \frac{f}{k^2} \left\{- \frac{3}{\sqrt{2}} \left\langle S \left| h \cdot j \right| D \right\rangle + \frac{3}{4} \left\langle D \left| h \cdot j \right| D \right\rangle \right\} - \]
\[- 2 \alpha_p \alpha_n \frac{K^2}{k^2} \cos \theta \left\{- \frac{3}{\sqrt{2}} \left\langle S \left| f \cdot j \right| D \right\rangle + \frac{3}{4} \left\langle D \left| f \cdot j \right| D \right\rangle \right\} - \]
\[- 2 \alpha_p \alpha_n \frac{K^2}{k^2} \left\{ \frac{3}{\sqrt{2}} \left\langle S \left| g\left(\frac{2}{15} j_0 - \frac{11}{21} j_2 + \frac{12}{35} j_4\right) \right| D \right\rangle - \frac{3}{4} \left\langle D \left| g\left(\frac{2}{15} j_0 - \frac{11}{21} j_2 + \frac{12}{35} j_4\right) \right| D \right\rangle \right\} + \]
\[+ \frac{1}{2} \alpha_p \alpha_n \frac{q^2}{k^2} \left\{ \frac{3}{\sqrt{2}} \left\langle S \left| g\left(- \frac{1}{7} j_2 - \frac{1}{7} j_4\right) \right| D \right\rangle - \frac{3}{4} \left\langle D \left| g\left(- \frac{1}{7} j_2 - \frac{1}{7} j_4\right) \right| D \right\rangle \right\} + \]
\[+ 4 \sqrt{2} \beta_p \beta_n \frac{K^2}{k^2} \left\langle S \left| f \cdot j \right| D \right\rangle - 2 \sqrt{2} \beta_p \beta_n \frac{K^2}{k^2} \left\langle S \left| g\left(- \frac{8}{15} j_0 - \frac{22}{21} j_2 - \frac{18}{35} j_4\right) \right| D \right\rangle + \]
\[+ \frac{1}{\sqrt{2}} \beta_p \beta_n \frac{q^2}{k^2} \left\langle S \left| g\left(- \frac{2}{7} j_2 + \frac{3}{14} j_4\right) \right| D \right\rangle - \]
\[- 2 (\alpha_p f_n + \alpha_n f_p) \frac{K}{k^2} \left\{ \sqrt{2} \left\langle S \left| f \cdot r\left(\frac{3}{5} j_1 - \frac{9}{10} j_3\right) \right| D \right\rangle - \left\langle D \left| f \cdot r\left(\frac{3}{10} j_1 - \frac{9}{20} j_3\right) \right| D \right\rangle \right\} \]

\[T_{10} = - T_{01} = - T_{-10} = T_{0-1} \]
\[- \sqrt{2} \beta_p \beta_n + \alpha_p \alpha_n \sin \theta \left\{ \frac{1}{\sqrt{2}} \left\langle S \left| f \cdot j \right| D \right\rangle - \frac{1}{2} \left\langle D \left| f \cdot \left( j_0 - j_2 \right) \right| D \right\rangle \right\} + \]
\[+ \frac{\sqrt{2}}{\sqrt{5}} \beta_p \beta_n + \alpha_p \alpha_n \sin \theta \left\{ \frac{1}{\sqrt{2}} \left\langle S \left| g\left( j_0 - \frac{1}{2} j_2 \right) \right| D \right\rangle + \left\langle D \left| g\left( j_0 - \frac{1}{2} j_2 \right) \right| D \right\rangle \right\} + \]
\[+ \frac{1}{\sqrt{2}} \left( \beta_p f_n + \beta_n f_p \right) \frac{q^2}{k^2} \left\{ \frac{1}{\sqrt{2}} \left\langle S \left| f \cdot r \cdot j \right| D \right\rangle + \left\langle D \left| f \cdot r \cdot j \right| D \right\rangle \right\} \]