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1. - INTRODUCTION

The subject of these lectures is to discuss the applications of the soft pion theorems to the interaction of pions with nuclei. I have assumed that you are not all familiar with these techniques and the first lecture will be devoted to explain the soft pion theorems and how they are derived.

The idea of considering the soft limit has first arisen for photons where it comes naturally as the long wave-length limit. In that limit, one obtains very simple results. The scattering of soft photons becomes a universal expression and does not depend on any detailed structure of the target. The Thomson formula for this scattering is

\[
\frac{d\sigma}{d\Omega} = \frac{1}{3} \left( \frac{e^2}{mc^2} \right)^2
\]

(1.1)

where \( e \) is the charge and \( m \) the mass of the target.

This universality is quite understandable, the infinite wave-length of the photon makes it impossible to explore the shape and structure of the target. In the words of Sens \(^1\): "If you want to explore the shape of your hand you pour sand over it, rather than marbles or apples."

There exist other soft photon theorems. Consider a process where a soft photon is emitted. It may be emitted by an external line, or by an internal line.
In the external emission graph, the intermediate particle to which the photon is attached is on the mass shell since the photon does not carry momentum in the soft limit. The propagator is zero; this means that this graph is divergent, the singularity is of order $k^{-1}$. These external graphs constitute the dominant part of the amplitude. On the other hand, the electromagnetic current is conserved and this imposes a gauge condition on the amplitude. This gauge condition makes possible $^2$ the determination of the amplitude from the external emission graph not only to the leading order $k^{-1}$ but to the next order $(k)^0$. These properties are the essential content of the soft photon theorems.

Is it possible to obtain similar results for other projectiles, pions for example for a pion or a kaon?

These particles have certain similarities with the photon. They are the quanta of the nuclear field, as the photon is the quantum of the e.m. field. But on the other hand they are completely different objects; they are strongly interacting particles. But in spite of these differences the answer is yes. In the infinite wave-length limit one finds very similar results $^3$--$^8$, the same universality for the scattering and analogous result for the emission.

Now what kind of wave-length are we talking about? Is it sufficient to take the ordinary wave-length infinite, i.e., a pion at rest? Certainly not because a pion at rest has still a finite Compton wave-length which enables it to explore the features of the target.

What we should do is to immerse the target in an infinite wave-length so that there is no variation at all on the target size. Not only we should have the pion at rest $q = 0$ but also the fourth component of the pion momentum should be taken to zero $q_0 = 0$. 
But what kind of pion are we considering? It is not the pion that we know which has a mass of 140 MeV. This is a new object, which is called a soft pion, which has a zero mass, \( q_2^2 = q_0^2 - q^2 = 0 \). And the fact that the soft amplitudes we are calculating do not apply to the physical pion but are off shell amplitudes raises a problem which is not present in the photon case.

There is another reason why the soft pion theorems are not mere repetitions of the results obtained for photons. If you take a process involving a photon, the difficulty in calculating the amplitude arises from the large number of graphs that may contribute. The soft limit greatly simplifies the problem since the dominant contributions arise only from the external emission graphs. But we know what is the interaction of the photon, we know what to write for the coupling of a photon with a nucleon or with a pion.

For pions instead, we do not know what is the basic interaction. The \( \pi \) nucleon coupling for instance may be taken to be pseudoscalar or pseudovector, to keep only the simplest ones. There is no way of testing these couplings from a perturbation expansion. This is why the soft pion theorems require more than just the softness assumption but need another hypothesis in order to fix the coupling. This hypothesis is that of the partial conservation of the axial current, PCAC. In certain cases, when more than one soft pion is involved (like in the scattering problem), or when the emission takes place in the presence of an e.m. perturbation (like in photoproduction), the commutation relations of current algebra are also used. These two hypotheses fix the nature of the coupling of the pion.

We shall then start by discussing the fundamental ingredient of the soft pion theorem, PCAC.

PCAC

This hypothesis \(^9\) applies to the axial current of the weak interactions. The hadronic current current of the weak interactions has a vector part \( V_\mu \) and an axial part \( A_\mu \), which have different
properties under space reflection. Here we are concerned only by the strangeness conserving part of the current.

In the $\beta$ decay of the neutron, the nucleons are subject to the strong interactions, for example pions can be exchanged between the weak and the strong vertex. The hadronic matrix element of the axial current is then modified over the value it would have for a point-like Dirac particle and from the Lorentz invariance properties one writes

$$\langle p| A^+_\mu | n(p) \rangle = \frac{\bar{u}_n(p)}{p_+} \left[ g_A(q^2) \sigma^\mu_\nu + g_N(q^2) q^\mu q^\nu \right] \sigma^+ u_p(p) \tag{1.2}$$

with $q = p_2 - p_1$.

From the decay rate of the neutron it was found that the axial form factor is close to one, $g_A = 1.23$.

There is a special significance about the number one. In the similar expansion for the vector current

$$\langle p| v^+ | n(p) \rangle = \frac{\bar{u}_n(p)}{p_+} \left[ g_V(q^2) \sigma^\mu_\nu + g_N(q^2) q^\mu q^\nu \right] \sigma^+ u_p(p) \tag{1.3}$$

in principle the vector form factor should also be renormalized by the strong interactions. But the conservation of the vector current (which is a part of the C.V.C. hypothesis) implies that there is no renormalization effect and that $g_V = 1$.

The value found for $g_A$ close to one had first suggested that maybe the axial current was also conserved. But it was quickly realized that if this would be the case, the pion would live for ever, or nearly ever, for seconds. For the main decay mode of the charged pion $\pi^\pm - \mu^\pm + v_\mu$ would be suppressed. The hadronic matrix element of this process is that of the axial current between the pion state and the vacuum $\langle 0| A^\pm_\mu | \pi^\mp \rangle$. The vector current does not contribute to this decay because we cannot build a quantity of pseudovector character with the pion momentum. On the other hand, the axial current acting on
a pseudoscalar object gives a quantity of vector character, as the pion momentum $q_{\mu}$ and we write

$$\langle 0 | A^{\mu}_0 | \pi^- \rangle = i \frac{p_{\pi}}{q_{\mu}} q_{\mu}$$ (1.4)

$f_\pi$ is called the decay constant of the charged pion. From the decay rate of the pion $f_\pi$ is found to have the value 0.94 $m_\pi$.

Take now the matrix element of the divergence of the axial current $D = \gamma^\mu A^{\mu}_0$ between the same states. It amounts to multiplying the previous matrix element (1.4) by $-i q_{\mu}$ and thus

$$\langle 0 | D^\mu | \pi^- \rangle = \frac{p_{\pi}}{q_{\mu}} q_{\mu} = \frac{p_{\pi}}{q_{\mu}} m_{\pi}^2$$ (1.5)

The conservation of the axial current implies $\gamma^\mu A^{\mu}_0 = 0$ and therefore from the relation (1.5) $f_\pi$ should be zero, which is not the case.

But the smallness of the renormalization effect in the axial coupling constant led to the idea of a partial conservation of the axial current, a somewhat misleading terminology which is explained in the following.

Take the matrix element of the divergence between two states A and B. This matrix element is made of all the objects that can be exchanged between D and the system A B. The divergence D is a pseudoscalar quantity. It has the quantum numbers of the pion. Among the objects that can be exchanged, certainly figures the pion and it is the lightest of these objects.
The matrix element is then singular when the exchanged pion is on the mass shell, i.e., when the momentum transfer is the pion mass
\[ t = (p_B - p_A)^2 = m_{\pi}^2. \]

Beside the pion pole what are the other singularities? When \( A \) and \( B \) are nucleon states, the next singularity comes from the exchange of three pions. But this is a complex system which has internal degrees of freedom; the pions can move around their c.m.

This singularity is then not a pole but a cut, which starts at \( t = 9m_{\pi}^2 \).

Writing
\[
\langle B(p_z) | D | A(p_z) \rangle = D(q^2) \bar{u}(p_z) i \gamma^5 u(p_z) \tag{1.6}
\]

an unsubtracted dispersion relation for \( D(q^2) \) gives
\[
D(q^2) = \text{pion pole} + \frac{i}{\pi} \int_{9m_{\pi}^2}^{\infty} dt \frac{g(t)}{t - q^2} \tag{1.7}
\]

The spectral function \( g(t) \) is a complicated object but if the momentum transfer \( q^2 \) is not too large, say \( |q^2| < m_{\pi}^2 \), the denominator of the dispersive integrand is large \( t - q^2 > 8m_{\pi}^2 \) and it is reasonable to assume that the contribution of the dispersive integral is negligible and to retain only the pion pole. Thus when \( A \) and \( B \) are nucleons we assume that the matrix element is dominated by the pion pole, not only at the pole but in a neighbouring region. With the expression of the \( \pi N \) vertex
\[
\langle B(p_z) | j^+_{\pi} | A(p_z) \rangle = \sqrt{2} g_{r}(q^2) \bar{u}(p_z) i \gamma^5 \sigma^z u(p_z) \tag{1.8}
\]

where \( g_{r}(q^2) \) is the \( \pi N \) coupling constant with its form factor, this dominance is expressed in the following formula
\[
D(q^2) = \frac{\sqrt{2} g_{r} m_{\pi}^2}{m_{\pi}^2 - q^2} g_{r} \tag{1.9}
\]

Here \( g_{r} = g_{r}(m_{\pi}^2) \) is the renormalized \( \pi N \) coupling constant.
This formula is approximate since we have neglected the contribution of the cut but it is a relation between measurable quantities $D(q^2)$ and $g_r$ and the degree of validity of this approximation may then be tested. This test, as we will see later, is made in the form of the Goldberger-Treiman relation and it is valid to about 10%.

Now what about nuclei? Let us try to apply the pole dominance assumption and see how it works. In a formula (1.9) as the one written for nucleons, the residue of the pole is taken to be a fixed quantity. All the variation in the momentum transfer $q^2$ is contained in the denominator which gives a slow and universal variation. But we know that this cannot possibly be valid for nuclei. The form factors in the space-like region can be obtained from $\mu$ capture; the variation with momentum transfer depends on the nuclear size. For large nuclei, this variation is very rapid

$$F_n(q^2) = F_n(0) \left[ 1 - \frac{|q|^2 <r^2>}{6} \right]$$  \hspace{1cm} (1.10)

where $<r>$ is the r.m.s. radius.

The reason for this behaviour is the singularity structure of the matrix element. In nuclei the next singularity after the pion pole is not the three-pion cut. Because the nucleus has a small binding energy, it can virtually decay into a nucleon plus another nucleus

This is the anomalous threshold singularity. The position of this cut depends on the binding energy but it is located somewhere around $2$ or $3m^2$. This singularity is not much further away than the pion pole and it is not possible to neglect its influence. How can we account for it?
We had written previously the pole dominance expression

\[ \langle B | \mathcal{D}^\pm | A \rangle = \frac{\rho_{\pi}}{m_\pi^2} \frac{\langle B | \mathcal{D}^\pm | A \rangle}{m_\pi^2 - q^2} \]  

(1.11)

where \( j_\pi \) is the pion source, the inhomogeneous term of the Klein-Gordon equation \((\Box + m_\pi^2)\psi = j_\pi\). In this relation, the residue of the pole was taken as a fixed quantity. For nuclei, we allow for the variation with momentum transfer of the pion source matrix element \( \langle B | j_\pi | A \rangle \). This amounts to take into consideration the spatial distribution of the pion source and therefore its form factor.

\[ \langle B(r) | \mathcal{D}^\pm | A(r) \rangle = \frac{\rho_{\pi}}{m_\pi^2} \frac{\langle B(r) | \mathcal{D}^\pm | A(r) \rangle}{m_\pi^2 - q^2} \]  

(1.12)

This relation can be condensed in the following proportionality relation of Gell-Mann Levy \(^9\) between the divergence and the pion field \( \psi \)

\[ \mathcal{D}^\pm = \frac{\rho_{\pi}}{m_\pi^2} \langle B(r) | \mathcal{D}^\pm | A(r) \rangle \]  

(1.13)

which is equivalent to the previous one (1.12).

In such a formula, the full singularity structure is in principle taken into account but we may enquire about the meaning of such a relation since on one side we have a measurable quantity \( \mathcal{D}(q^2) \) but on the right-hand side, the pion field or the pion source is a physical quantity only at \( q^2 = m_\pi^2 \). Without a model for the pion source, this relation is then just a definition of the extrapolating pion field. On the other hand, with a model for the pion source this relation acquires predictive power. In nuclear physics, this model is obtained from an optical potential. Of course, this relation is again approximate since the model in itself is an approximation, but the approximation is here different than in the pole dominance hypothesis. For nuclei, we have no choice, we cannot use pole dominance. Otherwise the scattering of pions by nuclei, to take an example, would be found to be independent of the nuclear size, in flagrant contradiction with experiments.
2. SOME EXAMPLES OF SOFT PION THEOREMS

Our first example is the celebrated Goldberger-Treiman relation. Take the matrix element of the divergence $D$ between two nucleonic states $\langle p_2 | D^+ | n(p_1) \rangle$.

We have written previously (1.2) the matrix element of the axial current. By multiplying by $i q \mu$ we get that of $D$:

$$\langle p_2 | D^+ | n(p_1) \rangle = \bar{u}_{p}(p_2) \left[ 2 m_g \alpha(q^2) + q^2 \beta(q^2) \right] i \gamma_5 \gamma^+ u_n(p_1)$$  \hspace{1cm} (2.1)

We may also write this matrix element from the PCAC relation $D^+ = f \pi m^2 \phi^+$:

$$\langle p_2 | D^+ | n(p_1) \rangle = f \pi m^2 \frac{m_{\pi}^2 \alpha(q^2) \bar{u}_{p}(p_2) i \gamma_5 \gamma^+ u_n(p_1)}{m_{\pi}^2 - q^2}$$  \hspace{1cm} (2.2)

Equating the two expressions at zero momentum transfer gives the Goldberger-Treiman relation between the strong pion nucleon coupling constant $g_{\pi}(0)$ and the constants of weak interactions $f \pi$ and $g_A$:

$$2 m_{\pi}^2 g_A(0) = \sqrt{2} f \pi g_{\pi}(0)$$  \hspace{1cm} (2.3)

Here we have not neglected the form factor of the $\pi N$ coupling constant and this relation is exactly valid at zero momentum transfer, i.e., for pions of zero mass. It is a typical example of a soft pion theorem which gives an exact relation between a strong interaction quantity, involving a pion of zero mass, and weak interaction constants (or amplitudes). However, as all the soft pion theorems, it does not provide a relation between physical quantities and before it can be applied to physical quantities, approximations or extrapolations have to be made.

Another example is the photoproduction of soft pions or the inverse process, the radiative capture of pions:

$$\pi^\pm + i \leftrightarrow \sigma + \rho$$
The transition matrix is
\[ S = -i \delta^4(k+p_\gamma - q-p_i) F \]
\[ F = \langle \frac{p_\gamma}{q^+} | j^{\pm}_{\pi} (0) | i \rangle \]
(2.4)

\( F \) is the invariant amplitude. We have taken here for all the states the boson normalization

\[ \langle p_i | p_f \rangle = (2\pi)^3 2p_\gamma 6(p_\gamma - p_i) \]
(2.5)

\( p_i, p_f \) are the initial and final state momenta, \( q \) and \( k \) those of the pion and photon. We keep the spatial momentum \( \mathbf{q} \) of the pion to be zero \( \mathbf{q} = 0 \) and we vary the time component \( q_0 \) by varying the difference \( (p_\gamma)_0 + k_0 - (p_i)_0 \). The invariant amplitude \( F \) is a function of \( q_0 \). The physical amplitude is obtained for \( q_0 = m_\pi \) and the soft one for \( q_0 = 0 \).

The soft amplitude is calculable from PCAC \(^{10)-12})\). The quickest way to obtain it is to use a form of PCAC modified in the presence of the e.m. interactions by the minimal coupling. The minimal coupling makes the usual replacement of \( \mathcal{A}_\nu \) by \( \mathcal{A}_\nu \pm ie \mathcal{A}_\nu \) where \( \mathcal{A}_\nu \) is the e.m. field. The \( \pm \) sign applies to \( \pi^\pm \) annihilation and PCAC then writes

\[ \frac{p_\gamma m_\pi}{m_\pi^2} m_\pi^2 \mathcal{V}_\nu^\pm = (\mathcal{A}_\nu \pm ie \mathcal{A}_\nu) A_\nu^\pm \]
(2.6)

Taking then the matrix element between the states \(| fi >\) and \(| i >\) gives

\[ \frac{p_\gamma}{m_\pi^2} q^i \mathcal{V}_\nu^\pm \langle fi | j^\pm_{\pi} | i \rangle = iq \mathcal{V}_\nu^\pm \langle fi | A_\nu^\pm | i \rangle \pm ie \mathcal{V}_\nu^\pm \langle fi | A_\nu^\pm | i \rangle \]
(2.7)
By making $q_0 = 0$ (we have already taken $q = 0$) we get on the left-hand side the invariant amplitude for soft pions. By taking $q_0 = 0$ on the right-hand side, we get thus the expression of the soft pion amplitude. Since we work to first order in the e.m. coupling constant we can replace in the term $\langle f \gamma^\nu A^\pm_\nu | i \rangle$ the e.m. field by the outgoing one which annihilates the outgoing photon and

$$\langle g^\nu A^\pm_\nu | i \rangle = \langle g | e^\nu A^\pm_\nu | i \rangle \quad (2.8)$$

A priori one is tempted to say that, on the right-hand side of Eq. (2.7) there is no contribution from the first term which is proportional to $q_0$. However, we have to be more careful because there are Born terms in the multiplying factor $\langle f \gamma^\nu A^\pm_\nu | i \rangle$. In these Born terms, the axial current is attached to an external line and these terms diverge when $q_0 \to 0$ since the intermediate particle is then on the mass shell. The product of these Born terms with $q_0$ may have a finite limit which should be included in the soft amplitude.

Born graphs

But you see in analogy with the photon case that the soft pion amplitude is completely calculable from the Born terms plus the direct term $\langle f | e^\nu A^\pm_\nu | i \rangle$.

It turns out that the direct term is the only contribution to the soft pion amplitude because the Born graph contribution vanishes. This can be shown if we take all the momenta $p_1$, $p_2$, and $k = 0$ and if we consider the Coulomb gauge where the fourth component of the photon polarization vector is zero. The coupling of the photon then occurs with the current and not the charge, but the current is zero since the particles are at rest. Thus the soft amplitude is simply
\[ F(0) = \pm \frac{ie}{g_\pi} \langle \pi \mid e \cdot A^+ \mid i \rangle \] (2.9)

In the same gauge there is no contribution to the soft amplitude of the pseudoscalar part of the axial current since
\[ \xi \cdot (p_f - p_i) = \xi \cdot \kappa = 0. \]

If the states \( i \) and \( f \) are nucleonic states we can use the Goldberger-Treiman relation \( (g_A/f_\pi) = (g_\pi)/(\sqrt{2}m_N) \) and the soft amplitude becomes
\[ \langle \pi \mid j^+ \mid n \rangle = \pm \frac{ie}{12m_n} \bar{u}_f \gamma^\mu \frac{g_\pi}{f_\pi} \sigma_\mu \sigma_5 \gamma^5 u_i \] (2.10)
or in the non-relativistic limit
\[ \pm \frac{ie}{12m_n} \chi_f \sigma_5 \xi_5 \sigma_\mu \xi^\mu \chi_i \]

where \( \chi_f \) and \( \chi_i \) are two components Pauli spinors.

This is the Kroll-Ruderman theorem \(^{13}\). It can also be obtained as a soft photon theorem. The amplitude is the sum of the three Born graphs, and in this case the coupling of the photon is pseudoscalar. Here the direct term replaces the Born graphs with pseudoscalar coupling. The two methods are equivalent.

The expression obtained relates the radiative capture to the axial current matrix element which appears in \( \mu \) capture \( \mu + i \to f + \nu_\mu \) or in \( \beta \) decay \( i \to f + e + \nu \). In these processes, the hadronic matrix element is that of the weak current. In this matrix element, there is a vector part (Fermi transition) and the axial vector part (Gamow-Teller transition). The radiative capture is related to the Gamow-Teller part. For the transitions which are pure Gamow-Teller the three processes can be linked
directly. This is a very interesting result which links a strong interaction amplitude to a weak amplitude. This link was first observed by a different method, the impulse approximation. In the impulse approximation the nuclear process is assumed to take place on one nucleon which is related to the other by the wave function but apart from this is free. The process \( i + \pi \rightarrow f + \sigma \) is thus related to the same process on a free nucleon \( N + \pi \rightarrow N + \sigma \).

For the free nucleon, the transition matrix element is in the non-relativistic limit

\[
\pm i e \frac{g_\pi}{\sqrt{2} m_N} \sigma \cdot \pi \sigma^\dagger
\]

For the nuclear transition, in the impulse approximation, the current is taken to be the sum of one nucleon current

\[
A^\pm_\mu = \sum_i i e \frac{g_\pi}{\sqrt{2} m_N} \sigma_i \cdot \pi \sigma_i^\dagger \delta(x_i - x_f) \tag{2.11}
\]

The transition amplitude for radiative capture is then proportional to the nuclear matrix element

\[
\langle f | \sum_i \sigma_i \cdot \pi \sigma_i^\dagger \delta(x_i - x_f) | i \rangle \tag{2.12}
\]

while for \( \mu \) capture or \( \beta \) decay the axial matrix element is proportional to

\[
\langle f | \sum_i \sigma_i \cdot \pi \sigma_i^\dagger \delta(x_i - x_f) | i \rangle \tag{2.13}
\]

Apart from the difference in the momentum transfer in the three processes the nuclear matrix element is the same and the three processes are related. The derivation of this link with soft pion technique is at a deeper level, it does not require the impulse approximation.
We know that the representation of a nuclear current by a sum of the individual contribution nucleons is only approximately true. There are exchange effects where the current interacts simultaneously with two or more nucleons and these effects are included in the soft pion treatment while they are not in the impulse approximation.

\[
\frac{\pi^+}{\pi^-}\]

The third example of a soft pion result is the elastic scattering of pion \(\pi^\pm\). The derivation is rather lengthy and at this stage we will only quote the results obtained by Weinberg \(7\) and Tomozawa \(8\).

When both the incident and the scattered pions become soft, the amplitude for elastic scattering becomes universal. This reminds us very much of the similar result for soft photons. In both cases we find universality since the target is seen by the soft pion or the soft photon as a point object. The scattering length depends in that limit only on the isospin of the target. The difference of the scattering lengths for \(\pi^\pm\) vanishes at \(q_0\), the pion mass, and it is related to the decay constant of the pion \(f_\pi\).

\[
a_{\pi^-} - a_{\pi^+} = 2(z-N)a^z = 2(z-N)\frac{g_0}{4\pi \rho^2}
\]

where \(z-N\) is the proton excess of the target. The sum for \(\pi^-\) and \(\pi^+\) amplitudes is of order \(m^2_{\pi}\) and it was thought to be small. This value will be discussed in detail in Section 5. If it is possible to take the sum \(a_{\pi^-} + a_{\pi^+}\) to be zero, these two results can be condensed in one formula:
\[ a = \frac{9\hbar}{4\pi f_{\pi}^2 T \cdot t} \]  

where \( T \) and \( t \) are the target and pion isospin operators.

The derivation of Weinberg's expression requires not only PCAC but also the equal time commutation relations of current algebra, because it involves two soft pions.

To summarize these examples, what is so remarkable about the soft pion result is the simplicity and the universality of the interaction. The elastic scattering amplitude depends only on the target isospin while the photoproduction one depends only on the spin isospin structure. The soft pion amplitudes are expressible as weak or e.m. amplitudes.

3. WHAT TO DO WITH SOFT PION RESULTS?

We have derived nice and simple expressions for processes involving soft photons and the question naturally arises: what should we do with these results? They apply to pions of zero mass and are exact results for these pions but they are off-shell quantities and they do not apply as such to the physical pion. How should we use them? What is retained of the simplicity of the interaction when we reach the physical mass?

The simplest application that could be made of these results is to forget that they have been derived for pions of zero mass and to apply them to the physical pion. Could it be possible that nature does not distinguish between these two pions so that there is no difference between the physical and the off-shell quantities? The answer is that it depends who is looking at the pion, what target is interacting with it. You see, all is a question of scale. To come back to the picture that we used in the first lecture that you should use sand to map the shape
of your hand and not marbles. But marbles are good for a giant's hand and sand is too coarse for Tom Thumb.

Let us first consider the case when the target is a nucleon and let us see what gives the application of the soft result to the physical case.

We had first derived what I would call the "soft Goldberger-Treiman relation" which applies to a pion of zero mass.

\[ g_{\pi}(0) = \frac{\sqrt{2} m_N g_A(0)}{f_\pi} \]  

(3.1)

If we replace the off-shell coupling constant \( g_{\pi}(c) \) by the on-shell one \( g_{\pi}(m_\pi^2) \) we get a relation between physical quantities

\[ g_{\pi} = \frac{\sqrt{2} m_N g_A(0)}{f_\pi} \]  

(3.2)

With \( f_\pi = 0.94 \) m_\pi, \( g_A = 1.23 \) we obtain \( g_{\pi} = 2.4 \) while the experimental value \( g_{\pi}^2/4\pi = 14.6 \) gives \( g_{\pi} = 13.5 \). The agreement is within 10%, quite a good one and the extrapolation in the pion mass seems to be smooth. Let us check other cases because there is always the possibility of an accidental agreement.

How does Weinberg's formula works for the scattering of pions on nucleons? When the target is a nucleon, the scattering lengths are obtained from the extrapolation at zero energy of the phase shifts obtained from the analysis of the scattering experiments.

The different analyses agree on the isospin antisymmetric amplitude \( a^- \) (related to the difference for \( \pi^- \) and \( \pi^+ \)). The value of \( (1 + m_\pi/m_N) a^- \) is close to 0.10 m_\pi^- in good agreement with the soft pion value \( L = m_\pi^- /4\pi f_\pi^2 = 0.09 \) m_\pi^- . For the isospin symmetric amplitude \( a^+ = \frac{3}{2}(a^- + a^+) \) the different analysis disagree, even on the sign \( (1 + m_\pi/m_N) a^+ \) ranging from -0.014 to
+ 0.02 m$^{-1}$\textsuperscript{16-18}), but there is a consensus on the fact that $a^+$ is sizeably smaller than $a^-$, as predicted by the soft pion result.

Similarly in the photoproduction case, the amplitude for charged pions is well predicted by the soft pion method. And the amplitude for neutral pion photoproduction is much reduced compared to that for the charged pions, as expected.

There is thus evidence that the amplitudes extrapolate smoothly between zero and the physical mass when the target is a nucleon.

Consider now the nuclear case. There are practically no scattering experiments of low energy pions on nuclei from which scattering lengths may be obtained. But there is an indirect knowledge of these scattering lengths from the energy shifts of the atomic levels of π mesic atoms\textsuperscript{19).} For a pion bound in a Bohr orbit, the strong interaction modifies the energy as compared to the Bohr value by an amount $\Delta E$, the energy shift. The 1s level shift is related to the s wave phase shift since both represent the amount by which the external wave function is perturbed, one in the negative energy, the other in the positive energy region. When the shift is small, the relation is\textsuperscript{20)}

$$\frac{\Delta E_{1s}}{E_{1s}} = -4 a / B$$

(3.3)

where $B$ is the Bohr radius. From this relation, the measured shift of 20 KeV in \textsuperscript{16}O corresponds to the scattering length $a^+ \approx -0.40 \text{ m}^{-1}$, far from the small value that it has for a zero mass pion.

Similarly the isospin odd amplitude may be obtained from the energy shift for nuclei of non-zero isospin. They give values which are not universal. As examples for nucleon numbers $A = 11$ and 18 we get $(a^-)_{A=11} = 0.057 \text{ m}^{-1}$; $(a^-)_{A=18} = 0.047 \text{ m}^{-1}$ which are smaller than the soft value $0.09 \text{ m}^{-1}$. The isospin odd amplitude is thus decreased and the even one increased (in magnitude) compared to
their soft values. The smoothness condition necessary direct application of the soft pion theorems to the physical amplitude is not valid in the nuclear case.

The difficulties of application to the nuclear case were known since the introduction of these soft pion theorems and it was even feared that the soft pion techniques would be useless in nuclei because there would be little relation between the physical and the soft amplitude and that all the simplicity of the interaction would be gone when the physical pion mass is reached.

In fact, this pessimistic attitude is not justified. One can find \(^{21)}-^{23}\) a dictionary to translate the soft pion expressions into values meaningful for the physical pion.

The first thing that we have to do in order to find this dictionary is to obtain a relation between the soft and the physical amplitude. Let us take the example of the radiative capture and go back to the expression we had derived with the minimal coupling method

\[
\frac{\rho^\pi m_\pi^2}{m_\pi^2 - q^2} \langle \bar{g} u | d_\pi \bar{d} | \nu \rangle = \pm i e \langle \bar{g} u | \bar{v} A_\nu^\pm | \nu \rangle + i q^2 \langle \bar{g} u | A_\nu^\pm | \nu \rangle
\]  

(3.4)

On the right-hand side, the first term \(\pm i e \langle \bar{g} u | \bar{v} A_\nu^\pm | \nu \rangle\) is the soft amplitude, \(F(0)\), and this equation relates the amplitude for a given pion mass \(q_0\) to the soft one. However, it is not possible to use this relation in this form for the physical pion since there are poles at the physical value of the pion mass \(q_0 = m_\pi\). One has to go through the step of removing these divergences which is easily done.

On the left-hand side, the pole is apparent. On the right-hand side it is contained in the matrix element of the axial current between the states \(\nu\) and \(f\), the pole is contained in the pseudoscalar part of this matrix element.
\[ i q \langle g \sigma | A^\pm | i \rangle = \frac{g}{m^2 - q^2} \langle g \sigma | j^{\pm} | i \rangle + i q \langle g \sigma | A^\pm | i \rangle \]  

(3.5)

where the barred quantity means that the pion pole has been removed.

Note that instead the pseudoscalar term of the axial current matrix element between \( i \) and \( f \) occurs at \((p_1 - p_f)^2 = m^2 \) and not at \((p_1 - p_f - k)^2 = q_0^2 = m^2 \) and it does not have to be removed.

Thus when the pion pole at \( q_0^2 = m^2 \) is removed, one obtains the following generalized Goldberger-Treiman relation which can be used for pions on and off the mass shell.

\[ \frac{g}{m} \langle g \sigma | j^\pm | i \rangle = \pm i e \langle g | \epsilon^0 A^\pm | i \rangle + i q \langle g \sigma | A^\pm | i \rangle \]  

(3.6)

In general, the contribution of the barred quantity cannot be evaluated exactly, except in the soft limit where it arises from the Born terms. One has to find approximations for this barred quantity. In order to do so the best way is to further reduce in the photon variable. As usual there appears a retarded commutator, of the e.m. field with the axial current.

\[ \langle g \sigma | A(\partial) | i \rangle = i \int d^4 x e^{ikx} \Box_x \langle g | \Theta(x) [\epsilon, \partial \phi] A_o(x) \rangle | i \rangle \]  

(3.7)

We displace the Klein-Gordon operator \( \Box_x \) inside the bracket. When it acts on the e.m. field, it gives the e.m. current. But we have to be careful because it also meets the \( \Theta \) function and we also get derivative of the \( \Theta \) function, i.e., equal time commutators. The final result is

\[ \langle g \sigma | \overline{A(\partial)} | i \rangle = i \int d^4 x e^{ikx} \left\{ \langle g | \Theta(x) [\epsilon, \partial \phi] \overline{A(\partial)} \rangle | i \rangle 
- i k_0 \langle g | \delta(x_0) [\epsilon, \partial \phi] | \overline{A(\partial)} \rangle | i \rangle 
+ \langle g | \delta(x_0) [\epsilon, \partial \phi] | \overline{A(\partial)} \rangle | i \rangle \right\} \]  

(3.8)
We insert now a complete set of intermediate states in the retarded commutator and perform the time integration, and we get the relation between the physical and the off-shell amplitude in the form of a sum rule which will be given in Section 6.

On this example, there appears the general features of the mass extrapolation. It first introduces equal time commutators which are not those of current algebra and on which we have very little information and which require extra assumptions. A second general feature is that the relation between the physical and the soft amplitude is obtained in the form of a sum rule which we do not know how to estimate exactly and which has to be truncated.

Note that these features are not very troublesome in the nuclear case since we can use the nucleon case as a test of the validity of the assumptions made for the equal time commutators or for the truncation of the sum.

In the nucleon case, Fubini-Furlan (F.F.) \(^{24}\) and de Alfarossetti \(^{25}\) have derived an elegant method which increases the convergence of the sum and makes it possible to saturate it with low lying states only, namely nucleon states and nucleon states with one pion. Their idea is to include two amplitudes in one single sum rule.

In the scattering problem, these amplitudes are those for negative and positive pions. They consider separately isospin symmetric and isospin antisymmetric combinations which have different transformation properties under crossing.

Similarly for the radiative capture they add the amplitudes \(F\) and \(\bar{F}\) for the two processes

\[
\pi^- + p \rightarrow \gamma + n
\]

\[
\gamma + p \rightarrow \pi^+ + n
\]
These two amplitudes are related by crossing \( F(s,t,u) = F^*(u,t,s) \). The problem is how to choose the kinematical variables in such a way that the sum rule involves an amplitude at one single point and not at two different points, which would be of no interest.

The invariants \( s, t, u \) and \( \bar{s}, \bar{t}, \bar{u} \) for the two reactions are

\[
\begin{align*}
\bar{s} &= (p_i + q)^2 \\
\bar{t} &= (p_f - p_i)^2 \\
u &= (p_f - q)^2
\end{align*}
\]

The observation of these authors is that in the Breit frame, i.e., in the frame where the momenta \( p_i \) and \( p_f \) of the initial and final states are opposite, \( p_i = -p_f \) these two amplitudes are related by crossing since

\[
\begin{align*}
\bar{s} &= \bar{s} \\
\bar{t} &= \bar{t} \\
u &= \bar{u}
\end{align*}
\]

And the sum rule then involves the quantity

\[
F[s, t, u] + F[\bar{s}, \bar{t}, \bar{u}] = F[s, t, u] + F^*[u, t, s]
\]

In the sum there survive those of the invariant amplitudes which are symmetric under crossing (\( u \leftrightarrow s \)), while in the difference remain only those which are odd.

The F.F. method of saturation of equal time commutators is absolutely equivalent to combining two amplitudes as obtained by the usual reduction techniques, although it may look quite different. It is maybe easier technically, while the reduction method might be conceptually clearer.
In the nuclear case, the F.F. method is well adapted to the scattering problem. For photoproduction it can be used only when the states $i$ and $f$ are members of an isomultiplet. In this case only, the crossing relations relate the physical to the unphysical region. This restricts the transitions and we will not make use of this method but we will work in one amplitude, as obtained from the reduction techniques.

4. - ELASTIC SCATTERING CHARGE ANTISYMMETRIC AMPLITUDE

The starting point of the F.F. method for this amplitude is the equal time commutator of two axial charges. The axial charge $Q_A(x_o) = \int d^3x A_0(x)$ is not a constant of motion since the axial current is not conserved and it depends on the time.

The hypotheses of current algebra give the equal time commutators of the two axial charges $Q_A^+$ and $Q_A^-$, proportional to the vector charge

$$\left[ Q_A^+(x_o), Q_A^-(x_o) \right] = 2 Q_V^+(x_o)$$  \hspace{1cm} (4.1)

The vector charge is associated with the isovector part of the e.m. current $j_{\mu}^{em V}$

$$j_{\mu}^{em V} = j_{\mu}^{em S} + j_{\mu}^{em V}$$  \hspace{1cm} (4.2)

The expectation value of this commutator is taken between the nuclear ground state at rest, the matrix element of the vector charge is proportional to $T_3$, the third component of the nucleus isospin. We insert now in this commutator a complete set of states $\Sigma_I |I> \langle I| = 1$ and we get the following sum rule
\[ (2\pi)^2 \delta(p-p')^2 M T_j = \langle B | Q^{+}_A | I \rangle \langle I | Q^{-}_A | B \rangle - c.t. (4.3) \]

The factor $2M$ where $M$ is the nuclear mass comes from the normalization of the nuclear state that we take to be that of a boson state

\[ \langle \rho | \rho' \rangle = (2\pi)^3 y \rho_0 \delta^3(\rho - \rho') \]  

(4.4)

This sum rule has quite a different appearance from the sum rule which relates the physical to the soft amplitude like the one we had obtained in the photoproduction case. The amplitude for nuclear scattering does not appear at all and it is our next task to bring this sum rule into a form which displays explicitly the amplitudes. To do this we use the PCAC relation, but this time in a reverse order. In the usual reduction techniques, we replaced the pion field by the divergence; here instead we start for the axial charge and transform it into a pion field. In order to do so, we relate the matrix element of the axial charge and that of the divergence. The axial charge matrix element may also be written as

\[ \langle B | Q_A(x) | I \rangle = \int d^3 \zeta \langle B | A(\zeta, x) | I \rangle = \frac{i}{i(E_B - E_x)} \int d^3 \zeta \langle B | \partial_\zeta A(\zeta, x) | I \rangle \]  

(4.5)

We assume that the axial current is localized, there is no current flow at infinity. The space integral

\[ \int d^3 \chi \langle B | A | I \rangle = 0 \]

therefore vanishes and we can add it to the previous relation to obtain the relation
\[ \langle B | Q_R(x) | I \rangle = \frac{1}{i(E_B - E_x)} \int d^3x \langle B | D(x, x) | I \rangle \]
\[ = \frac{(2\pi)^3 \delta^3(p_B - p_x)}{i(E_B - E_x)} \langle B | D(o, x) | I \rangle \] (4.6)

By applying PCAC, the divergence is transformed into a pion field. The sum on the right-hand side of Eq. (4.3) can thus be visualized as a sum over the processes where a pion is absorbed, leading to all possible final states. This pion is in general not on its mass shell. Its spatial momentum is zero because of the presence of the factor \[ \left[ \delta^3(p_{\pi} - p_B) \right] \] and its energy is the energy transfer between the states \( B \rangle \) and \( | I \rangle \). In certain cases, this pion is on the mass shell when the energy difference \( E_I - E_B \) is equal to the pion mass \( E_I - E_B = m_\pi \). This can occur in two ways: (i) the intermediate state can be a (highly excited) nuclear state without a pion; for instance, two fast nucleons are emitted in opposite directions, as occurs in the pair absorption process of the physical pion; or, (ii) the intermediate state is a nuclear state with one pion \( | I \rangle = | B \pi \rangle \). The condition \( E_I - M = m_\pi \) is satisfied when the intermediate nucleus is the ground state and the pion is at rest.

But whenever the pion is on the mass shell, the matrix element of the divergence is infinite. The sum rule has thus diverging terms and we are faced again with the situation encountered in photoproduction. Equation (4.3) is not directly applicable in this form but the divergences have first to be removed. This is done in the following manner \(^{24}\): we take the matrix element of the two equal time commutator divergences (integrated over space)

\[ \int d^3x d^3x' \langle B | [D(x, o), D(x', o)] | B \rangle \] (4.7)

This commutator is expanded in the same way with a complete set of states
\[
\int d^3x \int d^3x' \langle B| D^+(\mathbf{x},0), D^-(\mathbf{x}',0) |B\rangle \\
= \sum_\mathbf{I} \int d^3x \int d^3x' \langle B| D^+(\mathbf{x},0)|\mathbf{I}\rangle\langle \mathbf{I}| D^-(\mathbf{x}',0) |B\rangle - c.t. \tag{4.8}
\]

From the relation between \( Q \) and \( D \), the sum on the right-hand side is also
\[
\int d^3x \int d^3x' \langle B| D^+(\mathbf{x},0), D^-(\mathbf{x}',0) |B\rangle \\
= \sum_\mathbf{I} (E^I - E^0)^2 \langle B| Q^{+}_A |\mathbf{I}\rangle\langle \mathbf{I}| Q^{-}_A |B\rangle - c.t. \tag{4.9}
\]

We subtract from the first sum rule the second one divided by \( m_\pi^2 \). All the terms of the original sum are then multiplied by the factor
\( 1 - (E^I - E^0)^2/(m_\pi^2) \) which is zero when the energy difference is the pion mass and we have now a new sum rule which does not contain poles
\[
\langle B| [Q^+_A, Q^-_A] |B\rangle - \frac{1}{m_\pi^2} \int d^3x \int d^3x' \langle B| D^+(\mathbf{x},0), D^-(\mathbf{x}',0) |B\rangle \\
= \sum_\mathbf{I} \left[ 1 - \frac{(E^I - E^0)^2}{m_\pi^2} \right] \langle B| Q^{+}_A |\mathbf{I}\rangle\langle \mathbf{I}| Q^{-}_A |B\rangle - c.t. \tag{4.10}
\]

The removal of the divergences has been done at the price of introducing a new commutator which is not given by current algebra, on which we have to make model dependent assumptions.

In the models where the divergence is proportional to the pion field, the equal time commutator of two divergences is the equal time commutator of two fields which vanishes. We make here the assumption of the vanishing of this commutator. The fact that it does not seem to give any appreciable contribution in the nucleon case is a justification for this assumption in the nuclear case. With this assumption, the subtracted sum writes
\[
\langle B| Q^v |B\rangle = \int d^3x \int d^3x' \sum_\mathbf{I} \left[ \frac{1}{(E^I - E^0)^2} - \frac{1}{m_\pi^2} \right] \langle B| D^+(\mathbf{x},0)|\mathbf{I}\rangle\langle \mathbf{I}| D^-(\mathbf{x}',0)|\mathbf{I}\rangle \\
- c.t. \tag{4.11}
\]
We have not yet explicitly displayed the threshold scattering amplitude. It is contained in the part of the sum where the intermediate state is the nuclear ground state with one pion.

In a matrix element $<B|D|^\pi|B^\pi>$ it can happen that the nucleus does not interact with the pion, it travels by its own, independently of the pion and, of course, in that case, it has to keep its momentum $0$. Since the intermediate nucleus and the pion have opposite momenta, the pion also is at rest. This piece is called the disconnected, and the rest is the connected part. If we combine a connected piece in one and a disconnected piece in the other, we obtain a semi-disconnected part and we get the following graph

\[
\begin{array}{c}
\gamma(0) \hline \\
D \quad \pi(0) \quad B(0)
\end{array}
\]

What is the momentum transfer in the connected piece $<B(0)\pi(0)|D|B(0)>$? It is the energy difference which is equal to the pion mass $(E_\pi - M)^2 = m_\pi^2$. We are exactly at the pion pole for the matrix element of $D$. The residue of the pole $<B(0)\pi(0)|j_{\pi}|B(0)>$ is the threshold amplitude for the scattering of the pion on the nucleus $B$. The pion propagator $m_\pi^2 - (E_\pi - E_B)^2$ which is zero at the pole does not give rise to any divergence because its effect is cancelled by the factor

\[
\left[ \frac{1}{(E_\pi - E_B)^2} - \frac{1}{m_\pi^2} \right]
\]

introduced by the subtraction.

One can consider the semi-disconnected piece either in the direct term or in the crossed term. There appears thus the two threshold amplitudes for $\pi^-$ and $\pi^+$ scattering and the initial commutator is chosen in order that they appear with the opposite sign, and we can rewrite the sum rule in the form
\[
\frac{T_\pi^- - T_\pi^+}{8} = (z-N)T^- = \frac{2m_\pi M}{\frac{8}{\pi^2}} (z-N)
\]

\[
- m_\pi^3 (2\pi)^3 \delta((E-z-p_y)\Sigma) \frac{\langle B|A_\pi^+(0)|m\rangle \langle m|A_\pi^-(0)|B\rangle}{(E_x-H)^4 \left[(E_x-H)^2 - m_\pi^2\right]} \quad \text{c.t.}
\]  

(4.12)

This looks now like the kind of sum rule that we had in the photoproduction, and indeed this is exactly the sum rule we obtain if we reduce directly in the pion variables in the combination of two amplitudes \( a_\pi^- - a_\pi^+ \) and then use PCAC. The first part on the right-hand side of Eq. (4.12) arises from the equal time commutator. It represents the soft limit, the value of the amplitude when both pions are soft. The relation

\[
T^- = \frac{8m_\pi M}{\frac{8}{\pi^2}}
\]  

(4.13)

is equivalent to Weinberg’s relation

\[
\left(1 + \frac{m_\pi}{M}\right) a^- = L \quad ; \quad L = \frac{m_\pi}{4\pi\frac{8}{\pi^2}}
\]

since the threshold amplitude \( T \) and the scattering length are related by

\[
a = \frac{T}{8\pi(M + m_\pi)}
\]  

(4.14)

All the terms on the right-hand side are the corrections to the soft pion expression which are introduced by the mass extrapolation. Our task is now to estimate and interpret these corrections.

The only obvious limitation to the number of states that contribute to the saturation of this sum is due to the negative parity of the axial charge which restricts the possible contributors to the states that have a parity opposite to that of the ground state B.
As usual in the practical estimate of the sum we cannot keep all its terms, we have to truncate the sum. In doing this truncation, we are guided by the results of Fubini-Furlan in the nucleon case. For a nucleon target, an excellent saturation is obtained with a small number of states, i.e., the nucleon with one pion. The states further away in energy than one pion mass play practically no role. It is likely that in the nuclear case the saturation will also be obtained with low lying states. If these states are, in analogy with the nucleon case, the nuclear states with one pion cannot be immediately answered. One may worry about the nuclear excitation spectrum: there was no equivalent for the nucleon case since then the nucleon intermediate state was forbidden by parity and its excited states had large energy. But the nucleus has a very dense excitation spectrum at very low excitation energies. There was always a great concern that this spectrum could completely alter the balance between the different terms of the sum, and would affect the relation between the physical and the soft pion amplitudes in a radical and intricate way. This question was cleared by the work of d'Auria et al. 26) who showed that this excitation spectrum was in fact unimportant.

When the scattering takes place on a nucleon, the only correction to the pion value arises from the intermediate states consisting of a nucleon and a pion (connected part) and it is a small correction. In nuclei instead, it is the essential correction to the soft amplitude; it is responsible for the large size effects in the scattering of pions by nuclei.

What we are going to do now is to discuss a sum rule where only the coherent rescattering is kept, i.e., the intermediate states consisting of the nucleus $B$ in its ground state and one pion $|B\pi>$. We ignore therefore the contributions from the nuclear excited spectrum with and without pion.

How close is this new sum from the original one, we do not know for the moment, and we will have to discuss this point, but let us assume that it is a good one, and discuss it. The new truncated sum rule is
\[(Z-N)^{-1} = \frac{g M}{3 \pi^2} (Z-N) \]

\[- \frac{m^2}{2\pi^2} \left\langle \frac{d\sigma}{d\omega} \right\rangle_{\pi^+} \left\langle B(q) \pi^0(q) \right\rangle_{\pi^+} B(q) \frac{E_q}{E_q - M} \left\langle \frac{1}{m^2_\pi} \right\rangle_{\pi^+} \left\langle \frac{1}{E_q - M} \right\rangle_{\pi^+} \]

We have used PCAC to express the matrix element of divergence \(D\) with those of the pion source \(j_\pi\); \(q\) is the intermediate pion momentum, \(E_q\) its energy, \(E_q^2 = q^2 + m^2_\pi\); the intermediate nucleus energy is \(E_q\), it will be approximated, in the following, by the rest mass \(M\).

The isospin indices of the intermediate pion have not been written, the different possibilities have to be summed over, the intermediate re-scattering may occur with or without charge exchange.

The pion represented by the source \(j_\pi\) is not on the mass shell. Its spatial momentum is zero (there is no momentum transfer between \(|B>\) and \(|B\pi>\), and its energy is the energy transfer \(E_q^2 - M \approx E_q\). It is off the mass shell by the amount \(E_q^2 - m^2_\pi = q^2\).

In spite of the drastic simplifications that we have made in the original sum rule, because of the appearance of these off-shell effects, the new truncated sum rule does not reduce to a relation between physical quantities.

The evaluation of the sum depends on the detailed structure of the source function, which is a dynamical problem.

To proceed further, one has two possibilities. The first one is to make approximations which reduce the sum rule (4.15) to a relation between physical quantities.

The other choice is to introduce a specific model of the source function.
The first possibility will be shortly discussed but it is less attractive for two reasons. First the physical quantities in question, namely the pion-nuclear scattering amplitudes, are not available from the experiments. Moreover without a model for the source function one has no possibility of estimating the errors involved.

The dynamical model we will use for the source function is derived from an optical potential. The Schrödinger equation in the presence of a potential

\[ \nabla^2 \Phi + 2m_\pi E \Phi = 2m_\pi \nabla \Phi \]

(4.16)
corresponds to the non-relativistic version of the Klein-Gordon equation and on the right-hand side the quantity \( 2m_\pi \nabla \Phi \) represents, in the non-relativistic domain, the source function \( j_\pi \).

We could therefore insert this value in the matrix element of the sum (4.15) and calculate the sum. It is, however, unnecessary to do so because we can show the similarity between the dependence of the relativistic mass dispersion relation on the off-mass shell matrix elements and that of the non-relativistic potential dispersion relation on the off-energy shell matrix elements. Thus in the non-relativistic limit this formalism leads to the distorted wave Born approximation with an identification of the soft amplitude as the Born amplitude.

To display this similarity we just have to write the two dispersion relations. The mass dispersion one is already written in (4.15). Performing the summation over the possible charges of the intermediate pion, we get

\[ T^- = \frac{g_{\pi \pi}}{\sigma_\pi} + m_\pi^3 \int \frac{d^3 q}{(2\pi)^3 2M} \frac{T_{0q}^- \left[ T_{0q}^- + 2 T_{0q}^+ \right]}{\epsilon_q^2 q^2} \]

(4.17)
where
\[ T_{oq} = \langle B(o) | d_{\pi}(o) | B(q) \rangle \]
is the off-shell amplitude for \( \pi \) nuclear scattering.

On the other hand, in potential scattering, the scattering amplitude is defined as
\[ \varrho = \frac{1}{2\pi i} \sum_{l} (e^{2i\delta_{l}} - 1) \varrho_{l}(\cos \Theta) \quad (4.18) \]
and for pions of zero momentum
\[ \left( 1 + \frac{m_{\pi}}{M} \right) \varrho = -\frac{1}{4\pi} \varrho_{o} \left| 2m_{\pi} V \right| \varphi_{o}^{+} \quad (4.19) \]
Here \( \varrho_{o} \) is the plane wave state and \( \varphi_{o}^{+} \) the outgoing wave for pion of zero momentum. They are respectively solutions of the Schrödinger equations
\[ H_{o} \varrho_{o} = E_{o} \varrho_{o} \quad \text{and} \quad \left( E_{o} - H_{o} + i\xi \right) \varphi_{o}^{+} = V \varphi_{o}^{+} \]
where \( H_{o} \) is the Hamiltonian for the free pion. The pion kinetic energy \( E_{o} \) taken to be zero is written here for the sake of clarity. The \( \pi \) nucleus interaction optical potential is denoted \( V \) so that the total Hamiltonian is \( H = H_{o} + V \); the outgoing wave \( \varphi_{o}^{+} \) satisfies the following Lippman Schwinger equation
\[ \varphi_{o}^{+} = \varrho_{o} + \frac{i}{E_{o} - H_{o} + i\xi} V \varphi_{o}^{+} \quad (4.20) \]
which can be transformed by using the identity
\[ \left[ 1 + \frac{i}{E - H} V \right] \left[ 1 - \frac{i}{E - H_{o}} V \right] = 1 \quad (4.21) \]
\[ \Psi^+_o = \chi_o + \frac{1}{E_o - H + i\varepsilon} \chi_o \]  

(4.22)

And the scattering amplitude is thus

\[ (1 + \frac{m_P}{M}) \mathcal{P} = \langle \chi_o | -\frac{2m_P}{4\pi} \frac{V}{q^2} | \chi_o \rangle - \langle \chi_o | \frac{2m_P}{4\pi} \frac{V}{E_o - H + i\varepsilon} | \chi_o \rangle \]  

(4.23)

To pursue the analogy with the mass dispersion relation, we introduce a complete set of outgoing states

\[ \sum_q |\Psi^+_q \rangle \langle \Psi^+_q | = 1 \]  

(4.24)

which leads to the following form of the amplitude

\[ (1 + \frac{m_P}{M}) a = (1 + \frac{m_P}{M}) a_{\text{Born}} + \frac{1}{8\pi^2} (1 + \frac{m_P}{M})^2 \int \frac{d^3q}{q^2} |p_{oq}|^2 \]  

(4.25)

where

\[ (1 + \frac{m_P}{M}) a_{\text{Born}} = \langle \chi_o | -\frac{2m_P}{4\pi} \frac{V}{q^2} | \chi_o \rangle \quad (1 + \frac{m_P}{M}) \mathcal{P}_{oq} = \langle \chi_o | -\frac{2m_P}{4\pi} \frac{V}{q^2} | \Psi^+_q \rangle \]

This form displays explicitly the Born part. The dispersive integral in Eq. (4.25) represents the distortion of the pion wave.

To bring the two dispersion relations into identical forms, we relate the invariant relativistic amplitude to the non-relativistic one. In the non-relativistic domain,

\[ T = 8\pi M (1 + \frac{m_P}{M}) \mathcal{P} \]  

(4.26)
If the pion is taken to be non-relativistic in the mass dispersion relation, \( \epsilon_q = m_\pi \), this relation writes as

\[
\left(1 + \frac{m_\pi}{M}\right) a^- = \frac{m_\pi}{4\pi g_\pi^2} \int \frac{d^2q}{q^4} \frac{T_{oq}}{8\pi M} \left[ \frac{T_{q_0} + \frac{2}{8} T_{q_0}^+}{8\pi M} \right] \tag{4.27}
\]

Making abstraction of the isospin indices, the structure of this equation is identical to that of potential theory. We are therefore led to identify the soft pion expression and the Born amplitude

\[
\left(1 + \frac{m_\pi}{M}\right) a^-_{\text{Born}} = \frac{m_\pi}{4\pi g_\pi^2} = L \tag{4.28}
\]

The difference between the physical and the soft amplitude is then attributed to the pion wave distortion by the strong interaction.

Let us now clear up this question of isospin indices. The factor \( T^- \left[ T^- + 2T^+ \right] \) looks like a complex mixture of isospin odd and even amplitudes. It is in fact very easily understood. Let us take for simplicity the example of a target which has isospin \( \frac{1}{2} \), (with \( T_z = +\frac{1}{2} \)). The \( \pi^- \) nuclear system may be in the isospin state \( \frac{1}{2} \) or \( \frac{3}{2} \). The scattering lengths are different for these two isospin channels, call them \( a_1 \) (isospin \( \frac{1}{2} \)) and \( a_2 \) (isospin \( \frac{3}{2} \)). The \( \pi^- \) and \( \pi^- \) scattering lengths are related like in the nucleon case to \( a_1 \) and \( a_2 \) by

\[
\frac{a_{\pi^-}}{2} - \frac{a_{\pi^+}}{2} = \frac{a_1 - a_2}{3} \quad ; \quad \frac{a_{\pi^-} + a_{\pi^+}}{2} = \frac{a_1 + 2a_2}{3} \tag{4.29}
\]

The combination \( T^- \left[ T^- + 2T^+ \right] \) can also be written as

\[
T^- \left[ T^- + 2T^+ \right] = \frac{1}{3} \left[ T_1^2 - T_3^2 \right] \tag{4.30}
\]

On the other hand, on potential scattering, the potential depends on
the isospin channel, one has to write two different dispersion relations
in the two isospin channels

\[(1 + \frac{m_T}{M}) a_3 = (1 + \frac{m_T}{M}) a_{\text{Born}} + \frac{(1 + m_T/M)^2}{2 \pi^2} \int \frac{d^3q}{q^2} \left[ \frac{P_{1l}}{q_{1l}} - \frac{P_{2l}}{q_{2l}} \right]^2 \]  

(4.31)

Subtracting these two relations gives the isospin odd amplitude

\[(1 + \frac{m_T}{M}) a^- = (1 + \frac{m_T}{M}) a^-_{\text{Born}} + \frac{(1 + m_T/M)^2}{2 \pi^2} \int \frac{d^3q}{q^2} \left[ \frac{P_{1l}}{q_{1l}} - \frac{P_{2l}}{q_{2l}} \right]^2 \]  

(4.32)

as appears in the mass dispersion relation. This identity can be shown for any target isospin 27).

We have thus shown that by using an optical potential model as extrapolating method for the pion source, one is led to the identification of the soft pion amplitude and of the Born amplitude in potential scattering.

I insist, however, that at this stage, this conclusion is still tentative because a very important step is still missing: we have not yet shown that the truncated sum rule in which only the coherent re-scattering is kept as a correction to the soft pion amplitude is a good approximation to the original sum rule. Until we have done this, we have not achieved so much because in fact the result we got is rather obvious. If you look at Weinberg's expression:

\[ a = \frac{q_o}{4 \pi \rho_{p+n}^2} \frac{T \cdot t}{c} \]

you see that the threshold amplitude becomes infinitely weak for soft pions, \( a \to 0 \) for \( q_o \to 0 \). This means that the pion becomes like a neutrino, it interacts very weakly, i.e., it interacts only once.

The soft pion amplitude is therefore equivalent to a Born amplitude where the scattering occurs only once.
Now when we switch on the pion mass, many scattering can occur. The pion re-scatters in the nucleus many times.

What are the important re-scattering processes? When a pion re-scatters on the nucleons, it communicates a momentum transfer \( q \) to the nucleon. If the nucleus is to stay in the ground state in the intermediate state, the momentum that it can absorb is restricted to small values, roughly smaller than \( 1/R \), the inverse nucleus size. In keeping only the coherent re-scattering, we exclude the larger momentum transfers. A priori there is no reason why their effect should not be as important as that of the coherent re-scattering. Or we can say differently that when the pion transfers small momenta, it explores the gross feature of the nucleus, i.e., the size. For larger momentum transfers, it explores smaller details of the target, it explores the granular structure of the nucleus, it sees the pair correlation holes. We do not know a priori if it is a good approximation to neglect these last effects and to treat the scattering by the nucleus like the scattering by a piece of amorphous matter. The pion can also be absorbed and re-emitted and we have also to discuss how this absorption changes the scattering properties.

In the following, we discuss the effect of the remaining terms of the sum rule. Since the coherent re-scattering process was already treated, it is natural to proceed with the incoherent re-scattering processes where the nucleus is excited in the intermediate state. The question to be answered here is what happens when the pion re-scatters and the momentum transferred to a nucleon is too large to be absorbed by the nucleus in its ground state. This is discussed in the next paragraph.

4.1. Incoherent re-scattering

The sum rule with the incoherent re-scattering included writes
\[
(Z - N)^{-} = \frac{2N}{q^2} (Z - N) - \frac{m^3}{(2\pi)^4} \sum_{\text{p}} \int \frac{d^4q}{\sqrt{q^2}} \frac{\langle B(o)|\hat{J}_{\mu}(o)B(q)|n(q)\rangle \times \langle B(q)|\hat{J}_{\mu}(q)B(o)\rangle}{\left[\epsilon_q + E_q - M\right]^2 \left[\epsilon_q + E_q + m^2\right]}
\]

(4.33)

One can pursue in the non-relativistic limit the analogy with potential scattering since the potential \( V \) which describes completely the scattering of pion is non-diagonal. It acts not only on the pion variables but also on the nucleonic variables and it can connect the ground state to the excited states.

The elastic scattering amplitude is

\[
\left(1 + \frac{m^2}{M}\right)^0 = \langle B^\pi B\chi^+ \rangle \left(1 + \frac{m^2}{M}\right)^0 - \langle B^\pi B\chi^+ \rangle \left(1 + \frac{m^2}{M}\right)^0 \frac{V}{E_0 - H + i\epsilon} \quad \text{(4.34)}
\]

A complete set of intermediate states to be inserted into this equation has to include the excited nuclear states as well as the ground state. By doing this insertion, one gets

\[
\left(1 + \frac{m^2}{M}\right)^0 = \left(1 + \frac{m^2}{M}\right)^0 \sum_{\text{Born}} \frac{1}{i\pi} \frac{\langle B\chi\mid \varepsilon m^2 V \mid B\chi^+ \rangle \langle B\chi^+ \mid V \mid B\chi \rangle}{E_0 - E_\pi + i\epsilon} \quad \text{(4.35)}
\]

where \( E_\pi \) is the total energy of the intermediate state. This relation is the non-relativistic counterpart of the Fubini-Furlan sum rule. Thus, the identification should be made between the soft amplitude \( f(0) \) and the Born amplitude for the full non-diagonal potential \( V \). However, to describe the elastic scattering, one does not use this potential which would require to solve a system of coupled channel equations, but one uses instead an effective or optical potential for which the existence of the excited states may altogether be omitted. They are incorporated in the potential itself so that the scattering equation becomes
\[ T = V' + V' \frac{P_0}{E'_0 - H' + i\epsilon} T \quad (4.36) \]

where \( P_0 \) is the projection operator on the ground state. The relation between the original potential and the effective one is

\[ V_{\epsilon_0} = V + V \frac{P_{\text{exc}}}{E'_0 - H' + i\epsilon} V_{\epsilon_0} \quad (4.37) \]

\( P_{\text{exc}} \) projects on the excited states. The idea of the soft pion techniques is to make an expansion in the pion mass. The charge exchange potential \( V \) is of first order in the pion mass. The second term of the right-hand side of this equation is at least of second order. If we keep only second order terms in the pion mass

\[ V_{\epsilon_0} \approx V + V \frac{P_{\text{exc}}}{E'_0 - H' + i\epsilon} V \quad (4.38) \]

The Born amplitudes of the potentials \( V \) and \( V_{\text{eff}} \) are different. One cannot make therefore the identification of the soft amplitude and of the Born amplitude \( (a_{\text{eff}})_{\text{Born}} \) as deduced from the experiments. But one has instead the following relation between the Born amplitude of the effective potential and the soft amplitude \( L \)

\[ (1 + \frac{m^2}{M}) \left( a_{\text{Born}} \right)_{\epsilon_0} (z - N) = (z - N) L \]

\[ + \frac{m^3}{8\pi M} \sum_{e' B} \left( \frac{a^2 q}{(2\pi)^3} \right) \frac{< B(c)|H'_0|B'(q) \pi_0(q)> < B'(q) \pi_0(q)|H'_0|B(c)>}{[\epsilon_q + E_{B(q)} - M]^2[I_q + E_{B(q)} - M]^2 - m^4} \quad (4.39) \]

The intermediate pion isospin index \( \alpha \) has to be summed over (the values \( \alpha \) equal to \( -1 \) and \( 0 \)). Here \( E_{B',q} \) is the energy of the excited state \( B' \) for a momentum \( q \).

In order to evaluate this sum one makes the static approximation which greatly simplifies the expression (4.39). In this approximation, the nucleons are taken as massive, they move with very slow velocities and only the spatial correlations play a role.
The excitation energies which in a Fermi gas model are inversely proportional to the nucleon mass are neglected, all excited states are taken to be degenerate with the ground state and \( E_B'(q) \approx M \). The energy denominators are then independent of the particular state \( |B'\rangle \) and closure may then be used.

The matrix elements \( <B_o | j_{\pi^+} \pi^- | B'_q \pi_q > \) which appears in the sum of Eq. (4.38) should be estimated in the following manner. From the relation (4.38) between the effective and the non-diagonal potential this amplitude should be the Born amplitude as is seen by inserting plane wave states for the pion. On the other hand, since this amplitude is governed by the potential \( V \), it is legitimate to take for this matrix element the coherent sum of the individual nucleon contributions

\[
8 \pi M < B(o) | j_{\pi^+} \bar{\pi}^+(q) \pi(q) > = (1 + m_{\pi N}) \int d^3x e^{iQ \cdot x} < B | \Sigma(x) N^+ N^- \delta(x - x_0) | B' >
\]

\[
8 \pi M < B(o) | j_{\pi^+} \bar{\pi}^+(q) \pi(q) > = (1 + m_{\pi N}) \int d^3x e^{iQ \cdot x} < B | \Sigma(x) N^+ N^- \delta(x - x_0) | B' >
\]

where \( a_N^- \) is the \( \pi N \) scattering length. The off-shell effects in the \( \pi N \) amplitude have been neglected as well as the momentum dependence of this amplitude. The charge symmetric amplitude for \( \pi N \) scattering \( a_N^+ \) is very small and it may be neglected.

The relation between the Born amplitude and the soft amplitude becomes with these expressions

\[
\left( 1 + \frac{m_{\pi N}}{M} \right) (a_{\text{Born}}) (z - N) = (z - N) L
\]

\[
+ \frac{m_{\pi N}^3}{2 \pi \epsilon_0} (a^-)^2 (1 + m_{\pi N}) \int dq dq' d^3x\delta(x - x') e^{iQ \cdot x'} \sum_{\epsilon_0} \frac{\langle B | \Sigma \delta(x-x') \epsilon_0^+ | B' \rangle \langle B' | \Sigma \delta(x-x') \epsilon_0^- | B \rangle}{(E_{\epsilon_0} + \epsilon_{-N})^2 \left[ m_{\pi N}^2 - (E_{\epsilon_0} + \epsilon_{-N})^2 \right]}
\]

Using closure, the term to be estimated involves two summations on the nucleon labels \( i \) and \( j \). We may write symbolically

\[
\langle B | \Sigma_{i,j} | B > = \langle B | \Sigma_i | B > \langle B | \Sigma_j | B >
\]

\[
= \langle B | \Sigma_{i,j} | B > + \langle B | \Sigma_{i,j} | B > - \langle B | \Sigma_i | B > \langle B | \Sigma_j | B >
\]

\[
(4.42)
\]
The first piece with $i = j$ represents the self correlation effect. It is immediately calculable with the result

\[
(1 + \frac{m_{\pi}}{m}) \left( a^\text{Born}_{\delta g} - a^\text{soft}_{\delta g} \right) = 1 + \frac{m_{\pi}}{m} \left( \delta a^- \right)^\text{Born}_{\text{self correl}}
\]

\[
= m_{\pi}^2 (1 + \frac{m_{\pi}}{m_N})^2 \int \frac{d^2 q}{\pi^2} q^2 \epsilon_q^2 = \frac{2}{\pi} (1 + \frac{m_{\pi}}{m_N})^2 m_{\pi} (a^-)_c
\]

This correction is nothing else than the re-scattering on the individual nucleon.

This self correlation correction is a small positive effect, it increases the amplitude by only 6%. However, it is quite essential for the understanding of the link with multiple scattering; the result which is obtained at this stage is that the Born amplitude is proportional to the $\pi N$ nucleon amplitude. This approximation is the first order approximation of potential scattering where the potential is taken to be proportional to the $\pi N$ amplitude. This is somewhat disappointing: all this fancy theory to arrive at the starting point of a multiple scattering theory. But this is where we are now and in fact this result is more positive than it looks. First, in the soft pion technique, the remaining corrections appear in a more systematical way and in particular the role of the pair correlation will become very clear in this technique. Secondly the chosen example of the charge exchange amplitude is not the best to convince you of the interest of the method because here we can make an exact link with the multiple scattering theory. The charge symmetric elastic amplitude, or the photoproduction amplitude will be more interesting cases.

The remaining part of the incoherent rescattering

\[
<\theta | \Sigma_{\delta g} | \theta > - <\theta | \Sigma_{2} | \theta > <\theta | \Sigma_{2} | \theta >
\]

can be expressed in terms of correlation functions. These functions have a range smaller than the nuclear size and their effect is estimated with the local density approximation. Locally the correction is taken to be that of infinite nuclear matter with the local density at
the point considered. The model which is used in that picture will be that of an infinite Fermi gas where only Pauli correlations are present.

The pair term is expressed with two correlation functions.

We define

\[ G_+ (\alpha \alpha') = \langle \beta | \sum_{i \neq j} \delta (x_i - x_j) \sigma_i^+ \sigma_j^- | \beta \rangle \]

\[ - \langle \beta | \sum_j \delta (x_i - x_j) \sigma_i^+ | \beta \rangle \langle \beta | \sum_j \delta (x_i - x_j) \sigma_j^- | \beta \rangle \]  

(4.44)

and a similar expression for \( G_- \) where the order of the isospin indices is interchanged.

The effect of the pair term

\[ (\delta a^-)^{\text{Born}} \sigma_3 = \frac{m_3}{2 \pi} \left( 1 + \frac{m_3}{m_w} \right)^2 (a^-)_n \]

\[ \times \int d^4 q d^4 q' d^4 q'' \frac{e^{iq(x-x')}}{q^+ \epsilon_q^3} \left[ G_+ (\alpha \alpha') - G_- (\alpha \alpha') \right] \]  

(4.45)

Such correlation functions appear in the evaluation of the total muon capture rate, where the exclusion principle is known to have an important effect.

The role of the Pauli correlations can be visualized in the following. In a re-scattering process, the nucleon acquires a certain momentum transfer. This recoiling nucleon has to find an empty state, i.e., it has to lay outside the Fermi sphere. A certain fraction of the re-scattering processes is therefore forbidden by the exclusion principle. This fraction depends on the momentum transfer, the smaller the momentum the larger is this fraction.
Quantitatively the Fourier transform of the correlation function $G_{++}$ where a neutron is transformed into a proton is proportional to the neutron number $N$ and to the overlapping volume of two Fermi spheres displaced by the vector $q$

$$
\int d^3 x \int d^3 x' \frac{\sin q |x-x'|}{q |x-x'|} \frac{G_{+}(x,x')}{-i} = -N \left[ 1 - \frac{3}{2} \frac{q}{2p_F} + \frac{1}{2} \left( \frac{q}{2p_F} \right)^2 \right] \quad q < 2p_F \\
= 0 \quad q > 2p_F \quad (4.46)
$$

This value is well known in $\mu$ capture where it arises in the total capture rate calculation \textsuperscript{29}. Similarly, for the Fourier transform of the function $G_{+-}$

$$
\int d^3 x \int d^3 x' \frac{\sin q |x-x'|}{q |x-x'|} \frac{G_{+}(x,x')}{+i} = -Z \left[ 1 - \frac{3}{2} \frac{q}{2p_F} + \frac{1}{2} \left( \frac{q}{2p_F} \right)^2 \right] \quad q < 2p_F \\
= 0 \quad q > 2p_F \quad (4.47)
$$

The total effect of the pair term adds to the Born amplitude a piece

$$
\left( \delta a \right)_{\text{Born}}^{\text{pair}} = -\frac{e^2 N}{\pi} \frac{m_n^2}{m_H^2} \alpha_n^2 \left[ \frac{2p_F}{q_0} \int \frac{dq}{q} \left[ 1 - \frac{3}{2} \frac{q}{2p_F} + \frac{1}{2} \left( \frac{q}{2p_F} \right)^2 \right] \right] \quad (4.48)
$$

It is a negative effect which decreases the self-correlation. Combining the two parts of the incoherent re-scattering term, the self correlation and the pair correlation parts, the total effect is

$$
\left( \delta a \right)_{\text{Born}}^{\text{incoherent}} = \frac{2}{\pi} \frac{m_n^3}{m_H^2} \alpha_n^2 \left[ \frac{1}{p_F} \int_0^{p_F} \frac{dq}{q^3} \left[ 1 - \frac{q}{2p_F} - \frac{1}{2} \left( \frac{q}{2p_F} \right)^2 \right] + \int_{p_F}^{\infty} \frac{dq}{q^3} \right] \quad (4.49)
$$

The role of the exclusion principle is quite transparent on this expression. The re-scattering effects contained in the self-correlation part allow the intermediate nucleon to take any momentum transfers; the re-scattering occurs like for a free nucleon. The pair correlation term puts on this recoil the restrictions imposed by the exclusion principle. The total effect of the inelastic excitations is thus reduced compared to the nucleon case. For a given momentum transfer, it is proportional to the non-overlapping volume of the two Fermi spheres displaced by the distance $q$. 
The cut-off function $f(z)$

$$z = \frac{q}{2p}$$

increases from 0 (for $Z = 0$) to 1 ($Z = 1$).

The Fermi momentum is large compared to the pion mass, typically $p_F = 1.9 m_\pi$. The exclusion principle has then an appreciable effect, the incoherent re-scattering correction is reduced from a 6% effect (free nucleon value) to a 2% effect (nuclear value).

The over-all effect of the inelastic excitations might therefore be omitted and the soft amplitude is thus a better approximation to the Born amplitude than the $\pi N$ amplitude. Here the two quantities $L$ and $a^{-}_{NN}$ are nearly equal and the distinction is somewhat academic but this remark is important for the application that will be made to the charge symmetric case.

4.2. Pion absorption

There remains to be discussed the contribution to the sum of the terms where the intermediate states are the nuclear states not accompanied by a pion. A matrix element of the axial charge between two nuclear states represents, from PCAC, the absorption of a pion. This pion has zero spatial momentum (since the axial charge does not transfer momentum) and its energy is the energy transfer, i.e., the excitation energy of the state $B'$, $\epsilon_{B'}$. It is therefore off the mass shell by the amount $\epsilon_{B'}^2 - m_{\pi}^2$. 
When the excitation energy is of the order of the pion mass the pion is nearly physical and the absorption occurs predominantly by a two-nucleon process, the one-nucleon absorption is strongly suppressed by the energy momentum conservation. For low excitation energies instead $E_B' \ll m_\pi$ (small pion mass) the absorption is a one-body process. One may therefore distinguish approximately two regions of excitation, the low energy region where the dominant process is the one-nucleon absorption, the high energy region where the many-body mechanism dominates.

The contribution of the first region has been estimated by d'Auria et al. 26). They make the three assumptions which all follow from the consideration of the low lying states and the exclusion of the highly excited ones:

(i) the axial charge is taken to be the sum of single nucleon contributions $Q_A = \sum_i q_A^i$;

(ii) the multiplicative factor arising from the subtraction

$$1 - \left( \frac{E_B'}{m_\pi} \right)^2$$

is taken to be unity;

(iii) the axial charge for the nucleon is approximated by $q_A^\pm = (g_A/m_N) \Sigma B^\pm$ with a constant form factor $g_A$, which amounts to neglecting the pseudoscalar part in the expression of the axial current

$$\langle p(p_i)|n(p_i)|n(p_i)\rangle = \bar{u}_p(p)[g_A \sigma_5 + 2m_N g_A \frac{q_o}{m_N - q_o} \sigma_5] \sigma_i^+ u_n(p_i)$$

where $q_o$ is the energy transfer which is smaller than the pion mass and the pseudoscalar term may thus be neglected.

With these approximations the contribution of the one-nucleon absorption may be estimated immediately by closure. The result is related to the average kinetic energy $\langle \epsilon \rangle$ of the nucleon inside the nucleus

$$\left( \frac{\delta a^-}{\gamma_{\rm fermi}} \right)' \left( \frac{\delta a^-}{\gamma_{\rm Fermi}} \right) = - \frac{1}{2} g_A \frac{\langle \epsilon \rangle}{m_N}$$

(4.51)
Because of the smallness of the ratio $\langle E \rangle / m_N$, this is a small effect. It reduces the Born amplitude by $\approx 5\%$.

There remains to be discussed the two-nucleon contributions. Unfortunately, the estimate is more difficult than the previous one. The two-nucleon absorption is not easy to describe microscopically. One has of course the information of the capture rate for the physical pion, which gives a normalization in one point, for the physical pion. But to perform the off-mass shell extrapolation requires the knowledge of the reaction mechanism. This problem is at present not yet solved. For the time being, one assumes, as given by some crude estimates $^{30},^{31}$, that the effect is not larger in magnitude than the absorption itself $| (\delta a)^{\text{Born}}_{\text{pair abs.}} | \leq | \text{Im} a |$, which gives a small effect $(\delta a)^{\text{Born}}_{\text{pair abs.}} < 0.004 \text{ m}^{-1}$ that we have neglected. But more work remains to be done on the two-nucleon absorption effect.

4.3. - Conclusion and comparison with experiments

It has been shown that the main difference between the physical and the soft amplitude is the re-scattering correction of the potential scattering picture. The many-body nature of the nuclear system manifests itself as minor modifications.

Experimentally, the charge exchange amplitude is obtained from the energy shifts for nuclei with a different number of protons and neutrons. These energy shifts are isospin dependent and from this dependence the charge exchange optical potential is deduced. Its value as determined by Krell and Ericson $^{32}$ gives a Born amplitude

$$\left(1 + \frac{m_n}{M}\right)(a^-)^{\text{Born}} = 0.092 \text{ m}^{-1}$$

This value is in perfect agreement with the soft amplitude, $L = 0.09 \text{ m}^{-1}$ confirming the interpretation given here (in particular there seems to be no important effect of the two-nucleon absorption). This agreement gives
confidence in the mass extrapolation method which may then be applied in other cases where such a test is not possible (like in the charge symmetric case where the soft pion value is not known).

Before making this application to the charge symmetric amplitude, it is worth discussing in more detail how the method presented here is related to the more familiar treatment of multiple scattering.

4.4. - Link with the multiple scattering formalism

The aim of a multiple scattering technique is to relate the nuclear amplitude to the amplitudes for the individual nucleons. One is here in a unique situation to make the link between the two techniques. The reason is that the basic commutator which is involved in the Puppi-Puri method gives a vector charge. Its matrix element, from the e.m. current conservation, is exactly equal to the sum of the individual nucleon contributions. There are no exchange effects contained in such a matrix element. An exact correspondence between the two methods is therefore possible.

In a multiple scattering method, the potential is, in first approximation, proportional to the \( \pi N \) amplitude

\[
2m_{\pi} V(x) = -4\pi \left(1 + \frac{m_{\pi}}{m_{n}}\right) a_n \mathcal{G}(x)
\]  

(4.52)

Corrections to this first order approximation arise from the discrete structure of matter. The presence of one nucleon influences the local density of the other nucleons and the pion experiences influence this structure by virtually exciting the nucleus. In multiple scattering, this is the effective field correction \(^{33}\) which is shown below to be equivalent to the pair correlation term of the hard pion technique. This correction arises from the difference between the average field in the medium and the effective field which excites a scatterer.
The hole profile around a scattering at the point \( r \) is described by the nuclear pair correlation function \( g(r', r) \). Since one nucleon has been removed by the presence of the scatterer at the point \( r \), the normalization condition for the function \( g \) is

\[
\int d^3u \ g(u') g(r', r) = -1
\]  

(4.53)

The difference between the effective and the average field is the field radiated by the hole. In the long wavelength limit

\[
\phi_q \approx \phi_{\omega} - \alpha \langle \frac{1}{r} \rangle \phi_q
\]

\[
\langle \frac{1}{r} \rangle = -\int d^3u \ \frac{g(u')}{u'} g(u)
\]

(4.54)

where \( \langle 1/r \rangle \) is the average value of \( 1/r \) over the correlation function.

The replacement of the average field by the effective field amounts to replacing the \( \pi N \) scattering length by an effective value

\[
\alpha_{q} = \alpha - \alpha^* \langle \frac{1}{r} \rangle
\]

(4.55)

In the hard pion technique, we found the result that with the self correlation only, the optical potential was proportional to the \( \pi N \) amplitude. The inclusion of the pair term modified this result by an amount

\[
(\delta a)_{\text{pair}}^{\text{total}} = \frac{m_{\pi}^3}{2\pi^2} (1 + \frac{m_{\pi}}{m_N}) \alpha_N^* \int d^3q d^3r d^3r' \frac{e^q(u', u)}{q^2} G(u, u')
\]

(4.56)

The function \( G(r, r') \) has been defined previously (4.44). The isospins are here omitted not to burden the formulae. The function \( G \) is related to the hole profile function \( g(r, r') \) by

\[
G(u, u') = g(u) g(u') g(u, u')
\]

(4.57)
In the non-relativistic limit $\epsilon_q = m_\pi$ and with
\[ \int \frac{d^3q}{q^+} e^{iq\cdot r} g(r+u) g(u) = -2\pi^2 \left< \frac{d}{x} \right> \]

the modification introduced by the pair correlation is thus
\[ \left( \delta a \right)_{\text{pair}} = -a^2 \left< \frac{d}{x} \right> \]

which is the same as the effective field correction.

There is therefore a complete equivalence between the two theories concerning the effect of the nuclear polarization. However, in multiple scattering the starting point is the $\pi N$ amplitude where the re-scattering effect on the nucleon is already built in. The role of the pair correlation is to cancel a part of this re-scattering. While in the hard pion technique, because of this cancellation, the effect of the nuclear polarization may altogether nearly be omitted.

Finally, to make complete the link between the two approaches, I would like to say a word about the Fermi motion corrective term which arises in the soft pion method from the virtual absorption of the pion on single nucleons, leading to low lying states. This is, also in multiple scattering, a correction which arises apparently from a different origin \(^{33}\). The $\pi$ nucleon interaction has a $s$ and a $p$ wave component. The amplitude is
\[ \mathcal{F} = a + c \frac{k \cdot k'}{k^2} \]

$k$ and $k'$ are the momenta of the incident and scattered pion. In nuclei the motion of the nucleons induces a correction in the $p$ wave part which is proportional to
\[ c \: m_\pi \left[ \nu_\pi - \nu_N \right] m_\pi \left[ \nu'_\pi - \nu_N \right] \]

This induces a $s$ wave interaction which depends on the Fermi motion of the nucleons and is the same as the Fermi correction of the soft
pion approach. Finally, pion absorption by the pair mechanism rises
the same problems in a multiple scattering theory than by the soft
pion technique. Its influence has been quantitatively estimated in
a reliable way in none of the approaches.

There is thus a complete equivalence between the hard pion
technique and the multiple scattering method. The first one provides
in a quasi automatic manner the optical potential up to the second
order in the pion mass and gives a clear picture of the pair correla-
tion hole.

It is natural to ask the question : what would happen if
the saturation of the sum rule is not completely attained with these
low lying states only ? If it turns out that the saturation of the
sum rule in the nucleon case requires extra terms, they should also be
present in the nuclear case ; they are necessary to recover the expres-
sion of the \( \pi^0 N \) amplitude in the optical potential, and, therefore,
the equivalence with multiple scattering. There is no modification
of the physical picture which has been given here, but of course the
numerical value of the potential would be somewhat modified. The good
agreement between the experimental value of the \( \pi^0 N \) amplitude and
the extrapolated one indicates that the possible effects of extra
contributions are small.

5. - \textbf{CHARGE SYMMETRIC AMPLITUDE}

This amplitude is related to the \( \pi^\pm \) amplitudes by

\[
T^+ = \frac{1}{2} \left[ T^- + T^+ \right]
\]

(5.1)

The relation between the physical and the soft amplitude may be derived
either by the reduction formalism or by the Fubini-Furlan technique.
The starting point in the second method is the equal-time commutator of an axial charge $Q^+_A$ with the divergence of the axial current $D^-$, integrated over space.

$$\int d^3x \left[ Q^+_A(x), D^-(x,0) \right]$$

And this commutator is then expanded using a complete set of intermediate states

$$\sum_i \left< B | Q^+_A(0), D^-(x,0) \right> = \sum_i \left< B | Q^+_A(0), D^-(x,0) \right> = c.t. \cdot$$

The technique is completely similar to the one used in the charge exchange case. The threshold amplitude appears in the semi-disconnected terms. This time, the choice of the commutator changes the relative sign of the two amplitudes and there appears the sum of the amplitudes for $\pi^-$ and $\pi^+$, $T_{\pi^-} + T_{\pi^+}$. The elimination of the divergences requires the subtraction from this sum rule of a second sum rule obtained from the commutator $\int d^3x d^3x' \left[ D^+(x,0), D^-(x',0) \right]$. Here, we assume that we can use PCAC to make an estimate of the contribution of this commutator. The equal time commutator of a field with the time derivative of a field is just a c number. In that case there is no contribution of the equal time commutator $D^+,D^-$ from the subtraction, because if we do the things very carefully, we have to subtract the vacuum expectation value of the commutator which cancels this c number.

The physical amplitude is related to the equal time commutator by a sum rule

$$T^+ = \frac{1}{2} \left[ T_{\pi^-} + T_{\pi^+} \right] = -\frac{i}{\rho^1_{\pi^+}} \left< B | Q^+_A(0), D^-(0) \right> = c.t.$$

when the incident and outgoing pions become soft, $q_0 = 0$, there survives only the term arising from the equal time commutator.
\[ T^+(0) = -i \frac{\partial}{\partial \tau} \langle B | [Q^+_A, D^-(0)] | B \rangle \]  

(5.4)

What is this value? What is the expression of this commutator? The answer to these questions is that we do not know. There exist several models for this commutator, but we have to choose among them and the theoretical situation is actually quite confuse. Why should we proceed then? What is the point in trying to relate the physical to the soft pion amplitude if this last amplitude is not known? But exactly for this reason. This time we work in the reverse order. We do not know what is the soft amplitude but we can measure the physical amplitude and since we think that we understand the corrections quite well, maybe we can go back to the soft amplitude and learn something about this commutator \[ 34 \]. This is our aim, but let me first explain why this is an important object, on which we would like very much to have some knowledge.

The divergence \[ D = \mathcal{D}_o A_o + \mathcal{D}_a A_a \; ; \; i = 1,2,3. \]

We have seen that the spatial part does not contribute once it is integrated over space \[ \int d^3x \mathcal{D}_a A_a = 0. \] The commutator \[ \int d^3x [Q^+_A, D^-(x,0)] \] is then also the commutator of \[ Q \] with the time derivative of the fourth component of the axial current \[ \mathcal{D}_o A_o, \]

\[ \langle B | [Q^+_A, D^-(0)] | B \rangle = \int d^3x \langle B | [D^+(x,0), D^-(0)] | B \rangle = \int d^3x \langle B | Q A^+_o (x,0), D^-(0) | B \rangle \]  

(5.5)

This time derivative

\[ \mathcal{D}_o A_o = i [H, A_o] \]  

(5.6)

and thus

\[ \langle B | [Q^+_A, D^-(0)] | B \rangle = i \langle B | [H(x), A^+_o (x)], D^-(0) | B \rangle = i \langle B | [H(0), A^+_o (0)], Q A^{-}_o | B \rangle = i \langle B | [Q^+_A, [H(0), A^{-}_o (0)] | B \rangle \]  

(5.7)
The axial charge is not conserved and the Hamiltonian therefore does not commute with the axial charge, but the idea is that the axial current is nearly conserved. The total Hamiltonian has a part which commutes with $Q_A$ and a part, which we believe to be smaller, which breaks the conservation of the axial charge or the chiral symmetry. The knowledge of $T(0)$ may thus provide information on the way in which the chiral symmetry is broken.

To extract this information, it is really necessary to use nuclei and bother with all the complications of the nuclear problem. Why not, just consider the expectation value $\langle N | \sum_i \left[ \bar{q}_i^+, s_i^+, d_i^+, \bar{u}_i^+, \bar{d}_i^-, u_i^- \right] | N \rangle$ between nucleon states. In that case, the mass extrapolation problem is in principle much simpler, the correction due to the finite mass is just the re-scattering on the nucleon. It is expressed with off-shell $\pi N$ amplitudes. If the off-shell effects are neglected, the correction is easily calculable with the result

$$
\left(1 + \frac{m_\pi}{m_N}\right) a^+_N = \left(1 - \frac{m_\pi}{m_N}\right) a^+_N(0) + m_\pi \left(1 + \frac{m_\pi}{m_N}\right)^2 \left(a^{+2}_N + 2a^{-2}_N\right)
$$

(5.8)

One can thus trace back the value of $a^+_N(0)$ from that of the physical amplitude.

This program has been carried through in a systematic way by Von Hippel and Kim\textsuperscript{35)}. They have not considered only $\pi N$ scattering but $KN, \bar{K}N$ and $\pi \Sigma$ as well. Their result for the soft amplitude is

$$a^+_N(0) = -0.03 \frac{m_\pi^{-1}}{m_\pi^{-1}}$$

This value was questioned by a recent work of Cheng and Dashen\textsuperscript{36)}. By a completely different method they deduce instead a large value for the soft pion contribution $a^+_N(0) = -0.12 \frac{m_\pi^{-1}}{m_\pi^{-1}}$; the physical amplitude, however, remains very small. They do not comment on the origin of the discrepancy with Von Hippel and Kim\textsuperscript{35)} but it is natural to blame the neglect of the off-shell effects, particularly in the symmetric amplitude.
It is very interesting to use nuclei as a supplementary source of information which may help in settling the differences. Before explaining why I should mention that the result of Cheng and Dashen was criticized by Höhler et al. 37) who find by a similar method the value \( a^+(0) = 0.047 \ m^{-1} \) consistent with that of Von Hippel and Kim.

The reason why nuclei may be very useful is that the off-shell effects of the \( \pi N \) scattering amplitude are minimized since the incoherent re-scattering where these effects appear is largely suppressed by the exclusion principle.

The calculation is done here in complete analogy with the charge exchange case. We have seen that once the main distortion effect is taken into account, the Born amplitude is very close to the soft pion amplitude. The incoherent re-scattering represents only a minor modification. If the off-shell effects are neglected for the \( \pi N \) amplitude, the incoherent re-scattering integral is

\[
(1 + \frac{m_{\pi}}{M})(\sigma^+)^{\text{Born}}_{\text{mech.}} = A m_{\pi} \left( 1 + \frac{m_{\pi}}{m_N} \right) (\sigma^-) \left( \int_0^{2p_F} \frac{dq}{q+m_{\pi}^2} \left[ \frac{3}{2} \frac{q}{q+m_{\pi}^2} - \frac{1}{2} \frac{q}{q+m_{\pi}^2} \right] + \int_{2p_F}^{\infty} \frac{dq}{q+m_{\pi}^2} \right)
\]

\[= 0.0095 \ A \ m_{\pi}^{-1} \quad (5.9)\]

where \( A \) is the nucleon number, while in the absence of the exclusion principle it would have the value

\[
(1 + \frac{m_{\pi}}{M})(\sigma^+)^{\text{Born}} \quad (5.10)
\]

The exclusion principle reduces the importance of the incoherent re-scattering integral, and it is therefore more advantageous to work in nuclear matter in order to minimize the off-shell effects of the \( \pi N \) amplitude.

There remains to estimate the effect of pion absorption. The single nucleon absorption term is
\[
\left(1 + \frac{m_\pi}{M}\right) (\delta a^+)^{\text{Born}}_{\text{absorption}} = \left[ \frac{g_{A^1}}{8\pi\hbar^2 m_\pi^2} \left[ \Sigma (E_{B'} - M) < B | \Sigma \epsilon_z \epsilon_x \epsilon_y | B' > < B' | \Sigma \epsilon_z \epsilon_x \epsilon_y | B > 
+ (E_{B'} - M) < B | \Sigma \epsilon_z \epsilon_x \epsilon_y | B' > < B' | \Sigma \epsilon_z \epsilon_x \epsilon_y | B > \right] \right]
\]

The appearance of the energy factor \( E_{B' - M} \) prevents the use of closure. D'Auria et al. 26) have made the assumption that there is only one state \( B' \) which is strongly coupled to the ground state \( B \). (For \( 0^+ \) ground state this is supported by a calculation by Blomqvist and Molinari 38).)

With this assumption it is possible to factorize the energy \( (E_{B' - M}) \) and then use closure. D'Auria et al. get in this way a very small contribution

\[
\left(1 + \frac{m_\pi}{M}\right) (\delta a^+)^{\text{Born}}_{\text{absorption}} = 0.001 \pm m_\pi^{-1}
\]  \hspace{1cm} (5.12)

which may be ignored.

The two-nucleon absorption raises here the same problem than in the charge exchange case. With the assumption

\[
| (\delta a^+)^{\text{Born}}_{\text{pair absorption}} | \leq \frac{m a^+}{m_\pi} = 0.009 m_\pi^{-1}
\]

it should give a rather small contribution that we have neglected. The charge exchange case does not contradict the assumption of a small influence of the absorption but it would be very valuable to obtain a reliable estimate of the pair absorption effect in order to extract the soft pion amplitude in a completely reliable way.

The interpretation of the \( \pi^- \) mesic data by Krell and Ericson 32) has provided the value of the Born amplitude which is repulsive and proportional to the nucleon number \( A \)

\[
(1 + \frac{m_\pi}{M}) a^+_{\text{Born}} = -0.035 \ A \ m_\pi^{-1}
\]  \hspace{1cm} (5.13)
with

\[(1 + \frac{m_N}{M})(\delta a^+)^{\text{Born}} = 0.0095 A m_\pi^{-1} \quad (5.14)\]

we find

\[(1 + \frac{m_N}{M}) a^+(0) = -0.044 A m_\pi^{-1} \quad (5.15)\]

Extrapolating this value to a nucleon number \( A = 1 \), we obtain

\[a^+_N(0) = -\frac{0.044 m_\pi^{-1}}{1 + \frac{m_N}{m_\pi}} = -0.038 m_\pi^{-1} \quad (5.16)\]

which is in good agreement with the values of Höhler et al. and Von Hippel and Kim. It contradicts instead the result of Cheng and Dashen.

The estimate made here of the incoherent re-scattering is uncertain if there exist important off-shell effects in the \( \pi^- N \) amplitude. But there remains the fact that the exclusion principle in nuclei brings the Born amplitude towards the soft pion one and the consistency of our value and the one derived from the nucleon case suggests that the off-shell effects should not be too important. The fact that there is no evidence for large off-shell effects in the charge exchange case where there appears the combination \( f^-_q(f^-_q + 2f^+_q) \) supports this statement (a similar remark also applies to the photoproduction case where both the soft and the physical amplitudes are known). The mass extrapolation does not display any visible off-shell effects. We do not see therefore how the mass extrapolation could give a contribution to the Born amplitude as large as \( 0.09 A m_\pi^{-1} \) which is needed to understand the value of Cheng and Dashen which thus seems improbable.
6. - RADIATIVE CAPTURE

We have seen in details the technique of the mass extrapolation. I will therefore be extremely brief on the technical side for the radiative capture amplitude.

Similar to the elastic scattering case, the radiative amplitude $F(m_\pi)$ is related to the soft one $F(0)$ by a sum rule.

The soft amplitude is, as we have seen, proportional to the axial current matrix element. For charged pions,

$$F(0) = \pm \left( \frac{i e}{\hbar c} \right) \langle \phi | \varepsilon \cdot A^\pm | \chi \rangle$$

(6.1)

Neglecting an equal time commutator which vanishes in the exact chiral symmetry limit, the relation between the physical and the soft pion amplitude is, as obtained by the reduction technique:

$$F^\pm(m_\pi) = F^\pm(0)$$

$$+ m_\pi \sum_{n} (1)^{n} \delta(R_{n}-P_{f}) \langle \phi | \varepsilon \cdot j_{\text{em}} | m \rangle \langle m | j_{\text{em}}(0) | \chi \rangle \left( m_\pi + E_{f} - E_{n} \right)$$

(6.2)

One can show by techniques completely similar to those used for the elastic case that the main difference introduced by the mass extrapolation is the pion wave distortion. The physical amplitude gets thus renormalized compared to the soft one by a factor $C_{\pi}^2 < 1$ where $C_{\pi}$ represents the reproduction in the pion intensity due to the strong interaction.

In Fig. 1, taken from Krell-Ericson, the square of the pion wave function (related to the factor $C_{\pi}$) is represented with and without strong interaction potential, in the case of $^{16}O$. 

The question that comes to the mind when seeing this curve is the following: we had obtained the expression of soft pion amplitude in a model independent way. How much of a model dependence do we have to introduce for the physical pion? Do we have to know the distribution of the interaction in the nucleus? You see, in Fig. 1, that the pion intensity varies appreciably over the nuclear size. This variation is mainly due to the existence of a non-local part in the optical potential.

The reduction factor due to the strong interaction thus depends on the particular point inside the nucleus and in principle we should know the distribution of the interaction to account for the distortion in a completely correct manner. The case of the lighter nuclei is more favourable since the wave function has then a flatter profile. In a first stage, the test of the soft pion ideas should be made in very light nuclei (\( A \leq 12 \)).

The results for radiative capture are similar to those obtained for the elastic scattering.

There are some differences between the soft pion and the Born amplitude: they arise from different intermediate states. The most important ones are:

- the nucleon-antinucleon pair term;
- the effect of the vector meson (in particular the \( g \) );
- the incoherent re-scattering (like for the scattering problem, it is reduced by the effect of the exclusion principle);
- the nuclear states without pion (Fermi motion) give a negligible contribution.

With all the corrections, we derive an effective Hamiltonian which is

\[
H_{\text{eff}} = \pm \frac{ie}{\hbar c} \left[ \epsilon \cdot \mathbf{A} + g_A(\mathbf{q} \cdot \mathbf{r}) \mathbf{S} \cdot \mathbf{S} \right]
\]  
(6.3)
The first part on the right-hand side arises from the soft pion amplitude and the rest represents the finite mass corrections. They are here more important than in the elastic case. If the exchange effects are neglected in the soft amplitude, one gets a value for the effective Hamiltonian

$$\mathcal{H}_{\pi\pi} = \pm \frac{ie}{p_n} Q_n \left( \frac{1}{4} \gamma_5 \right) \Sigma_i \sigma_i \varepsilon \varepsilon_i^\pm \delta(\alpha \cdot \varepsilon)$$  \hspace{1cm} (6.4)$$

which is slightly renormalized (\sim5\%) over the one deduced with the impulsive approximation by the effect of the exclusion principle.

**Experiments**

The Born amplitude for the radiative transition is proportional to the quantity $<f | \mathcal{A} | i>$, which has to be estimated. The axial current matrix element is developed in form factors which have to be determined. Delorme 39 has developed a general method based on the link with the impulse approximation in order to eliminate those which give small contributions. The "axial" form factor which is the main contributor is, in certain cases, known at zero momentum transfer from the $\beta$ decay rate. To obtain its momentum dependence one makes the assumption that its variation is the same than that of the magnetic form factor $F_M$ which is obtained from the inelastic electron scattering leading to the isobaric analogue of the state $f$.

$$\frac{F_A(t)}{F_A(0)} = \frac{F_M(t)}{F_M(0)}$$  \hspace{1cm} (6.5)$$

This relation holds to a good precision in the impulse approximation and it accounts well for the observed $\mu$ capture rates. The hope is that this way of proceeding is an improvement over the impulse approximation since it should include, at least approximately, the exchange effects of the axial current matrix element. This is true if these effects vary slowly with the momentum transfer.
There exist measurements of the branching ratio for radiative capture for stopped pions, leading to a specific final state \( R = \Gamma_{\pi g} / \Gamma_{\text{tot}} \) where \( \Gamma_{\text{tot}} \) is the total absorption rate. If the pions were absorbed in the 1s orbit, the measurement of this ratio would lead directly to the knowledge of the radiative capture probability \( \Gamma_{\pi g} \) for s wave pions, i.e., would bring a test of the soft pion ideas. The total capture rate \( \Gamma_{\text{tot}} \) is known experimentally from the width of the 1s level.

Unfortunately a large fraction of the pions is absorbed in p state (typically of the order of 68% in \(^6\text{Li}\)). One has to take into account the radiative capture from the p states. This capture is not completely governed by the effective Hamiltonian given in (6.3). There are velocity dependent terms which have to be derived from the impulse approximation. Combining the soft pion technique for the 1s state and the impulse approximation for the 2p state, Delorme \(^{39}\) calculates a branching ratio for the transition \( \pi^- + ^6\text{Li} \to ^6\text{He} + \gamma \) to be compared with the observed one \(^{41}\) \( R = (1.0 \pm 0.1) \times 10^{-2} \). The general agreement is rather good but the absorption from the 2p state is responsible for \( \approx 50\% \) of this number. Definite conclusions cannot be drawn about the agreement for the 1s state alone.

Actually the best convincing case for the relation between \( \pi^- \) and \( \mu^- \) capture is the measurement of the Paroffsky ratio in \(^3\text{He}\): \(^{3}\)

\[
\mathcal{P} = \frac{\pi^- + ^3\text{He} \to \pi^- + T}{\pi^- + ^3\text{He} \to \gamma + T}
\]

which is measured to be \( 2.28 \pm 0.2 \).

The nuclei of \(^3\text{He}\) and \( T \) having the same quantum numbers as the proton and neutron, the form factor expansion of the axial current is the same as in the nucleon case, only the axial form factor has to be determined. Its value is known at \( t = 0 \) from \( \beta \) decay, \( F_A(0) = -1.21 \) and at \( t = -0.27 \text{ fm}^{-2} \) from \( \mu^- \) capture \( F_A(-0.27)/F_A(0) = 0.88 \). A linear extrapolation to the value \( t = -0.47 \text{ fm}^{-2} \) of \( \pi^- \) capture gives \( F_A(-0.47)/F_A(0) = 0.67 \).
The radiative capture probability for pion bounds in the 1s orbit is calculated $\Gamma_{1s} = 4.1 \times 10^{15}$ s$^{-1}$.

(The corresponding photoproduction cross-sections for $\pi^+$ are $d \sigma/dQ = 18.6 \mu b$, $d \sigma/dQ = 13.8 \mu b$). The absorption from the 2p state should not be important in such a light nucleus. But even if it is not negligible, the Panofsky ratio is not sensitive to the absorption from the 2p state, because both the radiative capture and the charge exchange are small in the p state.

The theoretical value of the charge exchange probability depends slightly on the estimate, impulse approximation or soft pion method.

In the first case the predicted Panofsky ratio is $P = 2.1$ while in the second one it is $P = 1.9$. This excellent agreement confirms the validity of the approach used but it would be valuable to have a direct measurement of the radiative capture from the s orbit. But it is already clear that the radiative pion capture will become a tool for measuring the axial current form factor.

We have thus seen that the soft pion technique allows a systematic study of the radiative capture process. It can be linked to the impulse approximation if the exchange effects are neglected but it goes beyond in the sense that it allows the study of the deviations from this approximation. To our knowledge this is the only method which makes this study possible.

We have seen that the deviations from the impulse approximation, the exchange effects are of two types: those which are contained in the soft pion amplitude, i.e., in the axial current matrix elements, and a renormalization effect introduced by the exclusion principle.

Note that instead for neutral pions the many-body effects play an essential role since the soft pion amplitude vanishes. This means that the impulse approximation is a very bad one.
7. - CONCLUSION

We have studied the relation between the physical and the soft pion amplitude. We have found that the main difference between the two is just the distortion of the pion wave, so that the soft amplitude appears as the Born amplitude. The distortion can be taken into account with an optical potential and in light nuclei it can be simply accounted for by an over-all multiplicative factor.

The composite structure of the nucleus manifests itself by small corrections to this picture. The physical pion sees the granular structure of the nucleus by virtually exciting the system. However, these effects are strongly suppressed in a nucleus by the exclusion principle which prevents the nucleons to recoil into an occupied state.

This remark has interesting consequences if one wants for instance to trace back the value of the soft pion amplitude from the physical one (as in the charge symmetric scattering amplitude).

In the photoproduction case this method provides an effective Hamiltonian the main part of which is independent of the impulse approximation. This method accounts in principle for the exchange effects which have two origins: those which are contained in the soft pion amplitude, in the axial current matrix element. The second is the effective field correction which arises from the exclusion principle, as mentioned before. The effective Hamiltonian gets thus renormalized in nuclei.

This interpretation of the mass extrapolation problem in nuclei seems to be well supported by the cases where the soft pion amplitude is theoretically known and the physical amplitude is experimentally measured, namely the charge exchange elastic amplitude and the radiative absorption process (transition $^3\text{He} \rightarrow \text{T}$). This good agreement gives confidence in the extraction of the soft pion amplitude from the physical one in the charge symmetric elastic scattering case.
The largest uncertainty actually arises from the estimate of the two-nucleon absorption effect. It would be extremely valuable to have a reliable estimate of this effect.

More experiments should be made, in particular an accurate determination of the isospin dependence of the scattering amplitude, in order to test with a high precision the charge exchange case. Other experimental tests for the radiative capture should be done, with the assurance that the pion has been absorbed from the $1s$ state.

A measurement of the scattering length of pions on deuterium would be welcome to give an idea of the importance of the two-nucleon absorption.

To summarize the essential ideas contained in these lectures, the idea of the soft pion technique is to make an expansion in the pion mass which is considered to be a small quantity. When we say that a quantity is small, we have to specify compared to what; what are the dimensionless parameters which fix the scale of the expansion? In the nucleon case there are two scales, and they are not associated with very small dimensionless parameters. The soft pion result is therefore not valid as such but one has to include the higher order terms of the expansion.

- The first scale is associated with the nuclear size. It is characterized by the dimensionless parameter $m_\pi R$, ratio of the nuclear radius to the pion Compton wavelength. This quantity is large even in light nuclei ($\approx 2$ or $3$). But the interpretation given here with an optical potential allows to keep all orders in this parameter.

- The second scale arises from the complex granular structure of the nucleus. The characteristic parameter is $m_\pi d$, the ratio of the correlation distance (inter-nucleonic distance) to the pion Compton wavelength. This parameter is moderately large ($\approx 1$) and we have kept only the corrections of order $m_\pi d$ and neglected the higher order ones.
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REFERENCES


27) A. Figureau - Private communication.
28) M. Ericson and M. Rho - To be published in Physics Reports.
34) W.T. Huang - Thesis, University of Maryland (1971);
    P. Gensini - University of Lecce Preprint (1971);
FIGURE CAPTION

Pion intensity

I. without strong interaction potential,
II. with strong interaction potential,
III. with the local part only.
NUCLEAR MODELS

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ABSTRACT

In these four lectures the aim will be to illustrate ways in which the nuclear physics aspects of various middle-energy phenomena have been treated. Wherever possible, the same phenomena will be discussed, as the nuclear physics approximations employed become more and more realistic. Firstly, we shall discuss the use of the so-called Elementary Particle Method of Kim and Primakoff in the particular case of the partial capture rate $\mu^- + ^{16}\text{O}(g.s.) \rightarrow ^{16}\text{N}(g.s.) + \nu_\mu$. Then we shall look at the corresponding total capture rate to illustrate the use of the Fermi Gas Model, Migdal theory, and the standard shell model.

INTRODUCTION

I realize that most of you are not interested in nuclear physics. You probably consider it as rather an unpleasant subject with which to get involved. But I hope to illustrate in these lectures the importance of this unpleasant subject. After all if you insist on bombarding nuclei with protons, pions, muons, then you have to at least admit the existence of nuclear physics. Of course, you might say that you will do experiments in such a way as to minimize the effects of nuclear physics. But then this means you have at some stage to calculate the nuclear physics effects, so that you can say and not just hope the corrections are small.
The theme in these lectures will be to consider nuclear models that become more and more sophisticated, which usually means more and more microscopic, and see how predictions or interpretations vary as the model improves.

Firstly, I shall discuss the Elementary Particle Method (E.P.M.)\(^1\) or Nuclear Physics in a Nut-Shell model in which most of the nuclear physics is dismissed as a series of form factors obtained from other experiments. In addition some general statements are injected from elementary particle physics in the form of CVC and PCAC. In many ways we can consider nuclear physics as an attempt to calculate these form factors from a more microscopic description of the nucleus. But that is not how results are usually expressed.

The first nuclear model I want to discuss is the Fermi Gas Model (F.G.M.)\(^2\), which merely confesses that the nucleus is made up of nucleons and that these nucleons obey the Pauli Principle. This model has been applied for example to total muon capture. In fact, the F.G.M. has proved to be a good starting point in understanding the nucleus itself. When the internucleon interactions are switched on the problem becomes that of infinite nuclear matter (I.N.M.). Over the last fifteen years or so the F.G.M. has shown us how to make the step between the very strong nucleon-nucleon interaction seen in two nucleon scattering and the comparatively weak effective two nucleon interaction seen in nuclear matter. As the name implies, I.N.M. considers an infinite system; but hopefully it represents to some extent the properties of nucleons at the interior of heavy nuclei. The adaption of the theory to finite systems introduces additional complications; but over the last five years or so most of these have been overcome. We can now say we have an effective interaction derived from the free space, two nucleon potential that can be used in finite systems.\(^3\)
At the same time as the above developments were being made, a model was proposed by Migdal (M.M.)\(^4\),\(^5\). He is pessimistic and has the viewpoint of Landau in his theory of Fermi Liquids (designed for liquid Helium) that it is too difficult to calculate the effective interaction to be used in these many body systems. Therefore he parameterizes the effective interaction to ensure a good description of some hopefully simple processes. This is more or less a case of expressing our original Kim-Primakoff (E.P.M.) form factors in terms of more 'basic' form factors, the parameters in the effective interaction. But then a theory-experiment disagreement raises similar questions as in the E.P.M., namely is our parameterization inadequate or is there something interesting taking place.

If we consider the derivation of the effective interaction reliable, as seems the case, we are led to what is usually expected to be the best of nuclear models - the shell model (S.M.) with realistic effective interactions.

**ELEMENTARY PARTICLE METHOD (E.P.M.)**

The Elementary Particle Method of Kim and Primakoff\(^1\) is best illustrated with an example:

\[
\mu^- + {^16}_O (g.s.) \rightarrow {^16}_N (g.s.) + \nu_\mu
\]

This capture rate can be expressed in terms of the matrix element

\[
<N^{16}_{\nu_\mu} | J_L \cdot J_B | O^{16}_{\mu^-}>
\]

where \(J_L \cdot J_B\) is the usual (Lepton current) \(\times\) (Baryon current) form of the weak inter-
action. This immediately reduces to the product
\[ \sum_{\mathbf{L}} <p_{\mu} | J^\lambda_{\mu} | p^- > \cdot < N^\mu | J^\lambda_{\nu} | 0^\nu > \]
where the first term is well known and will not be discussed further. The second term is more precisely
\[ < N^\mu (2^-) | J^\lambda_{\nu} | 0^\nu (0^+) > \]
displaying the angular momenta and parities of the two ground states. Treating the two nuclei non-relativistically (since we know nothing about their negative energy components) we can reduce the second term to the form
\[ \Theta^\lambda_{\mu} F_{\mu} (q^o) + \Theta^\lambda_{\nu} F_{\nu} (q^o) + \Theta^\lambda_{\rho} F_{\rho} (q^o) \]
where \( q^o \) is the four-momentum transfer given by
\[ q^2 = (p^0_{\mu} - p^0_{\nu})^2 = -m^2_{\mu} + 2m_{\mu} E_{\nu} \approx m^2_{\mu} + 2m_{\mu} \left[ m_{\mu} + M (0^\nu) - M (0^\mu) \right] \approx 0.8 m^2_{\mu} \]
The three functions \( \Theta^\lambda_{\mu}, \Theta^\lambda_{\nu}, \Theta^\lambda_{\rho} \) have magnetic, axial vector and induced pseudo-scalar character respectively. Each has a definite form. For example in the simpler case when both the initial and final nuclear states are \( 1^+_1 \) capture in hydrogen or \( \text{He}^3 \), we would have
\[ \Theta^\lambda_{\mu} = (0, 0) \]
the non-relativistic limits of \( \Theta^\lambda_{\mu} \) to lowest order in \( \frac{1}{M} \). In addition we have the vector term \( \Theta^\lambda_{\mu} = (0, 1) \) from \( \text{He}^3 \rightarrow 2^- \)
This vector term does not arise in a transition. It should be emphasized that the above \( \Theta^\lambda_{\mu} \) acts on the complete wavefunction, i.e. it acts in the space of the wavefunction and has nothing to do with any possible nucleon wavefunctions within this complete wavefunction. The problem remaining is the evaluation of the
which at this stage are completely arbitrary. They are estimated from related experiments with the aid of theoretical ideas taken from elementary particle physics. Firstly the $\beta$-decay

$$N^{16} (\text{j.s.}) \rightarrow O^{16} (\text{j.s.}) + e^- + \nu_e$$

involves the same type of nuclear matrix element as in the $p$-capture. This can be expressed simply as

$$\Theta^A \cdot F_A (q_1^2 \approx 0),$$

the usual GAMOW-TELLER form since the magnetic and pseudo-scalar terms $\Theta^A \cdot F_A$ are proportional to a higher power of the momentum transfer $q_1$, which is essentially zero for $\beta$-decay. This quenching of $\Theta^A$ with respect to $\Theta^B$ is also seen from the explicit expressions and given above for the transition. Therefore a knowledge of the $\beta$-decay rate immediately gives us $F_A (0)$. The second piece of experimental information comes from inelastic electron scattering off $O^{16} (\text{j.s.})$ which results in exciting the $2^-, T=1$ state at 13.85 MeV.

This is the isobaric analogue of the $N^{16} (\text{j.s.})$. When the electron is scattered through 180° the process is dominated by the magnetic contribution, and the nuclear matrix element is proportional to $\Theta^A \cdot \mu (q^2)$. Here $q^2$, the momentum transfer, depends on the energy of the incident electron beam and so this can be adjusted to obtain $\mu (q^2 = q_2^2 \approx 0.8 m_e^2)$. Similarly, observing the electromagnetic decay $N^{16} (2^-, T=1, 13.85 \text{ MeV}) \rightarrow O^{16} (\text{j.s.}) + \gamma$ gives a measure of $\mu (q^2 \approx 0)$. These transitions are summarized in fig. 1.
Knowing $F_{a,1}(q^2)$, $\mu(q^2)$, and $\mu(0)$, it remains to estimate $F_{M,A_1}(q^2)$. This is where the elementary particle ideas enter in the form of the CVC and PCAC hypotheses. CVC immediately tells us that $F_{M}(q^2) = \sqrt{2} \mu(q^2)$ for all $q$, since the weak interaction of $\mu$-capture and the electromagnetic interaction are considered to be the $\tau^\pm$ and $\tau^0$ components, respectively, of a single isospin current.

Fig. 1

$\Psi_a$ and $\Psi_b$ are isobaric analogues, i.e., $\Psi_a = T \Psi_b$ where $T$ is the isospin lowering operator.

Transition (1) is the desired $\mu^- + O^{16}(g.s.) \rightarrow N^{16}(g.s.) + \nu_{\mu}$

Transition (2) is the $\beta$-decay of $N^{16}(g.s.)$ giving $F_{a}(q^2 = 0)$. 
Transition \( \Theta \) is from inelastic electron scattering giving \( \mu(q_0^2) \).

Transition \( \Theta \) is from the electromagnetic \( M_2 \) decay giving \( \mu(0) \).

At this stage Primakoff and Kim have to introduce the impulse approximation in which the weak and electromagnetic interactions are considered to take place on the individual nucleons in the nucleus, mesonic effects being ignored, i.e.

\[
\theta_{n,m,p} = \sum_{a} \theta_{a}^{(\pi)}
\]

In this way they are able to show that CVC and PCAC at the nucleonic level could be extended to the nuclear level to give the relations:

\[ \frac{F_v(q^2)}{F_v(0)} \approx \frac{F_m(q^2)}{F_m(0)} \]  \text{(Nuclear CVC, not needed in this case)}

\[ F_p(q^2) = -\frac{m_n^2}{m_n^2 + q^2} F_m(q^2) \] \text{(Nuclear PCAC)}.

In the same approximation they also show that

\[ \frac{F_q(q^2)}{F_q(0)} \approx \frac{F_m(q^2)}{F_m(0)} \]

The hope is that these relations verified in the impulse approximation are indeed more general than this simple model. This seems to be the case in electron scattering, where the above type of ratios are found to be well given, even though the separate form factors are not. In other words, the variations of \( F \) with respect to \( q^2 \) should be reliable. Of course, it would be aesthetically more pleasing if the
impulse approximation did not have to be made. However, this
will only be possible when we can perform such experiments
as \( \nu + N \rightarrow \bar{A} \) between discrete nuclear
levels.

Collecting all these bits and pieces together, we
finally arrive at an estimate for \( F_{\text{\( n, n, p \)}} \)\( (q^2 = 0.8 m^2) \)
which gives us a \( \mu^- \) -capture rate of
\( (5.8 \pm 2.3) \times 10^3 \text{ sec}^{-1} \), to be compared with the
experimental value of \( (6.3 \pm 0.7) \times 10^3 \text{ sec}^{-1} \). In
Table 1 are given the other two published examples in which
this method has been applied.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>( \text{He}^3(\frac{1}{2}^+) \rightarrow \text{H}^3(\frac{1}{2}^+) )</th>
<th>( \text{C}^{12}(0^+) \rightarrow \text{B}^{12}(1^+) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.P.M.</td>
<td>( (1.51 \pm 0.04) \times 10^3 \text{ sec}^{-1} )</td>
<td>( (6.6 \pm 1.0) \times 10^3 \text{ sec}^{-1} )</td>
</tr>
<tr>
<td>Expt.</td>
<td>( (1.505 \pm 0.046) \times 10^3 \text{ sec}^{-1} )</td>
<td>( (6.75 \pm 0.10) \times 10^3 \text{ sec}^{-1} )</td>
</tr>
</tbody>
</table>

These results are very good, and so we may be led to ask
whether indeed nuclear physics can always be reduced to a
form factor type of analysis. The answer is that, in principle
'yes' but in practise 'no'. This approach has several draw-
backs, uncertainties, limitations,---.

1. Much experimental input is needed. The \( \beta^- \) -decay and
\( \gamma \) -transition measurements are not too demanding.
However, the inelastic electron scattering in the back-
derward direction at \( q^2 = 0.8 m^2 \) is more of
a problem and accounts for most of the 40% uncertainty in
the predicted $\mu$-capture rate. Indeed, the experiment
is not even possible in the case of a $\sigma^+ \rightarrow \sigma^-$ transition.

2. In writing down the original nuclear matrix elements for
the weak interaction, not all possibilities were included,
since we ignore second class terms such as the induced
tensor coupling. The existence of such terms in nuclear
$\beta$-decay is still uncertain. Of course the intro-
duction of more form factors would presumably require
more experiments before a prediction could be made.

3. That the relations connecting the various form factors at
different momentum transfers are true beyond the impulse
approximation is little more than an act of faith.

4. The method is limited to partial capture rates at the
most, since the original nuclear matrix element depends
on the angular momenta of the initial and final nuclear
states.

Of course, the fact that the predictions agree with the
experimental results is very pleasing and shows that the
fears expressed above do not always apply. However, if some-
one applies the technique elsewhere and does not get agree-
ment, what do we do? Do we suspect the experimental input,
the neglect of second class terms or the form factor relation-
ship? This method revels in success stories, but in the more
interesting situations where there is disagreement it does not
indicate what is at fault.
FERMI-GAS MODEL (F.G.M.)

Here we crack open the form factor nut and begin to admit the existence of nucleons inside the nucleus. Within a large nucleus each nucleon thinks and acts as if it were immersed in an infinitely large system of nucleons. This is because the nuclear forces are short ranged and so only the immediate neighbourhood of a nucleon is important in determining its motion. This is the so-called Local Density approximation and is the reason for studying the Fermi Gas Model (F.G.M.) of infinite extent in an attempt to describe finite nuclei. However, if we look at a large nucleus we immediately see that the evidence in favour of such an idealization is not overwhelming. For example, with a nucleus of about 250 nucleons about two-thirds of the nucleons are on the surface, so that in finite nuclei we expect large surface corrections.

In the F.G.M. we consider we have a large number $N$ of non-interacting nucleons in a large box ($\infty$) and that the ground state of our nucleus is represented by filling the lowest energy levels. This means the only correlation we introduce between the nucleons is that they obey the Pauli principle. The uniform density of this gas is then given by

$$\rho = \frac{N}{\mathcal{L}} = \frac{1}{\mathcal{L}} \sum_{k<k_p} \frac{\nu}{\mathcal{L}} \frac{\mathcal{L}}{(2\pi)^3} \int_{k<k_p} d^3k = \frac{2k_F^3}{3\pi^2}$$

where the factor of $\frac{\nu}{\mathcal{L}}$ arises from the fourfold degeneracy (neutron or proton, spin up or spin down) for each value of $\mathbf{k}$. In this infinite system Coulomb effects are ignored, until we return to a system of finite size. Here we see that the density, a local feature of the system, is well behaved even though both the number of nucleons and the size of the box are large or even infinite for our purposes.
We shall first apply the F.G.M. to the problem of total muon capture rates. The rate of such a process can be written in the impulse approximation as

\[
\sum_F \left( \frac{\nu_F}{\nu_p} \right)^2 \int \frac{d\mathbf{k}}{4\pi} |\langle F | \sum_{\alpha} \xi_\alpha^{(c)} e^{-i\mathbf{k} \cdot \mathbf{r}_\alpha} \times \right. \\
\times \left. \{ \mathbf{g}_\nu + \mathbf{g}_n \xi_\alpha + ... \} | \Gamma \rangle |^2
\]

where I is the initial nuclear state, \( F \) all the possible final nuclear states,

\[
\xi \nu_F = m_F c^2,
\xi \nu_F = m_F c^2 - (E_F - E_I),
\]

and \( g_\nu, g_n, ... \) are related to the more usual coupling constants e.g. \( g_\nu = 2\nu (1 + \frac{m_e}{2mc}) \).

In the F.G.M. we can immediately show that

\[
|\langle F | \sum_{\alpha} \xi_\alpha^{(c)} e^{-i\mathbf{k} \cdot \mathbf{r}_\alpha} \xi_\alpha | I \rangle |^2 = 3 |\langle F | \sum_{\alpha} \xi_\alpha^{(c)} e^{-i\mathbf{k} \cdot \mathbf{r}_\alpha} | I \rangle |^2.
\]

as follows.

In each case \( \xi_\alpha^{(c)} \) converts a proton in \( |I\rangle \) into a neutron to give \( |F\rangle \). Also in each case, if the struck proton has momentum \( \mathbf{k}_F \), the final neutron will have momentum \( \mathbf{k}_F - \mathbf{v}_F \), where by the Pauli Principle \( |k_F| < |k_F| \).

\[
|\mathbf{k}_F - \mathbf{v}_F| > |k_F|
\]

i.e. \( \frac{1}{\sqrt{2}} \exp(i\mathbf{k}_F \cdot \mathbf{r}_\alpha) \mathbf{p} \rightarrow \frac{1}{\sqrt{2}} \exp[i(\mathbf{k}_F - \mathbf{v}_F) \cdot \mathbf{r}_\alpha] \mathbf{n}.
\]

The operator \( |\rangle \) gives the value \( |\rangle \) independent of the spin of the struck proton, spin up and spin down being equally probable in the F.G.M. However the operator

\[
\xi_\alpha = \sqrt{\frac{2}{3}} \xi_\alpha^{(c)} \xi_\alpha + \sqrt{\frac{1}{3}} \xi_\alpha^{(c)} \xi_\alpha + \sqrt{\frac{1}{3}} \xi_\alpha^{(c)} \xi_\alpha
\]

in a spherical basis, when acting on a spin up \((\uparrow)\) proton gives a neutron with the spin combination \( \sqrt{\frac{1}{2}} (\downarrow \uparrow) + (\uparrow) \xi \). To calculate the transition
rate we need the square of this matrix element. Clearly, this same argument holds in the more general case of a doubly magic nucleus with \( N = Z \), when we assume that the ground states of such nuclei are represented by filled shell model orbitals.

If we now make the closure approximation the expression for the total capture rate reduces to

\[
G^2 \left( \frac{\overline{v}_e}{v_F} \right)^2 \int \frac{d\mathbf{r}}{4\pi} \langle \mathbf{r} \mid \mathbf{Z}_e^{(\nu)} \mathbf{Z}_e^{(\nu)} \rangle e^{-i \frac{\mathbf{Z}_e^{(\nu)} \cdot (\mathbf{r}_e - \mathbf{r}_t)}{\overline{v}_e}} | I >
\]

where \( \overline{v}_e = \overline{v}_e^0 + 3 \overline{v}_n^0 + \ldots \)

and \( \overline{v}_e \) is some average neutrino energy. In the case of no correlations, not even antisymmetrization, then the above requires that \( i = j \). This is because \( | I > \) would be simply a product wavefunction of all levels up to \( k < k_e \) and so the nucleon that \( \mathbf{Z}_e^{(\nu)} \) changes from a proton into a neutron must be the same one that \( \mathbf{Z}_e^{(\nu)} \) changes from a neutron into a proton. Otherwise we would not return to the original product wavefunction \( \langle I | \) . In general when \( i \neq j \) the above expression reduces to \( G^2 (\overline{v}_e / v_F)^2 \mathbf{Z} \) (where \( \mathbf{Z} \) is the number of protons) independent of the complexity of \( | I > \). When \( i \neq j \) a model is required for \( | I > \). If we use the F.G.M.

\[
| I > = \frac{1}{\sqrt{\mathbf{A}}} \begin{vmatrix}
\mathbf{Z} e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{p} \uparrow, & \ldots, & \mathbf{Z} e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{n} \downarrow, \\
\mathbf{Z} e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{p} \uparrow, & \ldots
\end{vmatrix}
\]

i.e. A nucleons in \( \mathbf{A} \) states \( k_e \mathbf{p} \uparrow, k_e \mathbf{p} \downarrow, k_e \mathbf{n} \uparrow, k_e \mathbf{n} \downarrow, \ldots, k_F \mathbf{p} \uparrow, k_F \mathbf{p} \downarrow, k_F \mathbf{n} \uparrow, k_F \mathbf{n} \downarrow \), where all momentum states with \( k \leq k_F \) are filled to give the ground state.

\( \mathbf{A} \) is the normalization volume. The matrix element of interest
\[ M_F = \langle F | \sum_{i=1}^{n} \frac{1}{a_i} \sum_{k=k_F} e^{-i \cdot k \cdot \cdot \cdot} | I \rangle \]

then reduces to
\[ \sum_{i=1}^{n} \frac{1}{a_i} \sum_{k=k_F} < \frac{1}{\sqrt{2 \pi}} e^{i \cdot k \cdot \cdot \cdot} | e^{-i \cdot k \cdot \cdot \cdot} | \frac{1}{\sqrt{2 \pi}} e^{i \cdot k \cdot \cdot \cdot} > \]
\[ = \frac{1}{a} \frac{\Delta}{(2\pi)^3} \int d^3 k \frac{1}{\sqrt{2}} (2\pi)^3 \delta(k-k'_F) = 1. \]

The particle with momentum \( k' \) is a neutron with a particular spin. In the closure approximation we now need to evaluate
\[ R = (\bar{v}_F / v_F)^2 \frac{\Delta}{(2\pi)^3} \int d^3 k' M_F^2 \]

where \( \bar{v}_F = \frac{\Delta}{2\pi} \int d^3 k' \) since at this stage we know that the nucleon involved is a neutron, but of either spin. With the restrictions \( k' = k - v_F \), \( k \leq k_F \), \( k' \geq k_F \),

this can be expressed as
\[ R = (\bar{v}_F / v_F)^2 \frac{\Delta}{(2\pi)^3} 2 \int d^3 k' \]

where \( V \) is the shaded volume in the figure.

(Both spheres of radius \( k_F \))

i.e.
\[ R = \left( \frac{\bar{v}_F}{v_F} \right)^2 \frac{\Delta}{(2\pi)^3} 2 \times 2 \pi \int d k \frac{k^2}{k - \bar{v}_F} \int d (\cos \theta) = \]
\[ \left( \frac{\bar{v}_F}{v_F} \right)^2 \frac{\Delta}{(2\pi)^3} \frac{3\pi^2}{2} \left( \frac{1}{k_F^3} \right) 2 \frac{2\pi}{2} k_F^3 \left[ \frac{\bar{v}_F}{k_F} - \frac{\bar{v}_F^3}{2k_F^3} \right] = \]
\[ \left( \frac{\bar{v}_F}{v_F} \right)^2 \frac{\Delta}{(2\pi)^3} \left( \frac{3}{2} \frac{\bar{v}_F}{2k_F} - \frac{1}{2} \left( \frac{\bar{v}_F}{2k_F} \right)^3 \right) = \]
\[ \left( \frac{\bar{v}_F}{v_F} \right)^2 \frac{\Delta}{(2\pi)^3} \left( \frac{3}{2} \frac{\bar{v}_F}{2k_F} \right) = \]
\[ \left( \frac{\bar{v}_F}{v_F} \right)^2 \frac{\Delta}{(2\pi)^3} \left( \frac{3}{2} k_F \right) = \]
\[ \left( \bar{v}_F / v_F \right)^2 \frac{\Delta}{(2\pi)^3} \left( \frac{3}{2} k_F \right) \quad \text{for} \quad \bar{v}_F \geq 2k_F \quad \text{, which is the} \]
case when the two spheres have separated and so the exclusion principle does not affect the transitions.

If we consider \( \frac{\hbar c V_p}{k_F} = 85 \) to 90 MeV, (corresponding to a final state energy of between 15 and 20 MeV, which would be appropriate if the capture were into the collective dipole states), we get that \( V_p \) ranges from (85 to 90) \( \times \) 0.7 fm\(^{-1}\) i.e. from 0.425 fm\(^{-1}\) to 0.450 fm\(^{-1}\).

\[
\frac{1}{140} \left( \frac{\hbar c V_p}{k_F} = \frac{1}{140} \right) \quad \text{is usually considered to be about 1.35 fm}\(^{-1}\), and so we see that \( R \) is considerably smaller than the uncorrelated value of somewhat less than 2. In \( ^{16}\text{O} \) we have the results shown in table 2.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>( \frac{\hbar c V_p}{k_F} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No exclusion principle</td>
<td>85 MeV</td>
</tr>
<tr>
<td></td>
<td>5.1</td>
</tr>
<tr>
<td>With exclusion principle</td>
<td>1.20</td>
</tr>
<tr>
<td>(( R ) in the F.G.M.)</td>
<td></td>
</tr>
</tbody>
</table>

Here we immediately see one of the drawbacks of the closure approach, namely, a strong \( V_p \) dependence. Even so reasonable values of \( V_p \) are not capable of reproducing the experimental value of \( R = 0.7 \), known to about 5% accuracy.

This reduction in \( R \) by a factor of four, when even the simplest of nuclear correlations, the exclusion principle, is introduced should serve as a warning to those who wish to
simplify the nuclear physics. As we shall see, refinements in
the nuclear wavefunctions are indeed capable of further
significant changes in \( R \).

One of the most useful applications of the F.G.M. is,
when it is taken as a theoretical playground in which to
understand finite nuclei. A good example of this is the
Moszkowski-Scott (M. and S.) separation method.\(^3\) This
illustrates how the strong repulsion, thought to be present
in realistic two nucleon potentials, is effectively
quenched in many-body systems, and so makes the success of
the shell model understandable.

In an infinite many-body system without interactions
the ground state energy (\( E_0 \)) is given by
\[
H_0 |\phi\> = E_0 |\phi\>
\]
where \( H_0 = \sum T_k \) and \( \phi \) is the
F.G. ground state of all levels filled upto \( k \leq k_F \).
When nucleon-nucleon interactions (\( V = \sum_{ij} V_{ij} \))
are included the energy is given by
\[
(H_0 + V) |\psi\> = (E_0 + \Delta E) |\psi\>.
\]
Therefore \( \Delta E = <\phi | V | \psi> \) with the
normalization \( <\phi | \psi> = 1 \). The
problem then reduces to evaluating two-body matrix elements
of the form
\[
<\phi(ij) | V_{ij} | \psi(ij)>
\]
where \( \phi(ij) \) is simply a plane wave and
\( \psi(ij) \) is the true solution in the relative coordinates
of particles \( i \) and \( j \). We now need \( \psi(ij) \).
To obtain $\Psi_{ij}$ Moszkowski and Scott show that it is best to consider the two regions of small $\xi = \xi_i - \xi_j$ and large $\xi$.

a) For small $\xi$ the free two nucleon potential $V_{ij}^{NN}$ is so strong that it introduces large momentum components into the wavefunction. Such momentum components are, to a first approximation, so large as to be unaffected by the presence of the Fermi sea. In other words, for small $\xi$, $\Psi_{ij}$ is simply the same as in the two-body problem - namely $\Psi_{ij}^{\text{free}}$, the solution of the Schröedinger equation with $V_{ij}^{NN}$.

b) For very large $\xi$, $\Psi_{ij}$ is simply $\phi_{ij}$, because of the boundary condition at infinity. As $\xi$ becomes smaller the potential $V_{ij}^{NN}$ tries to become effective. At first it can only introduce small momentum components into the wavefunction - but these are forbidden by the exclusion principle. In other words, even within the range of the potential, $\Psi_{ij}$ is essentially $\phi_{ij}$, the unperturbed wavefunction. Of course, as $\xi$ becomes even smaller this simple picture breaks down. This is where M. and S. say that, to lowest order,

$$\Psi_{ij} = \Psi_{ij}^{\text{free}} \quad r \leq d$$

$$= \phi_{ij} \quad r \geq d$$

Where the separation distance 'd' is determined so that $\Psi_{ij}$ and its first derivative are continuous at $r = d$. This situation is depicted in Fig.2.
\[ r = (r_i - r_j) \]

One of the complications is that 'd' is a function of 'k', the relative momentum. However, for the values of \( k \) needed in the ground state of the F.G., \( d \) is remarkably constant varying by about 0.05 \( fm \). Usually 'd' is approximately 1.0 \( fm \).

Now that we have a model for \( \Psi (\mathbf{r}; j) \), we can evaluate

\[ <\phi (\mathbf{r}; j) | V_{ij}^{NN} | \Psi (\mathbf{r}; j)> = <\phi (\mathbf{r}; j) | V_{ij}^{NN} (r \leq d) | \Psi (\mathbf{r}; j) > \]

\[ + <\phi (\mathbf{r}; j) | V_{ij}^{NN} (r \geq d) | \phi (\mathbf{r}; j) >. \]

But the first term on the R.H.S. is nothing more than the expression for the phaseshift due to the potential \( V_{ij}^{NN} (r \leq d) \). However, this is zero since it is continuous to the unperturbed wavefunction \( \phi (\mathbf{r}; j) \) at \( r = d \). This means we are simply left with
\[ \langle \phi(i) | V_{ij}^{\mu \nu} (r \geq d) | \phi(j) \rangle . \]

In other words, our effective interaction or reaction matrix is given, to lowest order, by

\[
V_{\text{eff}} = \begin{cases} 
0 & r \leq d \\
V_{ij}^{\mu \nu} & r \geq d,
\end{cases}
\]

which is effectively a weak interaction. Of course, there are several corrections to this basic model, but the above underlying features persist.

Comment

While on the subject of the F.G.M., I would like to question the utility of trying to introduce further correlations into this model. As an example of an attempt to do this, I quote the work of Wycech.\(^8\) He calculates the effect of the presence of other nucleons on the induced pseudoscalar coupling constant \( g_p \) in weak interactions. When we have only one nucleon, the origin of this induced coupling can be visualized as in fig. 3 giving \( g_p \approx 7 g_A \).

![Fig. 3](image-url)
Wycech then proceeds to calculate corrections to this of the form shown in fig. 4, where the bubble in the pion propagator is a particle-hole excitation of the nucleus treated in the F.G.M.

This results in a \( g_\rho \) (effective). This approach is fine provided we stay with the F.G.M. However, the temptation might be to use this value of \( g_\rho \) (effective) with a more realistic nuclear model. For example, it would be illegal to now use this \( g_\rho \) (effective) with wavefunctions derived from a diagonalization of the nucleon-nucleon interaction. Such a diagonalization would include for a second time the one pion exchange effects already incorporated in \( g_\rho \) (effective), and so we would end up with a double counting problem. Since diagonalization is able to take into account all components in the nucleon-nucleon interaction, not just O.P.E.P., and also gives wavefunctions with the correct angular momentum structure, it is more realistic than Wycech's approach. Using Wycech's \( g_\rho \) (effective) we find that the theoretical total capture rates from table 2 are reduced by only about 5%.
SHELL MODEL (S.M.)

In the E.F.M. nuclear states were treated as single particles with a definite angular momentum, and in this way a few \( \mu \) \( \mu \) capture rates could be successfully accounted for. However, in the F.G.M. the other extreme is applied, in which nuclei are treated as a gas of nucleons, uncorrelated except by the exclusion principle. Also the system is taken to be infinite, and so angular momentum requirements are replaced by linear momentum restrictions. Hopefully, the shell model (S.M.), which treats the nuclei as a finite group of nucleons, and also describes states with a given angular momentum, can bridge the gap between the successful but very phenomenological E.F.M., and the unsuccessful but microscopic F.G.M. The simplest type of wavefunction for the ground state of a doubly-magic nucleus is of the form

\[
|I\rangle = \frac{1}{\sqrt{N!}} \sum_{j_1, j_2, \ldots} \Psi_{(j_1 s_1 m_1, \ldots)} \psi_{(j_2 s_2 m_2, \ldots)}
\]

where

\[
\Psi_{(j s, j m)} (r) = \sum_{m, m} R_{m e} (r) Y_{m n e} \chi_{m s m} \ell s j \ell m e m \ell m
\]

being the radial wavefunction, the spin wavefunction, and the Clebsch-Gordan coefficient that couples the spherical harmonic together.

In the case of a doubly magic nucleus all the 'm' values, for the appropriate 'j' values, are filled. E.g., in \( ^{16}O \) the \( 1S_{1/2}, 1P_{3/2}, 1P_{1/2} \) are all
filled and the $|d_{5/2}, l_{3/2}, l_{1/2}, e \ldots \rangle$ are all empty. This means that $|I\rangle$ is a wavefunction with total angular momentum zero, since $M_j = 0$ and $J \pm |I\rangle = 0$, where $J^\pm = \frac{\hbar}{2}, \frac{3\hbar}{2}, \ldots$. It is not so easy, when we do not have a doubly magic nucleus. Then we have to ensure the valence nucleons, i.e. those nucleons outside the nearest closed shell, are coupled to good angular momentum and isotopic spin.

The above wavefunction is completely analogous to the earlier F.G.M. wavefunction. In fact, when $N = Z$ we still have such relations as

$$|\langle F| \sum_{x}^{n} \tau_i^{(r)} e^{-i \textbf{r}_i \cdot \textbf{s}_i} \frac{1}{2} |I\rangle|^2 = 3 |\langle F| \sum_{x}^{n} \tau_i^{(r)} e^{-i \textbf{r}_i \cdot \textbf{s}_i} 1 |I\rangle|^2.$$

The argument for this is very similar to the one in the F.G.M. The final states $|F\rangle$ are those that can be formed from $|I\rangle$ by the application of the one body operator $\sum_{x}^{n} \tau_i^{(r)} e^{-i \textbf{r}_i \cdot \textbf{s}_i} \text{exp} (-i \textbf{r}_i \cdot \textbf{s}_i)$. This requires the excitation of a proton in the closed core into a neutron outside the closed core. For example, if in $^16_8$ the proton is originally in the $1p_{3/2}$ orbital $\Psi_{(1/2)^{+}}^{p}$ and the final neutron in the $1d_{5/2}$ orbital $\Psi_{(5/2)^{-}}^{n}$ then

$$|F\rangle = \frac{1}{\sqrt{n!}} \begin{bmatrix} \Psi_{(1/2)^{+}}^{p} (\xi_1) \psi_1 \cdots \psi_{n} \psi_{1} \cdots \end{bmatrix}.$$
where the \( \left[ \psi_{\frac{1}{2}+} \right] \) column has been replaced by
the \( \left[ \psi_{\frac{1}{2}+} \right] \) column. At this stage it is
convenient to refer all states with respect to the above
do doubly-closed state \( |I> \) i.e. consider \( |I> \) as a
vacuum \( |0> \) . In this notation \( <F|=<1 \text{ particle} - \text{hole} |d_{5/2} - |p_{3/2} >| \).
We now see that \( <F| \) does not have good angular
momentum, so we should use those combinations of \( <F| \)
that have this property, since the observed states in the
final nucleus have good angular momentum.

i.e. \( <F|=<F(j_1, j_2...)| \rightarrow \sum_j c_j <F(j)| \)

This transformation is not necessary, when we use closure,
since then all the possible final states are considered to
be degenerate and so it does not matter which combination
of \( <F|'s \) we use. We then dive into angular momentum
algebra to calculate e.g.

\[
<F(j)| \sum_{a=1}^{A} \xi_j^{(a)} e^{-i\mathbf{r}_j \cdot \mathbf{q}} |0>
\]

by expanding

\[
e^{-i\mathbf{r}_j \cdot \mathbf{q}} \rightarrow e^{-i\mathbf{r}_j \cdot \mathbf{q}} = \sum_i \sqrt{\frac{2}{i}} \sqrt{\frac{2\pi}{i}} Y_{i0}(\mathbf{q}, \mathbf{r}_j) Y_{j0}(\mathbf{q}, \mathbf{r}_j).
\]

If we make the closure approximation, the total capture rate \( ^7 \)
becomes for \( ^{16} \)

\[
R = \left( \frac{\bar{v}_F}{v_F} \right)^2 \left[ 8 - \left( 8 + \frac{1}{2} \bar{v}_F b^* \right) e^{\frac{1}{2} \bar{v}_F b^*} \right]
\]

where \( \bar{v}_F \) is the average neutrino momentum assumed for
the closure and \( b \) is the oscillator length of the simple
harmonic shell model potential generating the radial wave-
functions. In the F.G.M., $k_F$ was the parameter that corresponded to $b$. This parameter is now fixed by fitting the R.M.S. radius from elastic electron scattering. ($b = 1.80 \text{ fm in } ^{16}\text{O}$.)

The first term in $R$ is simply $Z$, the number of protons. It dominates, when $\bar{v}_F$ is very large, since this is the limit in which the exclusion effect of the occupied $^{16}\text{O}$ orbitals is not effective. The struck proton becomes a highly excited neutron with very little overlap with the original core neutrons.

In $^{16}\text{O}$ we have the results shown in table 3 for the two values of $\bar{v}_F$ used in table 2. We see that the rates are more or

<table>
<thead>
<tr>
<th>Table 3</th>
<th>$R$ in the Shell Model $^7)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_c \bar{v}_F$</td>
<td>85 MeV</td>
</tr>
<tr>
<td>Closure</td>
<td>S.H.O</td>
</tr>
<tr>
<td></td>
<td>Finite well</td>
</tr>
<tr>
<td>Group summation</td>
<td>pure $p-h$</td>
</tr>
<tr>
<td></td>
<td>mixed $p-h$</td>
</tr>
</tbody>
</table>

less the same as for the F.G.M., and so still a long way from the experimental value of $R = 0.7$. We also show that this conclusion is not changed if the oscillator well is replaced by a more realistic finite well of the form

$$V = -V_0 \left[ 1 + \exp \left( \frac{r-R}{a} \right) \right]^{-1}.$$
As was pointed out by Bell and Løvseth, the agreement between the F.G.M. and the S.M. may be somewhat fortuitous since \( R \text{ (F.G.M.)} \propto \nu_F^3 \) whereas \( R \text{ (S.M.)} \propto \nu_F^4 \). However, for the values of \( \nu_F \) we expect to be most realistic we indeed get

\[
R \text{ (F.G.M.)} \approx R \text{ (S.M.)}.
\]

From table 3 we again see the drawback of the closure method - the strong \( \nu_F \) dependence. If this approximation is dropped and we explicitly carry out the summation

\[
R = \sum_f \left( \frac{\nu_F}{\nu_0} \right)^3 \left| \langle f \mid (p-1) \rangle \text{ (f)} \right| \sum_{i=1}^{A} \sum_{\alpha} \gamma_{\alpha} \left( -1 \right) e^{-\nu_F \bar{E}_i} \left| 0 > \right| i^2,
\]

we find that the main contribution of 1.29 (in the case of the oscillator and the finite well) comes from the dipole term in \( \sum_{\alpha} \left( -1 \right) e^{-\nu_F \bar{E}_i} \). This mainly excites p-h states in which the hole is in the l\text{p}-shell and the particle in the 2s-1d shell.

Such excitations correspond to \( \frac{\pi c \nu_F}{2} \) ranging from 80-90 MeV. The 1=2 multipole in \( e^{-\nu_F \bar{E}_i} \) mainly excites particles from the lp shell up to the 2p-1f shell, i.e. \( \frac{\pi c \nu_F}{2} \approx 75 \text{ MeV} \). Their contribution to \( R \) is about 0.18. These together give \( R = 1.47 \) as shown in table 3. This confirms our range of \( \nu_F \) in the closure approximation.

A simple improvement on this summation estimate can be made by using linear combinations of \((p-h)\) states for \( \langle f \rangle \). These combinations can be obtained by diagonalizing the interaction between the various p-h states. The effect of this is to localize the main contribution to the capture.
rate into a narrower energy range. In $^{16}O$ if we look at the $L = 1$ capture, which as we said dominates, then we find instead of the capture being more or less uniformly spread over the various possible $p-h$ states ranging from the

$^2s_{1/2} \, ^1p_{3/2}^{-1}$ at 12.3 MeV (in $^{16}O$) up to $^1d_{3/2} \, ^1p_{2/2}^{-1}$ at 22.7 MeV, that it is mainly localized into the combination

$$0.81 \left( ^1d_{5/2} \, ^1p_{3/2}^{-1} \right) + 0.41 \left( ^2s_{1/2} \, ^1p_{3/2}^{-1} \right)$$

$$+ 0.33 \left( ^1d_{3/2} \, ^1p_{1/2}^{-1} \right) + \ldots$$

at about 23 MeV.

In other words, we expect $\bar{\nu}_F$ to be smaller making the value of $R$ smaller. This indeed happens, and we get $R(2\pi - 1) \approx 1.0$ giving

$R(\text{all } l) \approx 1.2$.

The conclusion seems to be that, if we treat the $^{16}O$ ground state as being doubly magic either in linear momentum, as in the F.G.M., or in angular momentum, as in the S.M., then we fail by almost a factor of two in calculating the capture rate $R$.

This is not an isolated example. A very similar discrepancy exists in the understanding of the $\gamma$-absorption in light nuclei, and on closer examination we see that the two processes are related, as was shown by Fould and Walecka.

$$R(\text{capture}) \propto \sum \frac{\left( \nu_F/\nu_r \right)^2}{F} \left| <F|\bar{\nu} \times c_i|\Sigma> \right|^2$$

$$R(\text{dipole } \gamma\text{-absorption at energy } E) \propto \sum \frac{\left( E_F - E_r \right)^2}{\delta(E_F - E_r - E)} \left| <F|\bar{\nu} \times c_i|\Sigma> \right|^2$$
They showed that
\[ R(\text{capture}) \propto \int_0^{E_m} dE \left( \frac{E_m - E}{E_m} \right)^4 \frac{R(E, \gamma\text{-absorption})}{E} \]
where \( E_m \approx m_p c^2 \).

However, if we calculate \( R(E, \gamma\text{-absorption}) \) using the above types of model we find that
\[ R(E \leq 30\text{MeV}, \gamma\text{-absorption-experimental}) \geq \frac{1}{2} R(E \leq 30\text{MeV}, \gamma\text{-absorption-theoretical}) \]

where the experimental \( \gamma\)-absorption is only taken over the expected region of dipole dominance - namely \( E \approx 30\text{MeV} \). The remainder of the theoretical estimate is found experimentally to be spread up to \( E \approx 200\text{ MeV} \) - contrary to the above shell model interpretation, which condenses the whole effect into \( E \leq 30\text{ MeV} \).

Therefore, if in the expression connecting \( R(\text{capture}) \) and \( R(E, \gamma\text{-absorption}) \) we use the experimental value for \( R(E, \gamma\text{-absorption}) \) we gain a factor of \( \leq \frac{1}{2} \) compared with our earlier theoretical estimates. Of course, this is nothing more than nuclear physics in a nutshell again, and is analogous to the E.P.M. for partial capture rates. The main conclusion to be drawn from this work of Foldy and Walecka is that, since the two interactions involved, weak in \( \nu\text{-capture} \) and electromagnetic in \( \gamma\text{-absorption} \), are so different and yet suffer the same theory-experiment dilemma, it is the nuclear physics that is at fault. One problem that we have ignored in this comparison is that, in \( R(E, \gamma\text{-absorption-theoretical}) \) a \( \approx 40\% \) correction has been included to take into account meson exchange effects. It is not clear how such exchange effects should appear in the
\( \mu \) -capture, whereas the Foldy-Walecka approach assumes they are the same.

Until now we have taken the shell model configuration of our target nucleus to be as simple as possible e.g. the \( \Omega^{16} \) ground state is simply full \( 1s-1p \) shells with the higher shells completely empty. On such a model we would expect the next even parity levels to be up at about 20 MeV. However, experimentally we find the first excited state to be a \( \Omega^+ \). A reasonably successful model\(^{10}\) to explain this situation in the Oxygen and Calcium regions has been developed in which the magic core is broken. To cut a long story short, it is found that the ground state of \( \Omega^{16} \) can be written as

\[
\begin{align*}
0.87 \, [ & \Omega^{16}_{\Omega^+} ]^{\Omega=0, \Omega=0} \\
+ & \, 0.44 \, [ (1s_{1/2})^{2\Omega=0, \Omega=0} (1p_{3/2})^{0, 1} ]^{\Omega=0, \Omega=0} \\
+ & \, 0.18 \, [ (2s_{1/2})^{0, 1} (1p_{3/2})^{0, 1} ]^{\Omega=0, \Omega=0}.
\end{align*}
\]

A similar calculation for \( \Omega^{16} \) gives much less mixing and we find wavefunctions of the form

\[
1.0 \, [ \Omega^{16}_{\Omega^-} ] + 0.1 \, [ \Omega^{16}_{\Omega^+} ].
\]

A model of this sort is certainly needed to account for energy levels and electromagnetic transition rates, so why not in \( \mu \) -capture? Therefore, where we previously had the capture rate given by

\[
| \langle 1p-1h | \frac{\theta}{\Omega} \theta \frac{1}{\Omega} \rangle |^{2}
\]

it is now dominated by\(^{11}\)

\[
| \langle 1.0 \, [ \Omega^{16}_{\Omega^-} ] + 0.1 \, [ \Omega^{16}_{\Omega^+} ] \frac{\theta}{\Omega} \theta \frac{1}{\Omega} \rangle |^{2} \\
\approx 0.87^2 \, | \langle 1p-1h | \frac{\theta}{\Omega} \theta \frac{1}{\Omega} \rangle |^{2}
\]
giving \( R \approx 0.9 \)

easier our experimental goal of \( R = 0.7 \).

In this estimate it has still been assumed that

\[
| \langle \Phi | Z \frac{e^{i x \cdot r}}{r} \frac{e^{-i y \cdot r}}{r} | I \rangle |^2
\]

\[
= 3 | \langle \Phi | Z \frac{e^{i x \cdot r}}{r} \frac{e^{-i y \cdot r}}{r} | I \rangle |^2.
\]

However, it is not clear to what extent this is true when we are dealing with broken doubly magic shells.

The moral of this story seems to be that, if you play with nuclei you are advised not to attempt any short cuts with the nuclear physics. The problem is that often the more sophisticated models, such as the above multi-particle multi-hole approach, are not yet sufficiently well developed. Work has been done near the Oxygen and Calcium shells; but in heavier nuclei and in between the closed shells much work remains to be done especially on the theoretical side.

**MIGDAL MODEL (M.M.)**

In the shell model one of the main complications is the derivation of the effective nucleon-nucleon interaction \( V_{\text{eff}} \) to be used in the nucleus from the nucleon-nucleon interaction \( V_{\text{NN}} \) needed to account for two nucleon scattering in free space. Nowadays, this step is reasonably well understood\(^3\) and gives a more or less unique but somewhat complicated \( V_{\text{eff}} \). Once this \( V_{\text{eff}} \) is derived then, if we are to
be honest, we should live with what it forces on us in nuclei e.g. 1p - 1h mixtures, 2p - 2h configurations etc. However, about 7 or 8 years ago, before the $V_{NN}$ to $V_{eff}$ step was so well understood as it is today, Migdal proposed a theory for finite nuclei, which is an extension of the earlier Fermi liquid theory of Landau for liquid He$^3$. The philosophy behind this theory is that it is too hard to try and calculate many body effects starting with $V_{NN}$. Instead he parameterized $V_{eff}$ as

$$V_{0} \sum_{i} \left[ f_{x}^{i} + g_{x}^{i} \cdot \sigma_{i} \cdot \sigma_{i} + (f_{x}^{i} + g_{x}^{i} \cdot \sigma_{i} \cdot \sigma_{i}) \cdot \hat{\mathbf{e}} \cdot \hat{\mathbf{e}} \right] P_{e} \left( \omega \cdot \phi \right)$$

where $\sigma_{x}$ and $\hat{\mathbf{e}}$ are the spin and isospin operators for the two interacting quasi-particles, $P_{e} \left( \omega \cdot \phi \right)$ is the $e^{th}$ Legendre polynomial in the angle between the two initial momenta of the quasi-particles. The unknown parameters $f_{x}, g_{x}, f_{x}^{i}, g_{x}^{i}$ are adjusted to fit various experimental facts, which we think we understand such as the symmetry energy ($\beta$) compressibility ($K$) and effective mass ($M^*$) in nuclear matter:

$$\beta \propto (1 + 2f_{x})$$

$$K \propto (1 + 2f_{x})$$

$$M^* = M \left( 1 + \frac{2}{3}f_{1} \right), \text{ etc.}$$

It should be remembered that $V_{eff}$ is a reaction matrix, and so takes into account $V_{NN}$ to all orders. In addition he introduces various effective one-body operators, such as effective charges and effective $g$-factors for magnetic moments, which are fitted in 'simple' nuclei. Once we have fixed all these parameters and effective operators by looking at 'simple' processes in 'simple' nuclei we can start to look
at more complicated nuclei, assuming some mild $A$-dependence of the parameters. For example, for the total $\mu$-capture in $^{16}O$ one of the contributions is the axial vector term

$$e^{i(\sigma \tau)} \sum_F \left( \frac{v_F}{v_p} \right)^2 \left| \langle F | \sum_x e^{-iV_p \cdot x} \Sigma \Sigma \Sigma \Sigma | J > \right|^2.$$ 

The constant $e(\sigma \tau)$ in the effective one-body axial vector operator is taken to be $\approx 0.3$ by fitting magnetic moments. The initial wavefunction $|\Sigma\rangle$ is taken to be a doubly closed shell and the final wavefunctions and their corresponding $v_F$'s are given by diagonalizing $V_{eff}$ in the simplest $1p - 1h$ subspace. This procedure in $^{16}O$ yields a capture rate that is about one half of the value obtained with the usual shell model and so is in much better agreement with experiment.\textsuperscript{5)}

The renormalization of one-body operators is quite usual for the interpretation of transition rates in everyday nuclear physics. There we introduce effective charges, which is just another way of saying our bare nucleon is renormalized. Over the last few years much progress has been made in understanding the magnitude of these effects, by considering them as simulations of particular classes of excitations. But it is not clear what the microscopic meaning is behind $V_{eff}$. It is considerably different from what we get using realistic potentials. For example, in $^{16}O$ one of the most important parameters is $g_0'$. For this the M.M. needs $0.5$, whereas the effective interactions derived from the Reid and Bressel-Kerman nucleon-nucleon interactions give $-0.1$ and $-0.05$ respectively.\textsuperscript{5,12)} Presumably part
of the additional renormalization necessary to get the $V_{\text{eff}}$
of the M.M. is simulating, for example, 2p-2h correlations
in the initial wavefunction. Several attempts $^{12}$, $^{14}$ have
been made, with reasonable success, to understand these
differences.

The objections to the M.M. are similar to those for the
E.P.M. There we parameterized the process under consideration
directly in terms of some form factors $F_V, F_A$, -- which were
then obtained directly from experiment. We can, however, think
that the aim of theoretical nuclear physics is to calculate
these $F$'s from 'first principles' -- usually considered to be $V_{\text{NN}}$,
the free nucleon-nucleon interaction. Migdal agrees with the
aims of nuclear physics, but thinks that the 'first principles'
begin at the $V_{\text{eff}}$ level, which he parameterizes in terms of
f's and g's. In other words, he expresses the F's of the
E.P.M. in terms of his more microscopic parameters f
and g which he proceeds to get directly from experiment. In
this way the M.M. is able to discuss many more processes than
the E.P.M. But if at some stage a theory-experiment disagree-
ment arises, all that the M.M. is able to say is that its
parameterization is inadequate. Just as in the Fermi Gas
Model it is not easy to introduce more sophisticated nuclear
configurations at an advanced stage of the calculation, since
we then run the risk of double counting effects already
simulated by effective parameters.
References

1) C.W. Kim and H. Primakoff
P.R. **151** (1966) 1261 (giving earlier references)

2) J.S. Bell and J. Løvseth
N.C. **32** (1964) 433

3) G.E. Brown, Unified Theory of Nuclear Models and Forces
   (John Wiley and Sons, New York 1967)

4) A.B. Migdal
   N.P. **75** (1966) 441
   and lectures at the International School of Physics,
   Varenna, (1965)

5) M. Rho,
   P.R.L. **19** (1967) 248, P.R. **161** (1967) 955
   and lectures at Summer Institute of Nuclear and Particle
   Physics, McGill University, Montreal, 1967

6) J. Delorme and M. Rho,
   P.L. **34B** (1971) 238

7) H.P.C. Rood, H.A. Tolhoek, J.R. Luyten
   N.P. **41** (1963) 236, **70** (1965) 641 and 658
   H.P.C. Rood, Thesis University of Groningen (1964)

8) S. Wycech
   N.P. **B14** (1969) 133

9) L. Foldy and J.D. Walecka
   N.C. **34** (1964) 1026

10) G.E. Brown and A.M. Green
    N.P. **75** , (1966) 401
    W.J. Gerace and A.M. Green
    N.P. **93** (1967) 110

11) P. Erkkilä, M. Saarela and A. Kallio
    P.L. **34B** (1971) 5

12) Z. Bochnacki, I.M. Holban and I.N. Mikhailov
    N.P. **A97** (1967) 33

13) A.N. Gorbunov and V. Osipova,
    J.E.T.P. **16** (1962) 27
14) S.-O. Bäckman,
    N.P. A120 (1968) 593 and N.P. A130 (1969) 427
    R.S. Poggioli and A.D. Jackson
    N.P. A165 (1971) 582

PION-NUCLEON SCATTERING THEORY

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Introduction

These lectures contain a selection of the topics which may be of interest to experimenters or theorists who are concerned with the new, and projected, very accurate low energy πN scattering experiments.

The selection could have been different, but there certainly was not time for everything.

The topics are:

I Low Energy πN Parameters
II Coulomb Corrections
III The Use of Fixed t Dispersion Relations (Accurate Low Energy Phase Shifts)
IV Dynamics of Low Energy πN Scattering
V Information on $\sigma$.

The standard notation of πN dispersion theory is assumed. Much of the notation, and the elementary theory, can be found in "The Dynamics of Elementary Particles and the Pion-Nucleon Interaction," Courses A and B, Nordita Lecture Notes, Copenhagen, 1968 - 1970.
I Low Energy πN Parameters

I. Introduction

We shall discuss three topics which are relevant to high accuracy low energy πN experiments: i) parametrization of the S-wave phases, ii) sum rules and the Pomeranchuk theorem, iii) the determination of the coupling constant $f^2$.

I.1 Parametrization of the S-wave Phases

Years ago Cini et al. (1) pointed out that the expansion

$$\alpha_i = \alpha_i^0 + \frac{\alpha_i^1}{q^3} + \frac{\alpha_i^2}{q^5} + \cdots \quad (i = 1, 3)$$

is not a good way to fit the S-wave πN phases $\alpha_i^0 \quad (i = 1, 3)$. These series converge poorly, and using them is an easy way to find bad values for the scattering lengths $a_1$, $a_3$. Cini et al. also pointed out that crossing symmetry gives a strong hint about a good parametrization. Making use also of the dominance of $N_{33}^*$ at low energies, we can in fact find simple and useful formulae (2).

First we have to specify some symbols \(*\). Let $q_L$, $\omega_L$ and $q$, $\omega$ be the momenta and energies of the pion in the lab. system and the c.m.s. respectively. Then

$$\begin{align*}
\frac{q_L}{q} &= \frac{W}{M} \\
\omega_L &= \frac{E_{W} - M^2}{M}
\end{align*} \quad (1)$$


(2) J. Hamilton and W.S. Woolcock, Rev. Mod. Phys 35 p. 737 (1963)

\(*\) See for example ref. (2) for notation and equations.
where $W$ and $E$ are the total energy and the nucleon's energy in the c.m.s. We shall use $M$, $M$ for the (charged) pion and nucleon masses, and we often put $\mu = 1$ ($M = 6.7$). Also the usual invariants are

$$\begin{align*}
S &= M^2 + \mu^2 + 2M\omega_L \\
T &= -2q^2(1 - \cos \theta) \\
S + T + U &= 2M^2 + 2\mu^2
\end{align*}$$

(2)

where $\theta$ is the $\pi N \rightarrow \pi N$ scattering angle in the c.m.s.

The forward scattering amplitudes are $f_L(\omega_L)$ and $f(\omega)$ in the lab. and c.m.s. and

$$\frac{f_L}{q_L} = \frac{f}{q}.$$  

(3)

In terms of the invariant scattering amplitudes $A(s,t), B(s,t)$, we have

$$f_L = \frac{1}{4\pi i} (A + \omega_L B), \quad f = \frac{M}{4\pi i W} (A + \omega_L B)$$

(4)

**Charge Notation and Crossing**

We shall assume that the correct Coulomb corrections and the related small corrections for mass differences etc. have been made to the experimental data, and we shall assume (as seems to be the case) that charge independence is then obeyed to within a few percent. In what follows we shall use charge independent notation.

The $\pi N \rightarrow \pi N$ amplitudes are then written either as $f^{(3/2)}, f^{(1/2)}$, where the superscript denotes the isospin $T = 3/2, 1/2$, or
as \( f^{(+)} f^{(-)} \) where (+) and (-) correspond to isospin \( T = 0 \) and \( T = 1 \), respectively, in the t-channel. Also
\[
\begin{align*}
 f^{(+)} &= \frac{1}{3} f^{(u)} + \frac{2}{3} f^{(3)} \\
 f^{(-)} &= \frac{1}{3} (f^{(u)} - f^{(3)})
\end{align*}
\]

Crossing is the exchange \( s \leftrightarrow u \), \( t \rightarrow t \), and we have the symmetry
\[
\begin{align*}
 A^{(\pm)} (s, t) &= \pm A^{(\mp)} (u, t) \\
 B^{(\pm)} (s, t) &= \mp B^{(\mp)} (u, t)
\end{align*}
\]
(5)

For forward scattering \( t = 0 \), and by eq. (2)
\[
u = M^2 + \mu^2 - 2M \omega_L,\]
(6)
so by eq. (4)
\[
\int_{L}^{(\pm)} (\omega_L) = \pm \int_{L}^{(\mp)} (-\omega_L)
\]
(6a)

The forward amplitude in the c.m.s. is given by the
p.w. series
\[
f(\omega) = f_{\alpha}\chi_{s} + (2f_{1} \chi_{s} + f_{1/2}\chi_{s}) + (3f_{1}\chi_{s} + 2f_{1/2}\chi_{s}) + \ldots
\]
(7)
where \( \int_{L}^{(J=\pm 1/2)} \) are the partial wave amplitudes.
Thus eq. (6a) suggest that when the energy is so low that the P-waves can be neglected, a useful parametrization might be
\[
\frac{\mathcal{W}}{M+\mu} \frac{\sin(2\alpha_1) + 2\sin(2\alpha_3)}{2\mathcal{E}} = (a_1 + a_3) \left[ 1 + O(\mathcal{E}^{-1}) \right]
\] (8a)

\[
\frac{\mathcal{W}}{M+\mu} \frac{\sin(2\alpha_1) - \sin(2\alpha_3)}{2\mathcal{E}} = \mathcal{W}_L (a_1 - a_3) \left[ 1 + O(\mathcal{E}^{-1}) \right]
\] (8b)

where \(a_3, a_1\) are the S-wave scattering lengths for \(T = 3/2, 1/2\).

**Use of Forward D.R.**

We can put eqs. (8) in a firm basis and also considerably improve them by using the once subtracted forward dispersion relations

\[
\mathcal{f}_{L}^{(+)}(E_L) = \mathcal{f}_{L}^{(+)}(E) + \frac{f^2}{M(1-\lambda^2\mathcal{E}^2)} \frac{q_L^2}{\lambda^2 - \lambda^2\mathcal{E}^2}
\]

\[
+ \frac{q_L^2}{\lambda^2} \int \frac{d\omega'}{\omega'^2} \frac{\sigma^{+}(\omega')}{\omega'} \left[ \frac{1}{\omega' - \omega_L} + \frac{1}{\omega' + \omega_L} \right]
\] (9a)

\[
\mathcal{f}_{L}^{(-)}(E_L) = \mathcal{f}_{L}^{(-)}(E) - \frac{f^2}{\lambda^2} \frac{1}{\lambda^2 - \lambda^2\mathcal{E}^2} \frac{\mathcal{W}_L q_L^2}{\lambda^2 - \lambda^2\mathcal{E}^2}
\]

\[
+ \frac{q_L^2}{4\pi^2} \int \frac{d\omega'}{\omega'^2} \frac{\sigma^{+}(\omega')}{\omega'} \left[ \frac{1}{\omega' - \omega_L} - \frac{1}{\omega' + \omega_L} \right]
\] (9b)

Here \(\omega', q'\) are lab. system variables, and \(\sigma^{+} = 1/2 (\sigma_- + \sigma_+\) where \(\sigma_\pm\) are the total \(\pi^+_B\) cross-sections. Also

\[
\begin{align*}
\mathcal{f}_{L}^{(+)}(E) &= (1+\mathcal{W}_L) \left( \frac{1}{3} a_1 + \frac{2}{3} a_3 \right) \\
\mathcal{f}_{L}^{(-)}(E) &= (1+\mathcal{W}_L) \left( \frac{5}{3} (a_1 - a_3) \right)
\end{align*}
\] (10)

The units are such that \(f^2 = 0.081\).
Using eq. (7) and rewriting, eq. (9a) gives

\[
\frac{\delta m'(2\lambda_1) + 2\delta m'(2\lambda_2)}{2\lambda} \frac{W}{M + \mu} = (\alpha_1 + 2\alpha_2) + q_x^2 C^{(+)}(\omega_L) \tag{11a}
\]

and eq. (9b) gives

\[
\frac{\delta m'(2\lambda_1) - \delta m'(2\lambda_2)}{2\lambda} \frac{W}{M + \mu} = (\alpha_1 - \alpha_2) \omega_L + q_x^2 C^{(-)}(\omega_L) \tag{11b}
\]

where

\[
C^{(+)}(\omega_L) = \frac{3}{4\pi^2(1 + \lambda\eta)} \left[ \int_0^\infty \frac{d\omega'}{\omega'} \, \sigma^{(+)}(\omega') \left( \frac{1}{\omega' - \omega_L} + \frac{1}{\omega' + \omega_L} \right) \right. 
\]

\[
+ \frac{3 f^2}{(\lambda\eta)(1 - \lambda^2 \eta^2)(\lambda_1^2 - \lambda_2^2 \eta^2)} 
\]

\[
- \frac{M}{W(1 + \lambda\eta)} \text{Re} \left( \beta_{11} + 2\beta_{13} + 2\beta_{31} + 4\beta_{33} \right) + (D-\text{wave}) \tag{12a}
\]

\[
C^{(-)}(\omega_L) = \frac{3}{4\pi^2(1 + \lambda\eta)} \left[ \int_0^\infty \frac{d\omega'}{\omega'} \, \sigma^{(-)}(\omega') \left( \frac{1}{\omega' - \omega_L} - \frac{1}{\omega' + \omega_L} \right) \right. 
\]

\[
- \frac{6 f^2 \omega_L}{(1 + \lambda\eta)(1 - \lambda^2 \eta^2)(\lambda_1^2 - \lambda_2^2 \eta^2)} 
\]

\[
- \frac{M}{W(1 + \lambda\eta)} \text{Re} \left( \beta_{11} + 2\beta_{13} - \beta_{31} - 2\beta_{33} \right) + (D-\text{wave}) \tag{12b}
\]

Here \(\beta_{2T,2J}\) are the reduced P-wave p.w.a given by

\[q^2 \text{Re} \, \beta_{2T,2J} = \frac{\delta m'(2\lambda_{2T,2J})}{2\lambda} \]

where \(\lambda_{2T,2J}\) are the P-wave phase shifts \((T = 1/2, 3/2; J = 1/2, 3/2)\).
We might expect some simplifying features in the coefficients \( C^{(t)}_{(\omega)} \). The reason is that the S-wave amplitudes have almost no long range N-exchange interaction whereas the P-wave amplitudes at low energies are dominated by the long range N-exchange interaction (see § \( \overline{IV} \) below). If now we evaluate eqs. (12) roughly by:

(i) Assuming \( \sigma^{(t)} \) are dominated by \( N^*_{33} \), i.e. putting \( \sigma_+ = \delta n^2 \bar{F} \bar{P}_{33} \), \( \sigma_- = \frac{1}{3} \sigma_+ \).

(ii) Using the Chew-Low expressions *) for \( Re \bar{P}_{2\pi, 2\pi} \), then we find that \( C^{(t)}_{(\omega)} \) should show little dependence on \( \omega_L \) up to, say, 100 MeV lab. pion energy. The terms which have rapid energy dependence have cancelled out in eqs. (12). A preliminary evaluation of \( C^{(t)}_{(\omega)} \) by Woolcock (2) confirms this result.

1.2 Fitting the S-wave Data

Isospin 3/2

Estimates (2)(3) give \( C^{(t)}(\mu) \simeq C^{(t)}(\bar{\mu}) \simeq -0.09 \) (units \( k = c = \mu = 1 \)). Since \( C^{(t)}_{(\omega L)} \) are expected to vary slowly with \( \omega_L \), \( C^{(t)}_{(\omega L)} - C^{(t)}_{(\omega \bar{L})} \) should be small up to 50 MeV or higher. Eqs. (11) give

\[
X = a_3 - \frac{1}{3} (a_1 - a_3)(\omega_2 - 1) + \frac{1}{3} q^2 (C^{(t)}_{(\omega L)} - C^{(t)}_{(\omega \bar{L})})
\]

(13)

*) See ref. (2) p. 767-769 for the details

where
\[ X = \frac{W}{M + \mu} \frac{\text{Im}(2\omega_i)}{2q} \quad (13a) \]

Since the coefficient of the \( q^2 \) term is small the plot of \( X \) against \( \omega_L \) should be almost a straight line in the energy range up to 50 MeV lab. energy. This has been verified \(^4\), and it made possible a fairly accurate determination of the scattering length \( a_3 \) \( (a_3 = -0.091 \pm 0.005) \).

With improved experimental data eq. \((13)\) should be a useful form for parametrizing \( a_3' \). Writing \( q^2 = \omega^2 - 1 \), we see that the slope of the plot \( X(\omega_L) \) near threshold will be
\[ \frac{dX}{d\omega_L} \bigg|_{\nu} = -\frac{1}{3}(a_1 - a_3) + \frac{2}{3} \left( C^{(+)\omega_L} - C^{(-)\omega_L} \right) \quad (13b) \]

This quantity is of interest in another relation (cf. § I.3 below).

The basic dynamical reason why \( X \) has a very smooth low energy form is that the \( T = 3/2 \) S-wave scattering is mainly caused by a strong short range repulsion. There is some long range attraction due to \( \sigma \)-exchange and some medium range repulsion due to \( \rho \)-exchange, but the short range repulsion dominates (cf. § IV.3 below).

**Isospin 1/2**

Eqs. \((11)\) give
\[ Z = a_1 + \frac{2}{3}(a_1 - a_3)(\omega_L - 1) + \frac{1}{3} q^2( C^{(+)}\omega_L + 2 C^{(-)}\omega_L) \quad (14) \]

\(^4\) J. Hamilton, Phys. Letters 20, 687 (1966)
where
\[ Z = \frac{W}{\lambda + \mu} \frac{\sin \omega_1}{2 \omega_1} \]  

(14a)

We can write
\[ Z = a_1 + \left( \omega_L - 1 \right) \left( \frac{2}{3} (a_1 - a_3) + C^{(L)}(\omega_L) + 2 C^{(C)}(\omega_L) \right) + \frac{1}{3} \left( \omega_L - 1 \right)^2 \left( C^{(L)}(\omega_L) + 2 C^{(C)}(\omega_L) \right) \] 

(14b)

Since \( a_1 - a_3 \approx 0.275 \) and \( C^{(L)}(\omega_L) + 2 C^{(C)}(\omega_L) \approx -0.27 \), the coefficient of the linear term \( (\omega_L - 1) \) is small whereas the coefficient of the quadratic term is not small \( (4) \). For this reason it is in practice not easy to determine \( a_1 \) accurately. Certainly a linear fit to \( Z \) or \( \alpha_1 \), will lead to considerable errors. One should use a quadratic form in \( (\omega_L - 1) \) for fitting \( Z \) at low energies.

### 1.3 The \( C^{(1)} \) Relations

Now that accurate values of the cross-sections \( \sigma^L \) are available down to low energies, we should bear in mind the possibility of using accurate P-wave phase shifts in order to evaluate \( C^{(L)}(\omega_L) \) by eqs. (12) (ignoring the D-waves).

In particular for \( \omega_L = \mu \) we have \( (2) \)

\[ C^{(L)}(\mu) + \frac{1}{\gamma^2} (a_n + 2a_{13} + 2a_{21} + 4a_{33}) = \frac{3 \gamma^2}{\gamma^2 \omega_L^2 (1 - \frac{1}{n^2})^2} \]

\[ = \frac{3}{2 \pi^2 \gamma^2} \int_{\gamma}^{\infty} \frac{d\omega}{q} \frac{\omega}{q} \frac{\sigma^{(L)}(\omega)}{\omega^2 - 1} \] 

(15a)

\[ C^{(C)}(\mu) + \frac{1}{\gamma^2} (a_n + 2a_{13} - a_{31} - 2a_{33}) + \frac{6 \gamma^2}{\gamma^2 (1 - \frac{1}{n^2})^2} \]

\[ = \frac{3}{2 \pi^2 \gamma^2} \int_{\gamma}^{\infty} \frac{d\omega}{q} \frac{\omega}{q} \frac{\sigma^{(C)}(\omega)}{\omega^2 - 1} \] 

(15b)
where \( \gamma = 1 + \frac{1}{M} \) and \( \omega', q' \) are lab. variables. Here \( a_2, T, 2J \) are the P-wave scattering lengths.

With these relations we could connect the S-wave parametrization of § I.2 with the P-wave scattering lengths. Various other useful relations for \( a_2, T, 2J \) are given by H and W \(^{(2)}\).

I.4 Sum Rules and the Pomeranchuk Theorem

There is a famous sum rule \(^{(5)}\)

\[
\frac{1}{\mu} \left( 1 + \frac{1}{M} \right) \frac{2}{3} (a_i - a_j) = \frac{4f^2}{\mu^2} \frac{1}{1 - \frac{\mu^2}{M^2}} + \frac{1}{2\pi^2} \int_0^\infty \frac{d\omega'}{q'} (\sigma^{(\omega')} - \sigma^{(\omega)})
\]

(16)

It is derived by assuming \(^{*}\) (i) the convergence of the integral, (ii) \( Re f_L(\omega_L) / \omega_L \rightarrow 0 \) as \( \omega_L \rightarrow \infty \), (iii) the condition \( \sigma^{(\omega_L)} \omega_L \rightarrow 0 \) as \( \omega_L \rightarrow \infty \), and a smoothness condition on \( \sigma^{(\omega)} \) for \( \omega \rightarrow \infty \). These enable us to write the unsubtracted D.R.

\[
Re f_L(\omega_L) = \frac{2f^2 \omega_L}{\omega_L^2 - \frac{\mu^2}{M^2}} + \frac{\omega_L}{2\pi^2} \int_0^\infty \frac{d\omega'}{q'} \sigma^{(\omega')} \]

(16a)

(here \( \omega', q' \) are lab. variables). Now putting \( \omega_L = \mu \) gives eq. (16).

If the Pomeranchuk theorem is untrue and \( \sigma^{(\omega)} \rightarrow 0 \) as \( \omega \rightarrow \infty \), eqs. (16) and (16a) are invalid. But it is easy to modify eq. (16a) so as to get a result. Using Fig. 1


\(^{*}\) See, for example, Course A, p. 166 - 169
we write
\[ f_L^{(L)}(\omega) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} d\omega' \frac{f_L^{(L)}(\omega')}{\omega - \omega'} \]
\[ = \frac{1}{2\pi i} \int_{\{R^+ + \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_4\}} d\omega' \frac{f_L^{(L)}(\omega')}{\omega - \omega'} \]

Letting \( \omega \) tend to the real axis (from above) gives the new D.R.

\[ \text{Re} f_L^{(0)}(\omega_L) = \frac{2f^2_{\omega_L}}{\omega_L^2 - M_{\omega L}^2} + \frac{\omega_L}{2\pi^2} \int_{-\pi}^{\pi} \frac{d\omega'}{\omega' - \omega_L} \sigma^{(0)}(\omega') \]
\[ + \frac{1}{2\pi i} \int_{\mathcal{C}_R} d\omega' \frac{f_L^{(0)}(\omega')}{\omega - \omega'} \]  

(17a)

where \( R \) is some large fixed energy and \( M \leq \omega_L < R \).

Now letting \( \omega_L \to \mu \) gives the modified sum rule (6)

\[ \frac{1}{\lambda} \frac{1}{(1 + a_m)^{\frac{3}{2}(a_1 - a_3)} = \frac{4f^2_{\mu}}{\mu^2} \frac{1}{1 - a_{mM}^2} \]
\[ + \frac{1}{2\pi^2} \int_{\mathcal{C}_R} d\omega' \left( \sigma^{(0)}(\omega') - \sigma^{(0)}(\omega) \right) + \frac{1}{2\pi i} \int_{\mathcal{C}_R} d\omega' \frac{f_L^{(0)}(\omega')}{\omega - \omega'} \]  

(17b)

Choosing \( R \) sufficiently large, we can use some asymptotic analytic form for \( f_L^{(0)}(\omega) \) to give \( f_L^{(0)}(\omega) \) on \( \mathcal{C}_R \). This asymptotic form can violate Pomeranchuk's theorem, if we wish.

Moreover if we now let \( R \) go to \( \infty \), the divergence in the dispersion integral is cancelled by the integral over \( \mathcal{F}_R \), provided \( f_L^{(\ell)}(\omega) \) obeys certain very general conditions as \( |\omega| \to \infty \), (for example, we exclude an essential singularity in \( f_L^{(\ell)}(\omega) \) as \( |\omega| \to \infty \)).

There is a rough way of seeing that the very high energy behaviour (defined as \( \omega_L \geq 30 \text{ GeV} \)) indicated by the Serpukhov data (7) will have little effect on the value of \( (a_1 - a_3) \) deduced from eq. (17b). Choose \( R = 10 \text{ GeV} \). Then it is reasonable to expect that if \( f_L^{(\ell)}(\omega) \) is fitted to what is known about this amplitude from 5 GeV to 30 GeV, the values of \( f_L^{(\ell)}(\omega) \) on \( \mathcal{F}_R \) will be only a little influenced by the particular form chosen for \( f_L^{(\ell)}(\omega) \) in the very high energy region. This means that \( (a_1 - a_3) \) is little altered.

Detailed calculations by Lam and Truong (6) have shown that this is indeed true. These authors also emphasize that conversely we cannot use the sum rule in eq. (6) to obtain information on the behaviour of \( \sigma^{(\ell)}(\omega) \) at very high energies.

I.5 The Coupling Constant \( f^2 \)

In 1960 Spearman showed by fitting forward \( \pi^\pm \rho^- \) dispersion relations that the \( \pi^\pm \rho^- \) coupling constant \( f^\pm \) must lie in the range \( 0.075 < f^\pm < 0.085 \). Using the forward dispersion relation for \( B_+ (s,t=0) \) and effectively fitting the energy dependence of the Born term (nucleon pole) gave (2)

\[
 f^\pm = 0.081 \pm 0.003.
\]

In this method the dominant contribution

to the dispersion integral is from $0 < \omega - \mu \leq 300$ MeV, and in effect one is getting $\int f^2$ from the $N^{*}_{33}$ resonance. There have been several later analysis of the $\int f^2 \omega$ forward relations by Samarnayake and Woolcock (3),(9), the latest giving $f^2 = 0.0815 \pm 0.002$. Höhler et al. (10) get a similar result. Better data on forward scattering and cross-sections are needed, as well as improved theoretical methods, if we are to achieve much higher accuracy.

The presence of Pomeranchuk violating terms in the amplitudes at high energies may cause changes in some (but not all) of the above calculations of $\int f^2$. However, Elvekjær (11) using methods similar to those in § I.4 above, has found that the changes in the above values of $\int f^2$ due to taking account of the Serpukhov data (7), should be well within the errors quoted above. Of course it is desirable that any new techniques for calculating $\int f^2$ should suppress possible trouble from Pomeranchuk violating terms.

**Analytic Continuation**

The practical methods of analytic continuation developed by Cutkosky and Deo (12), and by Ciulli (13), should be used. Y.A. Chao and E. Pietarinen (14) have already used

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(10) G. Höhler et al., ZS für Phys. 229, 217 (1969)
(11) F. Elvekjær (Nordita, unpublished (1970))
such techniques to good effect in the more difficult case of $\kappa N$ scattering.

Let us speculate on how a modern determination of $f^2_L^{(+)}$ would be carried out *) . Suppose we use the forward lab. amplitude $f^2_L^{(+)}(\omega)$ (here $\omega = \text{lab. energy}$). By eq. (6a) $f^2_L^{(+)}(\omega)$ is an even function of $\omega$, and we shall write it as $\mathcal{G}(\omega^2)$. The singularities of $\mathcal{G}(\omega^2)$ are the nucleon pole at $\omega^2 = \kappa^2 M_\pi^2$ (cf. eq. (9a)) and the cut $\mu^2 \leq \omega^2 \leq \infty$. By eq. (6a), $\mathcal{G}(\omega^2)$ is regular on $-\infty \leq \omega^2 \leq 0$.

Suppose that we have fairly accurate measurements of $\text{Re } f^2_L^{(+)}(\omega)$ over the range $\mu \leq \omega \leq \omega^*_L$, and of $\sigma^{(+)}(\omega)$ over the (larger) range $\mu \leq \omega \leq \omega^*_\Pi$. Now we use a conformal transformation $\mathcal{Z}(\omega^2)$ which transforms the whole $\omega^2$-plane into the interior of the ellipse in the $\mathcal{Z}$-plane, as shown in Fig. 2. This transformation is chosen so that the upper and lower sides of the line $\omega^*_\Pi < \omega^2 < \infty$ map on to the upper and lower halves of the ellipse. Also the line $-\infty \leq \omega^2 \leq \omega^*_\Pi$ maps on to the major axis of the ellipse ($-\frac{1}{2} \leq \mathcal{Z} \leq \frac{1}{2}$).

It can be arranged that $\omega^2 = \mu^2$ and $\omega^2 =\omega^*_\Pi$ transform to $\mathcal{Z} = -1$ and $\mathcal{Z} = +1$ respectively.

Now write $\mathcal{G}$ as a function of $\mathcal{Z}$ and call it $\mathcal{G}(\mathcal{Z})$. Clearly $\mathcal{G}(\mathcal{Z})$ is regular inside the ellipse, except for the simple Born pole at $\mathcal{Z} = \frac{1}{2}$, and the cut $-1 \leq \mathcal{Z} \leq 1$. Thus the discrepancy function

$$\Delta(\mathcal{Z}) = \mathcal{G}(\mathcal{Z}) - \frac{i}{\pi} \int_{-1}^{1} \frac{\mathcal{G}(\mathcal{Z}')}{{\mathcal{Z}' - \mathcal{Z}}} d\mathcal{Z}'$$

(18)

*) I am indebted to E. Pietarinen for a discussion on this topic.
is regular inside the ellipse except for the Born pole.
We should use the function
\[ f(\bar{z}) = \left( \bar{z} + \bar{z}_g \right) \left( \bar{z} - \bar{z}_\alpha \right) \Delta(\bar{z}) \]  
(18a)
so as to remove the pole at \( \bar{z}_g \); also the factor \( \left( \bar{z} + \bar{z}_\alpha \right) \) will suppress the effects of the singularity at \( \bar{z}_\alpha \) and therefore of any violation of Pomeranchuk's theorem.

The function \( \tilde{F}(\bar{z}) \) will have singularities on the ellipse, but it is regular inside the ellipse. Thus we can use a Legendre expansion
\[ \tilde{F}(\bar{z}) = \sum_{n=1}^{\infty} a_n P_n(\bar{z}) \]  
(19)
to give \( \tilde{F}(\bar{z}) \) at any point inside the ellipse. The next problem is to determine the expansion coefficients \( a_n \).

We know \( \tilde{F}(\bar{z}) \) fairly accurately on \(-1 \leq \bar{z} \leq 1\) from the physical measurements, and we want to find \( \tilde{F}(\bar{z}_g) \), since that is proportional to \( \tilde{f}(\bar{z}) \). The errors in our knowledge of \( \tilde{F}(\bar{z}) \) on \(-1 \leq \bar{z} \leq 1\), will however have the effect that we cannot determine sensible values of the higher coefficients \( a_n \) in the expansion in eq. (19). This is what is called a stability problem.

In order to get the optimum expansion\(^{12}\)(13) we must cut off the series in eq. (19) at \( n = N \) where \( N \) depends inter alia on the size of the errors in the measured values of \( \tilde{F}(\bar{z}) \). (For an elementary account of this procedure see ref. \(^{15}\)). Thus we use
\[ \tilde{F}(\bar{z}) = \sum_{n=1}^{N} a_n P_n(\bar{z}) \]  
(19a)

\(^{15}\) J. Hamilton, "New Methods in the Analysis of \( \pi-N \) Scattering" in Springer Tracts in Modern Physics, Vol. 57 (1971)
This finite series gives the approximate continuation of \( F(\tilde{z}) \) from \(-1 \leq \tilde{z} \leq 1\). The smaller the errors in the data, the larger is \( N \).

Using eq. (19a) we can find \( F(\tilde{z} \sqrt{\tilde{b}}) \), and so \( \sqrt{\tilde{z}} \). Since \( \tilde{z} \sqrt{\tilde{b}} \) is close to -1, the continuation should work well. The error in \( \sqrt{\tilde{z}} \), as determined by eq. (19a), is easily estimated (12)(13).

II Coulomb Corrections

We shall only give a brief discussion of the principles and the main limitations of the methods at present used to obtain Coulomb, and related, corrections in low energy \( \pi N \) scattering.

II.1 Coulomb Scattering Notation *

Consider the non-relativistic Coulomb scattering of two particles of charge \( \pm e \). The radial wave equation for angular momentum \( l \) is

\[
\left\{ \frac{d^2}{dr^2} + \frac{1}{r^2} \left[ l(l+1) - \frac{2le^2}{r^2} \right] \right\} f(r) = 0
\]

(20)

where \( \tilde{q} \) is the momentum in the c.m.s. and

\[
\gamma = \frac{1}{q^2}
\]

where \( a = \frac{\hbar^2}{\kappa e} \) is a Bohr radius and \( \kappa \) is the reduced mass.

For \( \pi^0 p \), \( a = 158 \) units (\( \kappa = \mu = e = 1 \)) and is the Bohr

* There are many clear accounts. See for example Chap. XI and Appendix I of ref. (16)

radius of the \( \pi^- p \) mesic atom. In almost all cases of practical importance in \( \pi N \) scattering \( \gamma \ll 1 \).

The regular solution of eq. (20) is \( \bar{F}_e (\gamma, \rho) \) (where \( \rho = \rho_r \)) and its asymptotic form is

\[
\bar{F}_e (\gamma, \rho) \sim \frac{\sin (\rho - \ell \ln (\rho) + \gamma + i\frac{\ell \pi}{2})}{\rho \rightarrow \infty} \tag{20a}
\]

where \( \gamma = \arg \Gamma (l+1+i\gamma) \). We can also write

\[
\bar{F}_e (\gamma, \rho) = \frac{1}{2i} \left[ e^{i\gamma} u_e^{(+)} (\gamma, \rho) - e^{-i\gamma} u_e^{(-)} (\gamma, \rho) \right] \tag{21}
\]

where \( u_e^{(\pm)} \) are solutions of eq. (20) which have the asymptotic forms

\[
u_e^{(\pm)} (\gamma, \rho) \sim \exp \left[ \pm \theta (\rho - \ell \ln (\rho) - i\frac{\ell \pi}{2}) \right] \tag{21a}
\]

The wave function \( \Psi_e (\chi) \) corresponding to an incoming (distorted) plane wave can be written

\[
\Psi_e (\chi) = \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \exp \left[ i(\ell \chi - \ell \ln (\rho)) \right] \left[ u_e^{(-)} - e^{2i\gamma} u_e^{(+)} \right] \frac{\rho}{(\ell \pi - \ell \Theta)} \tag{22}
\]

where \( \Theta \) is the scattering angle.

Suppose that in addition to the Coulomb interaction there is a hadronic interaction described by a potential of finite range (range \( \lesssim 1 F \)). Suppose also that spin orbital effects are ignored for simplicity. In place of eq. (20) we have

\[
\left( \frac{d^2}{dr^2} + \kappa^2 \gamma - \frac{\ell(\ell + 1)}{r^2} - \frac{2k^2}{r} - \frac{2m}{\kappa^2} V(r) \right) f_e (r) = 0 \tag{23}
\]
The asymptotic form of the regular solution of this equation is

$$f_{\ell}(r) \sim \frac{\omega^\ell}{r^{\ell+2}} e^{i \frac{\ell+1}{2} \ln r + i \delta_x - \frac{i}{2} \ln r + i \delta_x} \quad (23a)$$

Here $\delta_x$ is the phase produced by the hadronic potential in the presence of the Coulomb potential. If $V(r) \equiv 0$, then $\delta_x = 0$.

Analogous to eq. (22) we can write the asymptotic form of the wave function as

$$\Psi_{Nc}(x) \sim \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} e^{i \frac{\ell+1}{2} \ln r + i \delta_x} \sum_{j=0}^{\infty} u_{\ell}^{(j)}(x) P_{\ell}(\cos \theta) \quad (24)$$

Hence

$$\Psi_{Nc}(x) \sim \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} e^{i \frac{\ell+1}{2} \ln r + i \delta_x} \sum_{j=0}^{\infty} u_{\ell}^{(j)}(x) P_{\ell}(\cos \theta) \quad (25)$$

Thus the differential cross-section for scattering is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad (26)$$

where

$$f(\theta) = f_c(\theta) + f_i(\theta) \quad (26a)$$

and $f_c(\theta)$ is the Coulomb amplitude:

$$f_c(\theta) = \frac{(-i)}{2\pi \sin^2 \theta} \exp \left[ 2i\sigma_0 - i\frac{1}{2} \ln \left( \frac{m^2}{\theta_0^2} \right) \right] \quad (26b)$$
while

\[ f' (\phi) = \frac{1}{2i \Omega} \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i \Omega \ell} \left( e^{i \phi_{\ell+1}} - e^{-i \phi_{\ell+1}} \right) P_{\ell} \left( \cos \phi \right) \]  

(26c)

The hadronic scattering is given by \( f'(\phi) \).

**Coulomb Interference**

Eqs. (26) show that

\[ \frac{d\sigma}{d\Omega} = \left| f_c(\phi) \right|^2 + 2\Re \left\{ f_c^*(\phi) f'(\phi) \right\} + \left| f'(\phi) \right|^2 \]  

(27)

The term \( \left| f_c(\phi) \right|^2 \) gives the pure Coulomb scattering and \( \left| f'(\phi) \right|^2 \) gives the pure hadronic scattering. The other term is the Coulomb interference, and since \( f_c(\phi) \) is known, this term can be used to give extra information on \( f'(\phi) \) over what is obtained from the \( \left| f'(\phi) \right|^2 \) term. On account of the very different ranges of the interactions, \( \left| f_c(\phi) \right|^2 \) dominates near the forward direction whereas \( \left| f'(\phi) \right|^2 \) dominates at the larger angles. (Fig. 5)

**II.2 Coulomb Corrections**

Now we look at the main problem. The phase \( f_c \) defined by eqs. (23), (23a) is the phase which is measured (eqs. (26)). However \( f_c \) is not identical with the purely hadronic phase \( f_{cN} \) which would be defined by the regular solution \( f_{cN}(r) \) of

\[ \left( \frac{d^2}{dr^2} + \beta^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{r^2} V(r) \right) f_{cN}(r) = 0 \]  

(28)

and the asymptotic condition

\[ f_{cN}(r) \sim e^{i \delta_{cN} \left( r - \frac{i}{2} \kappa r + \delta_{cN} \right)} \]  

(28a)
In order to investigate theoretical questions we need to know the hadronic phase \( \delta_{\ell N} \), therefore we must find the correction \( \left( \delta_{\ell} - \delta_{\ell N} \right) \). For example, the isospin \( T = 3/2 \) phases \( \delta^{(3/2)}_{\ell} \) measured in \( \pi^+ p \to n^+ p \) and in \( \pi^- p \to n^- p \) will not be identical, but (assuming charge independence) there is only one value \( \delta^{(3/2)}_{\ell N} \). Since charge independence is a good symmetry in \( \pi N \) physics, we shall use it in conjunction with the hadronic phases \( \delta^{(\tau)}_{\ell N} \).

We shall not repeat the history of the Coulomb corrections - there are good reviews available (17). As an illustration, we shall examine the \( S \)-wave \( \pi^+ p \) amplitude. From eqs. (23) and (28) we have (for \( \ell = 0 \)),

\[
\frac{d}{dr} \left\{ f_0(r) \frac{d}{dr} f_{0N}(r) - f_{0N}(r) \frac{d}{dr} f_0(r) \right\} = - \frac{2}{r} \frac{\alpha^*}{r_0} f_0(r) f_{0N}(r) \tag{29}
\]

Suppose that the hadronic interaction is negligible for \( r \gg \tilde{r} \), and also assume that \( \alpha^* a \tilde{r} \gg 1 \). Since \( \tilde{r} \approx 1 \tilde{r} \) this last condition is very reasonable. (Remember that \( T_{\pi} = 5 \) MeV lab is \( q \approx 0.25 \)). Then the asymptotic forms in eqs. (23a) and (28a) are valid for \( r \gg \tilde{r} \). On integrating eq. (29) we have

\[
\xi \sin \left( -\int_{0}^{r} \left( r - r' \right) \frac{f_0(r')}{f_{0N}(r')} \right) + \frac{1}{2} \alpha^* \left( \rho - \int_{0}^{r} \ln \left( r' \right) + \frac{\alpha^*}{2} + \delta_{\ell} \right)
\]

\[
= -2 \frac{\alpha^*}{r_0} \int_{0}^{r} \frac{f_0(r')}{f_{0N}(r')} \frac{dr'}{r'} \tag{29a}
\]

This holds for \( \tau \geq \bar{r} \). We can split the integral into the intervals \((0, \bar{r})\), \((\bar{r}, r)\) where \( r \) is large, and in the last interval use eqs. (23a), (28a). Now rewriting and letting \( r \to \infty \) gives

\[
\sin \left( -\tau \ln \left(2\pi \bar{r}^2 \right) + \sigma_0 + \delta_0 - \delta_{0W} \right) = -2\int_0^{\bar{r}} \frac{\partial'}{\partial' r} \frac{f_{0}(r')}{\ln(r')} \frac{\partial'}{\partial' r} \frac{f_{0W}(r')}{\ln(r')} \, dr'
+ \int_\bar{r}^{\infty} \frac{\partial'}{\partial' r} \cos(2\pi' - \tau \ln(2\pi') + \sigma_0 + \delta_0 + \delta_{0W}) \, dr' \tag{30}
\]

This is the basic Coulomb correction equation, using the non-relativistic method. It gives \((\delta_0 - \delta_{0W})\) independent of the "separation radius" \( \bar{r} \). The corrections for P-waves etc., and for \( \pi^- \rho \to \pi^- \rho \), \( \pi^- \pi \to n^0 n \), can be found by developments of the same method (17).

For \( \tau \ll 1 \), \( \sigma_0 = -\tau C \) where \( C = 0.557 \) is Euler's constant. Hence by eq. (30)

\[
\delta_0 - \delta_{0W} = O(\tau)
\]

If we write

\[
\eta = -\tau \ln \left(2\pi \bar{r}^2 \right) + \sigma_0 + \delta_0 - \delta_{0W},
\]

a good approximation to eq. (30) is

\[
\eta = -2\int_0^{\bar{r}} \frac{\partial'}{\partial' r} \frac{f_{0}(r')}{\ln(r')} \frac{\partial'}{\partial' r} \frac{f_{0W}(r')}{\ln(r')} \, dr' + \int_\bar{r}^{\infty} \frac{\partial'}{\partial' r} \cos(2\pi' - \eta + 2\delta_{0W}) \, dr' \tag{30a}
\]

This equation gives

\[
\frac{\partial^2}{\partial \bar{r}^2} (\delta_0 - \delta_{0W}) = \frac{1}{\bar{r}} \, O(\bar{r}^{-3})
\]
Working to $O(g)$, eq. (30a) yields

$$\delta_0 - S_{0N} = -2g \int_0^\infty dr' \frac{f_0(r') f_{0N}(r')}{r'}$$

$$+ \frac{1}{2} \ln(2g^2 r^2) + C - \cos(2 \ln(2g^2 r^2) + \sin(2 \ln(2g^2 r^2) \sin(2g^2 r^2))$$

(31)

The first term on the right of eq. (31) is the inner Coulomb correction\(^{(18)}\). The second term is van Hove's (outer)\(^{(19)}\) Coulomb correction. In the first term the factor $1/r'$ can be replaced by the actual potential - which is finite as $r' \to 0$

II.3 Comments

The method we have outlined has obvious defects:

a) It is non-relativistic, and van Hove\(^{(19)}\) suggests that this can be remedied in part on replacing $f$ by $f(1 - \beta^2)^{1/2}$ where $\beta$ is the relative velocity of the pion and nucleon in the c.m.s. Other corrections, such as magnetic moment terms, have been included by various authors\(^*)\).

b) It depends on using a model potential for the hadronic interaction. Of the various parts of the low energy $\pi$-$N$ interaction (cf. § IV below), the $\rho$-exchange, $N$-exchange, $N^*$-exchange, terms cannot be represented by a potential, except in a very rough fashion. Moreover, the Schrödinger

\(^*)\) See ref. (17) for references

(19) L. van Hove, Phys. Rev. 88, 1358 (1952)
equation is not appropriate except at very low energies, and the Klein-Gordon equation encounters serious difficulties.

It is obvious that a much improved theoretical treatment is needed. Some promising attempts are mentioned in ref. (17).

Using eq. (31) with model potentials, a number of Coulomb corrections have been estimated. Ref. (20) contains some recent results. I am afraid that we have almost reached the situation where the theoretical uncertainty in the Coulomb corrections sometimes exceeds the accuracy of the best experiments.

Other Corrections

In \( \pi^- p \) scattering it is necessary to allow for the channel \( \pi^- p \rightarrow \gamma n \), and also to remember that the pions and the nucleons have different masses in \( \pi^- p \rightarrow \pi^0 n \). These features affect the analysis of \( \pi^- p \) scattering.

It is usual to include the kinematic corrections due to the mass differences when analysing the data. In an interesting paper Oades and Rasche (21) have started an examination of the dynamical effects of the mass differences. These corrections are not negligible, and further investigations of the mass difference effects are required.

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(20) A.A. Carter et al., Nuclear Physics B26, 445 (1971)
III The Use of Fixed t Dispersion Relations
(Accurate Low Energy Phase Shifts)

The first use of fixed t D.R. to improve our knowledge of πN phases was by C.G.L.N. (22). However their method was severely limited by convergence problems (23). In another form, a similar difficulty occurs in the general fixed t D.R.; the Legendre series over the πN → πN partial wave amplitudes can only be used *) to express the absorptive parts which appear in the fixed t D.R. for \(-2\mu^2 < t < 4\mu^2\). Recently Steiner (24) has emphasized the same difficulty in relation to F.E.S.R. . Within these limits, fixed t D.R. have been much used +) to determine the small phase shifts from data on the large phase shifts.

Here we shall give a brief account of an ingenious method which has been developed and applied by Henry Nielsen (26).

III.1 Nielsen's Method

Henry Nielsen determines very accurate phase shifts in the low energy region, which is here defined as the range up to 270 MeV (lab. pion K.E.). He is interested in obtaining good values of the non-resonant phases, and in particular the

*) See also ref. (23), +) See for example ref. (25)
(22) G. Chew, M. Goldberger, F.E. Low and Y. Nambu, Phys. Rev. 106, 1337 (1957)
(23) D. H. Lyth, Rev. Mod. Phys. 27, 709 (1955)
D-waves. In this low energy region there is some doubt
about the accuracy of several of the partial waves in the
CERN phases\(^{(27)}\).

The input data is:

i) \text{up to } 270 \text{ MeV } (60 \mu^+ \leq s \leq 85 \mu^-)

a) the \(\pi^+p\) total cross-section which is now very accurately
determined\(^{(20)}\).

b) the \(T = 3/2\) S-wave phase \(\alpha_3\). This phase is smoothly
varying and is easy to determine accurately. The CERN values\(^{(27)}\)
will be used, and allowance will be made for the effect of
large inaccuracies in \(\alpha_3\).

c) the \(T = 3/2\), P wave phase \(\alpha_{31}\). This is quite well
enough known for the present purpose.

ii) \text{From } 270 \text{ MeV to } 2 \text{ GeV } (85 \mu^+ \leq s \leq 250 \mu^-)

The CERN phases\(^{(27)}\) and inelasticities are used as
input in this intermediate energy region. These values are
accurate enough for the present purpose in this region.

iii) \text{Above } 2 \text{ GeV}

An analytic continuation method is used to find the
contribution of this high energy region to the D.R. \textit{in a}
fashion which is independent of models, subtraction constants,
and the like.

The first step is to write the \(\pi^+p\) total cross-section
in the low energy region \((60 \leq s \leq 85)\) in the form

\(^{(27)}\) "CERN Experimental Solution" in UCRL-20030 (\(\pi N\) Scattering
Data), Berkeley Report (1970)
\[ \sigma^{(3)} = \frac{4\pi}{2} \left[ \sin^2 \alpha_3 + \sin^2 \alpha_{31} + 2 \sin \alpha_3 \sin \alpha_{31} + \cdots \right] \]  

(32)

Substituting the values of \( \alpha_3 \) and \( \alpha_{31} \) gives \( \alpha_{33} \) in the low energy region. The \( P_{31} \) partial cross-section is 1 - 2% of the \( P_{33} \) partial cross-section and only 10 - 20% of the \( S_{31} \) cross-section. Thus the \( P_{31} \) input is of adequate accuracy. If the \( S_{31} \) contribution to eq. (32) is increased or decreased by 20% the resulting value of \( \alpha_{33} \) does not change by more than 0.08. Such an increase or decrease in \( \alpha_3 \) is far greater than the quoted errors on \( \alpha_3 \). The D-wave contribution to eq. (32) is negligible.

Having \( \alpha_3 \) and \( \alpha_{33} \) in the low energy region, Henry Nielsen shows that the remaining partial waves in that energy region are uniquely determined by the fixed t D.R. (In fact this is also true for \( \alpha_{31} \)). He makes the very reasonable assumption that there is negligible inelasticity in this region.

III.2 The High Energy Terms

The amplitudes \( B(s,t) \) and

\[ A'(s,t) = A(s,t) + \frac{M(s-u)}{4M^2} B(s,t) \]  

(33)

are used. The crossing symmetric variable is \(^*\)

\[ \nu = \frac{1}{2M} (s-u) = \frac{i}{M} (s + \frac{1}{2} t - M^2 \nu^2) \]

\( A' (+)(\nu,t) \), \( B' (-)(\nu,t) \) are even functions of \( \nu \) and \( A' (-)(\nu,t) \), \( B' (+)(\nu,t) \) are odd functions of \( \nu \).

\(^*\) Beware of the factor 2 difference from some definitions of \( \nu \).
Neglecting subtractions, the fixed t D.R. are

\[ \text{Re} A'(\nu, t) = K \frac{G^2}{M} \nu \left\{ \frac{1}{\nu - \nu} + \frac{1}{\nu + \nu} \right\} + \frac{P}{\pi} \int_{\nu_{TH}}^{\infty} d\nu' I_m A'(\nu', t) \left( \frac{1}{\nu_{-\nu}} \pm \frac{1}{\nu_{+\nu}} \right) \]

(34a)

\[ \text{Re} B'(\nu, t) = \frac{G^2}{M} \left\{ \frac{1}{\nu_{-\nu}} + \frac{1}{\nu_{+\nu}} \right\} + \frac{P}{\pi} \int_{\nu_{TH}}^{\infty} d\nu' I_m B'(\nu', t) \left( \frac{1}{\nu_{-\nu}} \pm \frac{1}{\nu_{+\nu}} \right) \]

(34b)

with

\[
\begin{align*}

\nu_p &= \frac{1}{M} (-\mu^2 + \frac{1}{2} t) \\
\nu_{TH} &= 2\mu + \frac{t}{2M} \\
K &= \frac{2M^2}{4M^2 - t}
\end{align*}
\]

(35)

Let \( \nu \) correspond to \( s = 85 \mu^2 \) and \( \nu_p \) to \( s = 250 \mu^2 \). For values of \( t \) in the range \( -26 \mu^2 \leq t \leq 0 \), we have \( \nu_p \approx 30 \), \( \nu_t \approx 4 - 6 \) and \( 0 < \nu_{TH} \leq 2 \).

We shall use \( F(\nu, t) \) to denote any of \( A'(\nu, t), B'(\nu, t) \). We shall write

\[ \text{Re} F(\nu, t) = F_B(\nu, t) + \text{Re} F_{LE}(\nu, t) + \text{Re} F_{I,E}(\nu, t) + \text{Re} F_{H,E}(\nu, t) \]

(36)

where \( F_B(\nu, t) \) is the nucleon pole term, and

\[ \text{Re} F_{LE}(\nu, t) = \frac{P}{\pi} \int_{\nu_{TH}}^{\nu_p} I_m F(\nu', t) \left[ \frac{1}{\nu_{-\nu}} \pm \frac{1}{\nu_{+\nu}} \right] d\nu' \]

(37a)

\[ \text{Re} F_{I,E}(\nu, t) = \frac{P}{\pi} \int_{\nu_p}^{\nu_{TH}} I_m F(\nu', t) \left[ \frac{1}{\nu_{-\nu}} \pm \frac{1}{\nu_{+\nu}} \right] d\nu' \]

(37b)
We can if we wish regard $\Re \tilde{F}_{H,E}(\nu', t)$ as the contribution to $F(\nu, t)$ coming from a large circle $|\nu| = \nu_p$.

We should remember that in order to determine partial waves from $\Re F(\nu, t)$ in the range $\nu_p$ to $s = 85 \mu^2$, we need to know $\Re F(\nu, t)$ for $\nu_{Th} < \nu < \nu_i$, and $0 \geq t \geq -16 \mu^2$. We are therefore interested in the above equations for this range of $t$. For $\nu_i < \nu < \nu_p$, $F(\nu, t)$ lies in the physical $\pi N \to \pi N$ region when $0 \geq t \geq -16 \mu^2$. Therefore $\Re F(\nu, t)$ can be determined fairly accurately from the CERN phase shifts (27) in this region. Now eq. (36) is used to find $\Re \tilde{F}_{H,E}(\nu, t)$ for $\nu_i < \nu < \nu_p$, $0 \geq t \geq -16 \mu^2$ in the following way.

The term $\tilde{F}_{G}(\nu, t)$ is readily evaluated, using $f^2 = 0.081$. Eqs. (37a), (37b) give $\Re \tilde{F}_{LE}(\nu, t)$, $\Re \tilde{F}_{I,E}(\nu, t)$ for $\nu_i < \nu < \nu_p$, $0 \geq t \geq -16 \mu^2$. The values of $\Im F(\nu, t)$ needed in eq. (37b) are given by a convergent Legendre series in terms of the CERN phase shifts (27). In evaluating eq. (37a) only*) $s_m^2 \alpha_3$ and $s_m^2 \alpha_{33}$ are used to give $\Im F(\nu, t)$ ($\nu_{Th} < \nu < \nu_i$).

The functions $\tilde{F}_{H,E}(\nu, t)$ are analytic in the $\nu$-plane with cuts $-\infty \leq \nu \leq -\nu_p$, $\nu_p \leq \nu \leq \infty$. Knowing $\tilde{F}_{H,E}(\nu, t)$ on $\nu_i < \nu < \nu_p$ (and $-\nu_p < \nu < -\nu_i$), standard analytic continuation methods (cf. the refs. in § I.5) show how to find $\tilde{F}_{H,E}(\nu, t)$ on $\nu_{Th} < \nu < \nu_i$, $(0 \geq t \geq -16 \mu^2)$.

*) There is a small error here due to not using all the phases, but this disappears on iteration.
The results of the analytic continuation method turn out to be close to what we would get by the simple device of fitting $A^{(-)}_{\mu, \varepsilon}$ to the form $a + b\nu^2$, $A^{(\pm)}_{\mu, \varepsilon}$ and $B^{(-)}_{\mu, \varepsilon}$ to the forms $c\nu$, and $B^{(+)}_{\mu, \varepsilon}$ to a constant. As an example the results for $t = -10\mu^2$ are shown in Fig. 4.

The continuation to $(\nu_{TH}, \nu_f)$ should be very accurate since this interval is far from the cuts of $F_{\mu, \varepsilon}(\nu, t)$ and is small compared with the interval $(\nu, \nu_P)$ where $F_{\mu, \varepsilon}(\nu_P)$ is known.

It should be emphasized that this elegant method avoids many problems normally associated with the high energy region.

III.3 The Results

Now $\Re F(\nu, t)$ can be evaluated in the low energy region $(\nu_{TH} < \nu < \nu_f, 0 \geq t \geq -\mu^2)$ using eqs. (36), (37a), (37b). In the first run the CERN values(27) of the phases are used as input in $\nu_{TH} < \nu < \nu_f$, except $\alpha_3$ and $\alpha_{33}$ which are kept fixed (as described in § III.1). The $S_\mu$ amplitude is deduced from $S_{31}$ and $f_0(t)$; this is because $f_0(t)$ is poorly determined by the method on account of large cancellations among the fixed $t$ D.R.'s in this case.

In this way $\Re f_{\pm}(t)$ are found in the low energy region. These values are then unitarized by going to the nearest point on the unitary circle. The distance to the unitary circle is always small. Now we have the phases (step 1).

Next, these step 1 phases (plus $\alpha_3$ and $\alpha_{33}$) are used as input and the procedure is repeated. This gives the step
2 phases, and further iteration does not alter them. Also it was shown that the same final results can be reached by putting all phases, except $\alpha_3$ and $\alpha_{33}$ equal to zero in the initial run.

The input $\alpha_{33}$ values (from eq. (32)) are shown in Fig. 5. The resulting $D_{15}$ and $D_{35}$ phases are shown in Fig. 6, and $S_{11}$ in Fig. 7. Notice that $S_{11}$ is still poorly determined. The resulting $P_{13}$ phase is almost identical with that obtained in the experiments of Bugg et al.\(^{(28)}\) In a number of cases there are noticeable deviations from the CERN values\(^{(27)}\).

Henry Nielsen has pointed out that the D-wave results are particularly useful for precise investigations of the $\sigma$-exchange and $\rho$-exchange interactions in low energy $\pi N$ scattering (cf. § IV.1 below).

**IV Dynamics of Low Energy $\pi N$ Scattering**

We shall give a brief description of the exchange forces which are responsible for low energy $\pi N$ scattering. The nature of the interactions has been elucidated by analysis of the experimental data, and it has in various cases been confirmed by the predictions of low and medium energy partial wave amplitudes (p.w.a.) which are then made possible. For a more thorough survey see, for example, ref. \(^{(29)}\).

\(^{*}\) Of course this is mainly due to the large error assumed for $S_{31}$ and the fact that $S_{11}$ is given by $f_0(\pi^-$) and $S_{31}$

\(^{(28)}\) D.V. Bugg et al., Nuclear Phys. **B26**, 588 (1971)

It is hoped that more precise experimental results will provide further checks and refinements of this description. In particular, more work is required on the important $\sigma$-exchange process, and a better understanding of the $P_{11}$ (Roper) resonance is desirable.

IV.1 Singularities of P.W.A.

By studying p.w.a. we can obtain a good description of the low energy $\pi N$ dynamics because the left hand cuts (l.h.c.) of the p.w.a. give analogues of the potential description which is of such use in, for example, nuclear physics. It should however be emphasized at the outset that one cannot actually give potentials which represent the interactions - the $\pi N$ problem is essentially relativistic. The left hand cut terms describe the dynamics.

It is worth remembering that the $\pi N$ p.w.a. are essentially elastic up to 400 MeV pion lab. K.E., except for $P_{11}$. In $P_{11}$ the channel $\pi N \rightarrow \sigma N$ is possible for $\sigma$ in an S-state relative to N, so inelasticity appears from 300 MeV onwards, and it grows rapidly with energy. The only other candidates for inelasticity below 550 MeV are $D_{13}$ and $D_{33}$. Here $\pi N \rightarrow \pi N^*_3$ is possible with $\pi$ in an S-state relative to $N^*_3$. These amplitudes do indeed show some inelasticity in the region 400 - 500 MeV.

The singularities of a $\pi N \rightarrow \pi N$ p.w.a. $f_{\pm}(s)$ are shown in Fig. 8. The main features of the l.h.c. are:

(i) the cut $(M - M^2_H)^2 \leq s \leq M^2 + 2\mu^2$ gives the long range part of the nucleon exchange (N-exchange) term.
(ii) the cut \( 0 < s < (M - \mu)^2 \) gives the exchange of excited baryons, and in the cases we shall consider, it is mainly \( N^*_{33} \)-exchange (\( \Delta \)-exchange) which matters.

(iii) the circle \( |s| = \mu^2 \) gives the t-channel exchanges. These are \( \sigma \)-exchange and \( \rho \)-exchange. (cf. Fig. 9)

We should make a few more technical comments. The cuts which are near the physical threshold \( s_0 = (M - \mu)^2 \) give the longer range parts of the interaction (range \( \leq 1/F \)). The cuts, or parts of cuts, further off from \( s_0 \) give medium and short range parts of the interaction. By short range we will mean \( \leq 0.2 F \).

On the circle, we can only calculate the effect of \( \sigma \)-exchange or \( \rho \)-exchange for the arc \( |\text{arg} s| \leq 66^\circ \). This is because the Legendre series expansions used to relate the t-channel phenomena to the s-channel l.h.c. are not valid beyond this arc. It is useful to lump together the contributions from the remainder of the circle, \( |\text{arg} s| > 66^\circ \), and from the cut \( -\infty < s < 0 \). We call them the short range interaction.

As to the t-channel contributions, the \( T = 0 \) effects (e.g. \( \sigma \)-exchange) contribute only to the p.w.a. \( f_{\ell T}^{(+)}(s) \) for \( \pi N \rightarrow \pi N \), and the \( T = 1 \) effects (e.g. \( \rho \)-exchange) contribute only to \( f_{\ell T}^{(-)}(s) \).

IV.2 Reduced P.W.A.

Because we do not know much about the short range interactions it is very useful to suppress their effects as much
as possible. This can be done for orbital angular momentum \( \ell \geq 1 \), by making use of the centrifugal barrier. In place of \( f_{\ell \pm}(s) \) we write the partial wave D.R. for

\[
F_{\ell \pm}(s) = \frac{f_{\ell \pm}(s)}{q^2}
\]

By requiring \( F_{\ell \pm}(s_0) \) to be finite we have built in the centrifugal effect. Now \(|q^2|\) is large on all cuts which are far from \( s_0 \), so the l.h.c. contributions \( \text{Im} F_{\ell \pm}(s) \) are altered relative to the \( \text{Im} f_{\ell \pm}(s) \) in an interesting way. The ratio of the contributions from the short range cuts to those from the longer range cuts is much smaller in the case of \( F_{\ell \pm}(s) \) than when using \( f_{\ell \pm}(s) \). The larger is \( \ell \), the more marked is this effect.

This procedure has made it possible to predict the non-resonant P, D and F waves in the low energy region\(^{(29)}\). From information derived from an analysis\(^*\) of the experimental S-wave \( \pi-N \) data, the \( \sigma \) -exchange and \( \rho \) -exchange contributions to the l.h.c. of \( F_{\ell \pm}(s) \) were determined. The \( N \)-exchange and \( \Delta \) -exchange terms are also readily found. Now, relying on the fact that the shorter range contributions are well suppressed, the non-resonant amplitudes can be estimated.

Furthermore, a unitary sum rule can be used to give a fair estimate of the small short range interaction which was ignored in the first approximation, and an iteration method can give an estimate of the physical integral (rescattering)

\(^*\) See § IV.3 and V.I below
term. In this way the non-resonant amplitudes can be fairly accurately predicted up to 500 MeV (or higher), and the results agree well with the experimental phases.

In order to give some idea of the behaviour of the various l.h.c. interaction terms, we show in Fig. 10. the l.h.c. contributions to \( F_{\pm} (s) \) for the amplitudes \( P_{13} \) and \( D_{35} \).

**IV.3 Special Cases**

We shall briefly discuss some special cases which for one reason or another fall outside the scheme we have just discussed. They are the S-waves and the \( P_{11} \)-amplitude.

**S-Waves**

The low energy behaviour of \( S_{31} \) and \( S_{11} \) are mainly determined by i) \( \sigma \)-exchange and \( \rho \)-exchange, ii) a very strong short range repulsion. There is a negligible amount of long range N-exchange and not much \( \Delta \)-exchange (or other excited baryon exchange).

The strong short range repulsion (SR) has a range \( \approx 0.2 \text{ F} \), and it occurs in the amplitude \( f^{(+)}_o (s) \) but not in \( f^{(-)}_o (s) \).

In terms of \( \pi N \) isospin, we have

\[
\begin{align*}
\begin{cases}
    f^{(1)}_o (s) = f^{(+)}_o (s) + 2 f^{(-)}_o (s) \\
    f^{(3)}_o (s) = f^{(+)}_o (s) - f^{(-)}_o (s)
\end{cases}
\end{align*}
\]

(39)

The \( \sigma \)-exchange gives a strong long range attraction in \( f^{(+)}_o (s) \), and \( \rho \)-exchange gives a medium range effect in
\[ f'(s) \] so that \[ f'(s) \] is positive in the physical region.

Thus we can say schematically that the forces are

\[ T = \frac{1}{2}: \quad (\text{SR repulsion}) + (\sigma \text{ attraction}) + 2(\rho \text{ attraction}) \]

\[ T = \frac{3}{2}: \quad (\text{SR repulsion}) + (\sigma \text{ attraction}) + (\rho \text{ repulsion}) \]

The method of deducing these various interactions from the data on the S-wave phases makes use of the fact that the ranges of the interactions differ, and it especially depends on using crossing symmetry. Crossing symmetry can give us the values of \( f'(s) \) on the cut \( 0 < s < (M-\mu)^2 \) (cf. Fig. 8).

Since this is inside the circle, and the physical cut is outside, we can more or less isolate the circle contribution to \( f'(s) \) and so get the \( \sigma \) and \( \rho \) -exchange interactions. Now-a-days this is done by using modern analytic continuation methods. More accurate phase shifts would be very valuable in improving this method still further.

**The Amplitude**

In this case we would have to consider (for low energies) the channels

\[ \pi N \rightarrow \pi N \]
\[ \pi N \rightarrow \sigma \text{ N} \]
\[ \sigma \text{ N} \rightarrow \sigma \text{ N} \]

Multi-channel calculations are not easy, but the worst aspect

*) See § V.I below for more details, and a similar game with D-waves

is that $\sigma$ is such a wide "resonance". It is probably fair to say that there has not been a really good dynamical description of this amplitude, although on the other hand none of the observed features are difficult to understand in qualitative terms.

V Information on $\sigma$

There are at present three main methods for obtaining information on low-energy pion-pion scattering in the $T = 0$ $J = 0$ state. They depend on using: i) $\pi N$ p.w.a., ii) $\pi N$ backward scattering, iii) the process $\pi N \rightarrow \pi N N$. Each method involves analytic continuation. Experiments of high accuracy by low energy machines can contribute to the improvements of all three methods. Here we briefly survey these methods.

V.1 The Use of $\pi N \rightarrow \pi N$ p.w.a. \(^*\)

The singularities of a $\pi N$ p.w.a. $f_{\pm} (s)$, or $F_{\pm} (s)$ are shown in Fig. 8. From the scattering data we can find $f_{\pm} (s)$ on the physical cut $s_o \leq s \leq \infty$, and with the aid of crossing symmetry we can also find $f_{\pm} (s)$ on the cut $0 < s \leq (M - \mu)^2$.

Now consider the discrepancy function

$$\Delta^{(4)} (s) = f^{(4)}_{\pm} (s) - \frac{1}{\pi} \int_{s_o}^{\infty} ds' \frac{f_{\pm} (s')}{s' - s} - \frac{1}{\pi} \int_{s_o}^{(M - \mu)^2} \frac{f_{\pm} (s')}{s' - s}$$

$$- \frac{1}{\pi} \int_{(M - \mu)^2}^{(M - \mu)^2} \frac{f_{\pm} (s')}{s' - s}$$

The singularities of $\Delta^{(4)} (s)$ are the cuts \(151 = M^2 - \mu^2\) and

\(^*\) See for example ref. (29) and ref. (30)
the line $-\infty \leq s \leq 0$. It is regular elsewhere.

By using the experimental data we can determine $\Delta^{(\ell)}_{\frac{1}{2}}$ accurately on the low energy and medium energy part of the line $S_s \leq s \leq \infty$, and on a large part of the line $0 < s \leq (M-\lambda)^2$.

Analytic continuation of $\Delta^{(\ell)}_{\frac{1}{2}}$ from these two lines towards the circle $|s| = M^2 - \lambda^2$ gives the discontinuity in $\Delta^{(\ell)}_{\frac{1}{2}}$ across the circle (30).

The discontinuity in $\Delta^{(\ell)}_{\frac{1}{2}}$ across the circle is closely related to the absorptive part of the amplitude for the $t$-channel process

$$\pi\pi \rightarrow \Lambda \Lambda$$

in the isospin state $T = 0$. The analytic continuation will work best for the "front" of the circle (say, $|\arg\zeta| \leq 66^\circ$).

On that arc the discontinuity in $\Delta^{(\ell)}_{\frac{1}{2}}$ is predominately due to the absorptive part $\text{Im} \int_\lambda^0 (t)$ of the helicity amplitude $\int_\lambda^0 (t)$ for the $T = 0$, $J = 0^+$ channel of the process in eq. (41). On the arc in question the range is $4\mu^2 \leq t \leq 50\mu^2$, and $\text{Im} \int_\lambda^0 (t)$ has been determined (30) over this range, starting from the S-wave $\pi\Lambda$ amplitude $\int_\lambda^0 (t)$.

It is then possible to find $\text{Re} \int_\lambda^0 (t)$ over the same range by solving the partial wave D.R. for $\int_\lambda^0 (t)$ . The most recent way (30) of doing this employs an analytic continuation method to avoid possible difficulties from unknown far away contributions to the D.R. for $\int_\lambda^0 (t)$ . (this is the same sort of device as was used in § III.2 above).
Finally, extended unitarity\(^{(31)}\) gives

\[ f^0_+ (t) = \frac{1}{f^0_+ (t)} e^{i \delta^0_0 (t)} \]

(42)

where \(4 \mu^2 \leq t \leq 16 \mu^2\) and \(\delta^0_0 (t)\) is the \(T = 0, J = 0\), \(\pi \pi \rightarrow \pi \pi\) phase. Since there is little evidence of the process \(2\pi \rightarrow 4\pi\) up to much higher energies, eq. (42) is assumed to hold for our range \(4 \mu^2 \leq t \leq 50 \mu^2\). Now it is possible to deduce \(\delta^0_0 (t)\) for \(4 \mu^2 \leq t \leq 50 \mu^2\).

This whole procedure has been carried out by Henry Nielsen, J.L. Petersen and E. Pietarinen\(^{(30)}\) starting from S-wave \(\pi N\)-data. The resulting phase \(\delta^0_0 (t)\) does not pass through \(\pi/2\), but it is consistent with a very broad absorptive part in the \(T = 0, J = 0\), \(\pi \pi \rightarrow \pi \pi\) scattering amplitude. This agrees fairly well with what is usually called the \(\sigma\)-meson (\(m_\sigma \approx 700\) MeV, \(\Gamma_\sigma \gg 100\) MeV).

We shall not discuss the estimates of accuracy of the analytic continuations etc. (see ref. (30) for details, and ref. (15) for an elementary account of this work). It should however be emphasized that much more accurate low energy \(\pi N\) phases*) would be very welcome. They would make it possible to repeat the analysis with increased precision.

D-Wave Discrepancies

An interesting application of the low energy \(\pi N\) D-wave phases resulting from the fixed \(t\) D.R. described in \S \ III

*) Especially \(\alpha_1\)

above has been made by Henry Nielsen\textsuperscript{(26)}. He uses them to
derive discrepancy functions \( \Delta_2^+(S) \); these are defined as
in eq. (40), but using the reduced D-wave amplitudes \( F_2^+(S) \).
The values of \( \Delta_2^+(S) \) are shown in Fig. 11. The dots are
the values deduced from the CERN phases\textsuperscript{(27)}. The crosses
are the values deduced from Nielsen's new phases (they are
only shown where they differ from the dots).

Because of the factor \((q)^4\) in the definition of \( F_2^+(S) \),
the importance of the regions near \((M+\mu)^2 < S < \infty \)
and \(0 < S < (M-\mu)^2\) is enhanced. Also the portion of the
cut \(1/s = M^2-\mu^2\) near \(S = h^2-\mu^2\) has enhanced importance
(compared with the S-wave case discussed above). Therefore
this D-wave discrepancy should give a greater possibility
of distinguishing between the various \(T=0, J=0, \pi\pi \rightarrow \pi\pi\)
phases which have been proposed by many authors. For example,
the solid line in Fig. 11 shows the value of \( \Delta_2^+(S) \) calculated
from the values of \( \mathfrak{I}_m \int_0^0 f_+(t) \) given in ref. (30). [This
solid line should still be adjusted to include a small short
range term.]

V.2 Backward \( \pi N \rightarrow \pi N \)\textsuperscript{(30)}

Atkinson\textsuperscript{(32)} showed that the backward \( \pi N \rightarrow \pi N \) amplitude
and the backward \( \pi\pi \rightarrow \pi\pi \) amplitude are essentially the same
analytic function. Putting \( \cos \theta = -1 \) in eq. (2) gives
\( t = -4q^2 \), so the backward \( \pi N \rightarrow \pi N \) region is
\(-\infty < t < 0 \).

For \( \pi\pi \rightarrow \pi\pi \) the (pseudo) physical range is \( 4\mu^2 < t < \infty \).

Let \( F_2^+(t) (-\infty < t < 0) \) be the backward \( \pi N \) amplitude
in the (+) charge combination. If we know this well enough

\textsuperscript{(32)} D. Atkinson, Phys. Rev. 128, 1908 (1962)
in the low and medium energy range, then by analytic continuation we can find $F^{(+)}(t)$ on $4\mu^2 \leq t \leq \infty$ (or at least on the lower part of that range). In fact we get both the real and the imaginary parts of $F^{(+)}(t)$.

In a particular normalization(32), for $t \geq 4\mu^2$, we have

$$F^{(+)}(t) = -\frac{\xi^*}{M \beta_t^2} \sum_{J=0}^{\infty} (J + \frac{1}{2}) (2J + 1) \int_{+}^{J+\frac{1}{2}} f^{+}_{t}(t)$$

(43)

where $\xi_t$, $\beta_t$ are the pion and nucleon momenta in the c.m.s. for $\pi \pi \rightarrow NN$. Also $f^{+}_{t}(t)$ is the + helicity amplitude for $\pi \pi \rightarrow NN$ with angular momentum $J$.

Analogous to eq. (42) we have

$$f^{+}_{t}(t) = |f^{+}_{t}(t)| e^{i\delta^{+}_{o}(t)}$$

(42a)

for $4\mu^2 \leq t \leq 50\mu^2$. $\delta^{+}_{o}(t)$ is the $\pi \pi \rightarrow \pi \pi$ phase for the state $T = 0$ and angular momentum $J$. There is good reason to believe that $\delta^{+}_{o}(t)$ is small(33) in $4\mu^2 \leq t \leq 50\mu^2$. The same must be true for $\delta^{+}_{o}(t)$ with $J \geq 4$. Thus to a good approximation

$$\text{Im} F^{(+)}(t) = -\frac{4\pi}{M \beta_t^2} \text{Im} f^{+}_{t}(t), \quad (4\mu^2 \leq t \leq 50\mu^2)$$

(43a)

However, we cannot do this for $Re F^{(+)}(t)$; indeed in the case of $Re f^{+}_{t}(t)$ the series in eq. (43) converges

(33) G.C. Oades, Phys. Rev. 132, 1277 (1963)
slowly\(^{(30)}\). Let \(\theta_t\) be the scattering angle in \(\pi\pi \rightarrow N\bar{N}\). The nucleon pole occurs at

\[
\cos \theta_t = i \frac{\left(\frac{1}{2} t - \mu^2 \right)}{2 (\mu^2 - \epsilon_4)^{1/2} (\epsilon_4 - \mu^2)^{1/2}}
\]

For \(4.2 \mu^2 < t < 25 \mu^2\) this is close to the physical range of \(\cos \theta_t\). Thus in the Legendre series which gives eq. (43) we require to include terms up to \(J = 6\) to get 10% accuracy.

The difficulty can be overcome by removing the nucleon pole term and treating it explicitly. In works earlier than ref. (30) this was not done, and the results they found for \(\delta_0^{8/7} (t)\) are therefore suspect.

The remainder of the analysis is now straightforward. (for details see refs. (30) and (15)) \(\过\) \(\delta_0^{8/7} (t)\) is determined over \(4 \mu^2 < t < 50 \mu^2\), and eqs. (42) and (43a) then give \(\delta_0^{8/7} (t)\). The results are close to those obtained from the S-wave \(\pi N\) amplitude \(f_0^{8/7} (s)\), as described in § V.1 above.

V.3 The Process \(\pi N \rightarrow \pi\pi N\)

This method - the famous Chew-Low extrapolation - lies somewhat outside the scope of these lectures. Fig. 12a shows the general process \(\pi N \rightarrow \pi\pi N\), and by extrapolation in \(t\), the nucleon's momentum transfer, one can isolate the process shown in Fig. 12b, because it has a pole at \(t = \mu^2\). This extrapolation is to be carried out for each value of mass of the dipion system, so a very large amount of data is required. In order to keep the instability of the extrapolation
under control one should use good analytic continuation techniques and error estimates\(^{(12)}\)(\(^{(13)}\)). A nice example of using such methods to analyse pion production at 2.8 GeV/c is the work of Baton et al.\(^{(34)}\).

Figure Captions

Fig. 1 Contours in the $\omega$ plane. $C_{+}$ consists of two large semicircles of radius R. $C_{-}$ consists of contours around the physical cut from $\omega = \mu$ to $\omega = \rho$ and around the crossed cut from $\omega = -\rho$ to $\omega = -\mu$. $C_{3}$ and $C_{4}$ are around the nucleon poles at $\omega = \pm \lambda_{3,4}$.

Fig. 2 The conformal transformation of the whole $\omega$-plane into the interior of the ellipse in the $\bar{z}$-plane. The transform of the pole $\omega = \mu/4\lambda$ is $\mathcal{E}_{3}$.  

Fig. 3 Schematic figure showing the several parts of eq. (27)

Fig. 4 The dots show $F_{\mu E}(v, t = -10\lambda)$ calculated from eq. (36). The curves show the fits.

Fig. 5 The phase $\alpha_{33}$ deduced from eq. (32) is shown by the curve. The dots are the CERN values, ref. (27).

Fig. 6 The curves show the predicted phases $D_{15}$ and $D_{35}$. The dots and error bars are the CERN values, ref. (27).

Fig. 7 The phase $\alpha_{4}$. The error bars show the CERN values (27). The solid line is Nielsen's prediction with the CERN values of $\alpha_{3}$ as input. The dashed lines show the changes when the $S_{31}$ contribution to $\alpha(3\lambda)$ is changed by $\pm 20\%$ (see text).

Fig. 8 The singularity structure of $F_{\mu E}(s)$ or $F_{\bar{z}}(s)$.

Fig. 9 The process N-exchange, $\Delta$-exchange $\rho$-exchange, $\sigma$-exchange.

Fig. 10 The left hand cut contributions to $F_{\mu E}(s)$ in the case of $P_{13}$ and $D_{35}$. The values in the low energy region are shown.
Fig. 11 The discrepancy $\Delta_{2-}^{(+)}(s)$

Fig. 12 a) The process $\pi N \rightarrow \pi \pi N$

b) The pion exchange part

$t$ is the momentum transfer to the nucleon.
Fig. 2

z-plane

Fig. 3

\[ \frac{d\sigma}{d\Omega} \]

Coulomb

Interference

Hadronic
Fig. 6
Fig. 8

Fig. 9
Fig. 10
Fig. 12
PRODUCTION AND ABSORPTION OF PIONS ON NUCLEI; and
DISPERSION RELATIONS FOR ELASTIC NUCLEAR SCATTERING

M.P. Locher
SIN, Zürich

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I. PRODUCTION AND ABSORPTION OF PIONS ON NUCLEI

A. INTRODUCTION

A1. General Comments

In the first part of these lectures we deal with pion absorption on nuclei. Extensive literature on this subject exists both theoretically and experimentally dating back practically to the very discovery of the pion. We thought it worthwhile to present the material in a way which is complementary to the existing reviews and lecture notes. In particular, we refer for further background to the review by Koltun (Kol 69) who deals with the whole field of pionic interactions with nuclei. We can, therefore, allow ourselves to be somewhat more specific. We shall deal in considerable detail with the absorption of pions in flight (and the related production process) in the intermediate energy range which is of interest to pion factories, i.e., the range \( 0 < E_{\text{lab}}^p < 800 \text{ MeV} \). For threshold absorption phenomena related to mesic atoms we refer to the Herceg Novi Lecture by T. Ericson (Eri 69a) who surveys the experimental situation and explains the basic ideas.

In the present lectures we shall first treat in various ways the reaction \( \text{NN} \leftrightarrow \text{NN}^\pi \), fundamental for the dominating absorption channels in light nuclei. In Chapter B the phenomenological model, the non-relativistic Hamiltonian model and the relativistic approach using dispersion relations are discussed by means of three papers which we consider typical and instructive. In this way we will develop some feeling for the underlying physical processes which will turn out to be the same in technically different approaches. Any adequate description of our process is sensitive to essentially two ingredients: to pion rescattering (dominated by the \( N^* \) resonance except very near threshold) and to the short range NN interaction.

In the same chapter we comment briefly on soft pion techniques in connection with threshold absorption.
In Chapter C the knowledge of the two-nucleon process is used for the description of absorption in light nuclei. A few comments on related electromagnetic processes are made.

The introduction of dispersive methods in Chapter B is the formal link to the second part of the lectures. The status of the application of dispersion relations to light nuclei is shortly discussed. In Chapter D for the forward amplitude of \( \pi X \) and \( \pi X \) elastic scattering, in Chapter E for the full problem including the angular dependence.

Experimental results will generally be referred to in the context of the theoretical calculations. In most cases they are far from abundant and often very inaccurate as they frequently have an age of more than ten years.

A2. Some Background about Absorption

Whereas for most elastic processes the impulse approximation works at least qualitatively the salient features of pion absorption are not reproduced by the impulse term (Fig. A1).

\[
\begin{array}{c}
\pi \\
\hline
\xrightarrow[\text{-}]{\text{-}} \\
N \\
\end{array}
\begin{array}{c}
X \\
\hline
\xrightarrow[\text{-}]{\text{-}} \\
X - 1 \\
\end{array}
\]

Fig. A1

The absorption vertex \( \pi NN \) is not possible for a free nucleon by energy-momentum conservation. A large recoil momentum (\( \sim 500 \text{ MeV/c} \)) has to be absorbed by the nucleon which is bound in
the nucleus. Such a high momentum component is not readily available in the wave function of the nucleus. The one-nucleon exchange graph is therefore strongly suppressed. As far as it can be separated from the more important mechanisms we expect it to be very sensitive to the model nuclear wave function.

The dominant absorption channel is, therefore, expected to involve at least two nucleons (Fig. A2),

\[ \pi^- \rightarrow N N X - 2 \]

which should share amongst each other most of the pion rest mass, whilst the remaining fragments should have a small invariant energy (i.e., the excitation of \( X - 2 \) should be small).

We briefly present some material (see Kol 69) to convince ourselves that these ideas are born out by experiment. A first way to see the importance of two-nucleon processes is to measure the mean number of nucleons emitted in the absorption of a pion. For \( \pi^- \) absorption at rest Anderson et al. (And 64) observe a mean neutron number of two for a variety of nuclei after subtraction of evaporation nucleons. (Those result from a statistical process having typically an exponential energy distribution with a "temperature" around 2 MeV.) Evaporation contributes moderately for light nuclei (for C the mean neutron number is 2.8 including evaporation) and considerably for heavy nuclei (for \( U \) the mean number is 5). We shall not discuss evaporation here.
There is considerably more direct evidence for the fast two-nucleon process. Nordberg et al. (Nor 68) have measured the emission of \( \text{nn} \) and \( \text{np} \) pairs by a coincidence counter-experiment.

<table>
<thead>
<tr>
<th>Target</th>
<th>( N(\text{n-n}) )</th>
<th>( N(\text{n-p}) )</th>
<th>( N(\text{n-n})/N(\text{n-p}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^6\text{Li})</td>
<td>0.41 ± 0.17</td>
<td>0.11 ± 0.03</td>
<td>3.7 ± 1.0</td>
</tr>
<tr>
<td>(^9\text{Li})</td>
<td>0.69 ± 0.28</td>
<td>0.23 ± 0.06</td>
<td>3.0 ± 0.8</td>
</tr>
<tr>
<td>(^7\text{Li})</td>
<td>0.50 ± 0.21</td>
<td>0.13 ± 0.04</td>
<td>3.7 ± 1.0</td>
</tr>
<tr>
<td>(^9\text{Be})</td>
<td>0.33 ± 0.13</td>
<td>0.10 ± 0.02</td>
<td>3.3 ± 0.9</td>
</tr>
<tr>
<td>(^{10}\text{Be})</td>
<td>0.19 ± 0.08</td>
<td>0.08 ± 0.02</td>
<td>2.3 ± 0.8</td>
</tr>
<tr>
<td>(^{11}\text{B})</td>
<td>0.23 ± 0.10</td>
<td>0.05 ± 0.01</td>
<td>4.4 ± 1.3</td>
</tr>
<tr>
<td>(^{13}\text{C})</td>
<td>0.14 ± 0.06</td>
<td>0.06 ± 0.02</td>
<td>2.5 ± 1.0</td>
</tr>
<tr>
<td>(^{14}\text{N})</td>
<td>0.14 ± 0.06</td>
<td>0.04 ± 0.01</td>
<td>3.7 ± 1.1</td>
</tr>
<tr>
<td>(^{16}\text{O})</td>
<td>0.27 ± 0.11</td>
<td>0.08 ± 0.02</td>
<td>3.4 ± 1.1</td>
</tr>
<tr>
<td>(^{18}\text{O})</td>
<td>0.39 ± 0.16</td>
<td>0.10 ± 0.03</td>
<td>3.8 ± 1.0</td>
</tr>
<tr>
<td>(^{27}\text{Al})</td>
<td>0.07 ± 0.03</td>
<td>0.03 ± 0.01</td>
<td>2.4 ± 0.9</td>
</tr>
<tr>
<td>(\text{Cu})</td>
<td>0.08 ± 0.05</td>
<td>0.04 ± 0.01</td>
<td>2.0 ± 1.4</td>
</tr>
<tr>
<td>(\text{Pb})</td>
<td>0.06 ± 0.06</td>
<td>0.01 ± 0.01</td>
<td>4.7 ± 4.7</td>
</tr>
</tbody>
</table>

* Data taken with apparatus of finer angular resolution.

Table A1

The rate \( N \) of the two-nucleon mode is shown in Table A1 for several nuclei. It averages around 40%. That the observed two nucleons actually come from a process as shown in Fig. A2 is demonstrated by the distribution in the relative angle of the emitted pair. Figures A3 and A4 show the expected strong peaking around 180° for the \( \text{nn} \) and \( \text{np} \) pairs respectively.

Moreover, the proton energy spectrum in correlated emission peaks at 60 MeV (about half the pion rest energy) whereas the spectrum of uncorrelated proton emission prefers zero energy (not shown).

Whilst the two-nucleon process is clearly a dominating mode in pion absorption the remaining channels are not negligible. A sizable fraction of final states in an emulsion experiment (Vai 64)
Distribution in relative angle for $n$-$n$ emission in $\pi^-$ absorption by $^7$Li and $^{16}$O. From (Nor 68).

Fig. A3

Distribution in relative angle for $p$-$n$ emission in $\pi^-$ absorption by $^7$Li and $^{16}$O. From (Nor 68).

Fig. A4
contains fast deuterons and tritons, particularly near threshold, see Table A2. Note that the column marked \( p \) stands for fast \( np \) pair emission and that there are about four times as many \( nn \) pairs which are not observed in emulsion.

### Relative Probabilities (\%\) of Singly Charged Particles, \( E > E_{\text{min}} \) from \( \pi^- \) Absorbed in Emulsion (Vai 64)

<table>
<thead>
<tr>
<th>( E_{\pi}, \text{MeV} )</th>
<th>Target</th>
<th>( E_{\text{min}}, \text{MeV} )</th>
<th>( p )</th>
<th>( d )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>light</td>
<td>10</td>
<td>45</td>
<td>40</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>heavy</td>
<td>20</td>
<td>60</td>
<td>40</td>
<td>( \sim 0 )</td>
</tr>
<tr>
<td>300</td>
<td>all</td>
<td>—</td>
<td>96</td>
<td>4</td>
<td>( \sim 0 )</td>
</tr>
</tbody>
</table>

Table A2

In what follows, however, we shall restrict ourselves to the fast two-nucleon absorption channel.

**B. THE REACTION \( NN \rightarrow NN\pi \)**

**B1. Kinematics**

In this Section we collect the kinematical notation and formulae as needed for the description of the reaction \( NN \rightarrow NN\pi \). A very similar notation will be used for the elastic reactions in Section II.

The naming of the four momenta is as in Fig. B1. In the

![Diagram](image-url)
bound state channel \( p_3 \) and \( p_4 \) form a deuteron with four-momentum \( d \) and mass \( M \).

The invariant energy of the over-all system and the invariant momentum transfer to the pion are defined as

\[
S = (p_3 + p_4)^2 = (p_3 + p_4 + q)^2
\]

\[
t = (q - p_4)^2 = (p_3 + p_4 - p_2)^2
\]

And, similarly, for the momentum transfer to the nucleon \( p_2 \), i.e., the crossed momentum transfer \( u = (q - p_2)^2 \).

Denoting the components of the four vectors in the over-all c.m. system by

\[
p_3 = (\vec{P}_{in}, E_3), \quad p_4 = (\vec{P}_{in}, E_4)
\]

\[
p_3 = (\vec{P}_{3}, E_3), \quad p_4 = (\vec{P}_{4}, E_4), \quad q = (\vec{q}, \omega)
\]

one has the relations

\[
S = 4 E^2 \quad \text{and} \quad t = m^2 + \mu^2 + 2 \vec{Q} \cdot \vec{P}_{in} \cos \phi - 2 \omega E \]

where \( \phi = \phi(\vec{P}_{in}, \vec{q}) \). The magnitude of the three momenta are given by the invariant energy \( S \) and the masses \( \mu \), \( m \), \( M \) of the pion, nucleon and deuteron respectively

\[
P_{in} = \frac{1}{2} (S - 4 m^2)^{1/2}
\]

\[0 < Q < Q_{\text{max}} \text{ where } Q_{\text{max}} = \frac{[S - (\mu + M)^2]^{1/2}[S - (\mu - M)^2]^{1/2}}{2 S^{1/2}}\]
Experimentally, the energy is usually given by the nucleon kinetic energy in the lab. system $E_{\text{kin}}^{\text{lab}}$ from which

$$S = 4m^2 + 2mE_{\text{kin}}^{\text{lab}}$$  \hspace{1cm} (7)$$

Alternatively, the maximal pion energy $\omega_{\text{max}}$ or pion momentum $Q_{\text{max}}$ in the over-all c.m. system is used instead.

It is useful to remember that the N*(1238) or 33 resonance occurs at a nucleon energy of $E_{\text{kin}}^{\text{lab}} \approx 600$ MeV corresponding to a pion momentum $Q_{\text{max}} \approx 230$ MeV/c.

A frequently needed subsystem in the three-body final state is the centre-of-mass system of the pion with one of the nucleons, $p_3$, say. Obviously this system is most appropriate for introducing the 33 resonance.

In this connection we collect the following relations without proof. We will need them in the following section on the Mandelstam model.

Denoting the relative momentum between the final nucleons by

$$\vec{P} = \frac{1}{2} (\vec{P}_3 - \vec{P}_2) \hspace{1cm} \text{i.e.} \hspace{0.5cm} \vec{P}_3 = -\vec{P} - \frac{\vec{Q}}{2} \hspace{0.5cm} \text{and} \hspace{0.5cm} \vec{P}_2 = \vec{P} - \frac{\vec{Q}}{2}$$  \hspace{1cm} (8)$$

the final three-body state is characterized by $\vec{Q}$, $\vec{P}$ and the angle $\theta$ between $\vec{Q}$ and $\vec{P}$.

The relative momentum between the pion and the nucleon $\vec{P}_3$ is

$$\vec{Q}_r = \frac{m\vec{Q} - \mu \vec{P}_2}{\mu + m}$$  \hspace{1cm} (9)$$
The momentum of the other nucleon $\vec{P}_4$, denoted by $\vec{P}_c$, is just minus the $\pi^+N$ centre-of-mass momentum.

For the production or absorption process the natural resonance variables $Q_r$ and $P_c$ have to be expressed in terms of the over-all c.m. variables $Q$, $P$ and $\Theta$. These relations with some non-relativistic approximations in the nucleonic part are

$$ P_c = (P^2 - PQ \cos \Theta + \frac{1}{4} Q^2)^{1/2} $$

$$ Q_r \sin \alpha = QP \frac{\sin \Theta}{P_r} \quad [\alpha = \pm (Q_r, P)] $$

$$ Q_r \cos \alpha = \frac{A}{P_r} \left[ QP \cos \Theta - \frac{1}{2} Q^2 \right] + \frac{\omega P_c^2}{m + \omega + (P_r^2 + PQ \cos \Theta + \frac{1}{4} Q^2)} \left( \frac{A + \frac{P_c^2}{2m}}{P_r^2} \right) $$

If the resonance occurs in the other subsystem on the nucleon $\vec{P}_4$, the relative momenta in that system will be denoted by $Q_r'$ and $P_c'$. The dashed momenta are simply obtained by replacing $\cos \Theta$ by $-\cos \Theta$ everywhere.

Generally the three-body final state has five free kinematical variables. The best choice of parameters depends on the particular experiment to be discussed. The previous introduction of $Q$, $P$ and $\Theta$ is particularly suited for the resonance model in the next section as the $\pi^+N$ dynamics do not depend on the remaining two angles.

Finally, we mention that for the two-body reaction $NN \rightarrow \pi^+d$ time reversal invariance relates the unpolarized absorption and production process (principle of detailed balance) by

$$ \frac{d\sigma}{d\Omega} (PP \rightarrow \pi^+d) = \frac{3}{4} \frac{Q_{\text{max}}^2}{P_{\text{max}}^2} \frac{d\sigma}{d\Omega} (\pi^+d \rightarrow PP) $$

The factor of proportionality reflects the different flux factors and the different initial spin weights.
B2. Phenomenological Models

We have already stressed in the Introduction (Section A2) the importance of the fast two-nucleon channel in nuclear absorption of the pion. Hence, the best possible knowledge of the underlying process \( NN \rightarrow NN \pi \) will limit the understanding of nuclear absorption. The present Chapter B is completely devoted to this process.

In this Section B2, as a first step, we deal with the phenomenological description of

\[
NN \rightarrow NN \pi
\]

(12)

Early phenomenological papers date back to the fifties (WaB 51, Wat 52). Comparisons to experiment were made by Rosenfeld (Ros 54) and Gell-Mann and Watson (GeW 54). These authors already noted the dominance of the 33 resonance except very near threshold. This type of calculations has been extended by Mandelstam (Man 58). Recent extensions of the Mandelstam model have been reviewed by Amaldi (Ama 67), but, basically, the Mandelstam analysis is still valid and we shall discuss it in some detail, particularly as we want to develop some feeling for the complexity of the problem in an approximation stripped from non-essentials.

The procedure is roughly the following. One starts with a partial wave analysis of reaction (12) and retains only the lowest angular momenta. The 33 resonance, which is treated in a Chew Low approximation, contributes only to very few of them. Non-resonant transitions are discarded. The final state two-nucleon interaction is introduced for relative S states, where it is strongest. Apart from these two pieces of dynamical input the matrix elements are assumed to be independent of the momenta, the phenomenological constants of proportionality being fixed by experiment.

The present analysis restricts the initial state of reaction (12) to two protons, i.e., to the isospin \( I = 1 \) channel. Explicitly, we have
a) \( pp \rightarrow p n \pi^+ \), b) \( pp \rightarrow d \pi^+ \), c) \( pp \rightarrow pp \pi^0 \)  

\[ (13) \]

(The \( I=0 \) contribution for an \( np \) initial state is excluded anyhow in a resonance model since \( \frac{3}{2} \otimes \frac{1}{2} \) does not contain isospin zero.)

Allowing for \( S, P \) and \( D \) states in the initial \( pp \) state the only antisymmetric states are

\[ ^1S_0, \quad ^3P_2, \quad ^3P_1, \quad ^3P_0, \quad ^1D_2 \]

\[ (14) \]

the upper index refers to the spin and the lower to the total angular momentum \( J \) (Symmetric states \( ^3S_1, \quad ^1P_1, \quad ^3D_3 \),... are forbidden by the Pauli principle.) If only transitions to \( SS \) and \( Sp \) states are considered (small letters denote the pion angular momentum with respect to the over-all c.m. system) only four of them are allowed

\[
\begin{align*}
(a) \quad ^1S_0 & \rightarrow \ ^3S_1 \, ^3P_0 \\
(b) \quad ^3P_1 & \rightarrow \ ^3S_1 \, ^1S_0 \\
(c) \quad ^1D_2 & \rightarrow \ ^3S_1 \, ^3P_2 \\
(d) \quad ^3P_0 & \rightarrow \ ^1S_0 \, ^3S_0 
\end{align*}
\]

\[ (15) \]

(the last index is \( J \)). The intrinsic parity of the pion forbids transitions like \( S \rightarrow Ss, \quad P \rightarrow Sp \) and \( D \rightarrow Ss \). The conservation of \( J \) leaves us finally with the transitions \((a)\) to \((d)\). The deuteron can contribute to the triplet transitions \((a), \ (b)\) and \((c)\). The \( N^* \) resonance can occur only in reaction \((c)\) which is the only transition having a \( p \) wave pion and \( J \) larger than zero. The \( \pi^0 \) production is possible only in \((d)\) as the final protons have to be in an antisymmetric state. Early phenomenological analysis near threshold indicates that transition \((d)\) is small whereas \((c)\) is large.

The \textbf{Mandelstam model} keeps only the \textbf{resonant production}.

This means in the present terminology that only reaction \((c)\) is kept for final nucleons in a relative \( S \) state ("\textbf{S state production}"). However, for a reasonable description of the resonance region \( P \) \textbf{state production} can no longer be disregarded. For easy tracking
of the resonant contributions it is convenient to change the terminology of the angular momentum decomposition. First, the pion and one nucleon are combined into angular momentum \( \frac{3}{2} \) in their relative c.m. system and then the spin of the second nucleon is added to obtain a partial angular momentum \( j \) which is either two or one. Finally, the orbital angular momentum of the second nucleon is introduced (which is either S or P with respect to the overall c.m. system) to construct the total angular momentum \( J \). In this notation S state production allows again only for one transition \((j, J) = (2, 2)\) corresponding to the transition (c) in the old notation\(^*\). Resonant P state production \(^{**}\) allows now for five more amplitudes corresponding to

\[
(j, J) = (2, 2), (2, 1) \text{ and } (1, 2), (1, 1), (1, 0)
\]

(16)

where the initial pp state must be a \( \frac{3}{2}_p \) state (if \( \frac{3}{2}_p \) states are discarded) because the parity must be odd. We are thus left with \( 5 + 1 \) (complex) parameters, which is already too much in view of the limited experimental material to be explained. In the Mandelstam analysis the number of parameters is effectively reduced to two as described further below in the paragraph on numerical results. But before we do so we have to discuss the dynamical input, i.e., the treatment of the \( 3\overline{3} \) resonance and the \( NN \) final state interaction. The remainder is then simple isospin analysis and less simple angular momentum algebra.

\( NN \) and \( NN \) interactions

As in this model the production is assumed to occur exclusively through excitation of the \( 3\overline{3} \) resonance, each transition

---

\(^*\) The relation is not quite accurate due to the finite mass of the nucleons. In the old notation, e.g., the Ss transitions contain a small fraction of "displaced" p wave pions in the \( NN \) subsystem.

\(^{**}\) Alternatively, P state production contains in the old notation some displaced Ss and Sd production.
amplitude will be proportional to the resonating $\pi N$ amplitude which is taken to be of the Chew Low effective range form

$$f_{\pi N}(q_r) = \frac{q_r}{\omega_r(1 - r_{eff} \omega_r) - i \frac{4}{3} (f_{\pi N})^2 q_r^3}$$

(17)

the momentum and energy refer to the $\pi N$ c.m. system, see Eq. (9). The effective range of the $33$ resonance is $r_{eff} \approx 0.5 \mu^{-1}$ and the $\pi N$ coupling constant $f^2 \approx 0.08$. A standard Breit-Wigner form with a threshold factor in the width would be equally a good choice for the present calculation.

As far as the final state $NN$ interaction is concerned it is well known that particularly the strong $S$ wave interaction is of much longer range than the $\pi N$ interaction. It is, therefore, an important assumption of the model that the two interactions can be separated in space. In other words, the $NN$ interaction occurs after the pion has been produced and rescattered. The final state interaction then factorizes in the amplitude.

Nucleon-nucleon interaction for higher than relative $S$ states will be neglected. As the resonant transitions contain no initial $S$ states we need not consider initial state distortion. $S$ state scattering is strongly spin dependent. Triplet scattering at low energies is dominated by the deuteron pole, singlet scattering by a nearby virtual bound state. The final state interaction factor is then the ratio of the $NN$ scattering wave function (for $S$ states) with $NN$ interaction over that without $NN$ interaction taken at a distance close to the origin (due to the short range assumption for the pionic processes). For a potential model and in effective range approximation $^\ast)$ the magnitude of this ratio is of the form

$$R(P) = \left[ 1 + \frac{a + b P^2}{(P^2 + P_0^2)(1 - r_{NN} P_0^2 + \frac{1}{4} \lambda^2 P^2 + P_0^2)} \right]^{1/2}$$

(18)

$^\ast)$ Allowance can be made for the existence of a hard core by a suitable redefinition of the effective range.
$P$ is the relative NN momentum, Eq. (8). There is one set of parameters for triplet and singlet scattering each. The pole position for the triplet is given by the deuteron binding energy $\epsilon$

$$P_B^t = (m\epsilon)^{1/2} = 0.327 \, \text{MeV}$$

and

$$P_B^s = -0.056 \, \text{MeV}$$

for singlet scattering. The triplet and singlet effective ranges $r^t_0$ and $r^s_0$ are well-known experimentally.

Qualitatively one notices immediately that $R(p) \to 1$ for large $P$ whereas the poles dominate near threshold $R \propto (p^2 + P_B^2)^{-1/2}$. Near threshold, moreover, the term in $a$ dominates the term $bP^2$. These remaining (shape dependent) parameters $a$ and $b$ have been chosen to give similar values for $R$ as those obtained for the NN potential of Gammel et al. (Gam 57) from a numerical integration of the Schrödinger equation. Anyhow, a phenomenological constant will be set at threshold for $S$ state production which checks $a$ whereas $b$ is of lesser importance.

**Isospin analysis**

In order to keep properly trace of kinematical factors let us assume the resonance occurs on nucleon $P_3 = P - \frac{1}{2}Q$ (see kinematics). We are then left with the following charge channels

- **A**  $pp \rightarrow \pi^+ pn$
- **B**  $pp \rightarrow \pi^+ np$
- **C**  $pp \rightarrow \pi^0 pp$

(20)
where the first nucleon has momentum $\vec{P}_3$, the second $\vec{P}_4$. The
deuteron results as a special case in channels A and B. If the
amplitude for process A is called $a$ then the amplitude for
process B will simply be $-a/3$. This obtains, immediately from
the ratio of Clebsch-Gordan coefficients

$$\frac{A}{B} = \frac{(1 1 1,2 1,2 | 1,2 1,2) (3,2 1,2 1,2 -1,2 | 1,1)}{(1 1 1,2 -1,2 1,2) (3,2 1,2 1,2 1,2 | 1,1)} = -\frac{3}{3}$$

(21)

Similarly, reaction C has the weight $-a\sqrt{2}/3$. For the production
on the nucleon $\vec{P}_4 = \vec{P} - \frac{3}{2}\xi$ the amplitude gets a dash and the relative
weights are $-a'3/3, a'$ and $-\sqrt{2}/3a'$ respectively. Adding and squaring
the contributions, one obtains, for the cross-sections

$$\sigma_A \propto \left( a + \frac{1}{3} a' \right)^2$$
$$\sigma_B \propto \left( a' + \frac{1}{3} a \right)^2$$
$$\sigma_C \propto \left( \frac{\sqrt{2}}{3} a + \frac{\sqrt{2}}{3} a' \right)^2$$

(22)

the lower sign goes with the triplet state (note that $\sigma_C$ must
vanish for equal nucleon momenta in the triplet state). The constant
of proportionality in (22) is different for triplet and singlet states
for reasons of final state interaction.

If interference terms could be neglected the ratio of
charged to neutral production cross-sections becomes approximately

$$\frac{\sigma^+}{\sigma^0} \approx \frac{\frac{1}{3}a^2 + \frac{1}{3}a'^2}{\frac{1}{3}a^2 + \frac{1}{3}a'^2} = 5$$

(23)

for triplets and singlets, separately. This is the Peaslee ratio
(Pea 52).
Angular momentum analysis

The three-body angular algebra is rather more complicated. The first step is to express the Ψ \textit{N} c.m. variables $Q_r$ and $P_c$ in terms of the over-all c.m. system variables $Q$, $P$ and the relative angle $\Theta$. These relations are given in Eq. (10) of the kinematics. Part of the final angular phase space is trivial as $Q_r$ and $P_c$ do not depend on the directions of $Q$ (with $\Theta$ fixed) and the azimuth of $P$ with respect to $Q$. Therefore, to obtain the total cross-section, three angular factors can be integrated directly. The remaining integrations over $Q$ and $\cos \Theta$ have to be done numerically.

It is useful to consider the case of spinless nucleons first. In the $(j, J)$ classification of Eq. (16) the partial angular momentum $j$ will then always be equal to one whilst $J = 0$, 1 and 2. Instead of the five $P$ state transitions of Eq. (16) we have now three. Still there is one $S$ state production amplitude.

The cross-sections in Eq. (22) have direct terms $a^2$ and $a'(-\cos \Theta) = a^2(-\cos \Theta)$ as well as interference terms in $aa'$. For the purpose of illustration we give the results for the direct terms explicitly (without final state interaction)

$$I_o(Q, \theta) = |f(Q_r)|^2 S(Q)$$

$$I_{a0}(Q, \theta) = 3 P_c^2 |f(Q_r)|^2 \cos^2 \alpha S(Q)$$

$$I_{a1}(Q, \theta) = \frac{3}{2} P_c^2 |f(Q_r)|^2 \sin^2 \alpha S(Q)$$

$$I_{a2}(Q, \theta) = \frac{3}{40} P_c^2 |f(Q_r)|^2 (3 + \cos^2 \alpha) S(Q)$$

(24)
The first index refers to $S$ and $P$ state production respectively, the second index gives $J$. The angle $\alpha$ is between $\vec{q}_r$ and $\vec{F}_c$ and $S(Q)$ is a known phase space factor. The resonant amplitude $f(Q_r)$ is given in Eq. (17). The odd coefficients and angular factors in Eq. (24) result from the integration over the trivial angular dependence mentioned before. To obtain the cross-section a phenomenological constant of proportionality is to be inserted for each transition.

The four expressions for the interference terms are similar. They are proportional to $\text{Re} \left[ f(Q_r)f^*(Q_r)S(Q) \right]$ times more complicated angular functions containing the dashed momenta and angles for the other $\eta N$ subsystem. For our qualitative discussion we refrain from giving them here.

To apply the final state enhancement factor [essentially of the form (18)] one has to project out the $S$ wave part from the matrix elements before squaring.

If this is accomplished it is then possible to show that the case with spin is reducible to the spinless case. No new functions appear beyond those of Eq. (24) and the interference terms. The five $P$ state transitions for the full spin problem are linear combinations (with Racah coefficients) of those functions.

The total cross-section is finally obtained by integrating over $Q$ and $\cos \theta$, if it is not the energy distribution or the angular correlation of the outgoing particles which is required **).

The production with deuteron formation is easily obtained as a special case. Neglecting the deuteron internal motion we have to set $P = 0$ and $Q = Q_{\text{max}}$. Only the triplet transitions will contribute.

\[ S(Q) = \frac{1}{4} \frac{3}{2} P Q \left[ \sqrt{m + \frac{1}{2} T} - \text{\textbf{u}} \right] \quad \text{where} \quad T = \text{c.m. kinetic energy} = \sqrt{s} - 2m. \]

** The above treatment is not suitable for the angular distribution of the pion with respect to the incoming nucleon as the directions of the pion have been integrated over to obtain Eq. (24). The necessary modifications are given in the Mandelstam paper.
The dashed and undashed variables are equal, i.e., the interference terms are trivial $aa' = a^2$.

So far the unitarity limit for the total cross-section could have been exceeded. The total cross-section for any particular angular momentum $J$ must be less than $\pi(2J+1)/P^2_{in}$. These limits can be built into the theory by $K$ matrix unitarization, i.e., the previous results are considered to correspond to the $K$ matrix rather than to the $T$ matrix. In the present treatment this procedure has been reduced into a simple damping factor which enforces the unitarity limit.

We now turn to the discussion of the numerical results.

Results

In order to compare with experiment the determination of the phenomenological constants has to be discussed first. There are six parameters, one for $S$ state and five for $P$ state production. The $S$ state parameter is determined from $\pi^+$ production with deuteron formation near threshold. Earlier results are used (Ros 54) to subtract off the non-resonant $\pi N$ states with angular momentum $\frac{5}{2}$.

The $P$ state parameters are simplified drastically as follows. One coefficient is set again from the deuteron channel looking for a deviation from a pure $S$ state angular distribution which is $\frac{1}{2} + \cos^2 \Theta$. No deviation is found and the parameter is set to zero. One effective parameter is introduced for the rest (taking care of the dominant Clebsch-Gordan weights). This parameter is determined from the peak value of the total cross-section $pp \rightarrow \pi^+pn$ without deuteron formation (which is 11.4 mb). In this way, only one $P$ state parameter is actually contributing and the problem of knowing the relative phases does not even appear.

The total cross-section for $pp \rightarrow \pi^+d$ is shown in Fig. B2.
The lower curve shows S state production alone (including some non-resonant production near threshold) and the upper curve is S plus P state production. The difference between the two curves should not be taken seriously in the present model. The fit is satisfactory and stays so with the inclusion of newer data (see Section B3.1). Particularly the steep drop after 600 MeV occurs only with inclusion of the resonance factor Eq. (17) and is a first indication for the validity of the resonance model.

The total cross-section for \( pp \rightarrow \bar{\nu}^+ p n \) (no deuterons) which sets the effective P state parameter is shown in Fig. B3 left-hand side. The P state contribution (dash dotted) and the S state contribution are given separately. The first actual prediction is the cross-section for \( pp \rightarrow \bar{\nu}^0 p p \) which is shown on the right-hand side of Fig. B3. The agreement is good. Rosenfeld's non-resonant S state contributions are included in the dashed curve. They matter only for energies below 400 MeV. At 660 MeV the \( \bar{\nu}^+ / \bar{\nu}^0 \) ratio is predicted as 3.9 is in rather good agreement with experiment. The deviation from the Peaslee value of five Eq. (23), is partly due to
the interference terms and partly to the $\pi^+ \pi^0$ mass difference in the calculation of $Q_{\text{max}}$. Actually, the $\pi^0$ prediction is not sensitive to the choice of the effective P state parameter unless one assumes that the $J = 1$ transitions are dominating. In this case the $\pi^0$ cross-section would be reduced strongly by interference which is not born out by experiment. A contribution from D state production cannot be excluded. However, S and P state productions are still considerably below their unitarity limit.

With the same parameters as before (and some additional simplifications) the pion energy distribution of the differential cross-section $d^2\sigma/d\omega d\Omega_p$ has been calculated. For a pion lab. angle of 24° the results are shown in Fig. B4 for two energies, for 657 MeV on the left-hand side and for 556 MeV on the right-hand side ($\pi^+$ production). The abscissa is the pion energy. The dashed curves have no resonance factor (17) and are clearly incompatible with experiment. These distributions are therefore good evidence for the resonance model. The experimental data from (Mes 56) agree in shape with newer data. Normalization has not been calculated anyhow.
Similar agreement is obtained for $\pi^0$ production (where the decay $\gamma$ rays are observed at $0^\circ$.) The corresponding energy distribution is shown in Fig. B5. A minor discrepancy occurs at the pion high energy end. As the shapes for $\pi^+$ and $\pi^0$ are different the $\pi^+/\pi^0$ ratio has considerable momentum dependence whereas for the total cross-sections this ratio is approximately constant.

Similarly, angular correlations between the outgoing pion and nucleon can be calculated. We do not report them here.
To summarize we have learned that a fairly simple resonance model which has essentially two adjustable parameters describes the experimental situation quite well (total cross-sections and angular distributions). Final state interactions are important. No necessity for non-resonant contributions arises except near threshold. We have developed some feeling for the algebraic complexity which is mainly due to the presence of a three-body final state.

As mentioned before the Mandelstam model has been extended to higher energies, more partial waves (particularly initial F states) and more parameters. The recent situation has been reviewed by Amaldi Jr. (Ama 67). No qualitative conclusions in Mandelstam's initial analysis have to be changed.

B3. Dynamical models

One series of papers which goes beyond a phenomenological treatment uses Hamiltonian dynamics. The $\pi N$ interaction is supposed to be given in a non-relativistic form by a Hamiltonian and the $NN$ interaction (including the deuteron wave function) is described by a potential to various degrees of sophistication. Finally, these pieces are put together by a many channel scattering theory which typically involves off-shell amplitudes. The obtained prediction (which is never parameter free, however) is compared to experiment. Due to the complexity of the calculations not more than the worst simplifications are actually tested. We discuss one of the more complete analyses of this type in Section B3.1. It covers energies from threshold up and including the resonance region.

We then make a small insertion (Section B3.2) to comment on soft pion approximations. Their natural testing ground is the threshold region.

The resonance region has also been treated relativistically from the onset. The exploitation of analyticity properties (which
are deduced from the simplest Feynman diagrams) can replace some of the dynamics. Particularly, the off-shell problems are avoided this way. A partial wave dispersion relation calculation is described in Section B3.3 below.

B3.1 Non-relativistic models

We report in this section on the calculation of pp \(\leftrightarrow\pi^+d\) by Lazard et al. (Laz 70). The set of graphs collected in Fig. B6 is the basis for a dynamical scheme. The calculation is non-relativistic in the sense that the non-relativistic limit to the pseudovector NN coupling and non-relativistic propagators are used (of course, the pion is relativistic kinematically). As for the NN interaction a potential picture is implied. Although the model is particularly suited for the threshold region it has been pushed successfully to cover the resonance region. The \(\pi NN\) coupling constant \(f^2\) and some knowledge of the \(\pi N\) and NN elastic scattering amplitudes (essentially neglecting off-shell effects) are going to replace the phenomenological constants in the Mandelstam model.

The graphs that are considered are the nucleon exchange (Born term a), with nucleon rescattering b), the rescattering of pions to second order only c), (which includes the 33 resonance) and \(\pi N\) rescattering with final state NN interaction d). The graphs and some notations are collected in Fig. B6.

As mentioned already the vertex function \(V_1\) for the absorption or emission of a pion is taken to be the non-relativistic limit of the pseudovector coupling. Omitting a momentum conserving \(\delta\) function, \(\delta(\vec{k}_1 - \vec{k}_2 - \vec{q})\), the vertex function \(V_1\) has the form

\[
V_1(k_1, k_2, q) \propto f \left< \vec{\sigma} \cdot \left[ \vec{q} - \frac{\not{q}}{m} \right] (\vec{k}_2 + \vec{k}_1) \right>
\]

(25)

where the matrix element is between two component nucleon spinors.
The isospin labels and a normalization factor have been omitted for brevity here. The strength of the interaction is given by the \( \pi NN \) coupling constant \( f^2 = 0.08 \). The first term \( \mathcal{G}_1 \) in (25) is what remains in the static limit \( (m \to \infty) \) and gives the strong p wave scattering of pions. With inclusion of the second term the bracket \( [ ] \) is proportional to the velocity difference \( \vec{v}_\pi - \vec{v}_N \) (where \( \vec{v}_N \) is the average nucleon velocity). Therefore, the form of Eq. (25) is sometimes called Galilean invariant. Equation (25) contains now pion s waves as well.
The pion rescattering vertex in graphs c) and d) has the general form

$$V_2(k', q', k, q) \propto \left< f_1 + f_2 \frac{q}{\sqrt{2}} \cdot \bar{q} \right>$$

(26)

from invariance arguments. Again a factor $s(k' + q' - k - q)$ and a normalization have been omitted. The momenta $q'$ and $\bar{q}$ are in the c.m. of the $\pi N$ subsystem [see kinematics Eq. (9); $q'$ differs only in direction from $q$]. The invariant functions $f_1$ and $f_2$ depend on energy and angle in the $\pi N$ scattering. They are experimentally known in terms of $\pi N$ partial wave amplitudes. The present analysis considers only s and p waves. Introducing isospin (first index) and total angular momentum (second index) one has for $\pi^+ p \rightarrow \pi^0 p$ scattering, e.g.,

$$f_1 = S_{s_1} + 3 \cos \Theta_p \, P_{33}$$

$$f_2 = P_{51} - P_{33}$$

(27)

and similar expressions for the charge exchange reaction $\pi^+ n \rightarrow \pi^0 p$ (comp. the lectures of J. Hamilton).

Except for some kinematical factors the present calculation neglects that one of the pions and one of the nucleons in $V_2$ is off-shell.

Furthermore, it is assumed that the graphs where a $\pi^-$ is involved in the intermediate state (or a $\pi^+$ going backward in time) can be approximated by the same vertex function $V_2$.

The deuteron vertex $V_d$ is (apart from factors) given by the Fourier transform of the deuteron wave function. The short range behaviour of various parametrizations (generalized Hulthén forms) will be tested for the $3S_1$ part $u(r)$ and the $3D_1$ admixture $w(r)$, see (32) below.
With all these ingredients the matrix element for graph a) has the form

\[ T_a = \int d^3k_2 \ V_a(p_1, k_2, q) \ V_1(p_2, k_2, q) + \text{exch. term} \ (p_1 \leftrightarrow p_2) \tag{28} \]

and for the first of the graphs c) we have

\[ T_c = \int d^3k_1 d^3k_2 d^3q \ V_a(k_1, k_2, q) \ V_2(p_1, q, k_2, p_2) \ \frac{V_1(p_1, k_1, q)}{E_{tot} - k_1^2 - q^2 - p_2^2 + i\epsilon} \tag{29} \]

(the index zero denotes the energy) and a similar expression for the second of the graphs c) where two pions occur simultaneously in the intermediate state. Off-shell effects in $V_2$ are neglected. Again exchange terms with $p_1 \leftrightarrow p_2$ are to be added.

It remains to discuss the final state interaction of the two protons in graphs b) and d). In some of the papers on threshold absorption the two-nucleon scattering wave function $\Psi^*_\ell(Pr)$, where $P$ is the relative momentum, has been calculated from phenomenological potentials solving the Schrödinger equation, see e.g., (KoR 66). We have mentioned an effective range approximation earlier in connection with the Mandelstam model. (Note, however, that no relative $S$ waves occurred for the $pp$ states considered there and, therefore, the rescattering discussed here was neglected.) The present treatment simulates the rescattering effects at short distances rather crudely by introducing a cut-off function $C_\ell(r)$ into the scattering wave function

\[ \Psi^*_\ell(Pr) = r_\ell \exp(-i\delta_\ell) \left[ j_\ell(Pr) \cos\delta_\ell + C_\ell(r) n_\ell(Pr) \sin\delta_\ell \right] \tag{30} \]

where $\delta_\ell$ is the real part of the $pp$ phase shift and $r_\ell$ the corresponding inelasticity parameter. Regular behaviour for small separations $r$ is enforced by
\[ C_x(r) = \left[ 1 - \exp(-Zr) \right]^{\ell+1} \] (31)

where \( Z \) is treated as an adjustable parameter of the order \( Z = \mu \) or \( Z = 2\mu \), corresponding to a potential well with a range of the order 1.4 and 0.7 fm respectively.

The separation of the absorption process into the graphs a) to d) of Fig. B6 is not without problems of double counting. E.g., one might argue that the pion rescattering graph c) is already contained in the one-pion contribution of the final state interaction graph b). Lazard (Laz 70b) argues that this term contributes only to the \( P_{11} \) wave of the \( \pi N \) scattering part of the amplitude and this is small at least in the resonance region. Similarly, graph c) (the nucleon exchange part of \( V_2 \) at least) might be absorbed into the deuteron vertex function of graph a). As we are going to use explicit parametrisations for the deuteron wave function it is not easy to judge the approximation. Of course, already in the Mandelstam model we have advocated the longer range of the \( NN \) interaction to separate off the pionic processes, i.e., to factorize the interactions.

We now turn to a discussion of the numerical results.

Numerical Results

Apart from varying the input the calculations have been carried through in two different approximations. **Approximation I** consists in neglecting the ("retardation") factor \( \exp(-iq_2^2) \) which is, of course, the usual procedure in threshold calculations. It means a considerable simplification of the angular algebra. **Approximation II** is without this simplification. The radial wave functions for the deuteron are of a generalized Hulthén form (HuS 57). The radial \( \frac{3}{2}S_1 \) part has the form

\[ \psi(r) = N \sum_i^n c_i \exp(-\alpha_ir) \] (32)
with $\sum c_i = 0$ to ensure regularity at the origin and $n=5$ for most sets. A similar expression is used for the $^3D_1$ admixture $\omega(r)$. Various sets of parameters which fit mainly the deuteron-photodisintegration data (Gou 65, McQ 66) are used for the present calculation. To show the amount of variation two typical sets of radial wave functions are given in Fig. B7.

The pp phases for the NN wave function, Eq. (30), are experimental and the cut-off parameter $Z$ equals 1 or $2^{1/3}$.

Having described the input the results for the total cross-section are shown in Fig. B8. For one and the same deuteron wave function the approximations I and II are given as well as the dependence on $Z$. Approximation I is considerably less above the resonance. Note also that the slopes near threshold are different. The low energy behaviour in approximation I is of the form

$$G = \alpha \left(\frac{Q}{\mu}\right) + \beta \left(\frac{Q}{\mu}\right)^3$$  \hspace{1cm} (33)
Fig. B8 from (Laz 70)

for the pion production cross-section (corresponding to the usual threshold factors for $s$ and $p$ wave pions). The values for $\alpha \approx 0.25$ mb and $\beta \approx 0.9$ mb are consistent with experiment [and with the threshold calculation of $\alpha$ by Koltun (KoR 66) whereas the calculated value of Reitan (Rei 69b) is less by a factor of two]. However, Fig. B8 shows clearly that the experimental slope near threshold is not well defined yet.

From Fig. B8 we see that the cut-off parameter $Z$ easily can change the normalization of the cross-section by about 15% at the peak of the resonance. In all the following figures, from B9 to B11 the cut-off is fixed at $Z = 2 \sqrt{m}$ and the calculations are in Approximation II. In these figures the sensitivity to the deuteron wave function is tested for three sets of deuteron parameters (see Fig. B7 typically). Figure B9 gives the total cross-
section. As in the case of the cut-off dependence the effects are larger at the resonance and of the same order of magnitude.

A differential cross-section at the resonance with identical input is shown in Fig. B10. Due to the antisymmetry of the pp state it is a function of $\cos^2 \theta$. The sensitivity to the deuteron wave functions is not large at $\cos^2 \theta = 0$ where the momentum transfer is moderate. At $\cos^2 \theta = 1$, however, where large momentum components in the wave function matter the predictions are quite different. The remaining discrepancies to experiment might be due to the neglect of higher pion waves in the graphs which involve NN final state interaction.

For energies below 400 MeV the predictions agree with experiment (not shown). They are linear in $\cos^2 \theta$ as is Approximation I which has a wrong energy behaviour, however.
4π(dσ_{\text{prod}}/dΩ) for 622 MeV.

Fig. B10 from (Laz 70)

Finally, we remark that even the coefficient of pion asymmetry $\Lambda(\theta)$ measured in an experiment with polarized protons for three different energies (Aki 58) at 536, 616 and 654 MeV is surprisingly well predicted. The sensitivity to the deuteron wave function is not large. This might be due to the fact that the normalization essentially drops out in the definition of the polarization. One energy, 536 MeV, is shown in Fig. B11. The other two energies show similar agreement.

B3.2 Soft pion predictions in NN → NNπ

The threshold behaviour discussed shortly in the previous section can be tackled with a completely different method. There exists a number of restrictions on amplitudes involving pions which hold exactly in the limit of vanishing pion four-momentum $q \to 0$ necessitating $M \to 0$ as well. These relations should hold approximately for physical pions. Famous examples in particle physics are the Goldberger-Treiman relation for pion decay, Adler's self
consistency condition in $\pi N$ scattering and with some additional assumptions the Weinberg predictions for $\pi N$ scattering at threshold a variant of which is the Adler-Weisberger relation [comp., e.g., the textbook (AdD 68)]. Some of these topics are treated in the lectures of M. Ericson. The essential idea involved can be stated as a pole dominance approximation which says that the divergence of the axial vector current as a function of the pion mass $q^2$ is dominated by the pion pole in the region $0 < q^2 < \mu^2$. [In the language of dispersion relations this means that the contribution of three pion intermediate states (and higher) is a correction only and that no subtraction is necessary]. This approximation is called PCAC (partially conserved axial vector current).

While these ideas have worked quite well in the above-mentioned particle cases, even applications to processes involving nuclei seem to work in some cases. The corrections to the Weinberg value for the scattering length in $\pi$ nuclear charge exchange have been calculated successfully by M. Ericson et al. (Eri 69b). The pion finite mass correction is mainly due to nuclear rescattering [see (FuF 68) for the method]. The corresponding effect is much smaller in the nucleonic case.
For our present problem, the production amplitude $NN \leftrightarrow NN$, PCAC has been applied by Beder (Bed 68) and Schillaci et al. (Sch 68/9).

Most effectively the soft pion predictions have been stated by Adler and Dothan (AdD 66) in the form of a low energy theorem [which is similar to the Low theorem (Low 58) for low energy photon emission]. It says that for our production amplitude the term of zeroth order in $q$ is completely given by inserting into every external line of the elastic nucleon-nucleon scattering amplitude a factor

$$\frac{1}{2} \frac{q_{1} \cdot q_{2}}{p_{3} \cdot q - q^{2}}$$

(34)

(for an emission from nucleon $p_{3}$) which has exactly the form of pseudovector coupling. To lowest order one thus has a model independent prediction (if there were no pion mass corrections).

The agreement of such a calculation (without any corrections) for charged and neutral pion production near threshold ($300 < E_{p}^{lab} < 400$ MeV) is acceptable (Bed 68, Sch 68/9), particularly for the unbound production channel. The total cross-sections agree with experiment to within a factor of two. However, as we have mentioned already there are no data of real precision in this energy region.

Soft pion predictions have been tried out also in the $500 < E < 800$ MeV region by Grant et al. (Gra 69) and Schillaci and Silbar (ScS 70), keeping the pion "soft" in the over-all c.m. system i.e., $q = (\mathbf{M}, \mathbf{Q})$. Theoretical and experimental order of magnitudes of the cross-section do not agree. The curve marked GSS in Fig. B12 is the soft pion prediction from (Gra 69). (The three-momentum of the pion in the over-all c.m. system is denoted by $q$ in this figure.) The authors (ScS 70) blame the term linear in the four-vector $q$ for the discrepancy
\[ \gamma \approx a + bq \]

Fig B12 from (ScS 70)

as such a term is always present for a physical pion. The contribution of the resonance to the coefficient \( b \) of \( q \) is estimated and found to be large. This is of course what we expect from our experience with the Mandelstam and Lazard model. The discrepancy cannot be blamed on the pp phase shifts (which are the phenomenological input in this type of calculations). Errors in the phases are found to produce at most a 15\% effect. The formal manipulation of setting \( M = 0 \) in their soft pion expression leads to a drastic and probably senseless further reduction of the theoretical cross-section.

Recently, the Fubini Furlan mass dispersion technique has been applied to estimate the real pion mass corrections. Young (You 69) obtains an order of magnitude of 8\% for the correction to the process \( pp \rightarrow \pi^+ p n \) into \( I = 1 \) states near threshold. Similar results have been obtained by Banerjee et al. (Ban 71).

To summarize it should be clear that the soft pion predictions in these production processes have not been tested reliably yet.
B3.3 Relativistic or dispersive approach

In the present Section we will discuss a third approach to our basic reaction \( NN \leftrightarrow NN\pi \). We shall describe the application of partial wave dispersion relations to the deuteron channel \( pp \leftrightarrow \n^d \). Formally, this will appear to be quite different. We shall use a fully relativistic terminology and exploit some analyticity properties of the amplitude. Those will be deduced (not proved) by inspecting the simplest Feynman diagrams that are possible for our reaction. However, it is reassuring that the same physical processes (i.e., pion rescattering and the NN interaction) will turn out to be the important mechanisms as before.

The present Section serves as natural link to part II, Chapters D and E which comment on the status of the application of dispersion relations to elastic scattering in nuclei. We shall, therefore, describe briefly some of the basic ideas involved (comp., also the lectures by O. Steinmann and J. Hamilton) from a practical point of view.

The first step is to write down all sort of diagrams that can be imagined. Complicated ones as the one in Fig. B13 and

![Fig. B13](image)

simple ones, like the one-nucleon exchange of Fig. B14. The next

![Fig. B14](image)
step is to analyze the singularities of the amplitude $f$ in the
complex dynamical variables, energy $s$, momentum transfer $t$ (see
kinematics B1) and crossed momentum transfer $u$. Only two of them
are independent, $s$ and $t$ say (as we have $s+t+u = 2m^2 + M^2 + \mu^2$),
in this quasielastic process.

These singularities are given by a simple set of rules
(not so easy to exploit sometimes), the Landau rules (Lan 59). The
essential properties that arise are: the singularities of $T(s,t)$
as a function of complex $s$ and $t$ are either poles (graph B14) or
cuts (graph B13), the location of which can be calculated having all
internal particles on the mass shell. To calculate the position
of the singularities only the masses of all the particles involved
have to be known. (This may be complicated for complicated graphs).
The strength of the singularities is not determined so far.

Generally, if more lines are inserted into a given graph
the associated singularities are further away from the physical
region of $s$ and $t$ values. Therefore, those graphs are likely
to be of lesser importance. These properties of graphs have been
verified for a great number of examples and the experience is converted
immediately into a (heuristic) dynamical principle, "the principle
of the closest singularity". In practical terms it says that the
leading behaviour of the amplitude should be given by the singularities
of the simplest graphs.

Having determined in this way the region of analyticity
for a given amplitude one can apply the Cauchy formula to that region.
This is called a dispersion relation. In its conventional form it
links the real part of the amplitude to the imaginary part, known
for all energies. In the dispersive treatment one needs not to go
off shell with the intermediate particles to calculate the imaginary
part (which is the input). This is an advantage over the previous
methods which suffered from unknown off shell effects.
In simple cases like forward elastic $\pi N$ scattering the singularities occur only in the physical scattering region (but for a pole) where unitarity relates the imaginary part directly to the measurable total cross-section $\text{Im} f(s,0) = Q \frac{G(s)}{4\pi}$. By exploiting the dispersion relation one gets $\text{Re} f(s,0)$ and hence the complete forward amplitude (comp., Chapter D).

For the present break up process $\pi^+ d \to pp$ where we want to know the angular or $t$ dependence as well, the procedure is first to make a partial wave decomposition (which is reasonable as only a few waves contribute in our energy region) and then to calculate the amplitude from a dispersion relation in the energy alone for each partial wave.

In the following we describe briefly the calculation of $\pi^+ d \to pp$ by Schiff and Tran Than Van (ScT 66). According to the previous remarks the dominant graphs are expected to be those of Fig. B15 where the blobs stand for all possible internal insertions. For these graphs the singularities (imaginary parts) will have to be calculated which serve as starting point of the dispersive calculation. Graph C which corresponds to final $NN$ interactions will be neglected in the present calculation. Graph D is treated in the resonant approximation, i.e., the intermediate $\pi NN$ state is treated as

![Fig. B15](image)
NN*. The N* width is taken into account by introducing a Breit-Wigner or else a phenomenological mass distribution. The freedom in the choice of this function may be considered to represent some of the non-resonant background.

The singularities of $f(s,t)$ are summarized as follows.

**Singularities in $s$**

As Graph C is neglected *) the only remaining contributions are from the NN* intermediate states in D. Those start at the physical threshold $s_p$

$$s_p = (M + \mu)^2 = 208.4 \mu^2$$

(36)

as we have smeared out the resonance by a Breit-Wigner distribution.

**Singularities in $t$**

Because of antisymmetry in the pp state the $t$ and $u$ singularities are the same. We shall write them for $t$ only. Graph A evidently has the nucleon pole at

$$t = m^2$$

(37)

Graph B is actually a reduced form of D (or C) if the blob is replaced by a pion (Fig. B16), (the wavy intersection indicates that we are considering $t$ channel intermediate states). The corresponding

![Diagram](image)

*Fig. B16*

*) The position of the singularities from C are also shown in Fig. B17. They extend down to $s_0$ (the pp threshold) and even to $s_1$ (small anomalous $s$ cut).
singularities actually do not start at the normal threshold

\[ t_n = (m + \mu)^2 = 59.6 \mu^2 \]  \hspace{1cm} (38)

as one might expect but they extend to the lower threshold

\[ t_\alpha = 49 \mu^2. \]  \hspace{1cm} (39)

This extension is called an anomalous cut and is due to the deuteron's weak binding \[ \text{intuitively the deuteron has the possibility of an almost real decay, for more details see (ErL 70), e.g.}. \] The anomalous extension typically occurs in nuclear t channels. It is not large for the present break up reaction whereas in elastic πd scattering \[ t_\alpha = 1.7 \mu^2 \] very close to the scattering region \[ t \leq 0 \] (see Chapter E below).

Partial Wave Projection

The nucleon exchange pole or Born term A does not remain a pole under partial wave projection. Neglecting spin the \( \ell \)th partial wave \( f_\ell \) has the form

\[ f_\ell(s) \propto \int_1^{-1} \frac{P_\ell(\cos \theta)}{t - m^2} d(\cos \theta). \]  \hspace{1cm} (40)

This leads to an associated Legendre function which has cuts in the energy plane from \( s = 0 \) to \( s = -\infty \) and from \( s_1 \) to \( s_2 \) as is shown in Fig. B17. A similar case has been discussed in much detail in J. Hamilton's lectures.

Similarly, Graph B, called the "deuteron structure", leads to the egg-shaped cut in Fig. B17 \[ \text{the small length of cut on the real axis between } s_5 \text{ and the egg is due to the anomalous threshold (39)}. \] Moreover, a further cut from 0 to \(-\infty\) is induced.
To include deuteron structure a simple model is used for the npd vertex in Graph B. This consists of a few poles of Yukawa type \[ \sum_i \text{const}_i / (t_1 - t_i) \]. Compare Eq. (32) for co-ordinate space. One of the parametrizations used here has also been used by Lazard et al., see Section B3.1 to facilitate comparison. In this respect we are back to a non-relativistic deuteron model.

The corresponding partial wave branch points \( s_{c_1}, s_{c_2}, \) and \( s_{c_3} \) for a three-pole model of the deuteron are also shown in Fig. B17.

![Fig. B17 from (ScT 67)](image)

**Dispersion Relation**

For the full spin problem the partial wave \( f_J^L \) in Eq. (40) stands symbolically for \( f_J^{L' L'} \). Given the analytic properties sketched in Fig. B17 the Cauchy formula leads to the following dispersion relation

\[
\frac{F_{\ell}(s)}{F_{\ell}} = \frac{1}{2\pi i} \text{Res} \int_{s'} \frac{f_{\ell}(s')}{{s'} - s - i\epsilon} \, ds' + \frac{1}{\pi i} \int^{\infty}_{s'} \frac{\text{Im} f_{\ell}(s')}{{s'} - s - i\epsilon} \, ds'
\]

(41)
provided the amplitude decreases sufficiently at infinity. The first integral is over all the cuts to the left of the physical threshold $s_p$ with the discontinuities disc $f^i(s')$ across the cut. The second term extends over the physical energies. Ir principle (41) would stand for a system of coupled equations. This complication is presently avoided by neglecting the NN and $\pi d$ intermediate states right away.

We have sketched the input in the left-hand cut, namely, the deuteron structure, it remains to see how $\text{Im} f^i(s)$ is to be calculated for the right-hand cut. This can be done in terms of physical processes by exploiting the unitarity relation

$$\ln f^i_{fi} = \sum_n f^*_n f_n$$

where $i$ is the initial $\pi d$ state and $f$ the final $p p$ state. The sum includes all possible intermediate states, a summation over spin and an integration over phase space. Presently, the intermediate states will be restricted to $NN^*$. For partial waves $\text{Im} f^i_{fi}$, Eq. (42) is most conveniently exploited in treating the spin dependence in partial wave helicity amplitudes. The angular variables in (42) can then be integrated out.

For the actual calculation we need a model for the processes $\pi d \rightarrow NN^*$ ($f_{ni}^i$) and $pp \rightarrow NN^*$ ($f_{fn}^i$). The former is treated in one-nucleon exchange with the deuteron vertex as previously described and the latter in one-pion exchange with inclusion of a phenomenological pion form factor.

The bulk of the algebraic work consists now in transposing these models from the original spinorial expressions into invariant amplitudes, to calculate the helicity amplitudes, to project out the partial wave helicity amplitudes and finally to form the conventional partial wave amplitudes for which the dispersion relations are evaluated.
We turn to a discussion of numerical results. The total cross-section is shown in Fig. B18. The deuteron parameters are those which fit the differential cross-section best at the resonance. The dashed curve is from a calculation by Chahoud et al. (Cha 66) not discussed here. The sensitivity to the shape of the $N^*$ is not large (the effect is 10% at 900 MeV and less below). However, as expected we are sensitive to the dpn vertex function as is shown for the differential cross-section at 460 MeV. Four different deuteron parametrizations are given in Fig. B19. The one corresponding to the solid line fits best at the resonance, see Fig. B20. (This is one of the parametrizations used in the Lazard model, Section B3.1.) Below the resonance it is too flat (Fig. B19, solid line) while it is too steep above the resonance, Fig. B21.

The various partial wave transitions have been evaluated separately. There are three dominant transition amplitudes denoted by $b$, $c$ and $e$ in Table B1. In the old Rosenfeld notation, see Eq. (15), the amplitude $c$ corresponds to the resonant transition $^1D_2 \rightarrow ^3S_1p_2$ whereas $b$ and $e$ are transitions from $^1S_0$ and $^1G_4$ states, respectively. Their relative importance can easily be traced in Table B1. The nucleon exchange, including the structure (columns
Plot of the differential cross section, in the c.m. system, for the process $pp \rightarrow \pi^*d$ at an energy of the incident nucleon in the laboratory system $E_{\text{lab}}^N = 460 \text{ MeV}$. The process $\pi^*d \rightarrow pp$ at $E_{\text{c.m.}}^N = 140 \text{ MeV}$ which corresponds to $E_{\text{lab}}^N = 633 \text{ MeV}$. 

**Fig. B19 from (ScT 68)**

The process $\pi^*d \rightarrow pp$ at $E_{\text{c.m.}}^N = 180 \text{ MeV}$ which corresponds to $E_{\text{lab}}^N = 743 \text{ MeV}$. 

**Fig. B20 from (ScT 68)**

**Fig. B21 from (ScT 68)**
marked B) and the NN* contribution are given separately. As expected the NN* dominates the magnitude, particularly Im c. The real parts, however, are not resonance dominated. The different lines in Table B1 correspond to different deuteron parametrizations. We further note (not shown) that the D state admixture in the "Born" term is extremely important. It is 30 times larger than the S state contribution at 600 MeV. Clearly, the D state contributes the large momentum components needed for the Born graph. For the NN* intermediate state the deuteron D state contributes about 25% at the same energy.

Although the present calculation has not been pushed to quite the same amount of sophistication as the non-relativistic calculation by Lazard et al., reported in Section B3.1, the dispersive method has been proved to be working.

<table>
<thead>
<tr>
<th>Parameterization</th>
<th>( a^B )</th>
<th>( c^B )</th>
<th>Re ( \delta ) NN*</th>
<th>Re ( c ) NN*</th>
<th>Re ( e ) NN*</th>
<th>Im ( \delta ) NN*</th>
<th>Im ( c ) NN*</th>
<th>Im ( e ) NN*</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( 0.22 \times 10^{-1} )</td>
<td>-0.13</td>
<td>-0.11 \times 10^{-1}</td>
<td>-0.10 \times 10^{-2}</td>
<td>-0.37 \times 10^{-1}</td>
<td>-0.12 \times 10^{-2}</td>
<td>-0.59 \times 10^{-3}</td>
<td>-0.35 \times 10^{-2}</td>
</tr>
<tr>
<td>IV</td>
<td>-0.56 \times 10^{-1}</td>
<td>-0.13</td>
<td>-0.11 \times 10^{-1}</td>
<td>*a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>0.27 \times 10^{-1}</td>
<td>-0.11</td>
<td>-0.10 \times 10^{-1}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*a) We have not given the values of the NN* contributions for parametrizations IV and II; they differ at most by 3% from the values given for I.

Table B1 from (ScT 68)

C. PION ABSORPTION ON LIGHT NUCLEI

C1. Nuclear Wave Functions, Correlations

We are now prepared to increase the level of complication by one step and to discuss pion absorption in light nuclei in this Chapter. As was exposed in the Introduction, Section A2, fast two-nucleon absorption is one of the most important channels in nuclei. Therefore, we have gained some experience with the underlying absorption or production process on two nucleons in the previous Chapter B. The limitations in the present understanding of this basic process should have become quite obvious. Any microscopic approach to the
more general nuclear pion absorption has to be seen against this background. This is particularly critical for the long standing problem of nuclear correlations.

The use of the word correlation in our context is not really fixed in the literature. Sometimes, and loosely speaking, it is used for the behaviour of the NN interaction at short distances. As we have seen already the two-nucleon process of the previous chapter is sensitive to the deuteron wave function and the two-nucleon scattering wave function at short distances. Moreover, we have seen that the separation of these effects from the remaining (pionic) dynamical aspects is far from unique.

From now on we shall use the word correlation in the strict sense. Starting with the two-nucleon density \( \rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \) for the A nucleon wave function \( \Upsilon_A(\mathbf{r}_1, \ldots, \mathbf{r}_A) \)

\[
\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \int \Upsilon(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_A)^2 d^3 r_3 \ldots d^3 r_A
\]  

we subtract the product of the single particle densities to define the correlation function *) as

\[
C(\mathbf{r}_1, \mathbf{r}_2) = \rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) - \rho^{(1)}(\mathbf{r}_1) \rho^{(1)}(\mathbf{r}_2)
\]

For a product wave function C is then trivially zero. There is a strong non-dynamical correlation effect in a fermion system, the Pauli correlation, which follows from the antisymmetrization of the wave function. For a typical value of the Fermi momentum in nuclei its range is about 1.4 fm. The correlation which is left over

*) The conventional definition is adapted to the large A or nuclear matter case. For light nuclei one should include the c.m. constraint into the definition of C by introducing an additional \( \delta \) function \( \delta_{\frac{1}{A} \mathbf{r}_1 \mathbf{r}_2} \) into (44). For the deuteron this new correlation function is then trivially expressible by the single particle density, see (Ker 59), e.g.
after subtraction of the trivial Pauli correlations is called the (true) short range correlation. (A typical hard core radius is 0.5 fm.) It is this short range correlation one would like to discuss quantitatively. It should be clear already that there is no simple experiment in nuclear absorption which relates directly to nuclear correlations. Unfortunately, this is quite a general experience. Inelastic electron scattering, e.g., summed over all final states, involves the correlation function as well. However, any reasonable model calculation shows that short range correlations are a small effect as compared to the trivial and large Pauli correlations. Another possibility is to look into the diffraction minima in the elastic scattering of elementary particles on nuclei using a multiple scattering theory. This is sensitive to details in the nuclear wave function. The experience in this case is that it is hardly possible to distinguish the effect of short range correlations from minor adjustments in the single particle density.

Irrespective of this rather evasive problem of nuclear correlations we feel that a detailed description of nuclear pion absorption (phenomenological or not) is interesting in its own right. The next Section gives some illustrations.

C2. Light Nuclei

Pion absorption at threshold is discussed in several papers. A lesser number of calculations deals with absorption in flight. The situation is reviewed briefly by Koltun (Kol 69). The general idea is to factorize the amplitude into two parts. The first factor stands for the pion absorption on two nucleons (to be approximated by the free two-nucleon process) and the second factor involves essentially the overlap of the target nucleus wave function with the residual nucleus and the two absorbing nucleons. Very often the pionic part is treated rather crudely, i.e., in impulse approximation. However, as we have seen in the previous chapter the one-nucleon exchange is in most circumstances much less than the contribution which involves pion rescattering (see the discussion
of Table B1, in Section B3.3). Any definite conclusions about the nuclear part (particularly about "correlations") seem therefore at least difficult to interpret in those calculations.

After this general word of caution we shall restrict ourselves in the present section essentially to pion absorption on $^4$He. We shall take the natural extension of the deuteron absorption model of Section B3.1 to illustrate the procedure. The calculation we discuss is by Lazard et al. (Laz 69) and is designed to fit the needs of the experiment of Bressani et al. (Bre 69). Particularly, the following final state channels have been measured at 76 MeV pion energy

$$\pi^+ \; ^4\text{He} \rightarrow 2p\alpha$$  \hspace{0.5cm} (45a)

$$\pi^+ \; ^4\text{He} \rightarrow 8p\; n.$$  \hspace{0.5cm} (45b)

The idea is now to introduce the absorption on a nucleon pair into the absorption on $^4$He. In first approximation we have to consider the graphs of Fig. C1 for reaction (45a).

The experimentally, two protons are detected symmetrically around the pion direction with energies restricted to $40 < E_p < 130$ MeV.

The observation of the emission of two protons directly from the He vertex is strongly suppressed as their energy must be at least 40 MeV in this experiment.
Correspondingly, for the complete break up of $^4\text{He}$, reaction (45b), we have to consider the graphs in Fig. C2. Again, the emission of two fast protons directly from the $^4\text{He}$ vertex can be neglected.

These graphs are translated into algebraic expressions for the amplitude in the same (non-relativistic) way as in Section B3.1. One notices immediately that more information on the subamplitude $\text{NN} \leftrightarrow \text{NN}\pi$ is required than has been obtained in the previous Chapter. Particularly, graph D involves transitions from the isospin state $t'=1$. Denoting the cross-sections by $\sigma_{tt'}$, the knowledge of $\sigma_1^d$, $\sigma_{10}$ and $\sigma_{11}$ is required, where the index $d$ refers to the bound state deuteron channel. For any particular Born approximation graph the squared amplitude summed over spins is proportional to the differential two-nucleon absorption cross-section. For graph A we have, e.g.,

$$|f_\alpha|^2 \propto \frac{d\sigma}{d\Omega}(\pi^+d \rightarrow pp)_{c.m. pp} \cdot V_{^4\text{He}}$$

where $V$ stands for the $^4\text{He}$ vertex. The knowledge of the two-nucleon amplitude is not required.
In view of the poor experimental information on the two-nucleon process with unbound nucleons the following simplifications are made. For all isospin channels the shape of the differential distribution is taken from the deuteron channel. For all but the highest energies the form \( \text{const.} + \cos^2 \theta \) is a good approximation (see Section B3.1). Normalization is then obtained for each isospin channel by means of the total cross-sections \( \sigma_{10}^4, \sigma_{10}^5 \) and \( \sigma_{11} \). From the usual threshold factors their energy dependence is

\[
\sigma_{tt'} = \alpha_{tt'} \frac{Q}{\mu} + \beta_{tt'} \left( \frac{Q}{\mu} \right)^3.
\] (47)

For the unbound channels the coefficients \( \alpha \) and \( \beta \) have errors of about 30% as they refer to experiments back in the fifties.

The remaining input concerns the \( ^4\text{He} \) vertex. The overlap of the \( ^4\text{He} \) wave function with pairs of two-nucleon wave functions has to be calculated. In the present calculation the \( ^4\text{He} \) wave function is taken to be the conventional Gaussian whereas the deuteron parametrization is of the generalized Yukawa type as given in Eq. (32). For the two-nucleon scattering wave functions the simple parametrization with cut-off, Eq. (30) from the same section, has been used to simulate the short range effects.

The results of the calculation for the missing mass distribution of the experiment of Bressani et al., is shown in Fig. C3. The missing mass is the invariant kinetic energy of the unobserved particles. These are a neutron and a proton for reaction (45b) and a deuteron bound state for reaction (45a). The experimental resolution is not sufficient to separate the two channels. The theoretical predictions have therefore been added in Fig. C3 to compare with experiment. The sensitivity to the variation of the cut-off parameter \( Z \) is shown. The absolute normalization is not measured.

The corresponding theoretical separation into the two final states is shown in Fig. C4. The cut-off is fixed at \( Z = 2 \mu \).
For positive missing mass the cross-section is mostly due to the unbound channel. There, the calculated values are somewhat too small, see Fig. C3. This is possibly an indication that the neglected final state interactions are important for the total break up channel, Eq. (45b). However, one has to remember the considerable uncertainties in the (experimental) input of the calculation.

The predictions for angular and recoil momentum spectra agree with experiment quite well (not shown). Particularly, we mention that the events with negative missing mass (that are made up mainly by the deuteron channel) are consistent with spin one exchange considering the Treiman-Young distribution. No further final state interaction seems to be needed in that case.

This type of analysis can easily be extended to reactions like

\[ \pi^+ b^\prime Li \rightarrow 2p \, ^4He \]  

(48)
and a variety of calculations exist [for references see (Kol 69) and (Laz 70)].

However, further absorption channels on $^6$Li like the 2p2d final state which has four bodies are more difficult to handle even in these simple models. Already in "Born" approximation some graphs would involve pion absorption on four-exchanged nucleons which goes beyond the present approach.

To understand simple things first it would therefore be highly desirable to isolate the two-nucleon channel (48) with $^4$He in a definite state of excitation from the remaining more complicated final states by a kinematically complete experiment.

C3. **Comments on related processes**

Pion absorption is closely related to the electromagnetic process of nuclear **photodisintegration**. The simplest example would be the photodisintegration of the deuteron

$$\gamma d \rightarrow pn$$ (49a)
and more generally, for the nucleus $X$, a reaction of the form

$$\gamma X \rightarrow \sum X_i$$  \hspace{1cm} \text{(no pions)} \hspace{1cm} (49b)$$

To make the analogy with pion absorption we just have to replace the pion absorption vertex by the proton vertex everywhere (Fig. C5). The photon being a boson like the pion this vertex has some similarity. To compensate for the zero rest mass of the photon the pionic threshold absorption has to be compared with photodisintegration at $E_{\gamma} \geq 140$ MeV, of course. Furthermore, the spin-isospin algebra is somewhat more involved as the photon has spin one and as the electromagnetic current is moreover a mixture of isospin one and zero. Otherwise, the treatments of Section B3 have their exact corollary. Taking

Fig. C5

the deuteron as an illustration we have to consider the graphs of Fig. C6.

Fig. C6

A $N^*$ resonance model for the upper part in graph b) (which corresponds to photoproduction of pions) proves necessary for energies $E_{\gamma} > 100$ MeV. Photodisintegration turns out to be as sensitive to the deuteron structure and $NN$ interaction as the
pion absorption is. Similarly, the pion rescattering graph is important even at low energies. The study of these photonic processes, similar to the pionic absorption as they are, but also sufficiently different, gives valuable complementary information. For references we refer to (Laz 70b), for example.

Of course, photons occur as well in the final states of pionic absorption (radiative capture). For $^2\text{He}$, e.g., the pnn channel is large as expected (about 60%), the radiative channels $^3\text{H}\rightarrow\pi^-$ and $\Delta\rightarrow\pi^+$ are considerable and take about 10% of the total rate (Zai 65). For heavier nuclei this fraction is somewhat less. We shall not discuss radiative capture here. But we mention that the soft pion ideas introduced in Section B3.2 for the reaction $\text{NN} \leftrightarrow \text{NN}\pi$ have also been applied to radiative capture and its inverse, to $^3\text{H} \leftrightarrow ^3\text{He}$, e.g., for basics see M. Ericson (Eri 70).

Capture from bound state orbits in pionic atoms involves $\gamma$ rays of quite a different nature coming from the transitions between the lowest Bohr orbits. The pion orbits are 280 times smaller than the electron orbits. Still the overlap with the nuclear wave function is small enough that the pion nucleus strong interaction is only a perturbation to the atomic energy levels. The atomic energy level shift is, therefore, in first approximation the product of the unperturbed atomic wave function of the pion (at the origin) with the elastic (strong) pion nucleus scattering amplitude

$$\Delta E \approx \sigma_{\text{Bohr}}^{(E=0)} \cdot \sigma_{\pi X}^{(E=0)}.$$  

(50)

The real part of this equation gives the scattering length of elastic $\pi X$ scattering and is not our concern here. The imaginary part, however, the broadening of the atomic level due to strong interaction, gives directly the total absorption rate. Experimental information is available for many nuclei and gives a valuable check on the many calculations that exist for threshold absorption of pions. A review of the situation in mesic atoms is given in the Herceg Novi lectures by T. Ericson (Eri 69a) where all the basic ideas are explained as well.
II. DISPERSION RELATIONS FOR ELASTIC NUCLEAR SCATTERING

D. FORWARD DISPERSION RELATIONS (FDR)

D1. Method

In Section B3.3 we have already introduced dispersive methods for a description of the pionic disintegration of the deuteron. We shall now apply the method of dispersion relations to the (usually) simpler problem of elastic nuclear scattering and we shall deal with the particularly simple problem of forward scattering first. The technical advantages for this case will be discussed below. From a more formal point of view the forward amplitude is also simpler (see the lectures by O. Steinmann). The analytic properties of the (forward) scattering amplitude \( f(E) \) which are equivalent to the validity of dispersion relations can be proved in field theory for a number of elementary particle scattering amplitudes (NN scattering is not included). Although no formal proofs exist so far for our case we feel relatively safe in postulating nuclear forward dispersion relations. The singularities of the forward amplitude are usually quite easy to establish from a practical point of view whereas for the absorption problem, Section B3.3, or any partial wave dispersion relation the technical complications are considerable.

The status of the applications of FDR to nuclear scattering is reviewed by Ericson and Locher (ErL 70) where some minimal background is collected also. We shall shortly restate the essentials and discuss some newer results on nd scattering together with a few previous applications.

The basic assumption (provable in some cases as mentioned before) is the analyticity of the forward amplitude in the upper half of the complex energy plane. (This property is roughly speaking a consequence of causality.) The Cauchy integral formula can then be written as a relation between the real and the imaginary part of the
amplitude, the dispersion relation*)

$$\text{Re} f(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} f(\omega')}{\omega' - \omega} \, d\omega'$$  \hspace{1cm} (51)

provided \( f(\omega) \) decreases sufficiently at infinity. Note that the amplitude need not be decomposed into partial waves.

The further advantages of the \textit{forward} case are twofold. First, the spectrum of the input function \( \text{Im} f(\omega) \) can be determined directly for \( \omega \geq m \) (where \( m \) is the mass of the incident particle) by measuring the total cross-section. The optical theorem reads

$$\text{Im} f(\omega) = \frac{Q}{4\pi} \sigma(\omega)$$  \hspace{1cm} (52)

[and similarly one gets \( \text{Im} f(\omega) \) for \( \omega \leq -m \) from the total cross-section in the exchange or u channel by the crossing relation. This region is, however, unimportant due to the suppression by large denominators in Eq. (51) there].

The second important point is that we need not know the singularity structure in the t or momentum transfer channel as \( t \) is fixed (\( t = 0 \)). These singularities are particularly complicated in the nuclear case as they reflect the nuclear wave function (see Section B3.3 and Chapter E below). In the \textit{forward} case they enter only through the subtraction constant which normalizes the real part of the amplitude at some energy chosen at will. This is a measured \textit{input} quantity.

The principle of the present \textit{applications} is then as follows:

*) In this Chapter the total \textit{laboratory} energy of the incident particle is denoted by \( \omega \).
1) feeding into the FDR Eq. (51) the (experimental) information on \( \text{Im}f(\omega) \) in the physical region \( \omega > m \) using Eq. (52);

2) one compares the thus obtained truncated real part with experimental real parts;

3) to determine the strength parameters in the close by unphysical region \( \omega < m \).

It is in this third part where some dynamics enter. The particles that can be exchanged in the unphysical region are known and determine the position of poles and cuts. The dynamical quantities obtained by dispersion relations are then the pole residues or some effective residues. A famous example is the determination of the \( \pi^N \) coupling constant (the residue of the nucleon pole).

The "experimental" real parts referred to under point 2) are deducible from Coulomb interference (preferably) or from phase shift analysis (if the energies are low). Unfortunately, Coulomb interference has so far only been measured in one case at medium energies (\( \pi^C \) scattering in the resonance region at the CERN SC). Most of the present applications are therefore limited to energies below some 70 MeV. As soon as the experimental material will improve (total cross-sections as well!) far more applications will become possible.

We shall briefly illustrate the method on three examples \( \text{nd, n}^4\text{He and } \pi^{49}\text{Be elastic scattering.} \)

D2. Pole extrapolations in \( \text{nd and } \pi \text{He scattering} \)

The best illustrations for the application of FDR with incident nucleons are at present \( \text{nd} \) and \( \text{n}^4\text{He} \) scattering. They provide clean evidence for the importance of nuclear exchange (or heavy particle pick-up) in nucleon nucleus scattering.
Forward nd scattering

There are only two poles in low energy nd scattering that are close to the elastic scattering threshold. These are the direct triton pole in the doublet amplitude at -9.4 MeV and the proton exchange pole in both the doublet and quartet amplitude at -1.1 MeV (see the graphs in Fig. D1). Note that the proton exchange pole occurring in the $u$ channel enters the FDR as a simple pole as no partial wave projection is required. As we shall use, unpolarized total cross-sections to determine the imaginary part of the amplitude we have to consider the particular combination

$$
\varphi(E) = \frac{1}{3} \varphi^0(E) + \frac{2}{3} \varphi^+ (E)
$$

(53)

of doublet and quartet amplitudes ($E$ denotes the lab. kinetic energy).

The spectrum of the imaginary part is shown in Fig. D2.

* We neglect recoupling of spin and orbital momentum, which is an acceptable approximation for $E < 20$ MeV.
The mesonic exchange effects are at least 70 MeV below threshold (pπ° exchange) and the energy variation induced in the real part for 0 < E < 20 MeV should, therefore, be unimportant. The bulk of the mesonic effects is automatically included as the amplitude is normalized at threshold by the subtraction constant (i.e., the scattering length). This simultaneously improves convergence of the integral. The PDR for the lab. amplitude has the form

\[
\text{Re } f(E) = f(0) + \frac{r_t E}{(E_t - E)E_t} + \frac{r_p E}{(E_p - E)E_p} + \\
+ \frac{2}{\pi} \int_0^{\infty} \frac{G_{\text{na}}(k')}{k'^2 - k^2} \, dk'
\]

(54)

where k is the lab. three-momentum of the neutron. This relation is approximate insofar as the cut to the left of -70 MeV is neglected [except for its contribution to f(0)].

Following the procedure outlined in the previous section we first evaluate Eq. (54) for the physical region only, i.e., we omit the two-pole terms. The resulting curve (see Fig. D3) is compared with "experimental" real parts (diamonds and circles in Fig. D3) that are obtained from phase shifts *) forming the linear combination (53). The difference between the two curves, the discrepancy function Δ(E), should, therefore, be equal to **)

\[
\Delta(E) = \frac{2}{3} \frac{r_t E}{(E_t - E)E_t} + \frac{2}{3} \frac{r_p E}{(E_p - E)E_p}
\]

(55)

the sum of the poles in the unphysical region. Note that due to the

*) As the phase shifts describe the Coulomb interference region in pd scattering rather well the forward amplitude, obtained in summing partial waves should be independent of ambiguities in the individual phases.

**) The c.m. amplitude in Fig. D3 is \( f_{\text{cm}}(E) = \frac{k}{k_{\text{cm}}} f(E) \approx \frac{2}{3} f(E) \), leading to the factor 2/3 in Eq. (55).
Fig. D3 from (Loc 70)

subtraction [i.e., $r_t/(E_t-E) \rightarrow r_t E/(E_t-E)E_t$ etc.] the pole contributions in $\Delta$ are zero at threshold and reach a maximum at a distance of the order $E \sim |E_{pole}|$.

Having thus isolated the contribution from the unphysical region, the explanation as a sum of two poles is tested in Fig. D4. There are only two free parameters in (55), the pole residues $r_p$ and $r_t$ or in other words, the tdn and the dpp coupling constants. These are moreover both positive in the present notation which reflects the known parity of the exchanged particles [see (ErL 70), e.g.]. In order to determine $r_p$ and $r_t$ we subtract from $\Delta$ the proton pole

$$\Delta_t = \Delta - \frac{2}{3} \frac{r_p E}{(E_p-E)E_p}$$

(55')

Evidently $E/\Delta_t$ should be linear in $E$ if it represents the triton pole. The upper part of Fig. D4 confirms this. The proton residue has been varied to obtain an extrapolation through the correct
Tests for the pole behavior of the unphysical region $\Delta$. Use left-hand scale for the effective single pole plot $E/\Delta$. A similar plot for the triton contribution $\Delta_t$, eq. (55'), is shown, with the proton residue $r_p$ as parameter. Errors shown for $E/\Delta_t$ are calculated assuming a typical error of $\pm 0.28$ fm in $\Delta$. Such error is smaller than symbol size in the effective-pole plot $E/\Delta$.

Fig. D4 from (Loc 70)

triton pole position at $E = E_t = -9.4$ MeV. The slope determines then the triton residue $r_t$ as well. The best values are

$$r_p = 0.081 \pm 0.002$$  \hspace{1cm} (56)

$$r_t = 0.382 \pm 0.040$$  \hspace{1cm} (57)

(These are dimensionless coupling constants like $r_{\alpha}^2_{NN}$.)

There is a strong consistency check on the value for the proton residue from another process, namely triplet $np$ scattering, see Fig. D5, which has the deuteron pole just below the $np$ elastic scattering threshold.

Fig. D5
The connection to the triplet effective range $R_{\text{eff}}$ is

$$
\rho_p = \frac{\alpha}{m(1 - R_{\text{eff}}^2)} = 0.0818 \pm 0.0005 \quad \text{for} \quad R_{\text{eff}} = 1.747 \pm 0.004 \text{fm}
$$

(58)

where $\alpha^2 = mB_d$, $m$ is the nucleon mass and $B_d$ the deuteron binding energy. This agrees with our value (56) for $\rho_p$ and strengthens the confidence in the triton residue $r_t$.

There is a unique relationship between the pole residue and the normalization of the asymptotic wave function for large neutron separations [for the deuteron this is exploited in (58)]. Choosing for the purpose of illustration a Hulthén wave function for the triton *) this relation can be used to express $r_t$ in terms of the Hulthén short range parameter $\beta_t$. One obtains from the value (57)

$$
\beta_t = 2.63 \quad \alpha_t = 1.18 \pm 0.12 \text{fm}^{-1}
$$

(59)

The PDR give probably the most direct determination of $r_t$ at present.

The lower curve in Fig. D4 shows an effective one-pole plot for the discrepancy function (55). The effective pole position turns out to lie between $E_t$ and $E_p$ as it must because of the mentioned positivity of both $r_t$ and $r_p$. Evidently, looking at such a limited energy interval $0 < E < 15$ MeV the two-pole behaviour cannot be distinguished from an effective one-pole approximation. For further details see (Loc 70).

*)

$$
\gamma_t = N Y_{tm} \left[ \exp(-\alpha_t x) - \exp(-\beta_t x) \right] / x
$$

where

$$
N^2 = \frac{2 \beta \alpha (\alpha + \beta)}{(\beta - \alpha)^2}.
$$

As $\alpha^2 = 2 \mu_t B_t$ with $\mu_t$ the reduced mass, the only free parameter is $\beta_t$. 
\[ n^4He \text{ scattering} \]

The same method has also been applied to \( n^4He \) scattering (ErL 70). There are no spin complications and there is no bound state in the direct channel (which has baryon number five). However, the exchange channel is somewhat more complicated (Fig. D6).

![Diagram showing exchange processes](image)

Fig. D6

The spectrum for the imaginary part consists of a \( ^3He \) exchange pole and of cuts from pd and ppn exchange which start at a similar distance as is shown in Fig. D7.

![Diagram showing structure of singularities](image)

Structure of singularities in the kinetic energy \( E \) for \( n^4He \) scattering.

Fig. D7 from (ErL 70)

The same procedure as in the nd case (Fig. D8) leads to a discrepancy function \( \Delta \) which is displayed in Fig. D9. An effective pole at about -10 MeV is a very good approximation for the unphysical region for energies from threshold to at least 40 MeV. If \( \Delta \) is fitted to a two-pole expression (not shown) the effective pole for the cut is about equal and opposite in strength to the \( ^3He \) exchange pole. In summary the \( n^4He \) scattering provides a clean separation of nucleonic exchange from direct processes. The nucleonic exchange contribution \( \Delta \) is shown to be very large (see Fig. D9) for low energy scattering.
D3. FDR for incident pions

Applications of FDR for isospin zero nuclear targets exist for the deuteron, $^4$He and $^{12}$C. They are reviewed in (ErL 70). The newest predictions for $\pi^{12}$C scattering are shown in C. Wilkins lectures. All indications are that no significant contributions arise
from the unphysical region for those isospin zero targets. The nuclear amplitude corresponds to the charge symmetric combination \( f^{(+)} \) of pion nucleon amplitudes where the nucleon pole gives a negligible contribution also.] This statement has explicitly been checked for \( \pi^{12}\text{C} \) by comparing FDR with real parts from Coulomb interference at and above the resonance (to be published by the BSS group at CERN, see also the lectures of O. Wilkin). The smallness of the symmetric amplitude is linked to soft pion theorems. (See the lectures by M. Ericson where methods for obtaining real pion mass corrections in the nuclear case are discussed also.)

For isospin \( \frac{1}{2} \) nuclei we have, in addition to the symmetric amplitude the antisymmetric (or charge exchange) amplitude *)

\[
f^{(-)} = \frac{1}{2} \left[ f^{+X} - f^{-X} \right].
\]  

This is a much better detector of the unphysical region than \( f^{(+)} \). Clearly, to evaluate the dispersion integral for \( f^{(-)} \) one needs pion nucleus total cross-sections for both pion charges.

The only case where marginally sufficient data exist is \( ^9\text{Be} \). Even there only the \( \pi^{^9\text{Be}} \) total cross-sections are measured in the resonance region and the \( \pi^{^9\text{Be}} \) cross-sections had to be obtained by scaling the Carbon cross-sections. The FDR for \( f^{(-)}(\omega) \) was then evaluated at threshold \( \omega = \mu \) (this is called a sum rule) to obtain the effective coupling constant for a pion coupled to the \( ^9\text{Be} \) nucleus **)

\[
\Gamma_{eff} = \frac{\mu^2}{2} \text{Re} f^{(-)}(\mu) - \frac{\mu^2}{\pi} \int_{\omega_{min}}^{\omega_{max}} \frac{1}{R_i^2} f^{(-)}(\omega) d\omega,
\]  

(presumably \( f^{(-)} \) needs no subtraction). Using some additional information from \( \pi \)-mesic atoms on \( \text{Re} f^{(-)}(\mu) \) and the absorption cut the value deduced in (ErL 70) is

\[
\Gamma_{eff} = 0.06 \pm 0.03.
\]

*) The definition assumes \( X \) having neutron excess.

**) We denote the total lab. pion energy by \( \omega \) in this section.
rather close to the \( \pi NN \) coupling constant \( f^2 = 0.08 \), comp. the lectures of J. Hamilton. [For the connection of Eq. (61) with nuclear soft pion theorems see the lectures by M. Ericson.] Actually, such a value results in impulse approximation using closure over nuclear states (which are all near \( \omega = 0 \) as the nuclear excitation energies can be neglected compared to the pion rest mass), see (ErL 70).

It should be evident from the previous remarks that the experimental input information is very incomplete yet. Most needed are total cross-sections on isospin \( \frac{1}{2} \) nuclei for pions of both charges covering the resonance region. Apart from the completion of the \( ^9 \text{Be} \) data (\( ^9 \text{Be} \) has the disadvantage of being \( \frac{3}{2}^- \)) interesting candidates would be the mirror nuclei \( ^3 \text{He} \) and \( t \).

E. PARTIAL WAVE DISPERSION RELATIONS

Whereas FDR have been shown to be technically simple dispersion relations in two variables for the full scattering problem which includes angular variation are much more involved (quite apart from formal questions). This is true even in the case of (strongly bound) elementary particles. In the nuclear case there is the additional difficulty of dealing with nuclear structure which produces singularities in the momentum transfer channel that are very close to the scattering channel physical region. (We have met a similar situation already for pionic deuteron disintegration in Section E3.3.) Therefore, applications exist so far only for \( \pi d \) and \( N \bar{d} \) elastic scattering. The group of Rinat-Reiner, however, is going to tackle more complicated targets.

E1. \( \pi d \) elastic scattering

We briefly describe the calculation by Schiff and Tran Than Van (ScT 67). A dispersion relation in each partial wave is set up and solved by methods which are closely analogous to the previously discussed pionic deuteron disintegration which has
been calculated by the same authors (Section B3.3). We shall stress here mainly the new features which occur in the elastic case.

The first step is to find the simplest Feynman diagrams that are likely to dominate the scattering and to establish their singularities. These are the impulse diagrams (see Fig. E1).

![Impulse Diagrams](image)

Fig. E1

Only the two-body intermediate states consisting of NN and NN* will be retained.

Their \( s \) channel singularities are:

- the elastic scattering threshold at \( s = (M + \mu)^2 \) (from there on the amplitude is calculable from physical processes);

- a normal threshold at \( 4m^2 = 180.96 \mu^2 \) (and an anomalous one slightly below at \( s_0 = 180.84 \mu^2 \)) see Fig. E3 below.

These singularities will appear in each partial wave.

The \( t \) channel singularities before decomposition into partial waves are the following:

- from the triangular graph c) in Fig. E2 which is a reduction of the impulse graphs a) or b) one has an anomalous cut in \( t \) starting at [see (ErL 70) for relevant formulae, e.g.]

\[
\begin{align*}
  t &= 16 \alpha^2 = 1.73 \mu^2, \\
  \alpha^2 &= m_{\text{B}_d}^2 
\end{align*}
\]

(63)

with \( B_d \) denoting the deuteron binding energy. This is extremely close to the physical region \( t < 0 \) and is due to twice the occur-
rence of the weakly bound dpn vertex in the triangular graph c). [Comp. Eq. (39) for the break-up process where the deuteron vertex occurs only once.]

Fig. E2

After partial wave projection this leads to a cut in the energy plane between $s_1 = 156.6 \mu^2$ and $s_2 = 201.6 \mu^2$, comp. Fig. E3 where all the singularities for the partial waves are collected.

πd ELASTIC SCATTERING

Fig. E3 from (ScT 67)

The more complicated binding and multiple scattering graphs, see Fig. E4, have anomalous cuts in $t$ also. However, they start further out at $t = 11 \mu^2$ and $t = 28 \mu^2$ respectively.

The only further contribution which is taken into account in the $t$ channel are the two pion intermediate states. They lead to a normal threshold at $t = 4 \mu^2$. In the present calculation they are approximated by an effective $\sigma$ meson, graph f), Fig. E5. The
two-pion states give the circle \(|s| = M^2 - \mu^2\) in the partial wave energy plane, Fig. E3.

\[ \Gamma \text{ meson} \]

Fig. E5

The \textit{u channel} singularities will be neglected altogether as they are far from the physical region.

With these simplifications and an analytic structure of the partial waves as shown in Fig. E3 one writes again the Cauchy formula for the partial wave \( f_\ell \) (different \( \ell \)'s are decoupled as the \( \pi d \) intermediate states are neglected)

\[
\frac{f_\ell}{L} = \frac{1}{2\pi i} \int \frac{ds' \phi_\ell(s')}{s' - s - i\epsilon} ds' + \frac{1}{\pi} \int \frac{|\mu| \phi_\ell(s')}{s' - s - i\epsilon} ds'. \quad s' = (M^2 - \mu^2)
\]

All the singularities to the left of \( s \) are denoted by \( L \). Equation (64) is immediately modified by enforcing correct threshold behaviour \( k^{2\ell} \). We shall not pursue the formal details further.
The remainder of the calculation is very similar to the treatment of pionic deuteron disintegration, particularly as regards the dynamical approximations in the input:

- the \( \pi d \rightarrow NN \) and \( \pi d \rightarrow NN^* \) amplitudes which go into the calculation of \( \text{Im} f^\ell \) are obtained from one-nucleon exchange (the \( N^* \) having a Breit-Wigner distribution);

- the dpn vertex function consists of a few Yukawa type poles as before, comp. Eq. (32);

- and newly, a coupling constant for the effective \( \rho \) pole with \( m_{\rho} \approx 400 \text{ MeV} \), is obtained from fitting the total cross-section at low energies.

The numerical results are discussed next.

In practice, Eq. (64) is evaluated with a cut-off in the integral at about 450 MeV where the imaginary part becomes unreliable. To include some of the high energy contribution the real parts are enforced to vanish with increasing energy.

As a typical result we show the differential cross-section at 85 MeV pion lab. energy (Fig. B6). It is found that the impulse approximation at this energy (solid line) is not modified much by the correction terms (binding, multiple scattering and \( \rho \) term), see the dashed line. The dash dotted line is for an extra left-hand phenomenological pole which is introduced to improve the high energy asymptotic behaviour.

The sensitivity to different deuteron parametrizations with fixed D wave admixture is not large whereas the D wave contribution itself is not negligible. It is 5, 10 and 20% of the total cross-section at 90, 180 and 360 MeV, respectively.

The \( \rho \) term is apparently not important at 85 MeV. However, at higher energies it improves agreement with experiment at backward angles (not shown). The angular distribution is reproduced very well at 142 MeV which is to be expected as this energy is close to the resonance.
In conclusion this relatively crude evaluation of partial wave dispersion relations proves to be quite successful. Improvements are certainly possible in many respects for the price of a considerable complication in the technicalities.

E2. \( np \) elastic scattering

In this last section we comment briefly on the application of partial wave dispersion relations to \( np \) scattering (PhB 68, Rei 69 and Ebe 69). The problem is algebraically very involved due to the spin one of the deuteron. Even in forward direction there are as many as six invariant amplitudes (the full problem has 12).

We restrict our remarks to \( S \) waves. In this case, there are only doublet and quartet amplitudes. The partial wave
singularities are those corresponding to the triton and proton pole of Section D2. In addition, there are (anomalous) t channel singularities. The closest one coming from two-nucleon exchange occurs again at $t_a = 1.73 \mu^2$.

Partial wave projection transforms the proton exchange pole into a cut which is close to the physical region (from -1 to -9 MeV kinetic energy). The crudest evaluation (PhB 68) consists now in retaining only the triton pole (one parameter for the residue) and an effective pole for the proton exchange cut (two parameters for the pole position and the residue). These three parameters are fixed by the requirement to reproduce the doublet and quartet scattering length and the quartet effective range. With the above input a dispersion relation for the doublet S wave amplitude is evaluated and gives reasonable agreement when compared with phase shift information.

The above treatment disregards deuteron break-up in the physical region. A more complete treatment which incorporates these three-body unitarity contributions approximately and also includes the two-nucleon exchange contributions to the left-hand cut calculates the lowest partial waves $J = 0, 1, 2$ by dispersion relations. The energy range extends from threshold to about 25 MeV for nd scattering (Avi 69) and up to 95 MeV for pd scattering (Ebe 69). The reader is referred to those references for details.
REFERENCES

Laz 70b C. Lazard, thèse, Orsay 10/4/70, série A N°611.
Roz 54  A.H. Rosenfeld, Phys.Rev. 96, 130, 139 (1954).
Vai 64  A.O. Vaisenberg et al., Zh.Exp.Teor.Fiz. (USSR) 47, 1262 (1964); [Sov.Phys.JETP (USA) 20, 854 (1965)].
You 69  J.E. Young, MIT prepr. CTP95 (1969).
Zai 65  O.A. Zaimidoroga et al., Sov.Phys. JETP 21, 848 (1965) and 22, 1111 (1967).
BASIC NOTIONS OF RELATIVISTIC SCATTERING THEORY

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INTRODUCTION

In these lectures I intend to place the phenomenology of elementary particle reactions into a general theoretical context - quantum field theory - offering some hope of understanding what is going on. I shall show how the notion of the S-matrix and its often quoted properties are related to the general background ideas. In particular I have in mind here the most mysterious of these properties: the analyticity properties leading to crossing symmetry, dispersion relations, etc. I will not talk about dynamics, that is about specific models or interactions. I shall only talk about properties that are true in any model, provided, of course, it fits into my context. I am also not going to talk about applications of the general results, of dispersion relations for instance. About this you will hear from the other lecturers. What I wish to do is showing what general properties can be derived from what assumptions, and what motivates these assumptions. This shall help you to decide to what extent the said properties can be trusted, how seriously they should be taken.

Before starting on my business I should give you a note of warning. Since I am interested in the general ideas underlying the relations of field theory rather than in their applications, I shall allow myself to take a somewhat cavalier attitude towards normalisation constants and other such things without fundamental significance. $2\pi$ - factors and the like will be consistently ignored. Therefore my equations should not be taken too literally.

The first question presenting itself in my enterprise is: why choose field theory as framework in elementary particle physics? Well, mainly because at the moment we have nothing
better. Among the presently existing approaches field theory is in my opinion the only one that has any chance of ever becoming a full theory. This judgement is, of course, highly subjective, and you have the right not to be convinced by it. Fortunately there are also some more positive indications. For one thing, the spectacular successes of quantum electrodynamics show that field theory has probably something to do with reality, that it cannot be entirely wrong. Numerical successes of a less spectacular nature exist also in other areas (weak interactions, dispersion relations in strong interactions, etc.) But there are also some qualitative arguments. An elementary particle reaction looks like this: a high energy particle in an accelerator beam hits another particle sitting in a target, and out come any number of particles, not necessarily of the same species as the original two. Now, where the physics really happens is in the interaction region in the target, at the moment the original particles have met and the final particles have not yet separated. If we want to get hold of the dynamics of the process we must get down to looking at this interaction region, at least theoretically. (It would be very helpful if we could do this also experimentally. Of course, I have not the faintest idea of how to do such a thing.) But in this vital region particles cease to be the simple objects we are used to seeing in them. They loose their stability and even their identity completely: witness the fact that the emerging particles may be different in type and in number from the original ones. Therefore the notion of particle is not appropriate to the description of what goes on in the interaction region. What goes on there has obviously many more degrees of freedom than are accounted for by particles, things look more like a jelly than like an assembly of billiard balls. And for the description of such a structure field theory is the only means we have at hand.

After we have decided that it is sensible to have a look at field theory, let us proceed to a discussion of its basic ideas.
THE POSTULATES OF RELATIVISTIC QUANTUM FIELD THEORY

1st postulate: quantum mechanics. We want the theory to fit into the general framework of quantum mechanics. This means that states of the system under consideration are represented by vectors, or more generally by density matrices, in a Hilbert space $\mathcal{H}$, observables and other physical quantities by operators in $\mathcal{H}$. Presumably, nobody will quarrel with this assumption, so that I need not lose any words on its motivation. We shall work throughout in the Heisenberg picture. This is the most appropriate in the relativistic case since it treats space and time on an equal footing.

2nd postulate: field theory. We assume the fundamental quantities of the theory are fields $A^1(x), \ldots, A^\ell(x)$, where $\ell$ is usually (though not always) assumed to be finite. A field is an operator valued function $A'(x)$ defined on 4-dimensional space-time, i.e., $x$ is a point in Minkowski space.

The word 'function' has here to be taken with a grain of salt. From the properties that we shall postulate for our fields later on, it can be deduced that the value $A(x)$ in a point $x$ is not defined. What is defined is only the integral $\int dx \, A(x) \, \varphi(x)$ of $A$ over a sufficiently smooth test functions $\varphi : A(x)$ is what is called a 'distribution'. (The best known example of a distribution is the Dirac function $\delta(x)$.) However, I shall ignore this complication and treat the $A$ as functions in the usual sense. I shall be careful to make only statements that can be translated into distribution language and remain true under this translation. This is not the case for just any statement. For example, distributions can not be squared, hence field equations of the form $(\Box + m^2) A(x) = g \left[ A(x) \right]^2$, or similar, do not make sense a priori. That is where the
divergence difficulties of canonical field theory come from. I shall avoid these difficulties by never talking about non-linear field equations or similar suspect notions.

That the fields are the fundamental quantities means the following. We assume that all physical quantities, in particular the observables, are functions of the basic fields. Moreover, observables corresponding to experiment carried out in a bounded region $\mathcal{B}$ of space-time are functions of the $A^i(x)$ with $x \in \mathcal{B}$. Here the association of experiments with space-time regions refers simply to the fact that experiments take place in a finite region (in a laboratory, say) and last for a finite interval of time. The second part of the assumption states that the $x$ occurring in the argument of $A^i$ has something to do with the localization of events in space-time. This assumption is sometimes referred to as the assumption of locality. Unfortunately the term 'locality' is used in field theory with many different meanings. I shall reserve it for a different property to be stated later.

The motivation for this field theory assumption has already been shortly discussed in the introduction, and I will not say more about it.

3rd postulate: relativistic invariance. We assume the theory to be invariant under translations by a 4-vector $a$ and under Lorentz transformations $\Lambda$. This means that we have in $\mathcal{H}$ a unitary representation $U(\Lambda, a)$ of the inhomogeneous Lorentz group which transforms observables in the obvious way: if the operator $\mathcal{O}$ represents an observable, then $U(\Lambda, a) \mathcal{O} U^*(\Lambda, a)$ represents the transformed observable (the same observable measured in the new reference frame). This far the assumption has nothing to do with field theory. It is perfectly general and acceptable to everybody. The connection with field theory comes through the postulated transformation behaviour of the fields.
Consider first the translations. The restriction \( T(a) = U(1, e) \) of \( U \) to the translation group gives a unitary representation of this group. We demand

\[
A^i(x+\mathfrak{a}) = T(\mathfrak{a}) A^i(x) T^*(\mathfrak{a}) \quad \text{.} 
\]  

(1)

This property expresses the homogeneity of space-time and is hardly to be questioned, at least as long as one is allowed to forget about general relativity.

The behaviour under homogeneous Lorentz transformations is somewhat more complicated. We assume that \( A^i \) transforms under a finite-dimensional representation of the Lorentz group. That means that the field \( A^i \) has a finite number of components \( A^i_\mathfrak{q}, \quad \mathfrak{q} = 1, \ldots, n \), and that

\[
A^i_\mathfrak{q}(\Lambda x) = \sum_\mathfrak{q} L_{\mathfrak{q}\mathfrak{q}'}(\Lambda) U(\Lambda, 0) A^i_{\mathfrak{q}'}(x) U^*(\Lambda, 0) \quad \text{.} 
\]  

(2)

where \( L_{\mathfrak{q}\mathfrak{q}'} \) are the matrix elements of a finite dimensional representation of the Lorentz group. The parity operation may or may not be included in this scheme, depending on whether parity is conserved or not. Time-reversal is a bit more complicated. It is represented by an anti-unitary operator rather than by a unitary one and will not be discussed here.

A field satisfying (2) is said to be covariant. The simplest example is a scalar field, for which \( L_{\mathfrak{q}\mathfrak{q}'} \) is the trivial identity representation, i.e. \( n=1 \), so that the lower index can be dropped, and

\[
A^i(\Lambda x) = U(\Lambda, 0) A^i(x) U^*(\Lambda, 0) \quad \text{.} 
\]  

(3)

The next complicated example are the Dirac 4-component spinors if parity is admitted, or the Weyl 2-component spinors if parity is disregarded. The \( L_{\mathfrak{q}\mathfrak{q}'} \) are in this case the
2-valued representations known from the theory of the Dirac equation.

What can be said about the motivation of property (2)? Why do we, for instance, not assume that all fields are scalars? Because we know that covariant non-invariant quantities exist in nature. More specifically, we shall see later on how fields can be associated with particles, and in this association the transformation character of the field is directly connected with the spin of the particle. Scalar fields describe spinless particles, 4-spinors describe particles with spin 1/2 and so on. Since spinning particles exist we must have fields transforming under non-trivial representations. That we restrict ourselves to finite-dimensional representations is dictated by a desire for simplicity. The occurrence of infinite representations cannot be excluded a priori, but as long as the finite ones seem capable of doing the job it is reasonable to work with them exclusively. At the moment we have no indications that the infinite representations are necessary for describing reality.

For most of what I am going to say the transformation property of $\Lambda^\dagger$ under $\Lambda$ is irrelevant. I shall therefore ignore Eq. (2) to a large extent and shall correspondingly drop the lower index $\varphi$ in most of my proceedings. It will be reintroduced in the few places where it is really needed. The translation condition (1) is, however, absolutely vital!

$4^{th}$ postulate: spectral conditions. According to a generalization of Stone's theorem, the so-called SNAG theorem, we can write the translation operator $T(a)$ in the form

$$T(a) = \exp \, ia \, P$$

where the $P_{\mu}$ are self-adjoint operators. The index $\mu$ runs
from 0 to 3. The $P_\mu$ are the energy-momentum operators, as can be seen from the infinitesimal form of (1), i.e. by introducing (4) into (1) and letting $a$ tend to zero:

$$\frac{\partial A(x)}{\partial x^\mu} = i [P_\mu, A(x)] .$$

This is simply Heisenberg's equation of motion.

We demand that there is a unique eigenstate $\mathcal{\Omega}$, called the vacuum, of the energy operator $P_0$ of minimal energy. From invariance considerations it follows then immediately that $\mathcal{\Omega}$ must belong to the eigenvalues $p = 0$, i.e.

$$P_\mu \mathcal{\Omega} = 0$$

(5)

for all $\mu$. All other simultaneous eigenvalues of $P_\mu$ must lie in the forward cone $V$, which is defined by the requirements $p_0 > 0$, $p^2 = p_0^2 - \sum p_i^2 > 0$.

(The $P_\mu$ commute with each other and can therefore be diagonalized simultaneously.)

The existence of a vacuum (an empty world) is clearly a reasonable assumption within the context of special relativity. That it be the state of lowest energy in all frames of reference is motivated by our desire to have it stable. If states of negative energy existed, $\mathcal{\Omega}$ could split into such a state and one of positive energy, and this possibility would make the world quite unpredictable. The uniqueness of the state $\mathcal{\Omega}$ is not universally accepted. But again we stick to the simplest possibility as long as its inadequacy has not been demonstrated.

In order to establish a field-particle connection we must, unfortunately, place more stringent conditions on the energy-momentum spectrum, namely we assume that the spectrum is that
of a theory with massive particles only. This is the new assumption: the $P^\mu$ -spectrum consists of the isolated vacuum eigenvalue $p = 0$, of one or several isolated one-particle hyperboloids, and a continuum corresponding to two- or more-particle states which is separated from the 1-particle states (see Fig. 1).

![Diagram of one-particle states and vacuum](image)

**Fig. 1**

Somewhat less strictly: it is allowed that a 1-particle hyperboloid lies inside a continuum, if the respective states are separated by a superselection rule. For instance, in the $\pi$ - N case the one-nucleon states lie inside the $2\pi$ -continuum, but this does not matter because of the different fermion number and the different baryonic charge.

This assumption of non-vanishing masses only is made for what is usually termed 'technical reasons'. As yet it has not been possible to formulate a rigorous scattering theory for massless particles. The problem is the problem of the infrared divergences well known from electrodynamics. Whether it is really only a technical problem caused by
inadequate methods of proof, or whether it has a deeper significance, is not known at present. I personally think that the introduction of new assumptions will be necessary. A closer look reveals that massless particles like the photon are a rather dubious crowd and should be treated with circumspection. Of course, we know how to handle the problem in perturbation theory, but perturbation theory has a lot of regularities that we cannot expect to be shared by an exact theory, certainly not by just any exact theory. These problems being unsolved we shall ignore the massless case. Unfortunately this excludes quantum electrodynamics from our consideration, that is the case in which field theory has had its greatest triumphs. The weak interactions are also excluded (neutrino!) but as long as we stick to the strong interactions, i.e. to hadron physics, we are safe. It is hoped that these difficulties are temporary.

With the help of the vacuum assumption introduced above we can now give a new formulation of the requirement that our fields describe the theory completely. In postulate 2 we formulated this requirement by demanding that the observables are functions of the fields. In this form the assumption is difficult to apply in praxi. We introduce therefore a form which lies the stress on the states rather than the observables. We interpret $A^i(x)$ as an operator creating a field excitation in the point $x$. That all physical states are characterized by the state of the fields can then be achieved by demanding that the states

$$ A^i(x_1) \ldots A^i(x_n) \Omega $$

$n$ running from 0 up to arbitrarily high values, are total in $\mathcal{H}$. (A set of vectors in $\mathcal{H}$ is called total if every vector in $\mathcal{H}$ is the limit of a sequence of finite linear superpositions of vectors in the set.) This assumption is called the completeness assumption. Since it is really only a new formulation of a part of postulate 2 I shall not give it a number of its own.
5th postulate: local commutativity. We assume that any two fields \( A_i^1(x) \), \( A_j^1(y) \) commute or anticommute in spacelike separated points:

\[
[A_i^1(x), A_j^1(y)]_+ = 0 \quad \text{if } (x-y)^2 < 0 \quad (6)
\]

\([......]_+\) is the anticommutator, \([......]_-\) the commutator.

Assuming that this alternative holds it can be shown that the + applies if both fields are fermi fields, the - if one or both of the fields are bose fields. A field \( A_i^1 \) is called a bose field if the representation \( \Lambda \) of the Lorentz group occurring in the transformation equation (2) is one-valued, a fermi field if \( \Lambda \) is 2-valued. Bose fields are connected with particles of integer spin, fermi fields with particles of half-integer spin.

The assumption of local commutativity, or locality for short (this is the meaning in which I shall use the word 'locality'), originated in the canonical formalism for field theory. This formalism is obtained as a formal generalisation of the rules of ordinary quantum mechanics to systems with an infinite number of degrees of freedom. The values of the field \( A^i(t,\vec{x}) \) and their time derivatives at a fixed time in all points \( \vec{x} \) are taken as independent dynamical variables (the q's and p's of quantum mechanics) and are assumed to satisfy the corresponding canonical commutation relations. In particular, \( A_i^1(t,\vec{x}) \) and \( A_j^1(t,\vec{y}) \) being independent variables if \( \vec{x} \neq \vec{y} \), their commutator (or anticommutator) vanishes, whence (6) follows by invariance. That this procedure is not above doubt you see already from the necessity of introducing anticommutators in some cases, a thing that never occurs in ordinary quantum mechanics. More serious difficulties exist: the notorious divergence problems. Therefore 'axiomatic'
field theory, the point of view I am taking here, has given up most of the canonical framework. The locality condition (6) is, however, retained because of its eminent usefulness and because it seems to be innocent of the divergence problems. Some sort of motivation of a non-historical character for locality can be found in its connection with relativistic causality. Relativistic causality demands that observations carried out in space-like separated regions should not influence each other. More explicitly this means the following. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two bounded regions in space-time which are relatively space-like, i.e. which are situated such that no signal travelling not faster than light emitted in any point of one of the regions can reach any point of the other region. Let two measurements represented by the operators $\sigma_1$, $\sigma_2$ be carried out in the two regions. Because of the impossibility of the exchange of signals between the two regions the two measurements cannot disturb each other, which means in quantum language that their operators commute:

$$[\sigma_1, \sigma_2] = 0.$$  

The field themselves are in general not observables (fermi fields, for instance, cannot be observables), so that locality does not directly follow from this causality consideration. Locality is, however, the easiest way of satisfying causality: if we assume $\sigma_1$ and $\sigma_2$ to be functions of the fields in $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, containing only even powers of fermi fields, then they commute as a consequence of (6). But the reverse is not true. More complicated possibilities of satisfying causality may exist. One such possibility has been around for some years under the name 'parastatistics'. These parafield theories seem, however, not to be really more general than the local ones, but are only convenient ways of reformulating certain ordinary field theories satisfying locality. But this does not exclude the possibility that genuinely more general theories satisfying causality exist.
The causality problem is anyway an intricate and confused one, and prudent policy requires to say as little about it as possible. Therefore let me just say that the locality assumption (6) is a very convenient working hypothesis, vaguely justifiable by causality arguments, which will be retained as long as it is useful, but can be dropped if it should turn out to lead to inconsistencies or other trouble. We must have some sort of causality assumption, however, and locality is the most clearly defined and practical one yet proposed.

This completes the list of postulates, as far as they pertain to pure field theory. (An additional postulate dealing with the particle aspect will be introduced in the next section.) Now, what has all this to do with the scattering of elementary particles? How can field theory, the theory of a continuum, describe discrete objects like particles? That it can is clearly a quantum effect. It defies our intuition, which is not yet adapted to the requirements of the quantum age. Since intuition is deserting us we must have recourse to mathematics. You may find this deplorable, but it cannot be helped. Most of the results which I am going to explain in the remaining sections are obtained as the result of mathematical reasoning and cannot be derived by what is called 'physical arguments'. There are people who nevertheless give you such arguments. I must confess that their intuition is superior to mine. Of course, I have neither the time nor the desire to develop the mathematics of field theory in any detail. I shall mainly state results and give just a few indications about the general ideas underlying their derivation.

The connection field-particles will be established in the next section with the help of the LSZ-formalism, named after its creators H. Lehmann, K. Symanzik and W. Zimmermann.
THE LSZ FORMALISM

We start from the fact that the relation field - particle is well understood in the case of free fields. I do not have the time to develop the theory of free fields here, but must assume it to be known. Let me just remind you of a few basic facts. A free field to mass \( m \) is in \( p \)-space of the form

\[
\tilde{A}(p) = \theta(p^0_m) \delta(p^2 - m^2) a(p) + \theta(-p^0_m) \delta(p^2 - m^2) \tilde{a}^*(-p).
\]  

(7)

Here \( \theta \) is the step function: \( \theta(t) = 0 \) for \( t < 0 \), \( \theta(t) = 1 \) for \( t > 0 \). The operators \( a \) and \( \tilde{a} \) and their adjoints \( a^* \), \( \tilde{a}^* \) satisfy certain commutation relations, for instance in the case of scalar fields (spin 0):

\[
[a(p), a^*(q)] = [\tilde{a}(p), \tilde{a}^*(q)] = 2 \omega(p) \delta^3(p - q),
\]

all other commutators = 0,

(8)

with \( \omega(p) = \sqrt{p^2 + m^2} \). The operators \( a^*(p) \), \( \tilde{a}^*(p) \) can be interpreted as creation operators of a particle or an antiparticle respectively with momentum \( p \). \( a(p) \) and \( \tilde{a}(p) \) as the corresponding destruction operators, i.e. the state

\[
a^*(p_1) \ldots a^*(p_n) \tilde{a}^*(q_1) \ldots \tilde{a}^*(q_m) \mathcal{N}
\]

(9)

can be interpreted as a state with \( n \) free particles with momenta \( p_1, \ldots, p_n \), and \( m \) free antiparticles with momenta \( q_1, \ldots, q_m \). Application of \( a(p) \) to this state removes a particle with momentum \( p \) if one is present. Otherwise the state is annihilated. This interpretation is possible because an operator \( N \), called 'number operator' exists, for which the state (9) is an eigenstate with eigenvalue \( n + m \). If we form states of the form (9) with interacting fields we
find that the states with different $n, m$ are no longer orthogonal to each other, hence cannot be eigenstates of a number operator: a particle interpretation is only possible for free fields. Even in this case we see that particles are defined via their p-space properties rather than their behaviour in x-space: they are things with a more or less sharp momentum, and especially with a strictly sharp mass $m^2 = p^2$. (Note that (9) is not a normalisable vector, hence not properly admissible as a state. In order to be rigorous we ought to work with wave packets, which are not eigenstates of the momentum operators. However, a normalisable 1-particle state of the form \[ \int d^3 \psi f(\vec{p}) \ a^*(\vec{p}) \ \mathcal{N}, \] f a wave function, is still an eigenstate of the mass operator $M^2 = p_0^2 - \vec{p}^2$.) That this definition has anything to do with the phenomenological notion of a particle as a thing with a good spatial localization is not at all trivial. I do not have the time to discuss this point, since the definition of particle localization is rather involved.

The main idea of the LSZ approach is the following. If we look at a particle reaction, e.g. a scattering event, we see that the particles are interacting essentially only during a finite time. Sufficiently long before and after the event the particles are so far apart from each other that their interaction can be neglected: they can be treated as free particles. But free particles we can describe field theoretically with the help of free fields. This is, then, what we do in the asymptotic regions. In between, where the interaction takes place, we talk only about fields, not about particles. That this is an eminently reasonable attitude I have already tried to make clear in the introduction.

The going-over of interacting fields into free fields for large times is mathematically treated with the so-called
asymptotic conditions, which I wish now to introduce. For a
free field we can form the creation operator of a wave packet
with wave function \( \hat{a}(\vec{p}) \):

\[
\hat{a}^*(\vec{f}) = \int \frac{d^3p}{2\omega(p)} \hat{f}(\vec{p}) \hat{a}^*(\vec{p}) = \int d^4p \hat{f}(\vec{p}) \hat{a}^*(\vec{p}) \quad (10)
\]

(Remember the form (7) of a free field.) Translated into \( x \)-
space by a Fourier transform this becomes

\[
\hat{a}^*(\vec{f}) = -i \int d^3x f(x) \frac{\partial}{\partial x^0} A^*(x) \quad (11)
\]

here

\[
f(x) \quad \frac{\partial}{\partial x^0} A(x) = f(x) \frac{\partial A^*(x)}{\partial x^0} - \frac{\partial f(x)}{\partial x^0} A^*(x),
\]

and

\[
f(x) = \int d^4p \ e^{-ipx} \hat{a}(\vec{p}) \ \delta(p^2 - m^2) \quad (12)
\]

\( f \) is a solution of the same Klein-Gordon equation as the free
field \( A(x) \):

\[
(\Box + m^2) f(x) = 0.
\]

Because of the condition \( p_0 > 0 \) in (12) we call \( f \) a solu-
tion of positive frequency. Note that the integral (11) runs
only over the space components of \( x \), the time component
\( x^0 \) being kept at a fixed value \( t \). It turns out that the
integral does not depend on the value of this parameter \( t \),
as a consequence of \( f \) and \( A \) both being solutions of the
Klein-Gordon equation. If we take \( A(x) \) to be an interacting
field, we can still form the expression (11), but it will now
depend on \( t \):
\[ e^{i\theta(f,t)} = -i \int_{x^2 = t} d^3x f(x) \frac{\partial}{\partial x^0} A^*(x). \]  

Actually it turns out that this integral diverges for interacting \( A \), it exists only after integration over a smooth function in \( t \). But this is one of the distributionistic subtleties that I am ignoring for simplicity.

Before stating the asymptotic condition I must yet introduce a new technical assumption on our fields. Let us consider a particular particle, e.g. the pion. I assume that among our fields \( A^i \) there is one, called \( A^\rho \), that can be interpreted as a \( \mathfrak{g} \)-field. All that is needed for this is that the matrix element \( \langle \Omega | A^\rho(x) | \Omega \rangle \) of the field between the vacuum and the 1-pion states do not vanish. (A 1-pion state is an eigenstate of the mass operator belonging to the 1-pion hyperboloid in the energy-momentum spectrum as discussed in postulate 4.) If this matrix element is different from zero it can be made equal to the corresponding matrix element for a free field by multiplying \( A^\rho(x) \) by an appropriate constant. Such a renormalisation does not change the relevant properties of \( A^\rho \) and is therefore always allowed. We assume that this trivial change has been effected in \( A^\rho \). The assumption of the existence of a \( \mathfrak{g} \)-field (or whatever particle we are considering) in the above sense is made for simplicity only. As a matter of fact it is sufficient to have any local function of the fields for which the critical matrix element does not vanish, and the existence of such a function is guaranteed by the completeness assumption. However, the asymptotic condition and subsequent formulae become much more complicated in this general case, and I shall refrain from stating them.
After this preparation I can now formulate the asymptotic condition (the LSZ asymptotic condition, to be more exact: there exist other formulations). It states the following, where I take the pion case as an example: there exist in \( H \) two free pseudoscalar fields \( A^{\text{in}}(x) \) and \( A^{\text{out}}(x) \) to mass \( m_\pi \), such that

\[
\lim_{t \to \pm \infty} \langle \phi \mid a^{*} (\vec{r}, t) \mid \Psi \rangle = \langle \phi \mid a^{\text{in, out}} (\vec{r}) \mid \Psi \rangle, \quad (14)
\]

the superscript 'in' holding in the \(-\infty\) case, 'out' in the \(+\infty\) case. This condition can be proved from our earlier postulate, if the wave function \( \phi \) is sufficiently smooth and the vectors \( \phi, \Psi \) are taken from a certain dense set in \( H \) which I am not going to specify: the asymptotic condition is not an assumption, as is sometimes stated in the literature!

Exactly the same relations hold also for spinning particles, but it is good to remember that in this case certain auxiliary conditions hold. Consider, for instance, the nucleon, i.e. a particle of spin 1/2. We disregard isospin and consider only a single nucleon, let's say the proton. The proton system is described by two Dirac spinors \( \Psi \) and \( \bar{\Psi} \), each with 4 components. \( \Psi^{\text{in}}, \bar{\Psi}^{\text{in}} \) describe, however, only four different states (proton and antiproton, each with two spin states), not eight. This is so because \( \Psi \) satisfies the Dirac equation \( (p^\mu \gamma^\mu - m) \Psi = 0 \), which projects out the relevant two components. The asymptotic condition for \( t \to -\infty \) becomes then

\[
(p^\mu \gamma^\mu \rightarrow 0) \\
(p^\mu \gamma^\mu \rightarrow 2m \gamma^0)
\quad (15)
\]

in the sense stated above. The same holds, of course, for \( \bar{\Psi} \).
and the analogue for $t \to + \infty$. Formulating the same thing in another way we can say that $\psi \to 0$ on wave functions of the form $(p_y^2 + m) \psi(p)$, $\psi \to \psi^\ast$ on wave functions of the form $(p_y^2 + m) \psi(p)$. The interacting field $\psi$ does of course not solve the Dirac equation.

We introduce now a final postulate.

6th postulate: asymptotic completeness. By the above procedure we associate with every stable particle occurring in the theory under consideration two free fields $A^j_{\text{in, out}}$, the index $j$ numbering the different types of particles. Note that this implies that we treat the stable particles on an equal footing, no difference between 'fundamental' and 'composite' is made. In particular, nuclei are as good as nucleons, a fact that will be used in Dr. Locher's lectures. In practice one restricts himself, of course, to a reasonable set of particles. If we are interested in $\eta - N$ scattering we are certainly allowed to forget about the existence of lead nuclei, even though, strictly speaking, they ought to be taken into account. (In Feynman graph language: lead nuclei can occur as virtual particles, but the graphs containing them are not likely to be important.)

We demand now that the asymptotic fields $A^j_{\text{in}}$, taken for all stable particles, already create the full Hilbert space $\mathcal{H}$. The same is then true for the $A^j_{\text{out}}$, due to CTP-invariance, which is a consequence of the earlier postulates. In other words the assumption means that the states

$$\psi^\ast_{\text{in}}(p_1) \ldots \psi^\ast_{\text{in}}(p_n) \mathcal{N} = \prod_{i=1}^{n} \bar{f}_{i_j} \bar{f}_{i_{n_j} \text{in}},$$

$n = 0, 1, \ldots$, from a complete basis of $\mathcal{H}$. 
Physically this condition means that a state is completely characterized by its asymptotic form. There are no states showing the behaviour that we are used to from classical non-quantized field theory, where excitations occurring at a certain time in a finite region as a rule spread out over the whole space as time goes on and become diluted to unrecognizability. In contrast to this we assume that the energy, or whatever quantity you like to consider, of a scattering state remains concentrated in a finite number of packages, the particles which remain always observable (i.e. the local matter densities remain above the detection threshold.) Complete characterization of a state by its asymptotic form means that the time development of a state can be calculated uniquely, at least in principle if not in practice, if its limiting in- or out-state is known. Of course, such a calculation is only possible if the theory is fully known, i.e. if the interaction fields $A_j^i$ are known. The asymptotic fields $A_{in}^j$ alone, being free fields, do not contain any dynamical information. (Dynamical information is, however, contained in the relation between $A_{in}^i$ and $A_{out}^i$. Since the $A_{in}^j$ form a complete set of operators, the $A_{out}^j$ can be expressed as functions of them, and the form of these functions depends on the dynamics.)

As a consequence of asymptotic completeness we can expand any state $\psi \in \mathcal{H}$ with respect to asymptotic states:

$$|\psi\rangle = \sum_{n=0}^{\infty} \sum_{(i_1, \ldots, i_m)} \int \frac{d^3p_1}{i\omega(p_1)} \cdots \frac{d^3p_m}{i\omega(p_m)} |\mathbf{p}_1, i_1; \ldots; \mathbf{p}_m, i_m\rangle \langle \mathbf{p}_1, i_1; \ldots; \mathbf{p}_m, i_m|\psi\rangle. \quad (17)$$

The states $|\ldots\ldots\rangle_{in}$ occurring in this expansion are the states (16) multiplied with appropriate normalization constants which I will not write down explicitly. They depend on the numbers of particles of each separate species being present in the state.
THE S-MATRIX

We come now to the central quantity of scattering theory, the S-matrix. Consider a process with \( n \) incoming particles with wave functions \( \hat{f}_1(p_1), \ldots, \hat{f}_n(p_n) \), and \( m \) outgoing particles with wave functions \( \hat{g}_1(q_1), \ldots, \hat{g}_m(q_m) \). (For simplicity I drop the indices referring to particle species. You may consider them to be included in the variables \( p_i \), \( q_i \)). What is the probability that said in-state is scattered into said out-state? We represent our states as

\[
\phi_{\text{in}} = a^*_{\text{in}}(\hat{f}_1) \ldots a^*_{\text{in}}(\hat{f}_n) n
\]

\[
\phi_{\text{out}} = a^*_{\text{out}}(\hat{g}_1) \ldots a^*_{\text{out}}(\hat{g}_m)
\]

and assume \( \| \phi_{\text{in}} \| = \| \phi_{\text{out}} \| = 1 \). This normalization condition is the reason why we work with wave packets and not with sharp momentum states. The latter are not normalizable. The rules of quantum mechanics tell us that the probability of state \( \phi_{\text{in}} \) becoming \( \phi_{\text{out}} \) is the square of the amplitude

\[
( \phi_{\text{out}}, \phi_{\text{in}} ) = \left[ \int \frac{d^3p_i}{2 \omega(p_i)} \hat{f}_i(p_i) \right] \left[ \int \frac{d^3q_j}{2 \omega(q_j)} \hat{g}_j(q_j) \right]
\]

\[
\times \left( \mathcal{N}, a^*_{\text{out}}(q_1) \ldots a^*_{\text{out}}(q_m) a^*_{\text{in}}(p_1) \ldots a^*_{\text{in}}(p_n) \mathcal{N} \right).
\]

The kernel

\[
S(q_1, \ldots, q_m; p_1, \ldots, p_n) = ( \mathcal{N}, a_{\text{out}}(q_1) \ldots a^*_{\text{out}}(p_n) \mathcal{N}) \tag{19}
\]

is called the S-matrix element for the process \( n \) in-particles \( \rightarrow m \) out-particles. We define an operator \( S \) by stipulating the expressions (19) to be its matrix elements in an in-basis:
\[
\begin{align*}
\{ a_{in}(q_1), \ldots, a_{in}(q_m), S a_{in}^*(p_1), \ldots, a_{in}^*(p_n), \Lambda \} \\
= S(q_1, \ldots, p_n) \\
\end{align*}
\]

(20)

It follows easily from this definition that \( S \) transforms in-fields into out-fields and that it leaves the vacuum invariant:

\[
a_{out}(p) = S^* a_{in}(p) S, \quad S\Lambda = \Lambda.
\]

(21)

These two relations can actually be used as an alternative definition of the \( S \)-operator, or \textit{scattering operator}.

Two important properties of \( S \) that can be derived easily from our postulates are invariance and unitarity.

\textbf{Invariance.} It can be shown that the covariance equations (1) and (2) hold for the asymptotic fields \( A_{in}^{\Lambda} \), with the same \( \Pi(\Lambda, a) \) as for the interacting fields. Translation invariance (Eq. (1)) implies after transformation into \( p \)-space that \( S(q_1, \ldots, q_m, p_1, \ldots, p_n) \) contains a factor

\[
\delta^4(\sum_i q_i - \sum_j p_j)
\]

expressing energy-momentum conservation. Here the energy parts of the vectors \( q_i, p_j \) are defined by

\[
q_{io} = + \sqrt{q_i^2 + m_i^2}, \quad \text{the same for } p_{jo},
\]

with \( m_i \) the mass of the \( i \)th particle. In order to discuss the behaviour under Lorentz transformations we reintroduce the lower Lorentz indices dropped after the discussion of the invariance postulate. Eq. (2) written for \( a_{in, out} \) and applied to \( S_{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n}(q_1, \ldots, p_n) \) yields

\[
S_{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n}(q_1, \ldots, p_n) = L_{\mu_1 \nu_1} \ldots L_{\mu_m \nu_m} \ldots S_{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n}(q_1, \ldots, p_n). \quad (23)
\]
Upper indices indicating particle type should be added in the appropriate places. In particular, the representations $L$ depend on the spin of the particle in question. $\Lambda q_i$ is the space-part of the 4-vector $\Lambda q_i$, where $q_i$ is obtained from $q_i$ by adjoining the 0-component (22).

For the $S$-operator the relation (23) means simply

$$ S U(\Lambda) = U(\Lambda) S : $$

(24)

$s$ commutes with the Lorentz group.

Unitarity. Let $\phi, \psi$ be two vectors in $H$. The completeness relation (17) gives

$$ \langle \phi | \psi \rangle = \sum \int \frac{d^3p_1}{2\omega(p_1)}\cdots\frac{d^3p_n}{2\omega(p_n)} \langle \phi | p_1;\cdots;p_n \rangle_{in} \langle p_1;\cdots;p_n | \psi \rangle, $$

(25)

where the type-indices $j_i$ have again been incorporated into the $p_i$. Exactly the same relation holds for 'out' replacing 'in'. If we set

$$ \langle \phi | = \langle \phi_{in} | S^* , | \psi \rangle = S | \psi_{in} \rangle, \text{ with } \phi_{in} , \psi_{in}$$

two vectors of the asymptotic in-type introduced before, we obtain

$$ \langle \phi_{in} | S^* S | \psi_{in} \rangle = \sum \int \cdots \langle \phi_{in} | S^* | p_1;\cdots;p_n \rangle_{in} \times \langle p_1;\cdots;p_n | S | \psi_{in} \rangle $$

$$ = \sum \int \cdots \langle \phi_{in} | p_1;\cdots;p_r \rangle_{out} \times \langle p_1;\cdots;p_n | \psi_{in} \rangle $$

$$ = \langle \phi_{in} | \psi_{in} \rangle. $$
or

\[ S^* S = 1. \]  \hspace{1cm} (26)

Similarly we prove

\[ S S^* = 1, \]  \hspace{1cm} (27)

i.e. \textit{\textbf{S is unitary.}} The reader who is acquainted with the theory of the representations of the canonical commutation rules will notice that this result follows already from the relations (21).

It is customary to introduce a new operator \textbf{T} called 'transition matrix' via the decomposition

\[ S = 1 + iT. \]  \hspace{1cm} (28)

(Definitions vary somewhat from author to author. The \textbf{T}'s introduced in the literature differ from ours by real numerical factors, possibly with a simple energy dependence.) The operator \textbf{T} describes the events where something nontrivial happens. If you shoot a particle at another one, then most often nothing of interest will happen because you miss the target. Your projectile passes at a distance at which interactions are negligibly small, and the two particles will never notice each other's presence. These trivial events are contained in the identity part of (28).

The unitarity relation (26) becomes, if written in terms of \textbf{T}:

\[ S^* S = (1 - iT^*) \times (1 + iT) = 1, \]
which gives

\[ 2 \text{ Im } T = T^* T \quad . \] (29)

The imaginary part of the operator \( T \) is defined as \( \text{Im } T = \frac{T - T^*}{2i} \). Take the expectation value of relation (29) in an \( \phi_{\text{in}} \) in-state. You find

\[ 2 \text{ Im } \langle \phi_{\text{in}} | T | \phi_{\text{in}} \rangle = \langle \phi_{\text{in}} | T^* T | \phi_{\text{in}} \rangle = \sum_z |\langle \phi_{\text{in}} | z \rangle|^2 , \]

where the sum \( \sum_z \) extends over a complete set of intermediate in-states. But \( \langle \phi_{\text{in}} | T | z \rangle \) is the probability that \( \phi_{\text{in}} \) is scattered into the state \( |z'\rangle \) obtained from \( |z\rangle \) by replacing all in-fields by the corresponding out-fields, where trivial events of the type mentioned above are not counted. The \( z \)-sum gives then simply the probability that some non-trivial event takes place in \( \phi_{\text{in}} \), in other words, it is the total cross section (apart from a trivial numerical factor). Choose, in particular,

\[ \phi_{\text{in}} = |\vec{p}, \vec{q}\rangle_{\text{in}} \]

a 2-particle state. Then

\[ \langle \phi_{\text{in}} | T | \phi_{\text{in}} \rangle_{\text{in}} = F(\vec{p}, \vec{q}) \]

is the elastic scattering amplitude in the forward direction, so that we obtain

\[ 2 \text{ Im } F(\vec{p}, \vec{q}) = \sigma_{\text{tot}} \quad : \] (30)

the imaginary part of the forward scattering amplitude is essentially (apart from a numerical factor: remember my warning about 2\( \rho \) - factors!) the total cross section. This is the optical theorem.

To be discussed remains a third, less trivial, set of properties of \( S \): the analyticity properties. Before proceeding
to them I must yet do some preparatory work. Analyticity will come out as a consequence of the locality assumption. Therefore we must first find a convenient representation of $S(...;...) \lambda \phi$ (sometimes called the interpolating fields) so that locality can be applied. Such a representation is given by the reduction formula and is useful also in other contexts (see, e.g., the lectures by Mrs. Ericson.)

The reduction formula expresses $S$ in terms of time-ordered products of the fields $A^i$. The time-ordered product of the field operators $A^i(x_1), ..., A^i(x_n)$ is defined by

$$T \left[ \ A^i(x_1) \ ... \ A^i(x_n) \right]$$

$$= (-i)^{n-1} \sum_{\pi(1...n)} \pm \theta(x_{i_1}, ..., x_{i_n}) \ A^{i_{\pi(1)}}(x_{i_1}) \ ... \ A^{i_{\pi(n)}}(x_{i_n}) \ . (31)$$

The sum in this formula extends over all permutations $(i_1, ..., i_n)$ of the indices $(1, ..., n)$. The function $\theta(\ldots)$ is defined to be $1$ if its arguments are time-ordered (i.e., if $x_{i_1}^0 > x_{i_2}^0 > ... > x_{i_n}^0$), $\theta = 0$ otherwise. The sign in front of a term is positive if the permutation of the fermi-fields present is even, negative if this permutation is odd. The Bose fields have no influence on the sign.

The vacuum expectation value of $T$

$$\langle \phi^{<...<} \rangle(x_1, ..., x_n) = \langle \mathfrak{a}, T \left[ \ A^i(x_1) ... A^i(x_n) \right] \mathfrak{a} \rangle \ . (32)$$

is called the time-ordered function or Green's function.

This notation is not meant to imply that the type indices of all the fields necessarily different.
I shall drop the type indices of the fields for simplicity. Again, you may consider them to be included in the arguments \( x_1 \). For the moment I will, however, reintroduce the much neglected lower Lorentz indices:

\[
\tau_{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n) = (\mathbf{n}, T[A_{\mu_1}(x_1) \ldots] \mathbf{n}).
\]

This allows me to state the fact that the \( \tau \) are covariant:

\[
\tau_{\mu_1 \cdots \mu_n}(A_{x_1+a}, \ldots, A_{x_n+a}) = L^1_{\mu_1 \nu_1} \cdots L^n_{\mu_n \nu_n} \tau_{\nu_1 \cdots \nu_n}(x_1, \ldots, x_n) .
\] (33)

The \( L^i \) are the matrix elements of the Lorentz representation of the field \( A^i \). The covariance of \( \tau \) is not trivial; note that the function \( \Theta \) occurring in the definition of \( T \) is not invariant. But it turns out that the non-covariant parts of the various terms cancel due to locality. This cancellation is the reason for introducing the sign alternative in (31). I mention in passing that this is one of the points where the distribution character of the fields creates problems.

Formally the \( \tau \) are covariant, but a closer inspection reveals some flaws in the argument. I shall ignore this difficulty, even though it is by no means devoid of physical interest, witness the large amount of work that the current algebra people have invested in the problem.

From the translation invariance of \( \tau(x_1, \ldots, x_n) \) we can derive that its Fourier transform is of the form

\[
\hat{\tau}(p_1, \ldots, p_n) = \delta^4(p_1 + \ldots + p_n) \hat{\tau}(p_1, \ldots, p_n) .
\] (34)
The function \( \hat{f} \) is defined only in the points where \( p_1^+ \ldots + p_n = 0 \), and depends consequently only on \((n-1)\) 4-vectors. But it is not easy to decide which of the variables should be dropped. Therefore I retain them all, including one that is redundant. The \( \delta \)-factor in (34) expresses conservation of 4-momentum.

I can now derive the reduction formula for the S-matrix. I shall give only the formal skeleton of the derivation, interchanging limits freely. All the steps of the derivation can be rigorously justified for sufficiently well-behaved wave functions. (The admissible wave functions are dense in the space of all possible wave functions.)

Consider the S-matrix element (18) for an in-state with wave functions \( \hat{f}_1, \ldots, \hat{f}_n \), and an out-state with wave functions \( \hat{g}_1, \ldots, \hat{g}_m \):

\[
S = ( \mathcal{N}, a_{\text{out}}(\hat{g}_1) \ldots a_{\text{out}}(\hat{g}_m) a^*(\hat{f}_1) \ldots a^*(\hat{f}_n), \mathcal{N} ) .
\]  

(35)

Remember the definition (10) of \( a_{\text{in}, \text{out}}(\hat{f}) \) and the corresponding definition (13) of \( a(\hat{f}, t) \) for interacting fields. Remember also the asymptotic condition (14). We 'reduce' the particles in (35) one by one. Let us start with the \( \hat{f}_1 \)-operator and use the asymptotic condition:

\[
S = \lim_{t \to -\infty} ( \mathcal{N}, \ldots a_{\text{out}}(\hat{g}_m) a^*(\hat{f}_1, t) a^*(\hat{f}_2) \ldots \mathcal{N} ) .
\]

Consider the reverse limit of the same expression:

\[
S' = \lim_{t \to +\infty} ( \mathcal{N}, \ldots a^*(\hat{f}_1, t) \ldots \mathcal{N} )
= ( \mathcal{N}, \ldots a_{\text{out}}^*(\hat{f}_1) \ldots \mathcal{N} ) .
\]
The operator $a^*_{\text{out}}(\hat{f}_1)$ stands here to the right of destruction operators $a_{\text{out}}(\hat{g}_j)$. It can be commuted through to the left with the help of the commutation relations (8'), until it stands directly besides the vacuum $\mathcal{N}$. But $a^*_{\text{out}}$ acts to the left as a destruction operator, it annihilates the vacuum. The commutators appearing in the process give rise to terms of exactly the same form as the original $S$, but with one in- and one out-particle less, and the whole multiplied with the scalar product

$$ (\hat{g}_i, \hat{f}_1) = \int \frac{d^3p}{2\omega(p)} \hat{g}_i(p) \hat{f}_1(p) $$

For simplicity I assume that all the $\hat{f}_i$ are orthogonal to all the $\hat{g}_j$ so that these singular terms vanish. (They correspond to processes in which one of the particles does not participate in the interaction, but goes through behaving as a free particle.) With this assumption we have $S' = 0$, hence

$$ S = S - S' = \int_{-\infty}^{\infty} dt \left( \mathcal{N} \cdots \hat{a}^*(\hat{f}_1, t) \cdots \mathcal{N} \right), $$

the dot over $a^*$ denoting the $t$-derivative. Using the explicit form (13) of $a^*$ we obtain.

$$ \hat{a}^*(\hat{f}_1, t) = -i \int_{x = t} d^3x \left\{ f_1(x) \frac{\partial^2 A^*(x)}{\partial x^2} - \frac{\partial f_1(x)}{\partial x^2} A^*(x) \right\}. $$

We use the fact that $f_1$ is a solution of the Klein-Gordon equation:

$$ \frac{\partial^2 f_1(x)}{\partial x^2} = \Delta f_1(x) - m^2 f_1(x), $$

and integrate by parts. This yields

$$ \hat{a}^*(\hat{f}_1, t) = i \int_{x = t} d^3x \left\{ f_1(x) K_x A^*(x) \right\}, \quad (36) $$
with $K_x = \Box_x + m^2$, the Klein-Gordon operator. Thus
\[
S = -i \int d^4x_1 f_1(x_1) K_{x_1} (\not{\mathcal{L}}, \ldots, A^*(x_1) a^*_{in}(\hat{f}_2) \ldots \not{\mathcal{L}}).
\]

We proceed to the next step, the reduction of $\hat{f}_2$. Application of the asymptotic condition to $a^*_{in}(\hat{f}_2)$ gives
\[
S = -i \int d^4x_1 f_1(x_1) K_{x_1} \lim_{t \to \infty} \left( \not{\mathcal{L}}, \ldots, A^*(x_1) a^*_{in}(\hat{f}_2, t) \ldots \not{\mathcal{L}} \right).
\]

But from the definition of the $T$-product we see easily that
\[
-i \lim_{t \to -\infty} A^*(x_1) A^*(t, x_2) = \lim_{t \to -\infty} T \left[ A^*(x_1) A^*(t, x_2) \right],
\]
so that we can introduce a $T$-ordering in (37) without changing the result. In the limit $t \to \infty$ this $T$-product becomes
\[
-i a^*_{out}(\hat{f}_2) A^*(x_1). \quad \text{The $a^*_{out}$ stands again immediately to the right of the $a^*_{out}(g_{j})$ and can be commuted through as before.}
\]

This is the reason for the introduction of the $T$-ordering. Without it we obtain in the $t \to \infty$ limit the product
\[
A^*(x_1) a^*_{out}(\hat{f}_2), \quad \text{and this cannot be evaluated easily,}
\]
since the commutator of $A^*$ with $a_{out}$ is not known explicitly.

From here on we proceed exactly as in the first step and obtain
\[
S = i \int d^4x_1 d^4x_2 f_1(x_1) f_2(x_2) K_{x_1} K_{x_2} (\not{\mathcal{L}}, \ldots, \not{T} [A^*(x_1) A^*(x_2]) \ldots \not{\mathcal{L}}).
\]

Faithful repetition of this procedure for each of the remaining asymptotic fields, including the out-fields on the left, yields finally the result.
\[ S = i \sum_{y_1} \int d^4 x_j \prod_{i} \{ g^*_i(y_i) K_{y_i} \} \prod_{j} \{ f_j(x_j) K_{x_j} \} \mathcal{T}(y_1, \ldots, y_m, x_1, \ldots, x_n) \]  

(38)

This gives in p-space:

\[ S(q_1, \ldots, q_m; p_1, \ldots, p_n) = i \prod_{i} (q_i^2 - m_i^2) \prod_{j} (p_j^2 - m_j^2) \mathcal{T}(-q_1, \ldots, -q_m, p_1, \ldots, p_n) \]  

(39)

In words: in order to obtain \( S(q_1, \ldots, p_n) \), multiply \( \mathcal{T}(-q_1, \ldots, p_n) \) with the factors \( (q_i^2 - m_i^2) \) and \( (p_j^2 - m_j^2) \) in all variables. The result is called the amputated Green's function \( \mathcal{T}^{\text{amp}}(-q_1, \ldots, p_n) \). Then restrict all the variables \( q_j, p_i \) to their respective mass shell. This mass shell restriction is not zero despite the vanishing of the amputation factors \( (q_i^2 - m_i^2) \) etc. This is so because \( \mathcal{T}(-q_1, \ldots, p_n) \) contains a pole product \( \prod_{i} (q_i^2 - m_i^2)^{-1} \prod_{j} (p_j^2 - m_j^2)^{-1} \). \( \mathcal{T}^{\text{amp}} \) is the residue of this product.

Equation (39) is the desired reduction formula allowing to calculate \( S \) directly from the interacting fields, without a detour over the asymptotic condition.

**ANALYTICITY**

We are now finally in a position where we can discuss the analyticity of the S-matrix. This subject will occupy the rest of my lectures. The plan is to derive first analyticity properties of the Green's functions \( \mathcal{T} \) off the mass shell and
obtain from them properties of \( S \) by restriction to the mass shell. This is not the approach that is customarily used in the derivation of dispersion relations. Usually people stick as closely as possible to the mass shell from the very beginning and are very apologetic about the fact that slight excursions off the mass shell cannot be completely avoided. As a result the proofs are very technical from the start, and it is hard to perceive the simple general ideas underlying them. In the off-shell approach I intend to use the arguments are much more transparent, or so I think.

The main postulates underlying analyticity are locality and the spectral conditions. I wish to draw your attention especially on the latter, since their necessity is often passed over in silence in popular discussions of the subject. The spectral conditions are irrelevant for the discussion of the forward Compton scattering (if you are willing to forget the dubious character of the photon), but in all other cases they are essential.

To see where the analyticity of \( \hat{\mathcal{F}} \) comes from, let us first consider the simplest case, that of the 2-point function. This function is of little interest to the phenomenologist, but I use it here only as a model. The considerations for the more interesting 4-point function are essentially the same, just a bit more complicated.

We have already noted in Eq. (34) that \( \widehat{\mathcal{F}}(p,q) \) is of the form

\[
\widehat{\mathcal{F}}(p,q) = \delta^4(p+q) \hat{\mathcal{F}}(q), \tag{40}
\]

and it is the function \( \hat{\mathcal{F}} \) that is analytic. (In contrast to (34) I have here omitted one of the arguments in \( \hat{\mathcal{F}} \) as redundant).
Instead of $\hat{F}$ we consider first a closely related function, the retarded function. The \textit{retarded product} of two field operators $A(x), A(y)$ is defined as

$$R(x,y) = -i \quad \Theta(x-y) \quad [A(x), A(y)] . \quad (41)$$

If both fields involved are fermi fields, the commutator must be replaced by an anticommutator. For simplicity we restrict ourselves to the bose case. (We are still ignoring the indices indicating the field type).

The vacuum expectation value

$$r(x,y) = \langle \mathcal{N}, R(x,y) \mathcal{N} \rangle \quad (42)$$

is called \textit{retarded function}. It has the usual covariance properties. In particular it is invariant under translations: $r(x+a,y+a) = r(x,y)$, hence it depends only on the difference $\xi = x-y$ : $r(x,y) = r(\xi)$ . In analogy to (40) we have in $p$-space

$$\hat{R}(p,q) = \delta^4(p+q) \quad \hat{r}(q) , \quad (43)$$

and $\hat{r}$ turns out to be

$$\hat{r}(q) = \int d^4\xi \quad e^{iq\xi} \quad r(\xi) . \quad (44)$$

This formula can be used to define $\hat{r}$ for complex values of $q$. With $q = q' + i q''$, $q', q''$ real, we obtain

$$\hat{r}(q) = \int d^4\xi \quad e^{iq'\xi} \quad e^{-q''\xi} \quad r(\xi) . \quad (45)$$

Now, it is easy to see that $r(\xi)$ vanishes outside the closed forward cone $\xi \in \mathcal{V}_+$, partly because of the
\[ \text{\( \Theta \) -function appearing in the definition (41), partly because of the vanishing of the commutator \([A(x'), A(y)]\) for space-like separations \(x-y\). Hence the integral (45) extends only over the forward cone. But there the factor \( e^{-q''\xi} \) decreases strongly at infinity, if we take a \( q'' \) from the open forward cone \( \mathcal{V}_+ \).}

We introduce a new assumption of a technical nature, namely that \( r(\xi) \) does not increase stronger than a polynomial for \( \xi \to \infty \) in any direction. It is hard to describe the physical meaning of this assumption. There are no models known in which it is violated, so that its violation would presumably involve a rather perverse type of physics. In any case the assumption is not essential: all the results about analyticity on the mass shell can be derived without it, at the price of a considerable complication of the proofs.

Under this boundedness assumption the integral (45) converges for \( q'' \in \mathcal{V}_+ \), and the convergence is locally uniform (i.e. uniform in all bounded \( q \)-sets), from which it follows that \( \hat{\mathcal{A}}(q) \) is an analytic function in \( q'' \in \mathcal{V}_+ \). Note that we place no restriction on the real part \( q' \). This domain of analyticity of \( \hat{\mathcal{A}} \) is called the forward tube and is denoted by \( \mathcal{T}_+ \).

The real points \( q \) with \( q''=0 \) are not in \( \mathcal{T}_+ \), but they lie on its boundary. Hence the physically relevant function \( \hat{\mathcal{A}}(q') \) for real values of the arguments is a boundary value of the analytic function we have just introduced:

\[ \hat{\mathcal{A}}(q') = \lim_{q'' \to 0} \hat{\mathcal{A}}(q'+iq'') \quad (46) \]

This limit exists in the sense of distributions, i.e. after integration of both sides over a sufficiently smooth test
function in $q'$. In practice it is reasonable to consider $\hat{r}(q')$ as continuous apart from some poles that must be present for physical reasons, in which case (46) can be used as it stands.

Consider now the retarded function $r(y, x)$ with the order of factors reversed. Its Fourier transform is $\hat{r}(q, p) = \delta^4(p+q) \hat{r}(-q)$. According to the previous considerations $\hat{r}(-q)$ is a boundary value of the same analytic function as $\hat{r}(q)$, with a minus sign added to the argument, so that the analyticity domain is this time the backward tube $\mathcal{T}_-$ consisting of the $q$ whose imaginary part $q''$ lies in the backward cone $\mathcal{V}$.

I wish to show that these two functions $\hat{r}(q)$, $\hat{r}(-q)$ which are analytic in $\mathcal{T}_+$ and $\mathcal{T}_-$ respectively, are actually analytic continuations of each other. From the definition of $r$ we obtain immediately

$$r(x, y) - r(y, x) = -i \left( \mathcal{N} \left[ A(x), A(y) \right] \mathcal{N} \right),$$

or in $p$-space

$$\hat{r}(p, q) - \hat{r}(q, p) = -i \left\{ \mathcal{N} \left[ \tilde{A}(p), \tilde{A}(q) \right] \mathcal{N} - p \leftrightarrow q \right\}. \quad (47)$$

But $\tilde{A}(q) \mathcal{N}$ is a state with 4-momentum $-q$, hence it is different from zero only if $-q$ lies in the spectrum of the theory according to the 4th postulate. This state is orthogonal to the vacuum $\mathcal{N}$ because we demand that

$$\left( \mathcal{N}, A(x) \mathcal{N} \right) = 0.$$ This can always be achieved by subtracting a constant $c$-number from $A(x)$, hence the new requirement is no restriction of generality. I assume for simplicity that there is exactly one 1-particle state with the necessary quantum numbers, with mass $m$, and that the
continuum starts at mass $M > m$. Then the right-hand side of (47) is different from zero only if $q^2 = m^2$ or $q^2 > M^2$. Let me call this set $\mathcal{A}$, its complement $\mathcal{B}$. For $q$ in $\mathcal{B}$ we have

$$\tilde{r}(p,q) = \tilde{r}(q,p),$$

whence

$$\hat{r}(q) = \hat{r}(-q).$$

This hold for real $q \in \mathcal{B}$. We have the situation that in certain real points the boundary value of an analytic function in $\mathcal{T}_+$ coincides with the boundary value of an analytic function in $\mathcal{T}_-$. According to a mathematical theorem called the edge-of-the-wedge theorem the two functions are then analytic continuations of each other.*

In other words: there exists a function $F(q' + iq'')$ which is analytic in the union of $\mathcal{T}_+ \cup \mathcal{T}_-$ and a complex neighbourhood of the real points $\mathcal{B}$, and which coincides in $\mathcal{T}_+$ with $\hat{r}(q)$ and in $\mathcal{T}_-$ with $\hat{r}(-q)$. Both $\hat{r}(q)$ and $\hat{r}(-q)$ are boundary values of this one function $F$. They differ only in the prescription of how the boundary has to be approached, from $\mathcal{T}_+$ in one case, from $\mathcal{T}_-$ in the other.

We return to the object in which we are really interested, the time-ordered function. Elementary algebra gives the following relations between time-ordered and retarded products:

$$T(x,y) = R(x,y) - i A(y) A(x)$$

$$= R(y,x) - i A(x) A(y). \quad (48)$$

---

* The edge-of-the-wedge theorem is a generalization to several variables of the Schwarz reflection principle that is well known from the theory of analytic functions in one complex variable.
We take the vacuum expectation value of these relations, transform them into $p$-space, and apply to the second term on the right-hand side the same sort of spectral consideration that has already been used in connection with Eq. (47). We obtain:

$$\hat{\xi}(q) = \hat{\xi}(q) \quad \text{if} \quad -q \text{ is not in the spectrum},$$

$$\hat{\xi}(q) = \hat{\xi}(-q) \quad \text{if} \quad q \text{ is not in the spectrum}.$$  

In particular, $q$ is not in the spectrum (apart from the vacuum, which is irrelevant in the present context) if $q_o \not< 0$, hence

$$\xi(q) = \hat{\xi}(q) \quad \text{if} \quad q_o \not> 0,$$

$$\xi(q) = \hat{\xi}(-q) \quad \text{if} \quad q_o \not< 0. \quad (49)$$

The two alternatives together exhaust all the possibilities: $\hat{\xi}(q)$ is everywhere equal to either $\hat{\xi}(q)$ or $\hat{\xi}(-q)$, it is everywhere a boundary value of the analytic function $\xi(q)$. The boundary has to be reached from the appropriate tube, from $T_+$ if $q_o > 0$, from $T_-$ if $q_o < 0$:

$$\xi(q') = \lim_{q_o \to 0} \xi(q' + iq^o) \quad (50)$$

If $q'_o = 0$ it doesn't matter from which direction we reach our point, because the real points with $q'_o = 0$ belong to the analyticity domain of $\xi$.

Let me repeat what we know about the analyticity domain of $\xi$, because of its fundamental importance: $\xi(q' + iq^o)$ is analytic in the points for which either $q''^2 > 0$, or $q'' = 0$, and $q' < M^2, q'' \not= M^2$. This set is called the primitive domain of analyticity of $\xi$. (Remember that analyticity in
a point means analyticity in a complex neighbourhood of this point. I.e. \( F \) is actually analytic in a neighbourhood of the real points contained in the primitive domain.)

Things become a bit more familiar-looking if we make use of invariance. Let us start with the pion-propagator, i.e. assume both our fields \( A(x) \), \( A(y) \) to be pion fields. The \( \Phi \)-field is a scalar (pseudoscalar, actually, but the behaviour under the parity operation is irrelevant in the present context). Then all the functions under consideration, in particular \( \hat{f}(-q) \) and \( \hat{f}(q) \), are Lorentz invariant. It can be shown that as a consequence of this the analytic function \( F(q) \) is invariant even under complex Lorentz transformations and can be written as a function of the only available invariant \( q^2 \):

\[
F(q) = F(q^2). \tag{51}
\]

I wish to point out that the same is not true for the nonanalytic \( \hat{r}(q) \). If \( \hat{r}(q) \) were a function of \( q^2 \) only it would be invariant under the replacement \( q \to -q \), hence \( \hat{r}(-q) = \hat{r}(-q) \), hence \( r(x,y) = r(y,x) \), a false equation. In the case of \( \hat{r} \) we have more luck: \( \hat{r}(q) \) is indeed a function of \( q^2 \).

Consider the restriction of \( F \) to \( \mathcal{T}_+ \), the analytic function \( F_+(q) = \hat{r}(q) \). The mapping \( q \to q^2 \) maps \( \mathcal{T}_+ \) onto the complex \( q^2 \)-plane with a cut along the positive real axis, i.e. \( F_+(q^2) \) is analytic in this cut plane. The same is true for \( F_-(q^2) \), the restriction of \( F \) to \( \mathcal{T}_- \). For \( q^2_0 > 0 \) (to fix ideas) we have

\[
\hat{f}(q') = \lim_{\epsilon \to 0} F_+(q'^2 + i\epsilon),
\]

\[
\hat{f}(-q') = \lim_{\epsilon \to 0} F_-(q'^2 - i\epsilon).
\]
For space-like $q'$ we are in the analyticity region of both $F_+$ and $F_-$. Moreover we know that $\hat{\Lambda}(q') = \hat{\Lambda}(-q')$ for $q'$ space-like. Hence the two functions $F_+$, $F_-$ agree on the negative real axis, which suffices to make them agree everywhere: $F_+(q^2) = F_-(q^2) = F(q^2)$. The equality $\hat{\Lambda}(q') = \hat{\Lambda}(-q')$ is actually still valid for time-like $q'$ outside the spectrum, i.e. for $q'^2 < M^2$, $q'^2 \neq m^2$ (here $m$ is the $\rho$-mass, $M = 3m$, since the $3\rho$-state is the lowest-lying continuum state with the same quantum numbers as the pion).

In these positive real points the limits of $F$ from above and below coincide. The Schwarz reflection principle can be applied with the result that $F$ is still analytic in the said points: the cut starts only at $M^2$, not at zero. The resulting analyticity domain of $F(q^2)$ is shown in Fig. 2.

\begin{figure}
\centering
\begin{tikzpicture}
\begin{axis}[
    width=10cm,
    height=5cm,
    axis lines=center,
    xlabel={Re $q^2$},
    ylabel={Im $q^2$},
    xmin=-10, xmax=10,
    ymin=-10, ymax=10,
    xtick={-m^2, M^2},
    ytick={},
\]
\end{axis}
\end{tikzpicture}
\caption{Analyticity domain of $F(q)$}
\end{figure}

It can be shown that the singularity in $q^2 = m^2$ is a single pole: $(q^2 - m^2)^{-1}$. Its residuum is a simple power of $(2\rho)$ depending on the conventions chosen. It is fixed by the normalization condition for the matrix element $\langle \pi | A(x) | \alpha \rangle$ that we introduced earlier in connection with the asymptotic condition.
We use the information about the analyticity of $F$ for deriving the Källén-Lehmann representation of the $\Pi$-propagator, which is a simple example of a dispersion relation. Assume that $F(q^2)$ tends to zero faster than $1/q^2$ if $q^2 \to \infty$ in any complex direction. We can write a Cauchy formula for the contour given in Fig. 3.

\[ F(q^2) = \frac{1}{q^2 - m^2} + \frac{1}{2 \pi i} \int_{M^2}^{\infty} \frac{du}{q^2 - u} A(u), \quad (52) \]

with

\[ A(u) = F(u+i\epsilon) - F(u-i\epsilon) \quad (53) \]

the discontinuity of $F$ across the cut. (52) is the KL-representation in its unsubtracted form.

Subtractions are necessary if $F$ does not approach 0 at infinity, or if it does so but not fast enough. Assume for instance that $F(q^2)/q^2$ goes to zero faster than $1/q^2$, but $F$ itself does not. We introduce the new function $G(q^2) = F(q^2)/q^2$. $G$ has all the properties that $F$ had in
the non-subtracted case, except for the presence of an additional pole in the origin. Instead of (52) we obtain

$$G(q^2) = \frac{F(0)}{q^2} + \frac{1}{m^2(q^2-m^2)} + \frac{1}{2\pi i} \int_{M^2}^{\infty} \frac{du}{q^2-u} A(u)$$

and this becomes after multiplication with $q^2$:

$$F(q^2) = F(0) + \frac{q^2}{m^2(q^2-m^2)} + \frac{q^2}{2\pi i} \int_{M^2}^{\infty} \frac{cu}{u(q^2-u)} A(u).$$

This is the once-subtracted KL-representation. $F(0)$ is called a subtraction constant. In the same way we can treat the case where $F(q^2)$ is bounded at infinity by a polynomial, the number of subtraction constants increasing with the degree of the polynomial. The assumption that $F$ is polynomially bounded in $p$-space is not as harmless as the similar assumption we made earlier in $x$-space. It may well be violated in theories of physical interest. However, we need not be disturbed by this: the same difficulty will not manifest itself in the case that mainly interests us, the 4-point function on the mass shell.

In our 2-point case it is pointless to talk about the behaviour on the mass shell. The mass shell consists of the single point $q^2=m^2$, and there $\mathbf{f}^{\text{amp}} = (q^2-m^2)^2 \mathbf{f}$ vanishes, the 1-particle pole being overcompensated by the amputation factors.

Things are a bit more complicated for the nucleon propagator

$$\gamma_\phi(x,y) = (\mathbf{N},T [\gamma_\phi(x) \gamma_\phi(y)] \mathbf{N})$$

with $\gamma, \gamma$ 4-component spinors. $\hat{\gamma}_\phi(q)$, and there-
Therefore \( F_{\rho\sigma}(q) \), are not invariant but covariant; they transform like \( L^{+1} \otimes L \), the spin-1/2 representation. The trick we use here is to develop \( \hat{F}_{\rho\sigma} \) with respect to 'standard covariants', a complete set of simple polynomials in \( q \) with the correct transformation behaviour. In the present case there are two such covariants: \( \delta_{\rho\sigma} \) and 
\[
\gamma^\mu q_\mu \quad (\gamma^\mu \text{ are the Dirac } 4\times4 \text{ matrices}).
\]
It can be shown that \( F_{\rho\sigma}(q) \) can be decomposed as follows:
\[
F_{\rho\sigma}(q) = \delta_{\rho\sigma} F_1(q) + \gamma^\mu q_\mu F_2(q) ,
\]
where the functions \( F_{1,2} \) are invariant and have the same domain of analyticity as \( F \) itself. The invariance arguments from the \( \gamma \)-case can then be applied to \( F_1 \) and \( F_2 \), KL-representations can be derived for them, etc.

**ANALYTICITY OF THE SCATTERING AMPLITUDE**

The same sort of methods we used for the 2-point function can also be applied to the more interesting 4-point case \( \hat{\mathcal{F}}(p_1, p_2, p_3, p_4) \). Remember that \( \hat{\mathcal{F}} \) is of the form
\[
\hat{\mathcal{F}}(p_1, \ldots, p_4) = \delta^4(p_1+\ldots+p_4) \hat{\mathcal{F}}(p_1, \ldots, p_4) ,
\]
\( \hat{\mathcal{F}} \) defined on the manifold \( p_1+\ldots+p_4 = 0 \). It is the factor \( \hat{\mathcal{F}} \) that has analyticity properties.

As in the 2-point case one finds that \( \hat{\mathcal{F}} \) is everywhere equal to the Fourier transform of some retarded function.
retarded function is a function which vanishes in $x$-spaces in a sufficiently large region to acquire in $p$-space analyticity in a tube by the same mechanism that leads to the $\tau^-$-analyticity of $\hat{F}(q)$. Again the various retarded functions can be connected with the help of the spectral condition and the edge-of-the-wedge theorem. The result is a primitive domain of analyticity as given in the following statement.

There exists a function $F(p_1, \ldots, p_4)$ defined on the complex manifold $p_1 + \ldots + p_4 = 0$, which is analytic if the following requirements are satisfied:

$$(\text{Im } p_i)^2 > 0, \text{ or Im } p_i = 0 \text{ and } (\text{Re } p_i)^2 < m_i^2 \text{ for each } i = 1, \ldots , 4,$$

$$(\text{Im}(p_i + p_j))^2 > 0, \text{ or Im}(p_i + p_j) = 0 \text{ and }$$

$$(\text{Re}(p_i + p_j))^2 < m_{ij}^2 \text{ for each pair } (i, j).$$  \hspace{1cm} (58)

$m_i$ is the smallest mass of any state $\phi$ for which the matrix element $\langle \phi | A^i(x) | \Lambda \rangle \neq 0$. Remember that the vacuum $\Lambda$ is not a possible $\phi$. $m_{ij}$ is the smallest mass of any state $\phi$, apart from the vacuum, for which the matrix element $\langle \phi | A^i(x) A^j(y) | \Lambda \rangle$ is not identically zero. In both cases we can refine the result if the minimal mass state $\phi_{\mu_{\min}}$ is a 1-particle state. Such a 1-particle state will produce a simple pole $(p_i^2 - m_i^2)^{-1}$ or $((p_i + p_j)^2 - m_{ij}^2)^{-1}$, and apart from these poles we have analyticity up to $M_i^2$ and $M_{ij}^2$ respectively, where $M_i$, $M_{ij}$ are the masses of the lowest lying states $\phi$ with two or more particles.

We take $\varphi^--N$ scattering as an example, with $A^1$ and $A^3$ $\varphi^-$-fields, $A^2$ and $A^4$ nucleon fields. Then
\[ m_1 = m_3 = m_{\gamma}, \quad m_2 = m_4 = m_N, \]
\[ M_1 = M_3 = 3m_{\gamma}, \quad M_2 = M_4 = m_N + m_{\gamma}, \]
\[ m_{24} = m_{\gamma}, \quad m_{12} = m_{34} = m_N, \]
\[ M_{13} = M_{13} = 2m_{\gamma}, \quad M_{24} = 2m_{\gamma}, \quad M_{12} = M_{34} = m_N + m_{\gamma}. \]

The singularities occurring at \( p_i^2 > M_i^2 \), etc., are called the \( p_i^2 \) cuts, etc.

The function \( \tilde{F}(p_1, \ldots, p_4) \) for real values of \( p_i \) is everywhere the boundary value of \( F(p_1, \ldots, p_4) \), where the boundary has to be approached from within the part of the domain (58) in which all \( \text{Im}(p_i^2) \) and all \( \text{Im}((p_i+p_j)^2) \) are positive. This prescription is sometimes referred to as the \( \text{+i}\epsilon \) prescription.

We intend to obtain analyticity on the mass shell from off-shell analyticity simply by restriction: the restriction of \( \tilde{F}(p_1^2 - m_i^2) F(\ldots) \) to the mass shell \( p_i^2 = m_i^2 \) (remember the \( 1 \)-particle poles in \( F \)!) is analytic in the intersection of the complete analyticity domain with the mass shell. However, if we try to do this with the primitive domain (58) we are in for a disappointment: the intersection of (58) with the mass shell is empty. Before we can go over to the mass shell we must yet enlarge the off-mass shell domain in some way. How can we do this? It turns out that physics is of little help at this point, but fortunately we have more luck with mathematics. I can therefore no longer avoid mentioning some mathematical facts of a non-trivial nature.

\( F(p_1, \ldots, p_4) \) is an analytic function of several variables, and these functions have some strange properties which are
not present in the familiar case of one variable only. One of these properties is the possibility of analytic completion. In the case of one variable $z$ it is known that for any open set $\mathcal{O}$ in the complex $z$-plane there exists a function $f(z)$ which is analytic exactly in $\mathcal{O}$, i.e. it cannot be analytically continued across the boundary of $\mathcal{O}$ anywhere. For several variables it is no longer so: it can happen that all the functions which are analytic in a given domain $\mathcal{O}$ are in fact analytic in a larger domain $\overline{\mathcal{O}}$. The largest $\overline{\mathcal{O}}$ for which this is true is called the holomorphy envelope of $\mathcal{O}$. If $\overline{\mathcal{O}} = \mathcal{O}$, then $\mathcal{O}$ is called a natural domain of holomorphy. Now it turns out that the primitive domain (58) is not a natural domain, i.e. $F$ is still analytic in a larger domain, the holomorphy hull of (58). Unfortunately, the construction of $\overline{\mathcal{O}}$ for a given $\mathcal{O}$ is an entirely non-trivial problem. No simple general methods are available, but only some weak and cumbersome procedures applicable to special situations. Therefore the envelope of holomorphy of the domain (58) is not yet known. But we do know a part of it (i.e. a domain $\mathcal{O}'$ with $\mathcal{O} \subset \mathcal{O}' \subset \overline{\mathcal{O}}$) which is sufficiently large to give information on the mass shell.

I come, at last, to the analyticity of scattering amplitudes on the mass shell. The first problem presenting itself is that of finding a convenient parametrization of the mass shell, i.e. of finding appropriate variables in which to express functions defined on the mass shell. This is done most easily by using invariance. Let us take the case that all four $A^i$ are scalar fields (e.g. $\pi$-$\rho$-scattering). Then $F$ is invariant and can be written as function of a full set of basic invariants. In view of the condition $p_1^+ + \ldots + p_4 = 0$ we find that there are six independent invariants that can be formed out of the three independent
4-vectors on which \( F \) depends. The six invariants can be chosen at will from the set

\[
\mathcal{J}_i = p_i^2, \quad i=1,\ldots,4, \quad \text{Eq. (60)}
\]

\[
s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2.
\]

\( s, t, u \) are called \textit{Mandelstam variables}. The seven invariants \( \mathcal{J}_i, s, t, u \), are connected through the simple relation

\[
s + t + u = \sum_{i=1}^{4} \mathcal{J}_i. \quad \text{Eq. (61)}
\]

In order not to commit any injustice I write \( F \) and \( \hat{F} \) as a function of all seven invariants, even though only six of them are independent:

\[
F(p_1,\ldots,p_4) = F(\mathcal{J}_1,\ldots,\mathcal{J}_4,s,t,u) \quad \text{Eq. (62)}
\]

and similarly for \( \hat{F} \). I should mention that I am now talking about the amputated \( \hat{F} \)-function, i.e. the one with the 1-particle poles in the individual variables \( p_i \) removed (see Eq.(39)). The unamputated \( \hat{F} \) cannot be restricted to the mass shell. It should be noted, however, that the poles in the sums \( p_i + p_j \) are still present!

The mass shell restriction of \( F \) can now be very simply defined as

\[
F(s,t,u) = F(m_1^2,\ldots,m_4^2, s,t,u). \quad \text{Eq. (63)}
\]

Only two of the three variables \( s, t, u \), are independent:

\[
s + t + u = \sum m_i^2. \] The \( m_i \) are the masses of the particles involved.
In the case of $\eta$-$N$-scattering ($A^1$ and $A^3$ $\eta$-fields, $A^2 = \tilde{\varphi}_0$ and $A^4 = \varphi_0$ nucleon fields) we can again find a representation in standard covariants:

$$\hat{\mathbb{F}}_{\eta i}(p_1, \ldots, p_4) = \sum_i c_{\eta i}^i (p_1, \ldots, p_4) F_i(\gamma_1, \ldots, \gamma_4, s, t, u),$$

(64)

where the $c_{\eta i}^i$ are finitely many suitably chosen polynomials, and the $F_i$ are invariant and analytic in the same domain as the original $F$. The $c_{\eta i}^i$ are typically of the form

$$\delta_{\eta i}^j$$

or

$$\phi_{\eta i}^j p_{\eta i}^j$$,

e etc., with $p_{\eta i}^j$ a linear combination of the $p_j$. I am here not interested in the problem of finding an optimal set of covariants. All I care about is that the existence of a representation (64) allows to reduce the $\eta$-$N$-case to the invariant case by considering the functions $F_i$ instead of $F$. The same reduction can be achieved also in more complicated cases (scattering of particles with arbitrary spin), so that I can restrict myself henceforth to invariant amplitudes.

In order to relate the on-shell function $F(s, t, u)$ with scattering amplitudes I must yet make a few remarks on kinematics. What processes are described by the mass shell restriction $\hat{\mathbb{F}}(s, t, u)$ of $\hat{\mathbb{F}}(p_1, \ldots, p_4)$? Obviously various ones, depending on which $p_1$ lie in the forward cone and which in the backward cone. This is so because the momenta of the out-particles occur with a minus-sign in the reduction formula (39), hence are associated with variables in the backward cone. We must now remember the assumption that the 1-particle hyperboloids are isolated from the continuum in their superselection sector. This assumption is essential! It means that we consider only stable particles. (Remember, however, that we are ignoring weak and electromagnetic interactions, so that the pion is treated as being stable).
Under this stability assumption the momentum conservation \( \Sigma p_i = 0 \) can only be satisfied if two \( p_1 \) are in the forward cone (corresponding to in-particles), and two in the backward cone (out-particles). The process \( 34 \rightarrow 12 \) is trivially related to the process \( 12 \rightarrow 34 \) by CTP invariance (which is a consequence of our postulates), hence need not be considered separately. We can therefore arbitrarily fix \( p_1 \) to lie in the forward cone and are left with three essentially different possibilities.

One possibility is the process \( 12 \rightarrow 34 \), i.e. \( p_1 \) and \( p_2 \) are the momenta of the incoming particles, \( -p_3 \) and \( -p_4 \) the momenta of the outgoing particles. In this case \( s = (p_1 + p_2)^2 \) is the invariant mass of the scattering state, or, in other words, the square of the energy in the centre-of-mass system. In deference to this important role of \( s \) the process in question is called the \textit{s-channel}. \( t = (p_1 + p_3)^2 \) is the square of the momentum transfer from particle 1 to particle 3, \( u \) the squared momentum transfer from particle 1 to particle 4. The values of \( s, t, u \), that correspond to physically possible \( p_1 \) are called \textit{physical values}, the set of all physical values is called the \textit{physical region} in the \textit{s-channel}. It is easy to see that the physical values in the \textit{s-channel} must satisfy the conditions

\[
\begin{align*}
s & \geq (m_1 + m_2)^2, \\
t & \leq (m_1 - m_3)^2, \\
u & \leq (m_1 - m_4)^2,
\end{align*}
\]

plus, in general, relations involving two of the variables at the same time. Only if all four masses are the same are the conditions \((65)\) sufficient to characterize the physical region.
Analogously we define the $t$-channel $13 \rightarrow 24$ and the $u$-channel $14 \rightarrow 23$ and their respective physical regions. Because of the linear relation existing between $s,t,u$, we can plot the three physical regions in a plane, using an oblique system of coordinates. The result is the famous Mandelstam diagram (Fig. 4).

Fig. 4 Example of a Mandelstam diagram depicting the 3 physical regions in the $s$-$t$-$u$ plane.
An essential feature of any Mandelstam diagram is that the 3 physical regions do not overlap.

I devote the rest of the lecture to an enumeration of some of the results obtained so far about analyticity, without spending any more time on proofs. I shall mention first some results obtained by J. Bros, H. Epstein, and V Glaser by exactly the methods I have outlined above.

1st result: local analyticity of scattering amplitudes. The physical points themselves are not analyticity points of $F(s,t,u)$, since they lie, for example in the s-channel, on the cut $s \gg M_{12}^2$, etc., and these cuts do not disappear in the process of analytic completion explained above. (Obviously $M_{12}^2 \leq (m_1 + m_2)^2$, since the initial 1-2 state is a possible in the criterion determining $M_{12}$.) However: every physical point lies on the boundary of the analyticity domain of $F(s,t,u)$. Take the s-channel as example and write $F$ as a function of the two independent variables $s$ and $t$. Let $P$ be any point in the interior of the physical region (i.e. not on its boundary). Then there exists a complex $(s,t)$-neighbourhood of $P$ in which $F$ is analytic with the exception of the cut $s \gg M_{12}^2$. The physical scattering amplitude is the boundary value of $F$ obtained by approaching the s-cut from above, that is from the side where $\text{Im } s$ is positive. Nothing is yet said about the size of these neighbourhoods. But it is clear that the union of all these neighbourhoods for all points in the s-region forms a connected domain. This means that if we know $F(s,t)$ exactly in an arbitrarily small open set in the physical region, we can, in principle, calculate it everywhere by analytic continuation. In practice this method does not work because of the instability problems discussed in Professor Hamilton's lectures. Analogous results hold in the other two channels.
2nd result: crossing. In the first result it has been established that each of the three scattering amplitudes described by \( F \) is analytic in a certain domain. The crossing property states that these three domains are connected one to the other, i.e. \( F \) is analytic in a connected domain containing all three physical regions in its boundary. This implies that if the scattering amplitude in one channel is known, then the amplitudes corresponding to the other channels can be obtained from it by analytic continuation (in principle, in practice this is not easy to do).

Take the standard example, \( N-\pi \)-scattering in the s-channel, (the particles 1 and 3 are nucleons) and assume its amplitude to be known. Then the amplitude for the t-channel process \( NN \rightarrow \pi \pi \) can be calculated from it. (Note that in changing a particle from an incoming to an outgoing one we must replace the particle by its antiparticle. This is so because of the form (7) of the asymptotic fields, which combines a destruction operator of a particle with a creation operator of its antiparticle.) In the u-channel we have again \( N-\pi \)-scattering, so that analytic continuation from the s- to the u-channel, with the appropriate permutation of variables, must reproduce the original amplitude (I disregard the isospin complications). This property of the \( N-\pi \)-amplitude is called crossing symmetry. It implies the symmetry

\[
F(s,t,u) = F(u,t,s)
\]

(66)

of the analytic function \( F \), holding in its domain of analyticity.

The remaining results I wish to discuss have actually been obtained by techniques which differ somewhat from the ideas outlined above. Those techniques are, however, based
on exactly the same assumptions and are therefore equivalent in every respect. Nevertheless, it would be nice to have proofs along our more straightforward lines. Finding them is a task left for the future.

3rd result: fixed-t dispersion relations. While the two previous results hold for any values of the particle masses involved, as long as they are all larger than zero and all particles are stable, we come now to a result that holds only if the masses $m_i$, $M_i$, $M_{ij}$, etc., satisfy certain conditions which I will not state explicitly. A list of processes for which dispersion relations have been proved will be given later on.

The analyticity property underlying dispersion relations is the following. Consider the $s$-channel amplitude $F(s, t)$ for complex $s$ and a fixed, real, physical value of $t$ (i.e. $t \leq (m_1 - m_3)^2$, in the case of elastic scattering $t \leq 0$). If the mass conditions mentioned above are satisfied, then $F$ is, as a function of $s$, analytic in the full complex $s$-plane with the exception of some singularities on the real axis (Fig.5).

![Diagram](image-url)

**Fig.5** Domain of analyticity of $F(s, t)$ for fixed $t$. 
For the explanation of these singularities we go back to the primitive domain (58). We are, of course, only interested in the singularities concerning pairs $p_i + p_j$, since the $p_j^2$ have a fixed value. The s-cut in Fig. 5 comes from the cut $(p_1 + p_2)^2 \geq M_{12}^2$ in (58). If there are 1-particle states $\phi$ with $\langle \phi | \Lambda^1 \Lambda^2 | \Lambda \rangle \neq 0$ they give rise to poles below the s-cut. The u-cut comes from the cut $u = (p_1 + p_4)^2 \geq M_{14}^2$. Its position in the s-plane is given by the relation $s = \sum m_i^2 - t - u$; it extends from $u_0 = \sum m_i^2 - M_{14}^2 - t$ to $-\infty$. 1-particle poles can also come from the u-channel. The t-channel singularities are irrelevant because $t$ is fixed to a value away from them.

From this analyticity domain we can derive a dispersion relation in exactly the same way we derived the Källén-Lehmann representation of the 2-point function. If $F(s, t)$ is sufficiently well behaved at infinity we obtain an unsubtracted dispersion relation:

$$F(s, t) = \sum i q_s(t) \left( \frac{1}{s-s_i} \right) + \frac{1}{2\pi i} \int_{M_{12}^2}^\infty \frac{ds'}{s-s'} A_s(s', t)$$

$$- \frac{1}{2\pi i} \int_{-\infty}^{u_0} \frac{ds'}{s-s'} A_u(s', t). \quad (67)$$

The first term on the right is the contribution from the poles. The $s_i$ are the pole placements, the residue functions $q_i$ have a simple $t$-dependence depending on the spins of the particles involved. The second term is the contribution of the s-cut. $A_s$ is the discontinuity across the cut:

$$A_s(s', t) = F(s'+i\xi, t) - F(s'-i\xi, t) \quad (68)$$

It is called the absorptive part of the scattering amplitude. For elastic scattering it is simply the imaginary part of the
amplitude, hence can be very easily obtained from the amplitude. The same is true for inelastic scattering if the theory is PT-invariant or, equivalently, C invariant (C means charge exchange: the exchange of particles and antiparticles). For elastic scattering in the forward direction \( t = 0 \) we can apply the optical theorem (eq. (30)) : the absorptive part is given by the total cross section, which is directly accessible to experiment. At least it is so in the physical region of the cut given by the condition \( s \gg (m_1 + m_2)^2 \). If \( M_{12}^2 < (m_1 + m_2)^2 \) then we find an unphysical part at the beginning of the cut, where \( A_s \) can only be obtained by analytic continuation from the physical region. An example where this happens is K-K-scattering. Here the physical region starts at \( 4m_K^2 \), but \( M_{12}^2 = 4m_K^2 \), because 2\( \pi \)-states can have the same quantum numbers as a K-K-state. For non-forward scattering \( (t < 0) \) there is always a non-physical part of the s-cut present at the beginning of the cut.

The last term in (67) comes from the u-cut. Its integrand is defined by

\[
A_u(s',t) = F(s' - i\xi, t) - F(s' + i\xi, t),
\]

which becomes, if we write \( F \) again as a function of 3 variables:

\[
A_u(s',t) = F(s' - i\xi, t, u' + i\eta) - F(s' + i\xi, t, u' - i\eta)
\]

with

\[
u' = \frac{\xi}{4} m_i^2 - t - s'
\]

Hence \( A_u(s',t) \) is the absorptive part in the u-channel in the point \( (a', t) \). \( (u' \text{ is the invariant mass, } t \text{ the momentum transfer of the u-channel reaction. Remember that } F(s, t, u) \text{ is the u-channel amplitude for } (s, t, u) \text{ in the physical region of the u-channel.}) \)

In general it is necessary to use subtracted dispersion relations. They are derived in the same way we derived the
subtracted KL-representation (54). It can be proved from very
general assumption that two subtractions are always sufficient,
in other words that $A_s(s',t)/s'^3$ is always integrable at
infinity.

Whenever dispersion relations can be proved they can be
proved only for $t$ in a finite interval, never for arbitra-
riely negative $t$. Once a dispersion relation has been proved
for a certain $t$-range from locality and spectrum, this $t$-
range can then be enlarged by using the unitarity of the
$S$-matrix. The realization of this possibility is the most im-
portant progress in the field made in the last few years.
It is due to A. Martin.

There follows a list of rigorously proved dispersion re-
lations with the $t$-ranges in which they hold. $t$ is measured
in units of $m_\pi^2$ ($m_\pi$ the pion mass). The list is taken from
a recent review by G. Sommer. For the reference see the
literature section at the end of the lectures.

Table of exactly proved dispersion relations

<table>
<thead>
<tr>
<th>Process</th>
<th>Range of momentum transfer $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi \pi \rightarrow \pi \pi$</td>
<td>$-20 &lt; t &lt; 4$</td>
</tr>
<tr>
<td>$KK \rightarrow KK , { }$</td>
<td>$-21,8 &lt; t &lt; 4$</td>
</tr>
<tr>
<td>$\pi K \rightarrow \pi K$</td>
<td>$-34,9 &lt; t &lt; 4$</td>
</tr>
<tr>
<td>$\pi K \rightarrow KK , { }$</td>
<td>$-31,6 &lt; t &lt; 3,2$</td>
</tr>
<tr>
<td>$\pi \Lambda \rightarrow \pi \Lambda$</td>
<td>$-14,9 &lt; t &lt; 4$</td>
</tr>
<tr>
<td>$\pi \Sigma \rightarrow \pi \Sigma$</td>
<td>$-1,8 &lt; t &lt; 0,8$</td>
</tr>
<tr>
<td>$\pi \Lambda \rightarrow \pi \Sigma$</td>
<td>$-5,5 &lt; t &lt; 1,9$</td>
</tr>
<tr>
<td>$NN \rightarrow NN$</td>
<td>$-18 &lt; t &lt; 4$</td>
</tr>
</tbody>
</table>
The last process in this table is the only one directly related to the subject of this school. I have given the full list just to give you an impression of what is known. The fact that the people working in the field are all high-energy physicists explains why no processes involving nuclei are included in this list. This does not mean that dispersion relations cannot be proved for such processes.

*Not provable* with the presently known methods are dispersion relations for nucleon-nucleon scattering and for kaon-nucleon scattering. Of course, it has also not been proved that dispersion relations cannot hold in these and other cases, i.e. that \( f(s,t) \) does have singularities apart from the obvious ones collected in Fig. 5. It should be mentioned that a special type of such unorthodox singularities is known to occur in perturbation theory (in the analysis of Feynman graphs): there it sometimes happens that a cut, say the \( s \)-cut, starts earlier than at the expected threshold \( \sqrt{s} \). These additional singularities are known as 'anomalous thresholds'. They must be taken into account if dispersion relations are used for processes for which they have not been rigorously demonstrated.

**4th result: analyticity in** \( t \) **for fixed real** \( s \). We fix \( s \) at a real physical value and consider the \( s \)-channel amplitude \( f(s,t) \) as a function of the complex variable \( t \). Or rather it is customary to replace \( t \) by the cosine of the scattering angle \( \theta \) in the c.m.s. system. Consider the elastic scattering of two particles with masses \( m_1 \) and \( m_2 \). Then

\[
\cos \theta = 1 + t/2k^2, \quad k^2 = \frac{1}{4s} \left[ \frac{s-\left(m_1+m_2\right)^2}{s-\left(m_1-m_2\right)^2} \right]
\]

(69)
$K$ is the absolute value of the 3-momentum of either one of the particles in the centre-of-mass system. $F$ and the absorptive part $A_s$ can be written as functions of $s$ and $\cos \mathcal{J}$, and their analyticity for complex values of the variable $\cos \mathcal{J}$ can be examined. This information is needed in the traditional proofs of dispersion relations. It is also used to perform the analytic continuation necessary to obtain $A_s$ on the unphysical part of the cut (see above).

The first results obtained where the so-called Lehmann ellipses. Lehmann proved that both $F(s, \cos \mathcal{J})$ and $A_s(s, \cos \mathcal{J})$ are analytic in $\cos \mathcal{J}$ in ellipses with foci at $\pm 1$. $F$ is analytic in the small Lehmann ellipse with principal semi-axis

$$x_0 = \sqrt{1 + \frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{K^2(s - (M_1 - M_2)^2)}}.$$

$M_1, M_2$ are the masses of the lowest states with the same quantum numbers as the scattering particles, excepting the $1$-particle states themselves. $A_s$ is analytic in the large ellipse with principle semi-axis $2x_0^2 - 1$. This is not yet very satisfactory from the point of view of applications. In particular it is disturbing that the ellipses shrink to a stretch for $s \to \infty$. The results have been considerably improved by various people who have found much larger ellipses in some regions. For the exact results I refer to the review article by Sommer which I have already mentioned (see literature section). Let me just mention that sufficient $\cos \mathcal{J}$-analyticity is known to allow the necessary analytic continuations of the absorptive part in all exactly proved dispersion relations.

Finally I want to at least mention some further analyticity results that I did not have the time to consider.
There are results on simultaneous analyticity in $s$ and $t$ going much beyond what I have stated in the 1st result. I wish only to say that Mandelstam analyticity has not been proved for any value of the masses. (Mandelstam conjectured that the cuts $s \gg M_{12}^2$, $t \gg M_{13}^2$, $u \gg M_{14}^2$ and the 1-particle poles are the only singularities of $F(s,t,u)$.) Moreover, Martin has shown that it cannot possibly be proved with the methods presently available. This does not yet exclude the possibility that Mandelstam analyticity is a consequence of our postulates, since the presently known methods use the information contained in unitarity only in a rudimentary fashion.

Another thing I have not talked about are partial wave dispersion relations. Also not mentioned were the various bounds on the scattering amplitude that can be obtained from analyticity, like the famous Froissart bound. Several interesting results have been obtained recently in this context, but I cannot go into them.
LITERATURE

I have no intention of giving a complete or even representative list of references in the topics of my lectures. Rather I wish to mention some selected books and articles that might be helpful to the reader who is desirous to learn a bit more about these things.

The standard texts on axiomatic quantum field theory are:


Both these books are mainly concerned with the mathematical aspects of the formalism, the physical motivations and applications are not clearly expounded. In particular, they contain almost nothing about the main topic of my lectures, scattering theory (i.e. the asymptotic conditions, the S-matrix, etc.). On the other hand you find in them other important results like the CTP theorem and the spin-statistics theorem, which I have mentioned only in passing or not at all.

A discussion of the basic postulates and their motivation from a viewpoint slightly differing from mine, can be found in:

3) G. Raiton: Dispersion Techniques in Field Theory. W.A. Benjamin, Inc.: New York 1965

The basic ideas of the LSZ formalism can still be most profitably learned from the original papers:

A systematic treatment on a high level of mathematical rigour can be found in the Brandeis lectures (1965) of K. Hepp:


These lectures describe also the non-relativistic analogue of the LSZ formalism. This may be helpful to the nuclear physicist who does not feel entirely at home among the peculiarities of relativistic theory. The non-relativistic case is also treated extensively in

    An English translation of this paper is available in:

For the results on analyticity of scattering amplitudes see the recent review article by G. Sommer:


This article states all the presently known results, gives their proofs in part or in full, and contains an extensive list of references.

For the proof of dispersion relations you may also consult:

THE SCATTERING OF PIONS FROM NUCLEI

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1. INTRODUCTION

"It was said by Borgia of the expedition of the French into Italy, that they came with chalk in their hands to mark out their lodgings, not with arms to force their way in. I in like manner would have my doctrine enter quietly into the minds that are fit and capable of receiving it; for confutations cannot be employed when the difference is upon first principles and very notions, and even upon forms of demonstration". [Aphorisms*) 1.XXXV]

I have been charged in these lectures to review the theory, and thus to some extent the experimental results, of pion scattering from nuclei. Since Milan Locher is also talking about pions, we have agreed to divide the material so that I will only discuss processes where the pion does not vanish, i.e. elastic scattering, excitation of discrete nuclear levels, nuclear breakup, charge exchange -- both single and double. I will not consider much the very low-energy problem, which is inextricably tied up with mesic atoms. In general, the data tends to be old and of somewhat doubtful quality. Much the same can be said about the theory.

I shall keep returning throughout these lectures to the comparatively good data of the CERN-IPN group2) on the scattering of \( \pi^- \) from \(^{12}\text{C}\) in the energy range 120-280 MeV. Of course, some of the results will be peculiar to this nucleus, but it is of great use to have one fairly complete experiment with which to compare different theoretical approaches.

2. PHASE SHIFT ANALYSIS

Seemingly the first model-independent analysis one could perform on an elastic scattering distribution is a phase-shift decomposition. Since \(^{12}\text{C}\) has both spin and isospin zero, there is in this case only one amplitude

*) The aphorisms in this work are all taken from Ref. 1.
\[ f(\theta) = \frac{4}{k_x} \sum_{\lambda=0}^{\infty} (2\lambda + 1) f_{\lambda} P_{\lambda} (\cos \theta), \tag{1} \]

where the partial wave amplitude
\[ f_{\lambda} = \left( \eta e^{-i \delta_{\lambda}} - 1 \right)/2i \tag{2} \]
is expressed in terms of a real phase \( \delta_{\lambda} \) and an inelasticity parameter \( \eta_{\lambda} (0 \leq \eta_{\lambda} \leq 1) \). We shall use amplitudes normalized such that the differential cross-section
\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{3} \]
and by the optical theorem
\[ \sigma_T = \frac{4}{k_x} \int_m f(\theta). \tag{4} \]

Classically the number of important particle waves in a scattering is \( L \approx kR \), where \( R \) is some typical nuclear radius. This suggests that at 180 MeV we need at least five partial waves to describe \( \pi^{12} \). In addition the experimental data at large angles is, in general, non-existent. It is thus impractical to do a model-independent calculation. The Neuchâtel group\(^3\) took an optical model with free parameters to get predictions for the higher partial waves, \( \lambda \geq 2 \). This is similar in spirit to the \( pp \) phase shifters, who take the higher partial waves from a one-pion-exchange model. The radius \( R \) and the strength of the optical potential were fitted at each energy. As you can see from Fig. 1, the problem is not one of just fitting the data, but rather determining the uniqueness of the solution.

What have we learned from such an exercise? Apart from a sign ambiguity \( (d\sigma/d\Omega \text{ and } \sigma_T \text{ are both invariant under changes of sign of } \delta_{\lambda}) \),
Fig. 1 Prediction of the phase-shift analysis by Beiner and Huguenin\textsuperscript{3}) for $\pi^{-12}\text{C}$ scattering at 87.5 MeV. Data is from Ref. 29.

you can then predict the forward real part of the scattering amplitude. By demanding continuity in the energy, you can get round the ambiguity by linking up with $\pi$-mesic atom data. You can then compare your results with the analyses of the CERN-IPN Coulomb nuclear-interference measurements\textsuperscript{2}) (which I shall talk about later) or the forward dispersion relation predictions (which Locher has mentioned) and this is reasonable. One interesting point is that the optical potential r.m.s. radius $R$, which was necessary to give the higher partial waves, was found to vary quite strongly with energy. Its behaviour was roughly as

$$R^2 = R_0^2 + \frac{\beta^2}{R^2},$$

with $R_0 = 2.6 \pm 0.2 f$, $\beta^2 = 3.3 \pm 0.8$. One could also look for resonances in the $\pi\text{C}$ system, but I'm not very sanguine about this (it's not bloody
likely). Apart from this, you can compare the phases with the results of a dynamical model. This may be useful if you suspect that your dynamics is only getting part of the answer right. For example, you may just have trouble in the $s$-wave, and still get a terrible angular distribution. We shall come back to this later.

3. **IMPACT PARAMETER REPRESENTATION**

At higher energies there are many partial waves, and there seems to be nothing very special about any one of them. The phase shifts in a potential problem often show an equally smooth behaviour with respect to $\ell$ as to energy. It is for these reasons that the impact parameter representation is attractive. Now we can *always* represent the $^{12}$C amplitude in the form\(^4,\ 5\)

$$f(q) = i \frac{\hbar}{k} \int_0^\infty J_\ell(qb) b \, db \, \Gamma(b), \tag{6}$$

where $q$ is the momentum transfer, $q^2 = -t$ and $\Gamma(b)$ is the profile function, which of course is energy-dependent. This is not just a high-energy approximation, but the difficulty at low energies is that the inverse transform

$$\Gamma(b) = \frac{1}{2i \hbar} \int_0^{q_{\text{max}}} q \, dq \, \tilde{J}_\ell(qb) \tilde{f}(q) \tag{7}$$

involves only a finite range of integration, $q_{\text{max}} = 2k$. If we substitute the partial wave expansion into the integral of Eq. (7), it follows that

$$\Gamma(b) = -4i \sum (2\lambda + 1) \int_0^\infty \frac{J_{2\ell + \lambda}(q \hbar b)}{2 \hbar b} \, dq \tag{8}.$$

If $\Gamma(b)$ falls off fast for large $b$, then we can solve this equation approximately\(^6\) for $f_\ell$. 


\[ f_q = \frac{i}{2} \left[ \Gamma(b) - \frac{1}{8} k^2 \left( \frac{d^2 \Gamma}{db^2} \right) + O\left( \frac{1}{k^4} \right) \right] b = (l + \frac{1}{2})/k, \]  

(9)

You can see the usual identification of partial wave with impact parameter, \( kb = l + \frac{1}{2} \). At finite energies you need to know \( \Gamma(b) \) around this point to be able to find \( f_q \).

The basic advantage of the partial wave expansion is that unitarity is simple. If there is no inelasticity \( \eta_k = 1 \) and, in general, \( 0 \leq \eta_k \leq 1 \). Now to order \( 1/k^2 \) the unitarity relation in impact parameter space is

\[ 2 \Re \Gamma(b) = \varphi(b) + \left| \Gamma(b) \right|^2 + \frac{1}{4} \frac{d^2}{db^2} \left( \frac{b^2 \left| \frac{d}{db} \Gamma \right|^2}{b^2} \right), \]  

(10)

where \( \varphi \) represents the inelastic scattering. It is because of the derivative term that we do not have a simple relationship for \( \Gamma \). Thus if we write

\[ \Gamma(b) \equiv 1 - e^{i \chi(b)}, \]  

(11)

even if there is no inelastic scattering, \( \chi \) is not purely real; there is a small imaginary part of order \( 1/k^2 \). Providing \( \Gamma \) does not vary too fast, then you find that for no inelasticity

\[ \Gamma(b) \approx 1 - e^{i \chi(b) - \left[ \frac{\chi'(b)}{k^2} \right] / k^2}, \]  

(12)

satisfies the unitarity condition approximately, with \( \chi \) real. The important point to notice is that the corrections are always proportional to derivatives and are only significant in the regions where \( \chi \) varies fast.

So far the representation has been written down for the spinless case. Using the integral representation of the Bessel function, we have from Eq. (6)

\[ f(q) = \frac{i k}{2 \pi} \int d^2 b \, e^{i q \cdot b} \Gamma(b). \]  

(13)
When we later have to deal with spin, the only difference from the generalized formula (13) is that \( \Gamma(b) \) will depend upon the direction of \( b \) as well as its magnitude.

4. SIMPLE COLLECTIVE MODELS

In order to get a feeling for the elastic and inelastic scattering from nuclei, it is educational to get away from these formal considerations and take a very simple model for nuclear scattering. The oscillating liquid drop model has been studied extensively for electron scattering by Walecka\(^7\) and for strongly interacting particles by Blair\(^8\) and Inopin\(^9\), to whom I refer for more details. We assume that the drop is incompressible so that the radius may be written as

\[
R = a(1 + \sum q_{\ell m} Y(\theta, \phi) - \frac{\lambda}{4} \sum |q_{\ell m}|^2).
\]  

(14)

If the parameters satisfy \( q_{\ell-m}^* = q_{\ell m} \), then the volume is constant to order \( q^2 \), which is assumed to be a small quantity. The potential energy is provided by the surface tension. The drop just oscillates. The advantage of writing the radius in this way is that the classical Hamiltonian for the drop is diagonal in this representation

\[
\hat{H} = \sum_{\ell \geq 2} \left( \frac{\lambda}{2} B_{\ell} |P_{\ell m}|^2 + \frac{C_{\ell}}{2} |Q_{\ell m}|^2 \right).
\]

(15)

where in the absence of Coulomb forces

\[
B_{\ell} = \mu \frac{a_5}{\ell},
\]

\[
C_{\ell} = \sigma a^2 \ell (\ell - \lambda),
\]

(16)

with \( \mu \) the mass density and \( \sigma \) the surface tension. The generalized momenta are defined by
\[ p_{\ell m} = B_{\ell} q^+_{\ell m}. \]

Being a set of harmonic oscillators, it is trivial to quantize the system, introducing a set of creation and annihilation operators for these excitations, which are variously called phonons, surfons, etc. It was emphasized by members of the audience that this model is probably not too good for $^{12}\text{C}$, a permanently deformed description being much more appropriate. The lecturer was already aware of this, but since the final formulae are applicable to a range of deformed models (providing the $B_{\ell}$ and $C_{\ell}$ are treated as parameters to be determined), he preferred to present the prettier vibrational model. Taken strictly the model predicts the first excited state to be $2^+$ with energy $\varepsilon$ and the next as $3^-$ with energy $\sqrt{15/2} \varepsilon$. For $^{12}\text{C}$ the 4.4 MeV state is $2^+$ and the 9.6 MeV is $3^-$. The charge density is a constant inside the drop and zero outside

\[ \hat{\rho}(r) = \frac{3Z}{4\pi a^3} \theta(R-r) \]
\[ = \frac{3Z}{4\pi a^3} \theta(a \left[ 1 + \sum q_{\ell m} Y_{\ell m} - \frac{1}{4\pi} \sum q_{\ell m}^2 \right]. \tag{17} \]

After the quantization, the $\hat{q}_{\ell m}$ are to be interpreted as operators in the surfon space, as is $\hat{\rho}$. The form factor is the Fourier transform of the charge density

\[ \hat{F}(q) = \int e^{-i\frac{q\cdot r}{r}} d^3r \hat{\rho}(r) \tag{18} \]

and we have later to project out the required initial and final surfon states. Using a multipole expansion in Eqs. (17) and (18),

\[ \hat{F}(q) = \frac{3Z}{4\pi a^3} \int d\Omega \int_0^a \left[ 1 + \sum q_{\ell m} Y_{\ell m} - \frac{1}{4\pi} \sum q_{\ell m}^2 \right] \]
\[ \times \frac{4\pi}{\omega} \sum_{JM} \left(-1\right)^J j_J(qr) Y_{JM}(\Omega) Y_{JN}^{*}(\Omega - q). \tag{19} \]
The integrals can be done providing you expand the upper limit of integration in powers of \( q \). To lowest order

\[
\hat{F}(q) = \sum_{m} (-1)^m \sum_{\Omega q} \left( \mathcal{F}_{\Omega q} \right) \times \\
\times \left[ \frac{3 \tilde{f}^2(qa)}{q^2} \sqrt{4\pi} \sum_{\Delta c} + 3 \tilde{f}^2(qa) q^+_{\Delta m} \right].
\]

Projecting out the elastic scattering form factor (zero surfons initially and finally)

\[
F_o = Z \frac{\tilde{f}^2(qa)}{qa}
\]

and for the excitations

\[
F_L = 3 Z \frac{\sqrt{2 L+1}}{\sqrt{4\pi}} \frac{\Lambda}{(4 B_L C_L)^1} \tilde{f}^L(qa),
\]

where use has been made of the commutation properties of the \( q_{\Delta m} \). A different collective model would just give a different numerical value for \( q_{\Delta m}^+ \) and we will not in fact use this specific value. It would be a travesty to say that formulae (21) and (22) give good predictions for the \( ^{12}\text{C} \) form factors -- the nuclear edge is much too diffuse. It gives a better over-all description than the simplest particle-hole model since the low-lying states are extremely collective. Experimentally the radius for elastic scattering is smaller than that for the \( 2^+ \), which in turn is smaller than that of the \( 3^- \). As a general point, notice that for small momentum transfers \( F_L \sim q^L \). This follows from angular momentum considerations and we shall need it later.

Now we turn to the subject of our lectures, viz. pion scattering\(^1\). At 180 MeV the pion-nucleus amplitude is almost purely imaginary, very black\(^2\). It is therefore not too stupid to try describing the \( ^{12}\text{C} \) nucleus
by a black disc. In the impact parameter formalism this just means that
\( \Gamma(b) = 1 - e^{i \chi(b)} = 1 \) if \( b \) lies in the shadow of the nucleus and zero
otherwise. This corresponds to complete absorption, the phase-shift func-
tion \( \chi(b) \to i^{\infty} \) behind the nucleus. The partial wave formulation \(^8\) would
give a slightly different result, but we will not worry about such details
now.

The scattering amplitude is

\[
\hat{f}(q) = \frac{i \hbar}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty e^{i \frac{q}{\hbar} \hat{b}} b \, db
\]

We need the largest projection of the radius onto the plane perpendicular
to \( k \), and to lowest order in \( q \), this means we have to take the argument
of \( Y_{\lambda m} \) as \( \theta = \pi/2 \).

Now

\[
e^{i \frac{q}{\hbar} \hat{b}} = J_0(q \hat{b}) + 2 \sum_{n=0}^{\infty} \frac{(i \hbar)^n}{n!} J_n(q \hat{b}) \cos(n \phi)
\]

and expanding about the upper integration limit we can do the integrals

\[
\hat{f}(q) = i \hbar a^2 \left[ \frac{J_1(qa)}{qa} + \sum_{LM} (-1)^M \frac{(2L+1)}{\sqrt{4\pi}} J_M(qa) \left[ L \cdot M \right] \hat{q}^+ \right]
\]

with numerical coefficients

\[
\left[ L \cdot M \right] = \frac{i^L \left[ (L-M)! (L+M)! \right]^{1/2}}{(L-M)!! (L+M)!!}
\]

Projecting out in surfon space, the elastic scattering is

\[
\hat{f}_0(q) = i \hbar a^2 \frac{J_1(qa)}{qa}
\]
and the inelastic scattering is

\[ f_{LM}(q) = (-1)^L i \frac{\kappa a^2}{\sqrt{4\pi}} \frac{(2L+1)^{1/2}}{\sqrt{4\pi}} J_M(qa) \left[ L : M \right] \frac{A}{(4B_C C_L)^{1/4}} \]  

(28)

In Fig. 2 are plotted the quantities \([2J_1(x)/x]^2\)

\[ \sum_M \left( J_M(x) [2 : M] \right)^2 = \frac{1}{4} \left[ J_0^2(x) + 3 J_2^2(x) \right], \]

and

\[ \sum_M \left( J_M(x) [3 : M] \right)^2 = \frac{1}{8} \left[ 3 J_4^2(x) + 5 J_2^2(x) \right]. \]

The elastic form falls fast to a minimum (a zero) at \(x^2 \sim 15\), then has a maximum, minimum ad infinitum. The \(2^+\) has maxima where the elastic has minima and vice versa. It is completely out of phase with the elastic. The \(3^-\) is back in phase again. This Blair phase rule\(^8\) can be seen immediately from the properties of the Bessel functions. Experimentally the CERN-IPN \(2^+\) cross-section has a maximum and minimum just out of phase with the elastic which is encouraging. Secondly, the total cross-section of \(68.1 \text{ f}^2\) gives a radius \(a = 3.29 \text{ f}\). From this we deduce that the first elastic diffraction minimum should occur at \(q = 1.16 \text{ f}^{-1}\). Experimentally the figure is perhaps 72% greater than this. The difficulty comes when we study the angular distribution in detail -- the theoretical cross-section does not fall off quickly enough. To circumvent this difficulty, without losing the essential simplicity of the formalism, Inopin\(^9\) introduced the concept of a fuzzy black disc. Instead of having a sharp edge, he smeared this over another distribution, thus getting the convolution of two densities. This convolution is disentangled by the Fourier transform and the end result is that all the predicted cross-sections should be multiplied by the Fourier transform of the smearing function. This function is arbitrary, but should be well localized in \(r\)-space and once
Fig. 2  Plot of the squares of the Bessel functions required in the deformed black sphere model [Eq. (28)].

fixed for elastic scattering, it has to be used for the inelastic scattering. Using a Gaussian

$$\frac{d\sigma}{d\Gamma} = \sqrt{\alpha} e^{-|\beta_0^2|\Gamma} \left| \frac{J_1(q\alpha)}{q\alpha} \right|^2,$$

(29)

with $\beta^2 = 0.075$ (Note: $\beta^2 \ll 1$!), you get the predictions of Fig. 3. The slope at the origin is now in agreement as well as the height of the subsidiary maximum. The shapes of the $2^+$ and $3^-$ excitations are now completely determined and with adjusted normalizations they are shown in Fig. 4. The $2^+$ is very well fit. The $3^-$ is much poorer, but the data
Fig. 3 Prediction of the black sphere model for 180 MeV \( \pi^{-12}\text{C} \) elastic scattering\(^2\).

are also much worse. It was not always easy to separate experimentally the contribution of the \( 0^+ \) (7.6 MeV) level, which is unfortunate since different hypotheses will give quite different predictions for this excitation. Also when experimental resolution gets better, we may be able to learn what happens at small \( q^2 \), where measurements are swamped by the forward elastic diffraction peak. It would be interesting to see if there actually are oscillations at small angles caused by the different polarizations.

In principle, the absolute normalization of the excitation spectra is completely predicted from the electron scattering values of the \( \langle |q^+_{LM}| \rangle \). There is some difficulty with regards to the radius to be used
Fig. 4 Predictions of the deformed black sphere model for the excitation of the $2^+ (4.4 \text{ MeV})$ and $3^- (9.6 \text{ MeV})$ levels in $^{12}\text{C}$ with 180 MeV incident $\pi^-$. 

in interpreting the electron data, but taking the measured values for each $L$, we estimate pion cross-sections about a factor of two too low. However, the behaviour of the pion data around $q^2 = 1 \text{ fm}^{-2}$ determines the normalization and in this region the Inupin fudge factor is also about $\frac{1}{2}$. It could, therefore, be argued that there is no significant discrepancy between the electron and pion results. In summary the strong absorption model of collective excitations reproduces the 180 MeV $\pi^{12}\text{C}$ data quite well. Since the same can be said for $\alpha$-particle scattering, what can we learn from the results? The values of the radius, smearing factor, and the normalizations of the $2^+$ and $3^-$ predictions all depend upon the basic interaction as well as the properties of the nucleus. We have not calcu-
lated them. It does suggest though that when the interaction is so black, there are at most only four parameters we can determine from such data.

5. SINGLE SCATTERING APPROXIMATION

"The honour of the ancient authors, and indeed of all, remains untouched, since the comparison I challenge is not of wits or faculties, but of ways and methods, and the part I take upon myself is not that of a judge, but of a guide." [Aphorisms 1.XXXII]

After the strong absorption models, we now go to the opposite limit, where the pion-nucleon interaction is supposed so weak that the scattering of a pion from a nucleus can be thought of as occurring on only one nucleon\(^{11,12}\). The region of validity of this approach would appear small, very light nuclei and well away from the pion-nucleon \(\frac{3}{2}-\frac{3}{2}\) resonance. It is more important than this since the multiple scattering approaches that we discuss later will have this as the first term in their expansions.

Consider a simplified deuteron with no spin. Elastic \(\pi d\) scattering is to be thought of as the single scattering from either of the two nucleons bound in the deuteron. We must, therefore, examine graphs of the form of Fig. 5, where the blob is a pion-nucleon scattering.

![Fig. 5](image)

Now let us work in the brick wall frame of Fig. 6. This treats the initial and final states symmetrically, yet keeps the nucleus as non-relativistic as possible. We now write down a four-dimensional Feynman integral corresponding to this graph, and make the crucial approximation of putting the spectator nucleon (marked with a cross) on its mass shell.

\[
\frac{\uparrow}{E^2 - p^2 - m^2 + i\epsilon} \rightarrow \frac{\uparrow}{E} \delta \left( E - \sqrt{p^2 + m^2} \right).
\]  \hspace{1cm} (30)
Unfortunately we have to make some sort of approximation like this, since we know almost nothing about the behaviour of the d-np vertex (Fig. 7), when both of its nucleons are off their mass shells. However, if we can make a few non-relativistic kinematic approximations, then it is easy to see that with one leg on the mass shell, the diagram of Fig. 7 is a deuteron wave function in momentum space \( \phi(\vec{p} - \vec{q}/4) \). Feeding this into our Feynman integral we obtain

\[
F_{\pi \Delta d}(s, t) = \int d^3 \vec{p} \, \phi(\vec{p} - \vec{q}/4) \int d^3 \vec{q} \, \phi(\vec{p} + \vec{q}/4),
\]

where we are using invariant variables

\[
t = -q^2
\]
\[
s = (\pi d \text{ invariant mass})^2
\]
\[
s_a = (\pi p \text{ invariant mass})^2
\]
We have now implicitly assumed the impulse approximation so that we can forget the off-shell nature of the $\pi N$ amplitude and just take a physical amplitude at some value of $s_a$ and $t$. "Man, being the servant and interpreter of Nature, can do and understand so much and so much only as he has observed in fact or in thought of the course of nature. Beyond this he neither knows anything, nor can do anything." [Aphorisms 1.I]

The next approximation we shall make is to assume that the pion-nucleon amplitude is varying so smoothly that we can take it out of the integral with its energy evaluated at $\tilde{p} = 0$.

$$F_{\pi d}(s, t) = \int_{\pi p} F_{\pi p}(S_{a, t}) \int d^3p \phi(p - \frac{q}{4}) \phi(p + \frac{q}{4}).$$ (33)

By Fourier transforming the integral we find

$$F_{\pi d}(s, t) = \int_{\pi p} F_{\pi p}(S_{a, t}) \int d^3r \left| \frac{\vec{q}}{2} \right|^2 |\psi(r)|^2$$

$$= \int_{\pi p} F_{\pi p}(S_{a, t}) S(q/2),$$ (34)

where $S$ is the deuteron form factor. There is one point of difference with the usually quoted result, viz. the connection between $s$ and $s_a$.

In terms of the laboratory pion kinetic energies, this may at SIN energies be approximated by

$$T = T + \frac{q^2}{16} m^2,$$ (35)

where we have neglected the binding energy. Thus for a non-forward scattering, the pion-nucleon amplitude is required at a higher energy than the pion-deuteron. This is clearly necessary, because by simple kinematics, for fixed incident energy, the maximum value of $t$ is greater for $\pi d$ scattering than for $\pi p$ which would make nonsense of Eq. (34). The effect
is not large. For 180 MeV pion scattering through 90\(^\circ\) it is about 10 MeV. For heavier nuclei, the kinematics make it approach a factor of two larger.

Let us go back one stage and investigate the effects of Fermi motion\(^{13,14}\). If we know \(f_{np}^2\) as a function of \(s_{\alpha}\) and \(t\), then we can do the integral over \(p\). In view of the rapid decrease in the form factor with \(q\), one has to be very careful that the approximations in evaluating the integral do not destroy this feature. If the wave function is Gaussian

\[
\psi(p) \sim e^{-\frac{\alpha^2 r^2}{2}}
\]

\[
\phi(p) \sim e^{-\frac{\beta^2 p^2}{2}}
\]

then so is the form factor

\[
S(q/2) = e^{-\frac{q^2}{16} \alpha^2}
\]

In this special case, Eq. (31) reduces to

\[
\bar{F}_{\pi d}(s, t) = S(q/2) \int d^3p \left( \phi(p) \right)^2 \int d^3p \left( S_{\alpha, t} \right).
\]

Even though this factorization is only really valid for a Gaussian wave function, it is a very simple prescription which should roughly take into account the effects of Fermi motion for light nuclei. The smearing function for deuterium\(^{14}\) has a width of the order of \(\Gamma \sim 40\) MeV, significantly less than the width of the \(\frac{1}{2}^+ - \frac{1}{2}^+\) resonance. One should therefore still see resonances clearly in \(\pi d\) total cross-sections. The smearing tends to shift peaks by a couple of MeV, but this like binding effects is very model-dependent. We have chosen to put the "spectator" nucleon on its mass shell. If we choose to put the other internal lines on-shell instead, then we get a shift the other way. Since we have not attempted to treat the dynamics of the nuclear variables, we cannot say which is the better prescription. Experimentally some half-way stage may be preferred\(^{15}\).

The \(q^2\) shift of Eq. (35) also depends upon which nucleon is put on-shell.
The ambiguities could presumably be understood on the basis of a genuine three-body approach such as the Fadeev equations, but I do not think anybody has seriously tried it yet. There are small relativistic corrections when $q$ gets large, such as the Lorentz contraction of the deuteron, but I shall ignore these completely.

So far we have not discussed the effects of spin. The nucleon having spin $\frac{1}{2}$, there are two independent amplitudes for $\pi N$ scattering\textsuperscript{16}. Denoting the initial and final pion and nucleon momenta by $k_1$, $k_2$, $p_1$, $p_2$, the invariant amplitudes $A$ and $B$ are connected to the $T$-matrix by

$$T = A + \frac{\lambda}{2} i \gamma_\mu (p_1 + p_2) \gamma^\mu B.$$  \hspace{1cm} (39)

This has to be sandwiched between four component nucleon spinors $\tilde{u}_f(p_2)$ and $u_i(p_1)$. The unpolarized centre-of-mass differential cross-section becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi}\right)^2 \frac{\lambda}{s} \sum_{\text{Sum average}} |\tilde{u}_f T u_i|^2.$$  \hspace{1cm} (40)

In dealing with the scattering from nuclei, if we want to exploit the nuclear wave functions, we must reduce Eq. (39) in terms of two component Pauli spinors $\chi$. This reduction is usually carried out in the centre-of-mass frame\textsuperscript{16}, and for completeness I shall give these results.

If

$$M = f + i \sigma \frac{(\vec{k}_1 \wedge \vec{k}_2)}{|\vec{k}_1 \wedge \vec{k}_2|} \chi,$$  \hspace{1cm} (41)

where

$$\frac{d\sigma}{d\Omega} = \sum_{\text{Sum average}} |\chi_f M \chi_i|^2.$$  \hspace{1cm} (42)
then

\[ \tilde{f} = \tilde{f}_1 + \tilde{f}_2 \cos \theta \]

\[ q = -\tilde{f}_2 \sin \theta \]

\[ \tilde{f}_1 = \frac{E + m}{8 \pi W} \left[ A + \left( W - m \right) B \right] \]

\[ \tilde{f}_2 = \frac{E - m}{8 \pi W} \left[ - A + \left( W + m \right) B \right], \]

where \( W \) is the c.m. energy (\( = \sqrt{s} \)) and \( E \) and \( m \) are the energy and mass of the nucleon, respectively.

Now the reduction we need is not in this frame, but rather in the laboratory or brick-wall frames. In general the distinction between spin-flip and spin-non-flip is frame dependent. It is, therefore, important when we start sandwiching between non-relativistic wave functions that we work consistently in one frame.

For the brick-wall case (Fig. 6)

\[ M = \tilde{f} + i \sigma \left( \hat{q} \cdot \hat{k} \right) \]

\[ \tilde{f} = N \left[ A \sqrt{1 - \frac{4m^2}{W}} \right] + B \left\{ \frac{5 - m^2 - \mu^2 + \frac{1}{2}}{2m \sqrt{1 - \frac{4m^2}{W}}} \right\} \]

\[ q = N \sigma \frac{k \cdot q}{2m} \]

\[ \sigma \]

\[ (44) \]

\[ (45) \]
and the kinematic transformation factor is

\[ N = \frac{m}{8\pi w} \left[ \frac{\partial (\cos \theta_{\text{c.m.}})}{\partial (\cos \theta_{\text{brick-wall}})} \right]. \tag{46} \]

In the laboratory frame

\[ M = \gamma + i \frac{q \wedge k_\lambda}{|q \wedge k_\lambda|} \tag{47} \]

\[ \gamma = N \sqrt{1 - \frac{1}{4m^2}} \left[ A + B \left( \frac{S - m^2 - \mu^2 + \frac{1}{2}}{2m (1 - \frac{1}{4m^2})} \right) \right] \tag{48} \]

\[ q = -N \frac{q \cdot q}{2m} \frac{\sqrt{1 - \frac{1}{4m^2}}}{(1 - \frac{1}{4m^2})}, \]

with a normalization factor analogous to Eq. (46).

You can see that the brick wall and lab. frames give essentially the same result, but this is not the same as the c.m. due to the relativistic spin transformations. Examine the case of B = 0. The pion-nucleon phase shifts are immediately related to the c.m. amplitudes and so we have to be a little careful in using them.

The calculation of πd elastic scattering with spin is straightforward\textsuperscript{17,18}. Neglecting Fermi motion, etc.,

\[ F_{\pi d} = \langle M | f_{\pi p} (q) e^{i q \cdot \hat{r}/2} + f_{\pi p} e^{-i q \cdot \hat{r}/2} | M \rangle, \tag{49} \]

where |M\rangle represents the deuteron having spin-projection M. The spin averaged cross-section then becomes
\[
\frac{d\sigma}{d\Omega} = \left| f_p + f_n \right|^2 \left[ S_c (q_2) \right]^2 + \\
+ \left| q_p + q_n \right|^2 \left[ S_M (q_2) \right]^2,
\]

with charge and magnetic form factors

\[
S_c^2 = (S_h + S_2)^2 + (2S_3 - \frac{A}{\sqrt{2}}S_4)^2
\]

\[
S_M = S_h - \frac{4}{5}S_2 + \frac{A}{2\sqrt{2}}S_3 + \frac{A}{2}S_4,
\]

where

\[
S_h = \langle s | \phi_0 (q\tau) | s \rangle \\
S_2 = \langle d | \phi_2 (q\tau) | d \rangle \\
S_3 = \langle d | \phi_3 (q\tau) | s \rangle \\
S_4 = \langle d | \phi_4 (q\tau) | d \rangle
\]

and \(|s\rangle\) and \(|d\rangle\) are the radial s- and d-waves of the deuteron.

In Fig. 8 are the results of an experiment at 142 MeV compared with a calculation of Carlson¹⁸). He has included some small higher-order effects that I shall come to presently. His approximations are not quite the same as I have described (some frame differences) and he has neglected Fermi motion. The single scattering gives at least a qualitative description of \(\pi d\) cross-sections at this energy.

Carlson included also contributions where the pion scatters from both neutron and proton in the deuteron (Fig. 9). There are now two
Fig. 8  Prediction by Carlson\textsuperscript{18}) for 142 MeV nd elastic scattering.

extreme but very simplifying approximations in the literature\textsuperscript{19}). It would be best if we could put a complete set of interacting states for the np blob in the middle of Fig. 9, but that is in general difficult. The coherent approximation assumes that the only important intermediate nuclear state is the ground state (Fig. 10), which is the approach we shall consider when talking about simple optical potentials. Alternatively, we can use essentially a closure approximation, retaining all intermediate

Fig. 9

Fig. 10
nuclear states, but neglecting the energy difference between these and
the ground state. This is the same as neglecting the np intermediate
state interaction (Fig. 11). This is the philosophy implicit in Glauber
type. Using the latter approximation, Carlson\textsuperscript{18}) only kept the $\delta$-function
part of the pion propagator, discarding the principal part of the integra-
tion. At high energies you can show that this is not an approximation,
but is actually required for consistency of the theory\textsuperscript{20}). He is then

![Diagram](image)

Fig. 11

left with what amounts to a five-dimensional phase-space integral

$$\int d^3 p \int d^3 p' \delta (E_{n'} - E_n') \int \int \int \int \int$$

which he evaluated by expanding the amplitudes in partial waves, and then
doing a few numerical integrations. At 142 MeV, the double scattering
gives only 10% effects, which is negligible when compared with the very
poor quality of the experimental data. New experiments are clearly needed!

At high energies, you can approximate the energy-conserving delta
function by one conserving longitudinal momentum\textsuperscript{21}). This is then equiva-
 lent to Glauber theory. The shoulder in the angular distribution at
3.65 (GeV/c) [Fig. 12] is mainly caused by double scattering\textsuperscript{17}).

The easiest way to evaluate the double scattering integral at low
energies is to make a Fourier representation of the delta function. If
$\phi$ and $f$ are approximated as sums of polynomials and Gaussians, you can
carry out all but a one-dimensional integration analytically, assuming
that the target nucleons are non-relativistic.

It is easy to construct the single scattering approximation for
heavier nuclei, but the results are less and less believable as $A$ increases.
Fig. 12 Elastic $\pi d$ scattering at 3.65 GeV/c. The Glauber model predictions are from Michael and Wilkin.

For a spin zero nucleus, the spin-flip amplitude averages to zero and we obtain

$$F_{\pi A} = \left[ Z \int_{Wp}(q) + N \int_{Wn}(q) \right] S(q)$$

(54)

The form factor $S(q)$ is the Fourier transform of the nuclear density, assumed to be the same for neutrons and protons.

6. THE OPTICAL POTENTIAL

The simplest refinement in the single-scattering approximation is to assume that what we have calculated is not a scattering amplitude but rather the Born approximation to a potential and all we have to do to
improve the predictions is to iterate the potential is some wave equation. This means that after every scattering the nucleus goes back to the ground state -- the coherent approximation.

Non-relativistically the Born amplitude

\[
F_{\pi A}^B = -\frac{Z}{4\pi^3} \int e^{-i\mathbf{Q}\cdot\mathbf{r}} V(r) e^{i\mathbf{E}\cdot\mathbf{r}} d^3r,
\]

where \(\mu\) is the pion mass, can be inverted to give the potential

\[
V(r) = -\frac{\lambda}{2\pi^2(2\mu)} \int e^{-i\mathbf{Q}\cdot\mathbf{r}} \left[ Z f_{\pi p}^p(q) + N f_{\pi n}^n(q) \right] \times \times S(q) d^3q.
\]

A particularly simple result follows if we assume that the form factor \(S\) decreases much more rapidly with \(q\) than does \(f\) (i.e. if the nucleus is much bigger than the nucleon).

\[
V(r) = -\frac{\lambda \pi}{2\mu} \left[ Z f_{\pi p}^p(c) + N f_{\pi n}^n(c) \right] \rho(r).
\]

Equations (56) and (57) are often called the single-scattering approximation to the optical potential\(^2\). Although it appears to be essentially non-relativistic, it requires not much more effort to make the kinematics relativistic, suitable for a Klein-Gordon equation rather than a Schrödinger equation.

If the pion energy is high on the nuclear scale, viz.

\[
\frac{\mathbf{p}^2}{\hbar^2} \gg 2\mu |V(r)|
\]

\[
\mathbf{p}_A R_A \gg \Lambda,
\]

where \(R_A\) is the nuclear radius, and \(V\) does not have too sharp an edge, then we can make an eikonal approximation\(^3\) to Eq. (56) to get the scat-
tering amplitude. The phase-shift function in the impact parameter representation (Eq. 11) is then given by

$$\chi(b) = -\frac{\lambda}{kn} \int_{-\infty}^{\infty} V(b,Z) dZ$$

$$= \frac{\lambda}{2\pi \kappa n} \int_{-\infty}^{\infty} e^{-i\mathbf{q} \cdot \mathbf{b}} \left[ Z f_{n}(q) + N f_{n}(q) \right] S(q) dq .$$

(Eq. 60)

In the zero range approximation (Eq. 57)

$$\chi(b) = \frac{2\pi}{k n} \left[ Z f_{n}(0) + N f_{n}(0) \right] \int_{-\infty}^{\infty} \rho(b,Z) dZ .$$

(Eq. 61)

This last form was used successfully many years ago\textsuperscript{25}) to explain 700 MeV \( \pi^{+}A \) reaction cross-sections. This is defined as being the difference between the total and the elastic cross-sections, and using Eqs. (6) and (11) [at high energies]

$$\sigma_R = \sigma_T - \sigma_e$$

$$= 2\pi \int b db \left[ 1 - \left| e^{2i\chi(b)} \right|^2 \right] .$$

(Eq. 62)

In the analysis, the experimentalists\textsuperscript{24}) allowed the proton and neutron densities in the nucleus to have different radii, but they found no evidence for neutrons lying outside the protons in heavy nuclei. Their conclusions were confirmed\textsuperscript{25}) by calculating the amplitudes accurately, with a numerical integration of the Klein-Gordon equation. This merely proves that the eikonal approximation was adequate. The experiment is now being repeated at many more energies and with a better precision\textsuperscript{26}).

We have so far assumed that the optical potential, though energy-dependent, is local. This means that the Born amplitude depends upon the initial and final momenta only through the difference \( \mathbf{q} = (\mathbf{k}' - \mathbf{k}) \). At
low energies the πN amplitudes are dominated by just s- and p-waves, and if we neglect frame differences

\[ f = a_0 + a_1 \frac{k^2}{\mu} \cos \theta \]

\[ = a_0 + a_1 \hat{k} \cdot \hat{k} \]  \hspace{1cm} (63)

Now we measure this amplitude when the pions are on the energy shell

\[ k^2 = k^2 = 2\mu E, \]  \hspace{1cm} (64)

so that in this limit

\[ f = a_0 + a_1 \left( 2\mu E - \frac{q^2}{2} \right) \]  \hspace{1cm} (65)

is completely equivalent. When this is used in conjunction with Eq. (56) it leads to a local energy-dependent potential

\[ V_L (r) = -\frac{2\Omega A}{\mu} \left[ \alpha_0 \rho(r) + \alpha_1 \left( 2\mu E \rho(r) + \frac{1}{2} \nabla^2 \rho(r) \right) \right], \]  \hspace{1cm} (66)

where the Laplacian acts only upon the density. Here \( a_0 \) and \( a_1 \) are the appropriate isospin averages.

At the other extreme, suppose we insist that Eq. (63) is the appropriate representation both on and off the energy shell. Then we are going to be stuck with a non-local, momentum-dependent potential.

\[ V_K (r) = -\frac{2\Omega A}{\mu} \left[ \alpha_0 \rho(r) + \alpha_1 \nabla \rho(r) \nabla \right]. \]  \hspace{1cm} (67)

The Schrödinger equation then becomes the Kisslinger equation\(^{27,28}\):

\[ \left\{ \nabla (1 + W) \nabla + k^2 - \mu V \right\} \psi = 0, \]  \hspace{1cm} (68)
where

\[ W = -4 \tilde{\pi} a_1 \rho(r) \]

\[ V = -4 \tilde{\pi} a_0 \rho(r). \]

(69)

Although the Kisslinger potential is very appealing on the basis of multiple scattering theory, its real justification must lie in its success in interpreting experiment. It is said in the literature\(^{29}\) that a local potential is not capable of explaining, for example, 30 MeV \( \pi^{12}C \) scattering. However, this statement seems to refer to the zero range form (57), where only the forward part of the scattering amplitude is retained. This then gives a potential with too small a radius to agree with the data (does not fall off fast enough in q space). The local potential \( V_L(66) \) has a larger radius.

The other claimed disadvantage is that a local potential cannot produce a repulsion in the s-states of mesonic atoms with attraction in the higher waves\(^{30}\). I do not want to go into details about mesonic atoms. "Nor shall we thus be led to the doctrine of atoms, which implies the hypothesis of a vacuum and that of the unchangeableness of matter (both false assumptions); we shall be led only to real particles, such as really exist." [Aphorisms 2.VIII].

If we calculate the energy shifts in lowest order perturbation theory, we must evaluate matrix elements of \( V \) between Coulomb wave functions

\[ \mathcal{E} = \int \Psi_C^* V \Psi_C \, d^3r. \]

(70)

For the Kisslinger case, integration by parts yields

\[ \mathcal{E}_K = -\frac{2}{\mu} \int \left[ a_0 \rho \Psi_C^* \Psi_C + a_1 \rho \left( \vec{\nabla} \Psi_C^* \right) \left( \vec{\nabla} \Psi_C \right) \right] \, d^3r. \]

(71)
For the local case we must integrate by parts and use the fact that $\psi_c$ is a solution of the Schrödinger equation with the Coulomb potential

$$
\mathcal{E}_c = -\frac{2}{\mu} \int \left\{ \left( a_c \rho + 2 \alpha_1 \rho \mathbf{V}_c \right) \psi_c^* \psi_c + \alpha_1 \rho \left( \nabla \psi_c^* \right) \cdot \left( \nabla \psi_c \right) \right\} d^3r.
$$

(72)

The only difference between the Kisslinger and the local results is the term in $V_c$ which is an effective s-wave contribution. To get an order of magnitude estimate for the difference for the case of $^{16}\text{O}$ say, take $V_c$ out of the integral at some mean value

$$
V_c \sim -\frac{Z e^2}{R} \sim -\frac{8 \sqrt{2}}{137 \times 2.7} = -0.03 \mu.
$$

(73)

This gives an extra induced s-wave term $2\mu V_c a_1 = -0.06 \mu^2$ to $a_0$

$$
a_0^{\text{eff}} = a_0 - 0.06 \alpha_1 \mu
$$

(74)

Experimentally$^{30}$, $\text{Re} \left( a_0^{\text{eff}} \right) = -0.03 \mu^{-1}$, $\text{Re} \ a_1 = 0.22 \mu^{-3}$, so that the induced term would be some 40% of the required answer. Of course, this is only a rough calculation, but it does suggest that the two potentials are not too different at low energies, and an extra bit of s-wave repulsion could be useful.

The local potential has a series of practical advantages. It is much easier to solve numerically. If you consider that d-waves are important, then this changes the local potential slightly, but in the Kisslinger model it would mean having a rather intractable fourth order differential equation!

Returning to the Kisslinger equation, for the scattering from an isospin zero nucleus such as $^{12}\text{C}$, the $a_0$ and $a_1$ are supposed proportional to the isoscalar s- and p-wave $NN$ amplitudes at the incident energy $E = k^2/(2\mu)$. However, this optical potential neglects one very important phenomenon, viz. pion absorption$^{28}$. As Milan Locher will have impressed
upon you, at 100 MeV or so the reaction $\pi^{12}\text{C} \rightarrow \text{ nucleons}$ is significant. This reaction probably takes place as a two-step process on a pair of nucleons

$$\pi^- + (NN) \rightarrow (N^*N) \rightarrow 2N$$

rather than on a single nucleon with a huge Fermi motion. Despite the other lectures and seminars at the school, I think it is fair to say that this process is not too well understood, except possibly in deuterium. The modifications we have to make to the optical potential to include such possibilities are at best phenomenological. At low energies the extra potential may be assumed to be imaginary [objections from the audience] since pions are removed from the beam by the absorption, and proportional to the square of the nucleon density, since the absorption takes place on two nucleons. This latter follows if the basic process only occurs where the two nucleons are very close together. Then we have

$$W = -4\pi \left[ a_1 \rho(r) + i \text{Im} C_0 \rho^2(r) \right]$$

(75)

$$V = -4\pi \left[ a_0 \rho(r) + i \text{Im} B_0 \rho^2(r) \right].$$

If we do not want to use information from $\pi\pi$ scattering, this means that at each energy there are six unknown parameters to be determined, the real and imaginary parts of $a_1$ and $a_0$ and $\text{Im} C_0$ and $\text{Im} B_0$. This then becomes a hard problem. To describe the CERN-IPN $^{12}\text{C}$ data, Krell and Barmo took $a_0$ to be real and determined from $\pi$-mesic atom analyses, as were $\text{Im} B_0$ and $\text{Im} C_0$. Now, in principle, this is a very bad procedure, since the absorption mechanism at threshold is possibly not the same as at the $\frac{3}{2}^-\frac{3}{2}^+$ resonance. The $\pi^+d \rightarrow \text{pp}$ reaction is very strongly peaked in this region, and this effect has been neglected in the analysis. In fact if you examine the values of their absorption parameters, you will find that it gives a very small contribution to the potential at 180 MeV. [Since it is proportional to $\rho^2$ it just makes the centre of an already black nucleus a little bit blacker.] It probably does not matter very much what you do with the small parameters (within reason); it is
dominantly \( a_1 \) which determines the cross-section. Krell and Barmo searched upon the complex parameter \( a_1 \), or rather on \( c_1 = (1 + \mu/m)a_1 \), which takes account (to some extent) of the finite proton mass. For each energy they tended to find two solutions of generally comparable quality (see Fig. 13a). The ambiguity is roughly that of the sign of the real part of the forward scattering amplitude, as is seen in Fig. 13b. The predictions of the forward dispersion relations clearly favour one solution. Note that the real part changes sign below resonance, at about 165 MeV.

Similar results have been derived independently by Sternheim and Auerbach\(^{33} \)). They took \( a_0 \) from the \( \pi N \) amplitude (smeared over Fermi motion) with no absorption, and then varied \( a_1 \). Expressed in the same variable as Krell and Barmo, they have clearly got the same general solution (Fig. 14), except for the 150 MeV point, and even there Krell and Barmo agree that the odd point has the lower \( \chi^2 \).

The simplest method of solving such a non-local potential\(^{30} \)) involves replacing the \( \ell^{\text{th}} \) wave function \( \phi_\ell \), by a new one defined by

\[
\phi_\ell = \frac{\Lambda}{r [\Lambda + W(r)]^{1/2}} u_\ell(r).
\]  

We then end up with a Schrödinger equation with a strange-looking, but local potential. This is the local representation of a non-local equation. The transformation (76) is singular if \( (1 + W) \) vanishes, as is the new local potential. This disease was noted by early experimentalists\(^{29} \)) in the field who replaced \( (1 + W)^{-1} \) by \( (1 - W) \), whenever there was a chance of it blowing up. From the definition of \( W \) [Eq. (69)], since nuclear densities tend to be decreasing functions of \( r \), the dangerous region is

\[
\Lambda - 4 \pi \rho_0 \Re \alpha_\Lambda < 0
\]  

(77)

together with \( \Im a_1 \) small. In the Krell-Barmo analysis, the worst case is 120 MeV with \( \Re a_1 \approx 0.27 \mu^3 \), \( \Im a_1 \approx -0.09 \mu^3 \), \( \rho_0 \approx 0.45 \mu^3 \), slightly on the wrong side of condition (77). Mottershead\(^{34} \)) has found that the solutions are discontinuous in the parameter \( \Im a_1 \) in such circumstances. This is much more serious for the 24-70 MeV \( \pi N \) cross-sections
Fig. 13a Elastic $\pi^{12}C$ scattering at 260 MeV. The fit à la Kisslinger by Krell and Barmo$^{31}$ shows the typical pair of solutions.

Fig. 13b Predictions of Krell and Barmo$^{31}$ for the real part of the forward scattering amplitude for $\pi^{12}C$ elastic scattering, as obtained by fitting the angular distributions with a Kisslinger potential. The forward dispersion relation calculation$^{63}$ (marked EL) is in fact the old solution incorrectly normalized (see Fig. 28).
that he was analysing (due to the high central density). The origin of this spurious singularity is clear. By taking the p-wave contribution as $\mathbf{k'} \cdot \mathbf{k} a_1(E) S(q)$ for high densities, we may get a potential energy term which is sometimes larger than the kinetic energy term. People who play the Fadeev game with separable potentials\(^{35}\) are always careful that this does not happen and put in extra damping factors in the momentum variables

$$\int_1^\infty \left( \frac{\mathbf{q}}{(1 + \mathbf{q}^2 R_1^2)} \right) \left( \frac{\mathbf{p}}{4 + \mathbf{p}^2 R_2^2} \right) q \rho_1(E) S(q).$$

A value of $a \sim 0.4 \text{ f}$ has been quoted\(^{19}\). It is impractical then to transform to coordinate space, but you can actually solve the integral equation directly in momentum space and the spurious singularity is then absent\(^{36}\).
tive to the unknown parameter \( a \). It is just another manifestation of the problem that we only know the on-shell \( \pi N \) amplitude and there are a myriad of ways of going off-shell. Note that my pet local potential (66) does not have the spurious singularity even for very large densities].

There has been quite a lot of interest\(^3\) in the low-energy (24–70 MeV) \( \pi^\pm \alpha \) elastic scattering, in the hope of determining the pion electromagnetic form factor. The Kisslinger equation is modified by the introduction of the Coulomb potential

\[
V(q) \sim \frac{Ze}{q^2} \frac{e^\pi}{S(q)} \int_{\pi} f(q),
\]

(79)

where \( Ze \) is the nuclear charge and \( S \) and \( f_\pi \) the nuclear and pion form factors. In Fig. 15 you can see the result of Crowe's group\(^3\) at 60 MeV for \( \pi^\pm \alpha \) scattering, together with the fits using the Kisslinger equation\(^3\). Although they are probably large, the effects of pion absorption were neglected in the potential. Thus if we parametrize

\[
f_{\pi}(q) = \text{e}^{-q^2/\lambda_c^2},
\]

(80)

we have five parameters to determine, viz. \( a_0 \), \( a_1 \) and \( r_\pi \). The large difference between \( \pi^\pm \) at small angles is caused mainly by the interference between the real Coulomb-Born term and the real part of the strong interaction amplitude. This real part is therefore probably very well determined; you can see without doing a calculation that it is large and positive. The minimum around \( \cos \theta = 0.25 \) is caused by the Born amplitude vanishing in this region (the p-wave amplitude vanishes for \( \cos \theta^e = 0 \)). Of the potential parameters \( a_0 \), \( a_1 \) deduced from fitting the data\(^3\), the only one which is in reasonable agreement with the predictions of the \( \pi N \) phase shifts is \( \text{Re} a_1 \). It must in honesty be stated that the \( \pi N \) angular distributions upon which the phase shifts are based are poor, and not in too good agreement with Crowe's own measurements at 60 MeV\(^3\). In Fig. 16 you can see the predictions for \( (\sigma_- - \sigma_+) \) with \( r_\pi = 0 \) and \( r_\pi = 2.7 \text{ fm} \)\(^3\).
Fig. 15 Elastic $\pi \alpha$ scattering at 60 MeV. The mean of the $\pi^0$ results are compared with a fit using the Kisslinger equation.

I think it would be fair to say that the experimentalists are not very happy with the large size which the data suggest; other methods lead to a value much closer to that of the nucleon and to our theoretical prejudices (0.8 fm). It has been claimed that the discrepancy is due to genuinely relativistic effects (e.g. Fig. 17).

I think these experiments will be reanalysed in the future when

i) there are good measurements of the $\pi \alpha$ total cross-sections to give Im $f(0)$;

ii) there are dispersion relation predictions of Re $f(0)$;
Fig. 16 The cross-section difference between $\pi^+\alpha$ scattering at 60 MeV compared\(^{38}\) with a fit using the Kisslinger equation.

Fig. 17

iii) there are reliable $\pi N$ phase shifts from new $\pi N$ angular distributions;
iv) there is more understanding of the Coulomb problem in $\pi N$ scattering, which is actively being studied\(^{40}\).

7. **CORRELATIONS AND THE OPTICAL POTENTIAL**

So far we have taken the nucleus to be a collection of uncorrelated nucleons, so that the only intermediate state of the nucleus which is allowed is the ground state. You cannot excite the nucleus by striking
one nucleon and then de-excite it by hitting another nucleon if the nucleons are uncorrelated. We thus see a very strong connection between correlations and the influence of inelastic channels on elastic scattering. We could at this point set up a complicated multi-channel system, but this is hard work.

A simple description of correlations within a multiple scattering expansion can be found in lecture notes by Ericson\(^1\). I shall not follow his arguments in detail, but rather distort them in my own language. We have so far calculated only the triangle graph contribution (Fig. 5) to the optical potential with just s-waves

$$2 \mu V(r) = - 4 \pi a \rho(r)$$  \hspace{1cm} (81)

where the nucleon number is absorbed in the density. When this is iterated in the Schrödinger equation, the double scattering terms are as in Fig. 18

![Fig. 18](image1)

![Fig. 19](image2)

with only the ground state between the scatterings. If there are correlations, we can also have graphs of the form of Fig. 19, where the π scatters from two distinct nucleons. This can easily be evaluated in the closure approximation (free waves between scatterings) to give a contribution to \( V \) of the form

$$2 \mu V^{(2)} = 4 \pi a^2 \int \rho(r,r') \frac{\psi_r^* |\vec{r} - \vec{r}'|}{|\vec{r} - \vec{r}'|} \, d^3 r'. \hspace{1cm} (82)$$

It scatters once at point \( r \) with strength \( a \), propagates from \( r \) to \( r' \) and then scatters again. We must multiply by the probability \( \rho \) that a pair of nucleons are at positions \( r \) and \( r' \). Equation (82) defines a non-local potential -- we must really multiply on the right by \( \psi(r') \) before doing
the integration over $r$. Now we have clearly overcounted, since the complete set of intermediate states of Fig. 19 includes the ground state of Fig. 18, which is already included. This latter is proportional to the square of the one-body density, and subtracting it out we get

$$2 \mu V^{(2)} = 4 \pi a^2 \int \left[ \rho(r, r') - \rho(r) \rho(r') \right] x^i \frac{e^{i \mathbf{k} \cdot \mathbf{r} - \mathbf{r}'}}{|r - r'|} d^3 r^1. \quad (83)$$

In the absence of correlations, $\rho(r, r') = \rho(r) \rho(r')$, so let us define the correlation function $C$ by

$$\left[ A + C(r, r') \right] \rho(r) \rho(r') = \rho(r, r') \quad (84)$$

so that

$$2 \mu V^{(2)} = 4 \pi a^2 \rho(r) \int C(r, r') \rho(r') \frac{e^{i \mathbf{k} \cdot \mathbf{r} - \mathbf{r}'}}{|r - r'|} d^3 r^1. \quad (85)$$

The simplest contribution to such a correlation, apart from the all-pervading Pauli, is the repulsion at short distances. Because of the hard core, two nucleons in the nucleus will not approach each other closer than about $d = 0.8 \text{ fm}$. A nucleon is thus surrounded by a hole cut in the nuclear medium

$$C(r, r') = -1, \quad |r - r'| < d \quad \text{(86)}$$

$$C(r, r') = 0, \quad |r - r'| > d.$$ 

We are still left with a non-local potential, but if we work in the long wavelength limit ($\psi$ smoothly varying over distances $d$), then we get an approximately local term
\[
2\mu V^{(z)}(r) = 4\pi a^2 \langle \frac{A}{r} \rangle_{nclq}.
\]

Thus to second order in the interaction

\[
2\mu V = -4\pi p [a - a^2 \langle \frac{A}{r} \rangle_{nclq}].
\]

It has been shown by Ericson and Ericson\(^2^8\) (in rather a different formalism) that in this low-energy limit the effect of two-body correlations to all orders in the interaction is to give an effective potential

\[
2\mu V = \frac{-4\pi p a}{1 + a \langle \frac{A}{r} \rangle}.
\]

Since we only calculated \(V\) to second order in \(a\), we cannot distinguish between formulae (88) and (89). In practice the difference is quite negligible.

We should also consider the effects of spin and isospin. The double scattering may consist of a double charge exchange, so that at low energies (where \(a\) is expected to be very small) the largest term in \(V\) may well be of the double charge-exchange type.

We can carry through the same type of argument for the gradient potential -- I refer to Ericson\(^3^1\) for details -- but there is one very important difference. In the multiple scattering expansion derived by Foldy\(^4^2\) back in 1945, he found that the equations behaved worse and worse at \(r = 0\) as higher partial waves were introduced into the primary interactions. Now to calculate the effect of a two-particle correlation on the potential, write down a Feynman integral corresponding to Fig. 19 with \(p\)-wave interactions (it is easier to do this in \(p\)-space and transform to \(r\)-space) and subtract the iterated one-body terms. The resulting complicated integral can be evaluated in the long wavelength limit. The singularity at small \(r\) manifests itself by the integral not going to zero as the correlation volume goes to zero. In the limit of a zero range
hard core correlation, the integral gives a very simple correction to the one-body term

$$2\mu V^{(a)}(r) = -\frac{\lambda}{3} \nabla \left( \frac{4\pi a_1}{\rho} \right)^2 \nabla \rho.$$

(90)

Summing to all orders in the two-body correlation, Ericson-Erickson\textsuperscript{28}) obtain

$$2\mu V = \nabla \rho \left( \frac{\frac{4\pi a_1}{\rho}}{1 + \frac{4\pi a_1}{3} \rho} \right) \nabla \rho$$

(91)

the analogue of the Lorentz-Lorenz effect in the scattering of light. In magnitude the effect does not seem large, especially since in fitting data \(a_1\) is often allowed to float. It does, however, alleviate somewhat the disease caused by \((1 + W)\) vanishing [see Eq. (77)]. When isospin is included, then the potential analogous to Eq. (91) will contain terms proportional to single and double charge exchange. Hence even in Born approximation this potential can induce transitions with \(\Delta T_z = 2\). This is because we have already summed different orders of scattering in calculating \(V\) itself.

In summary, the evaluation of the correlation contribution [Eq. (90)] is only reasonable for very low energies, and only takes into account the marking of the singularity of the equation by short-range correlations. I suspect this singularity is not present if we start from the local potential [Eq. (66)].

8. GLAUBER THEORY

There has been quite a lot of doubt about applying this model to low-energy pion scattering. "But the best demonstration by far is experience, if it go not beyond the actual experiment. For if it be transferred to other cases which are deemed similar, unless such be made by a just and orderly process, it is a fallacious thing. But the manner of making experiments which men now use is blind and stupid." [Aphorisms 1.LXX].

I have been told to assume that the basic material\textsuperscript{43}) is known, so that I can proceed quite fast, merely recapitulating the main arguments.
First neglect spin and isospin. Consider the target nucleus as a collection of nucleons, located at the fixed points \( \mathbf{r}_1 \ldots \mathbf{r}_A \). If we know the pion nucleon amplitude, we can calculate a phase shift function \( \chi_j(b) \) for each scattering centre [Eq. (7)]. The Glauber ansatz is that the phase-shift function for the whole nucleus is just the sum of the phase shifts from each nucleon

\[
\chi_A(b) = \sum_j \chi_j(b - \mathbf{s}_j). \tag{92}
\]

Of course since the nucleons are not sitting on top of each other, the individual pion-nucleon phase shifts have to be fed in at displaced values of the impact parameter; \( \mathbf{s}_j \) is the two-dimensional part of \( \mathbf{r}_j \) in the plane perpendicular to the mean pion momentum \( \mathbf{k} \). Hence the nuclear profile function becomes

\[
\Gamma_A(b_1, \mathbf{r}_1 \ldots \mathbf{r}_A) = 1 - \frac{A}{A!} \sum_{j=1}^A \{ 1 - \Gamma_j(b - \mathbf{s}_j) \} \tag{93}
\]

and the scattering amplitude

\[
F(q, b_1, \mathbf{r}_1 \ldots \mathbf{r}_A) = \frac{i b}{2 \pi} \int e^{-i q \cdot b} \left\{ 1 - \frac{A}{A!} \left[ 1 - \Gamma_j(b - \mathbf{s}_j) \right] \right\} db. \tag{94}
\]

Note that Glauber theory is unitarity at high energies. From Eq. (12), if \( \chi_j \) is real and we can neglect the \( O(1/k^2) \) corrections, then Eq. (92) implies \( \chi_A \) is real and the nuclear amplitude is also unitary. If we expand the A-fold product in Eq. (94), we end up with a familiar multiple scattering series, the first term of which is the single scattering approximation we have already discussed [Eq. (54)]. Although the additivity of the phase shifts [Eq. (92)] may be derived from high-energy potential scattering, using an eikonal approximation, the form is probably much more general. No really satisfactory derivation exists for the real world.
The physical scattering amplitudes are obtained by sandwiching $F(q, \hat{r}_1 \ldots \hat{r}_A)\) between initial and final nuclear states

$$
F_{fi}(q) = \langle f | F(q, r_1 \ldots r_A) | i \rangle
$$

(95)

This is an extra physical assumption; at the very least it neglects the excitation energy of the intermediate nuclear state, so that we can then use the closure approximation\(^{44}\).

It is now merely a matter of using these simple formulae. The crudest ansatz one can make for elastic scattering is to take all the target nucleons to be completely independent, i.e. assume that the nucleon density is the product of individual nucleon densities

$$
\rho_A(\vec{r}_1 \ldots \vec{r}_A) = \rho(\vec{r}_1) \ldots \rho(\vec{r}_A),
$$

(96)

whence

$$
F(q) = \frac{i k}{2\pi} \int e^{i \vec{q} \cdot \vec{b}} d^3 b \left[ 1 - \frac{A}{4\pi} \left( 1 - \epsilon \langle \Gamma | (\hat{b} - \hat{b}_q) \rangle \right) \right].
$$

(97)

For a large nucleus the expectation of $\Gamma$ with respect to the single particle density $\tilde{\rho}$, $(\Gamma)$, decreases like $A^{-3/2}$ so that we can then replace $1 - \langle \Gamma \rangle$ by an exponential

$$
F(q) = \frac{i k}{2\pi} \int e^{i \vec{q} \cdot \vec{b}} d^3 b \left[ 1 - e^{-\sum_q \langle \Gamma_q | (\hat{b} - \hat{b}_q) \rangle} \right].
$$

(98)

But

$$
\sum_q \langle \Gamma_q | (\hat{b} - \hat{b}_q) \rangle \equiv A \Gamma(b) = \sum_q \int \tilde{\rho}(r) \Gamma_q(\hat{b} - \hat{b}_q) d^3 r,
$$

(99)
using the inversion formula (7)

\[
\mathcal{A} \mathcal{F}(b) = \int e^{-i \tilde{q} \cdot \tilde{b}} S(q) \left[ Z f_{n}^{P}(q) + N f_{n}^{P}(q) \right] d'q
\]

(100)

which is exactly the same approximate formula [Eq. (60)] we had before for the iterated single scattering term in the optical potential approach\(^{45}\). The reason is quite clear. Since there are no correlations in the density, there is only the nuclear ground state between scattering. This is not the whole story though, because to get the optical limit, we must also replace \(1 - \Gamma\) by \(e^{-\Gamma}\). This is connected with the so-called self-correlation effect\(^{46}\) which was emphasized so forcibly in Mme. Ericson's lectures. In the single scattering approximation to the optical potential, the pion can scatter more than once on the same nucleon (even if the other nucleons do not take part in the scattering!). We thus get an exponential in \(\Gamma\), which contains an infinite number of scatterings. Glauber does not allow the pion to strike a given nucleon more than once, and because there are \(A\) nucleons in the nucleus, he gets at most an \(A\)-fold scattering. The basic problem is that by Eq. (56) we have made the optical potential proportional to the pion-nucleon amplitude, which already contains an indefinite number of potential interactions. Glauber is more subtle\(^{23}\) and equates potentials (if potentials exist) and so gets his counting right. The effects of self correlations turn out to be relatively minor even for a nucleus as light as \(^{12}\)C, the optical limit, Eq. (98), giving results almost indistinguishable from the more correct Eq. (97) for 1 GeV protons\(^{47}\).

The first correction to the product wave function is to treat the nucleon recoil correctly. This means that the averaging [Eq. (95)] should be only over the internal degrees of freedom of the nucleus; you tie the centre of mass down (it does not matter where, but the origin is as good a place as any)

\[
\langle \mathcal{F} \rangle = \int d^{3}r_{1} \ldots r_{A} \, \delta(\vec{r}_{1} + \ldots + \vec{r}_{A}) \, \mathcal{F}(\vec{q}, \vec{r}_{1}, \ldots, \vec{r}_{A}) \, \rho(\vec{r}_{1}, \ldots, \vec{r}_{A})
\]

(101)
Even if we take a product density now, we can no longer factorize this integral into a product of A integrals because of the delta function. There is a useful result due to Tassie and Barker\(^{46}\), such that if the nucleus can be described by a set of harmonic oscillator states with lowest energy, then we can forget the delta function, providing we multiply the answer by \(\frac{q^2}{4A\alpha^2}\), \(\alpha\) being the spring constant. Then Eq. (101) becomes

\[
\langle F \rangle = e^{\frac{q^2}{4A\alpha^2}} \int d^{3}r_{1} \ldots r_{A} F(q, r_{1} \ldots r_{A}) P(r_{1} \ldots r_{A}).
\]

This prescription is satisfactory for the ground states of light nuclei, while for heavy nuclei the centre-of-mass correlation is unimportant. The problem is most serious for the excited states of light nuclei, but even there the resulting ambiguities are not large.

The next obvious correlation to include is the Pauli principle, which means taking a Slater determinant wave function rather than a simple product

\[
\Psi(r_{1} \ldots r_{A}) = (A!)^{-\frac{1}{2}} \det[\Psi_{m}(r_{n})].
\]

We then need matrix elements of the form

\[
\Gamma_{\lambda}(b) = \langle \Psi | 1 - \Pi_{f} (1 - \Gamma_{f}) | \Psi \rangle
\]

\[
= 1 - \langle \Psi | \Pi_{f} (1 - \Gamma_{f}) | \Psi \rangle.
\]

The operator is factorizable into operators which act in one- and only one-particle subspaces,

\[
\Pi_{f} (1 - \Gamma_{f}) | \Psi \rangle = (A!)^{\frac{1}{2}} \det[(1 - \Gamma_{n}) \Psi_{m}(r_{n})].
\]
Having antisymmetrized the ket, we can liberate the bra from this restriction to get
\[ \Gamma(b) = 1 - \det \left[ \delta_{nm} - \tau_{nm}(b) \right], \]  
with matrix elements
\[ \tau_{nm} = \int \psi_m^*(\vec{r}) \tau(b, f) \psi_n(\vec{r}) \, d^3r. \]

The determinant is of course a little more difficult to handle than the product, but it is still very easy for light nuclei. Without spin flip and charge exchange, the determinant for an \( I = J = 0 \) nucleus (such as \(^{12}\text{C}\)) becomes the product of four lower order (such as third) determinants. In practice the Pauli principle plays not too important a role in elastic scattering. As you can see from Fig. 20, for 1 GeV protons scattering from \(^{16}\text{O}\) \(^{49,50}\) correlations push up the subsidiary maximum by some 10-15%. (Note the logarithmic scale.)

**Fig. 20** Elastic p\(^{16}\text{O}\) scattering\(^{49,50}\) at 1 GeV. The Glauber predictions, showing the effect of the Pauli principle, are from Ref. 50.
From the identity
\[ \det (1 - T) = \exp \operatorname{tr} \log (1 - T) \] (108)

if \( T \) is small, we can expand the logarithm and keep just the first term
\[ \det (1 - T) \approx \exp \operatorname{tr} (-T). \] (109)

Using this in Eq. (106)
\[
\Gamma_A (b) \approx 1 - \exp \left\{ - \sum_n \Lambda_n (b) \right\} = 1 - \exp (- A^+ (b)),
\] (110)

which is the optical limit [Eq. (60)] once more. If the second-order terms are kept in the expansion of the logarithm, we end up with an extra term in the exponent of Eq. (110) proportional to the two-particle correlation function
\[
\int \mathrm{d}^3 r \mathrm{d}^3 r' \ C (r, r') \rho (r) \rho (r') \Gamma (b - \xi_1) \Gamma (b - \xi_2).
\] (111)

An amusing exercise for the interested is to compare this expression with that of Eq. (85) taken in the high-energy limit. Of course the result is much more general than the derivation of Eq. (111) -- dynamical correlations would lead to the same form\(^3\). Because of the Pauli correlation, the pion can hit a nucleon out of the s-shell into the p, and then knock a different p-shell nucleon back. This implies having intermediate excited nuclear states.

From a practical viewpoint, it is very advantageous to have a simple analytic parametrization of the pion-nucleon amplitudes. The most important features are the value in the forward direction and the angular variation there. At high energies it is normal to take just a diffraction peak
\[ f(q) = \frac{i k \sigma}{4 \pi} \left( 1 - i \varphi \right) e^{-\beta^2 q^2/2} \]  

(112)

However, low-energy \( \pi N \) scattering is dominantly just s- and p-wave, so that a form

\[ f'(q) = \frac{i k \sigma}{4 \pi} \left( 1 - i \varphi \right) \left( 1 - \beta^2 q^2/2 \right) \]  

(113)

looks better, but not obviously so. We need to calculate terms like

\[ f(b) = \int S(q) f(q) e^{-i b \cdot q} \]  

(114)

and in the standard Glauber treatment we merrily carry out the integration to infinite \( q \) even though this is very suspect beyond \( q^2 = 4k^2 \). If we use the polynomial form for \( f(q) \), then at low energies \( (kR \gg 1) \) there can be an appreciable contribution from the unphysical region (normally the form factor \( S \) damps these contributions considerably). It is computationally unpleasant to have to introduce a cut-off in the \( q^2 \) variable. I have done the calculation with both types of input [actually including a \( q^6 \) term in the polynomial of Eq. (113)] and the results are fortunately qualitatively similar. Probably a more important difference between the two forms is that \( \beta^2 \) was taken as real for the Gaussian, but allowed to be complex for the polynomial. The actual values of the parameters were obtained at each energy by a comparison with the predictions of the CERN phase shifts.

The most complete scattering data being the CERN-IPN-2 \( \pi^{-12} C \), we return to this again. We need some model for the wave function (or correlation functions) rather than just the form factor associated with the matter density. The simplest ansatz is that of closed s and \( p_{3/2} \) shells with harmonic oscillator wave functions,

\[ \psi_s \sim e^{-\alpha^2 r^2/2} \]

\[ \psi_p \sim r e^{-\alpha^2 r^2/2} \]  

(115)
The electron scattering value\textsuperscript{54}) is $\alpha^2 = 0.4 \text{ f}^{-2}$. Before we look at the pion data, let us consider the results with 1 GeV protons\textsuperscript{50,55}) (Fig. 21). With $\alpha^2 = 0.4 \text{ f}^{-2}$, the primary slope is well reproduced, but the minimum is not, and the subsidiary maximum is wrong by 50%. Changing the radius parameter to $\alpha^2 = 0.46 \text{ f}^{-2}$ does not improve the over-all agreement.

$^{12}\text{C}$ is a more deformed nucleus than $^{16}\text{O}$. Since these nuclei have spin zero, I must clarify this statement. The ground state, $2^+$, $3^-$ excitations of $^{12}\text{C}$ look like a collection of rotational states, implying that there must be a large mixing with higher shells, in addition to the $s$ and $p_{3/2}$, in the ground state. Considerably better agreement with the data can be obtained by taking a density distribution from a deformed well (oblate) and then projecting out a spherically symmetric density to describe the ground state\textsuperscript{56}). Another way of stating the problem, is that by taking the simple density, we are grossly underestimating the probability of the $2^+$ level as an intermediate state of the scattering. Because of this difficulty, it is hard to choose between the values of 0.4 and 0.46 for $\alpha^2$.

We now turn to the low-energy pion data\textsuperscript{2}). As can be seen from Fig. 22, the experimental total cross-section is largest at about 150 MeV,
and so is the prediction of the Glauber model. This is to be compared with 190 MeV for the pion-nucleon case. The reason for the shift in the theoretical calculation is that as the energy decreases, the pion nucleon slope parameter tends to get larger ($\beta^2 \approx l/k^2$). If you interpret $\beta$ to be the radius of the nucleon as measured by the pion, then the effective nuclear radius is a decreasing function of energy [cf. the empirical fits given by Eq. (5)]. When the energy drops below 190 MeV, the nucleus gets less black, but it also gets bigger, which compensates for some of this. One of the experimentalists in the audience pointed out that this was just the effect of the p-waves being further out at lower energies.

The total cross-sections and the angular distributions shown in Figs. 22 and 23 were computed from the Gaussian input [Eq. (112)]. At

![Graph showing total cross-section and elastic cross-section vs. energy](image)

**Fig. 22** Total cross-section$^2$) of pions on $^{12}$C. The Glauber predictions of Ref. 52 are based on a Gaussian input [Eq. (112)].
Fig. 23 Elastic $\pi^{-}^{12}C$ scattering$^{2}$). The Glauber predictions of Ref. 52 are based on a Gaussian input [Eq. (112)].
Fig. 23 cont.
120 MeV, there is only a small minimum, both experimentally and theoretically; the major discrepancy is the height of the subsidiary maximum. As the energy increases the minimum gets deeper until the resonance is past, when it starts getting shallower again. All in all there is quite good agreement, apart from the height of the subsidiary maximum. A very similar calculation by the Trondheim group\textsuperscript{57}) gets rather different answers. The main reason for this is their claim that Fermi motion washes out the angular variation of the \pi NN amplitude (i.e. they took $\beta^2 = 0$) so that they have too small a nucleon radius, especially at the lower energies. (Note that Figs. 23 and 24 were calculated with a small nuclear radius anyway, $\alpha^2 = 0.46 \, \text{fm}^{-2}$). A third calculation was done by Schmit\textsuperscript{58}), with quite similar input to mine, but leading to a much lower total cross-section (by about 20\%). Another calculation by an SIN group\textsuperscript{59}), based upon an approximate, but closed form expression, clearly favours my answer.

There is also a preliminary \pi NN angular distribution measured at 270 MeV (Fig. 24)\textsuperscript{52,60}). Because the authors showed a little uncertainty in the absolute normalization, I took the liberty of scaling it upwards uniformly by a factor of 1.3. This then brings it into line with the old...
total cross-section measurements of Ignatenko et al.\textsuperscript{61}) [At the school, John Domingo, on behalf of the experimentalists concerned, said that he expected the final results to be generally displaced upwards by about the predicted factor!] There are very preliminary data available from Virginia\textsuperscript{62}) at 220, 230 and 240 MeV and they promise numbers also at 170, 180 and 190 MeV in the near future. It is still too early to draw any firm conclusions from this experiment, but the best-looking data at 230 MeV show a striking resemblance to the Glauber predictions. We must be a little patient.

In Fig. 25 are the results of 280 MeV with the polynomial input as well as the Gaussian, with $\alpha^2 = 0.4$ and $\alpha^2 = 0.46 f^{-2}$. Fermi motion can be introduced roughly by smearing the input amplitudes over the momentum distribution in the nucleus, using Eq. (38). With the polynomial input, the effect of this on the total cross-sections is shown in Fig. 26. The top of the resonance peak is chopped off and and we start getting quite respectable agreement with experiment. The same is also true for the old $\pi^{16}O$ data\textsuperscript{61}). Angular distributions are also improved a bit by the inclusion of Fermi motion (see Fig. 27 for the 230 MeV data).

Finally in Fig. 28 we can compare the predictions for the real part of the forward scattering amplitude with those derived from forward dispersion relations\textsuperscript{63}). There are two curves associated with this latter approach, one being calculated with the new total cross-section input. The agreement for high energies is very good; even for low energies it has the right shape.

At lower energies, the theory does not do as well\textsuperscript{64}). For example, at 70 MeV (Fig. 29) the theory predicts a sharp dip where the experimentalists cannot seem to find one\textsuperscript{29}). This dip is close to $\theta^* = 90^\circ$, where the p-wave single scattering term is zero. It may be profitable to compare the predictions with the phase-shift analysis of Eiener and Huguenin\textsuperscript{3}). Although their s-wave is not too well determined, the main discrepancy between the two approaches is that the Glauber s-wave phase shift has far too small an imaginary part. Assuming that this is a real effect, and not due to the great uncertainty in the pion-nucleon input, then it may suggest either that the eikonal approximation is not valid here, or that extra absorption is required for small impact parameters. If we
Fig. 25 Elastic $\pi^{12}C$ scattering\textsuperscript{2}) at 280 MeV. The two Glauber calculations illustrate the exponential and polynomial inputs. (Note that the latter has been evaluated with a slightly larger nuclear radius.)

include a phenomenological potential to account for the two-nucleon absorption of the pion, it would push things the right way.

The energy range 150–200 MeV may well be the most interesting. When the real part of the $\pi N$ amplitude is zero, standard Glauber theory ensures that the real part of the $\pi^{12}C$ amplitude is also zero. This condition is roughly satisfied at 190 MeV, but other analyses and the Coulomb nuclear interference measurements suggest that for $\pi^{12}C$ this is about 30 MeV too high. People have put forward many mechanisms to displace the
Fig. 26 Total $\pi^{12}\text{C}$ and $\pi^{16}\text{O}$ cross-sections (the 140 MeV point of the latter has been normalized upwards such that their equivalent $^{12}\text{C}$ point is in agreement with the results of Ref. 2). The Glauber calculations are performed with a polynomial input, with and without the inclusion of Fermi motion and with two different radii for $^{12}\text{C}$.

total cross-sections and real parts. First of all the $\pi\text{N}$ input is not reliable (perhaps more reliable than the theory described here, remarked Professor Hamilton). The recent accurate $\pi\text{N}$ total cross-section measurements\textsuperscript{65} put the peak about 10 MeV lower than the older data, and this will have a corresponding effect on the real part. The $q^2$ dependence of the $\pi\text{N}$ input data [Eq. (35)], if it is to be believed, will push the structure to larger energies. (When the input forward amplitudes are decreasing functions of energy, the $q^2$ shift increases the effective value of $\beta^2$). There is the evidence from the quasi-elastic ($e$, $e'\text{p}$) reaction in $^{12}\text{C}$\textsuperscript{66},
that the mean kinetic energy of the struck nucleon is about 25 MeV. This would push the structure to lower energies. It has been suggested\textsuperscript{67} that, in analogy with the resonant scattering of light, the optical potential should be $V/(1 + V/k^2)$, where $V$ is the simple optical potential [Eq. (56)]. I am not completely convinced that the self-correlation is then taken completely into account, but I cannot disprove it! This mechanism would also shift the structure to lower energies. The discrepancy with the real part could be due to an inadequacy of the eikonal approximation at low energies. We only need a slight error in the phase of the amplitude for some of the large imaginary part to feed into the real part.
Fig. 28  Real part of the forward $\pi^{12}C$ elastic scattering amplitude. The points are an evaluation by Locher$^{75}$) from the data of Ref. 2 (the results are similar, but the error bars much more conservative than those quoted in Ref. 2). The predictions of the forward dispersion relations$^{63}$) are based on old$^{61}$) and newer$^{2}$) data for $\sigma_T$. The Glauber calculations show the effect of Fermi motion.
9. **THE INFLUENCE OF SPIN AND ISOSPIN**

"There are many things in this History which to common apprehension, or indeed to any understanding accustomed to the present system, will seem to be curiously and unprofitably subtle". [Aphorisms l.CXXI].

We now have to consider the effects of internal degrees of freedom of the target, in particular charge and spin$^{23,68}$. Both of these can at least approximately be incorporated into Glauber theory. Let us start with spin. I assert that for a pion-nucleon amplitude of the form

$$ F(q) = f(q) + i \sigma \cdot \left( \vec{p} \wedge \vec{q} \right) q(q), $$

(116)
there exists an impact parameter representation

\[ F(\vec{b}) = \frac{i \lambda_{1}^b}{2\pi i} \int d^2q \ e^{-i\vec{q}\cdot\vec{b}} \ \Gamma(\vec{b}). \]

To find the profile function, start from the inversion relation (neglecting the finite cut-off)

\[ \Gamma(\vec{b}) = \frac{\lambda}{2\pi i \hbar} \int d^2q \ e^{-i\vec{q}\cdot\vec{b}} \ F(q) \]

\[ = \frac{\lambda}{2\pi i \hbar} \int d^2q \ e^{-i\vec{q}\cdot\vec{b}} \ \left[ F(q) + i \ \tilde{S}(\vec{b} \cdot \vec{q}) \ g(p) \right] \]

\[ \equiv \Gamma_c(b) + \tilde{S}(\vec{b} \cdot \vec{q}) \ \Gamma_s(b), \]

(118)

where

\[ \Gamma_c(b) = \frac{\lambda}{i \hbar} \int J_0(qb) \ q \ dq \ f(q) \]

\[ \Gamma_s(b) = \frac{\lambda}{i \hbar b} \int q^2 dq \ J_1(qb) \ g(q). \]

(119)

The next observation is that if we have two potentials containing spin-orbit as well as central terms, then the phase additivity [Eq. (92)] is still valid at high energies providing the scattering angle is not too large\(^{23}\). Using this, the inclusion of spin into \( \pi \alpha \) scattering is straightforward. Taking the target nucleons all in s-states, we must calculate \( \langle \Gamma(\vec{b} - \vec{s}) \rangle \). Because of the spherical symmetry of the s-state, we can choose \( \vec{b} \) to lie in an arbitrary direction, for example along the y-axis. Then with \( \vec{k} \) in the x direction, the spatial averaging gives
\[ \langle \Gamma(b, S) \rangle = F(b) + \sigma_z \ G(b) \]

\[ F(b) = \langle \Gamma_c(b, S) \rangle \]

\[ G(b) = \langle k(b, -y) \Gamma_s(b, S) \rangle . \]

Now if we take a Slater determinant for the ground state wave function, then in the determinantal expression for \( \Gamma(b) \) [Eq. (106)] we only have diagonal elements

\[ \Gamma_a(b) = 1 - \left[ 1 - F - G \right] \left[ 1 - F + G \right] \left[ 1 - F - G \right] \left[ 1 - F + G \right] \]

\[ = 1 - \left[ (A - F(b))^2 - G(b) \right] . \]

(121)

You can write down a similar formalism for multiple charge exchange using a Slater determinant. There is, however, the very important difference that with an incident \( \pi^+ \) there can only be a double charge exchange if the neutron lies in front of the proton. (This is actually an extra approximation, which is valid providing the nucleons do not overlap too much). This introduces an extra constraint linking all the expectation values, but it may be resolved to just a counting problem. The result including both charge and spin is

\[ \Gamma_a = 1 - \frac{\lambda}{4\sigma} \left\{ (\mathcal{A}_p^+ \mathcal{A}_n^+)^2 - 2 (\mathcal{A}_p^+ \mathcal{A}_n^-)^2 \right\} \times \\
\times \left\{ (\mathcal{A}_p^- \mathcal{A}_n^-)^2 - 2 (\mathcal{A}_p^- \mathcal{A}_n^-)^2 \right\} + \\
+ \frac{\lambda}{2\mu} (\mathcal{A}_p^- \mathcal{A}_n^-)^2 (\mathcal{A}_p^+ \mathcal{A}_n^+)^2 \],

(122)
where
\[ A^\pm_p = 1 - F_p \mp G_p. \] (123)

It is to be expected that spin-flip terms should be most important for low energies. At high energies the small \( q^2 \), which are important for elastic scattering, become very small angles, and the spin-flip terms are constrained to vanish in the forward direction. In a recent investigation\(^{69}\), an attempt was made to introduce spin effects directly in a momentum space formalism. Approximations are made, which though valid for large \( q \), lead to spin effects which have the wrong sign for small \( q \). Fortunately, the authors find very little effect from spin in the high-energy region where they were working.

In Fig. 30, you can see the predictions for 60 MeV pions scattered from the \( \alpha \) particle\(^{37}\). The solid curve is a single scattering prediction, which is a very good first approximation, especially for the larger angles. Somewhat surprisingly, the dashed curve corresponding to the full multiple scattering theory, with spin and isospin flips, hardly differs at all from the single scattering model! The dot-dashed curve involves retaining only intermediate states with \( I = J = 0 \) (i.e. considering \( \pi^0 \) scattering but without spin and isospin). That this latter fills in the minimum may be accidental. Until Professor Bugg\(^{70}\) gives us reliable pion-nucleon amplitudes close to this region, we shall still be in the dark about such fine points. [The small angle discrepancy is certainly due to the poor input]. On the other hand this may be physics. In order to excite the \( \alpha \) particle to a state with either \( I \) or \( J \) non-zero, we must give the \( \alpha \) particle well over 20 MeV of energy. The pion would have to be deflected by more than 90° to give one nucleon this much energy, and so we must expect a large suppression of these contributions. Glauber theory assumes that all intermediate nuclear states are degenerate in energy with the ground state. If you let the energy decrease further, then you run into more serious contradictions -- the calculated total cross-section becomes less than the calculated elastic cross-section.

This is again connected with the incorrect treatment of intermediate phase space. Unless the inversion formula [Eq. (71)] is cut off at \( q^2 = 4k^2 \),
Fig. 30  Elastic $\pi\alpha$ scattering at 60 MeV; the experimental points are the mean of the $\pi^-$ results. The solid curve is a single scattering approximation; the dashed curve has multiple scatterings with spin and isospin flip; the dot-dashed curve has only $I = 0$ intermediate states.
we can get very large unphysical contributions, especially from the spin-flip terms. As a simple exercise, take the famous Glauber formula for the scattering from deuterium\textsuperscript{23)}

\[
F_{ud}(q) = \left( f_{\pi\rho}(q) + f_{\pi\eta}(q) \right) S\left( \frac{q}{2} \right) + \frac{l}{2\pi^2 k_0} \int S\left( q' \right) f_{\pi\rho}\left( \frac{q'}{2}, \mathbf{r}, \mathbf{t} \right) f_{\pi\eta}\left( \frac{q'}{2}, \mathbf{r} - \mathbf{t} \right) d^2 q'.
\]

(124)

If the \( \pi N \) amplitude is purely s-wave, show that the total cross-section calculated from Eq. (124) is positive definite if the integral is cut off at \( 4k^2 \), but may go negative otherwise. Take the deuteron size as a variable.

These sophistications can be extended to heavier nuclei such as \( ^{12}\text{C} \), but the counting problem for charge exchange is tedious. I therefore put forward the following ansatz. We have seen previously that the neglect of Pauli and centre-of-mass correlations was not very serious for \( ^{12}\text{C} \) or \( ^{16}\text{O} \). However, if there are no correlations such as to make the nucleus have a definite spin and isospin, there would not be much significance in multiple spin and isospin flips. For an \( I = J = 0 \) nucleus one can take this correlation into account by considering that for each spatial configuration there are four nucleons \( p^+ p^+ n^+ n^- \) coupled to \( I = J = 0 \). This \( \alpha \) particle, of course, has the spatial wave function appropriate for \( ^{12}\text{C} \). This model has the advantage that it is easily calculable; it has the structure

\[
\Gamma_{\alpha} \approx 1 - \left( 1 - \Gamma_{\alpha} \right)^3,
\]

(125)

where \( \Gamma_{\alpha} \) has the structure of Eq. (122).

In this approximation the effects of spin and isospin on the CERN-IPN data are very small. As an example, the total cross-sections at 180 MeV under various assumptions are:
Single scattering \( \sigma_T = 1.613 \text{ mb} \)

I = J = 0 intermediate states \( \sigma_T = 710 \text{ mb} \)

Standard Glauber (\( \pi^+ \)) \( \sigma_T = 714 \text{ mb} \)

Including charge exchange \( \sigma_T = 718 \text{ mb} \)

Including spin \( \sigma_T = 718 \text{ mb} \)

Including spin and charge exchange \( \sigma_T = 723 \text{ mb} \).

I shall not discuss spin flip for nuclei with spin, e.g. \(^{13}\text{C}\). For such a scattering \( F(q') \) depends upon the direction of \( q' \). The nucleon \( \Gamma_b \) can be split up in terms of \( \Gamma_c \) and \( \Gamma_s \), which depend only upon the magnitude of \( b \), but you do not need to do this. (It would be quite hard for \(^9\text{Be}\)).

10. COULOMB DISTORTION OF THE OPTICAL POTENTIAL

As we saw from low-energy \( \pi^+ \alpha \) scattering, at small angles the Coulomb amplitude and its interference with the strong interaction amplitude are very important. If we include this merely by the addition of a real Coulomb-Born amplitude (a good first approximation), then it is clear that the interference determines mainly the real part of the \( \pi \)-nuclear amplitude. The next approximation is to assume that we can just add the Coulomb potential to our strong interaction potential. Then the total phase-shift function is\(^{71} \)

\[
\chi(b) = \chi_N(b) + \chi_c(b),
\]

(126)

where the Coulomb phase shift is an integral over the potential [Eq. (59)]

\[
\chi_c(b) = -\frac{\lambda}{v} \int_{-\infty}^{\infty} V_c(b, z) d\bar{z}.
\]

(127)

To ensure relativistic kinematics, \( v \) is the relativistic velocity of the pion. Because of the long range of the Coulomb force, the integral diverges at its limits, so we introduce a screening radius\(^{71} \). After all the graphite in the target has electrons around the nuclei, which shield
the charge at large distances. The screened nuclear Coulomb potential is of the form
\[
V_{c}(r) = Z e^2 \int \frac{\rho_c(r')}{|r-r'|} C(r) \, d^3r',
\]
(128)
where \( \rho_c(r) \) is the nuclear charge density (including the sizes of pion and proton), and we take the screening function as
\[
C(r) = \begin{cases} 
1 & , \quad r \leq a \\
0 & , \quad r > a.
\end{cases}
\]
(129)
The following procedure is only meaningful if we can show that the choice of a shielding radius \( a \) does not affect the predictions. We shall take \( a \) so large on the nucleon scale that we may approximate Eqs. (127) and (128) by
\[
\chi_c(b) = -2n \int \rho_c(r') \left[ \log \left( \frac{2a}{|b-r'|} \right) \right] d^3r',
\]
(130)
with
\[
n = \frac{Z e^2}{\nu}.
\]
(131)
We can then split the phase shift into two parts
\[
\chi_c = \chi_s + \chi_p,
\]
(132)
where the shielding contribution is just an additive constant, independent of \( b \)
\[
\chi_s = -2n \log(2a)
\]
(133)
and
\[ \chi_p = 2n \int \rho_c(r^*) \log |1 \mathbf{b} - \mathbf{b}'| d^3r'. \] (134)

Thus given a reasonable nuclear density (polynomials and Gaussians) we can calculate the phase shift \( \chi_p \) in closed form. For a point charge
\[ \rho_i(r^*) = \delta(r^*) \]
and
\[ \chi_{pr}(b) = 2n \log b. \] (135)

The corresponding Coulomb amplitude is
\[ f(q) = f_{pr}(q) e^{i\chi_s} \]
\[ \approx -n^2 \frac{q_0}{q} \exp \{ -2i \left( \log (q/2) + \gamma \right) + i\chi_s \}. \] (136)

where \( \gamma \) is Euler's constant.

There is now a cunning trick\(^71\). The combined nuclear and Coulomb amplitude is
\[ F(q) = \kappa \int \left\{ 1 - e^{i\chi_N + i\chi_s + i\chi_p} \right\} J_0(qb) b db. \] (137)

Add and subtract the point Coulomb amplitude to the right-hand side
\[ F(q) = f_{pr}(q) e^{i\chi_s} - i \kappa \int J_0(qb) b db \times \]
\[ \times \left\{ e^{i\chi_N + i\chi_s + i\chi_p} - e^{-i\chi_N - i\chi_s} \right\}. \] (138)
But $\chi_s$ is a constant, independent of $b$

$$F(q) = e^{i\chi_s} \left\{ f_{P}(q) - i\frac{b}{p} \int_0^b dq b \left[ e^{i\chi_{N}} - e^{i\chi_{P}} \right] \right\}. \quad (139)$$

The shielding radius just gives an over-all arbitrary phase, which does not affect the predicted cross-section. If we had not done the subtraction, the integral [Eq. (137)] would have had important contributions from large distances.

In Fig. 31 you can see the predictions of the Glauber model for 260 MeV $\pi^{-12}$C scattering. The strong destructive interference for small angles is clearly seen, but one must be careful in that the plot is against $\theta$ and not $q^2$. There is a slight discrepancy in normalization between the theory and the preliminary results of the CERN-IPN group\(^2\) (which could be of theoretical or experimental origin), but the agreement in the shape strongly suggests that the experimental value for the forward real part is close to the theoretical one, $\text{Re} f(0)/\text{Im} f(0) = -0.35$.

Of course we could have taken the strong interaction phase-shift function $\chi_N(b)$ from a different form of optical potential. An obvious candidate is a phenomenological one fit to the actual data. Suppose that for small $q^2$ the strong interaction amplitude decreases exponentially

$$f_{P^{-N}}(q) = \frac{i k_0 \sigma}{4 \pi} (1-i\rho) e^{-q^2 R_N^2/6}. \quad (140)$$

The total cross-section $\sigma$ may be measured independently, and the strong interaction radius $R_N$ taken from sufficiently large angles that the Coulomb effect is negligible. The phase shift corresponding to Eq. (140) is

$$e^{i\chi_N(b)} = 1 - \frac{3 \sigma (1-i\rho)}{4 \pi R_N^2} e^{-3 b^2/2 R_N^2}. \quad (141)$$
Fig. 31 Elastic $\pi^{12}$C scattering\note{2} at 260 MeV, small angle points only. The Glauber curve predicts a total cross-section which is 5\% too large.

The amplitude with the Coulomb scattering added is, by Eq. (139)

$$F(q) = f_{p+}(q) + i\hbar \int d\beta\, J_0(q\beta) \left\{ e^{i\chi \rho_{1-}} - e^{-\frac{3\sigma(1-i\rho)}{4\mu TR_N} + \frac{3b^2}{2R_N^2}} \right\},$$

(142)
where the inessential phase is neglected. The cross-section can now be predicted as a function of the free parameter \( \rho \), and this can be adjusted to give a good fit to the experimental data. This analysis of course assumes that \( \rho \) is a constant, independent of angle, but the Glauber model suggests that this is roughly true.

Experimentalists prefer even simpler rules of thumb than that given by Eq. (142). It turns out that for small \( q^2 \), and not too large \( Z \), the Bessel transform in Eq. (142) may be approximated by

\[
i^2 \frac{e^{i\omega}}{\sqrt{2\pi R_N^2}} \int dB dB' f(qB) \left( \frac{3 \sigma(1+\rho)}{4\pi R_N^2} \right) e^{-3b^2/2} \]

\[
= \left( \frac{i\omega}{F_N(q)} \right) \cdot (143)
\]

The constant phase \( \omega \) is somewhat model-dependent. If you take a Gaussian charge form factor \( e^{-q^2 R_E^2/6} \), then

\[
\omega = i n \left\{ -\gamma + \log \left( \frac{8}{3} \left[ R_E^2 + R_N^2 \right] \right) \right\} \cdot (144)
\]

We can now divide out by the phase factor to get the final formulae

\[
\frac{d\sigma}{d\Omega} = \left| -n \frac{2k}{q} e^{i\phi} + F_N(q) \right|^2 \cdot (145)
\]

where the Bethe\(^{(72,73)}\) phase is given by

\[
\phi = -2n \left\{ \log(\sin \frac{\Theta}{2}) + \frac{1}{2} \gamma + \frac{1}{2} \log \left( \frac{8}{3} R_E^2 \left[ R_E^2 + R_N^2 \right] \right) \right\} \cdot (146)
\]

Although this phase is written in association with the Coulomb amplitude in Eq. (145), it is not just a property of the Coulomb scattering; it also includes the distortion of the strong interaction amplitude by the Coulomb potential. In his early work on the subject, Soloviev\(^{(74)}\) did not consider the nucleus or nucleon to have finite sizes \((R_E = R_N = 0)\). Since the formula for \( \phi \) must not contain the screening radius, then the whole of the last logarithm in Eq. (146) must be absent on dimensional grounds (except for possibly a small constant contribution). In Fig. 32 you can
Fig. 32 Elastic $\pi^{12}\text{C}$ scattering\(^2\) at 260 MeV. Using the Bethe\(^7\)\(^2\) approach, the curves\(^7\)\(^5\) show the dependence of the cross-section on the real part of the forward scattering amplitude (assuming a constant phase).

see the fits of Locher\(^7\)\(^5\) using formula (145) at 260 MeV; from the behaviour in the interference he infers that $-0.6 \leq \rho \leq -0.3$. As I have previously mentioned, this value is in good agreement with the forward dispersion relation predictions (Fig. 28).

One can try to improve upon this theory by adding up a few relativistic Feynman graphs, but such methods are fraught with difficulty. If done
properly I suspect that the answer will not differ too much from what we have given here. Another small difference is possible if the optical potential includes large contributions from virtual charge exchange. When the pion is in its neutral phase, the Coulomb force does not matter too much. However, it is probably the long range part of the Coulomb force that gives rise to important effects, and for this the pion has always the same charge. One could think about other dispersive corrections. In the Glauber predictions, one may be a little uneasy that the pN experimental input still has some effect of the Coulomb force in it. At the moment, the uncertainty due to this is in general much less than the uncertainty due to the input data as a whole!

To finish this section, I recommend that experimentalists fit their results with the optical amplitude generated by Eq. (142), rather than use the Bethe formula [Eq. (146)], especially for large nuclei.

11. INELASTIC SCATTERING TO DISCRETE LEVELS

"It would be an unsound fancy and self-contradictory to expect that things which have never yet been done can be done except by means which have never yet been tried". [Aphorisms 1.VI].

We should now like to discuss the excitation of discrete levels in the nucleus, within a microscopic theory (not a deformed black sphere!). In the way we have used Glauber theory so far, things are simple if the nucleus can be described by a set of independent particles, with only Pauli and centre-of-mass correlations. Unfortunately, the first 2\(^{+}\) level in \(^{12}\)C is by no means a simple particle-hole state; this gives much too low an electric transition moment B(E2)\(^{76}\). It is not surprising, therefore, that if we use the theory with such a simplified nucleon model, it grossly underestimates the excitation with pions and protons. It would be nice to have an approximation to the Glauber model, where explicit knowledge of wave functions is not required. The Distorted Wave Impulse Approximation (DWIA) only requires knowledge of form factors which are measurable in electron scattering. This can be written down without using the eikonal approximation, but in the impact parameter form\(^{77}\) used here, you can make a direct connection to Glauber theory. Only those terms in the theory are kept which are proportional to a direct transition
from ground to excited states; the rest of the multiple scatterings merely provides an absorption factor. Thus the DWIA amplitude for the production of a state with angular momentum \( l \) and projection (in the \( \hat{q} \) direction) \( m \) is

\[
F_{\ell}^m(\vec{q}) = A F_{rN}^0(0) \int \frac{d^3r}{r} \Psi_{\ell}^m(r) \Gamma^m(\vec{b}) d^3r,
\]

(147)

where we have neglected spin and isospin flips. \( \Gamma(\vec{b}) \) is an absorption factor for the scattering from \((A - 1)\) residual nucleons. In the optical limit [Eq. (60)]

\[
\Gamma^m(\vec{b}) = \exp \left\{ -\frac{2\pi i}{\hbar} (A - 1) F_{rN}^0(0) \int_{-\infty}^{\infty} \tilde{\rho}_{\ell}^m(r) dr \right\}
\]

(148)

In the above, \( \tilde{\rho}_0 \) and \( \tilde{\rho}_k \) are effective ground state and nuclear transition densities; the size of the nucleon has been folded in. Note the implicit assumption that the residual nuclear potential is just \((A - 1)/A\) times the original one. In the single scattering approximation, which would be reasonable for electrons

\[
F_{\ell}^m(\vec{q}) = A F_{rN}^0(0) F_{\ell}(\vec{q}) S_{\ell,0}
\]

(149)

with \( F_{\ell} \), the transition electric form factor.

From angular momentum considerations, the transition form factors behave like \( q^\ell \) for small \( q \) [cf. Eq. (22)], and it is found empirically that expressions of the type

\[
F_{\ell}(\vec{q}) = B_{\ell} q^\ell \exp \left( -\frac{q^2}{4\alpha_{\ell}^2} \right)
\]

(150)

reproduce the electron data\(^{78,79}\) for the \( 2^+ \) and \( 3^- \) excitations accurately out to \( q^2 \sim 6 f^{-2} \) with the constants \( B_2 = 0.284 \) \( f^{-2} \), \( \alpha_2^2 = 0.335 \) \( f^{-2} \), \( B_3 = 0.176 \) \( f^{\ell} \) and \( \alpha_3^2 = 0.266 \) \( f^{-2} \). The one correction to the simple formula, which we shall consider, is to take a centre-of-mass fudge factor \( \exp \left( q^2/4A\alpha_{\ell}^2 \right) \) as we did for elastic scattering in Glauber theory [Eq. (102)]. This has been included in the quoted values of the parameters.
The ground state density was taken as for elastic scattering (with $\alpha^2 = 0.40 f^{-2}$) and the pion-nucleon parametrization as Gaussian [Eq. (112)]. The resultant predictions\textsuperscript{77} are shown in Figs. 33 and 34 for the $2^+$ and $3^-$ levels at 180 and 260 MeV. For the $2^+$ case, the size and shape of the first diffraction maximum agrees quite well with the experimental one, and the variation with energy is encouraging. The $3^-$ curve tends to be high, and there is no real evidence for an experimental diffraction minimum. Since it is possible that this is due to background from the $0^+$ (7.6 MeV) level poorly resolved, this also has been calculated in the same way and added to the $3^-$ curve. When consideration is given to the experimental input uncertainty, the agreement is satisfactory.

![Graph](image_url)

Fig. 33 Excitation\textsuperscript{2} of the $2^+$ (4.4 MeV) level in $^{12}$C with pions. The curves\textsuperscript{77} result from a DWIA calculation.
Fig. 34: Excitation of the $3^-$ (9.6 MeV) level in $^{12}$C with pions. The curves result from a DWIA calculation. If the $0^+$ level (7.6 MeV) is included as well, the dashed curve is predicted.

It is very instructive to compare the predictions at 180 MeV with those of the deformed black sphere (Fig. 4). The $2^+$ curves are quite similar, with only slight differences in radii and fuzziness. The $3^-$ curves show marked differences, engendered mainly by the large experimental radius for the transition form factor, much larger than that suggested by the deformed sphere model. Experimentalists are to be encouraged to improve their resolution on the $3^-$ level so they can at least distinguish between these two models. It would be even nicer to get good data on the $0^+$ level, since there is a strong possibility that this may be excited in a two-step process, with the $2^+$ as an intermediate stage.
It is implicit in Eq. (150) that for electron scattering

\[ \log \left( \frac{F_z^2}{q^2} \right) = 2 \log B_\lambda - \frac{q^2}{2} \alpha_\lambda^2 \]  

(151)

is linear in \( q^2 \) out to \( q^2 \sim 6 f^{-2} \). Tibell\textsuperscript{80} noted that if a similar quantity

\[ Z_\lambda = \log \left( q^{-2} \frac{d \sigma_\lambda}{d \Omega} \right) \]  

(152)

is constructed for the excitation cross-sections of 186 MeV protons, then for a variety of nuclei \( Z_\lambda \) also seems linear until the first diffraction minimum is approached

\[ Z_\lambda \approx 2 \log B_\lambda^p - \frac{q^2}{2} \alpha_\lambda^p \]  

(153)

Furthermore, the ratio \( B_3/B_2 \) for electron scattering is quite close to the analogous ratio for protons \( B_3^p/B_2^p \) for \( ^{12}\text{C} \) (and a variety of other nuclei). The quoted pion data\textsuperscript{2} also have reasonably linear \( Z_\lambda \) (Fig. 35) and the \( B_3^\Pi/B_2^\Pi \) is consistent with being constant at a mean value \( 0.55 \pm 0.03 f \). This is not dissimilar to the electron scattering \( B_3/B_2 = 0.62 \pm 0.04 f \).

Now if theoretical Tibell plots are constructed from the results of the DWIA calculations, they show some slight deviation from linearity over an equivalent \( q^2 \) interval, but the introduction of random errors of the same order as the experimental ones would effectively obscure this. The deduced \( B_3^\Pi/B_2^\Pi \) is again roughly energy independent, but is slightly higher in the region of the larger pion-nucleon cross-sections. The mean value \( 0.70 \pm 0.02 f \) is, however, significantly higher than the electron scattering input. This seems perhaps reasonable, since the \( 3^- \) state is associated with a larger transition radius than the \( 2^+ \), and should therefore suffer less from the absorption. As was suggested by T. Ericson, during the lecture, that the experimental ratio is actually low could be due to complications with the \( 0^+ \) level changing the slope of the extrapolation.
The DWIA theory can of course predict the polarization of the $2^+$ level also, and hence the angular distribution of the $\gamma$-rays emitted in the decay of this state. A measurement of this excitation with 70 MeV $\pi^+$ has recently been carried out\textsuperscript{81).} The total rate (which depends slightly upon the assumed angular distribution of the $\gamma$-rays) is $14.5 \pm 3.0$ mb for $\pi^+$ and $17.9 \pm 3.1$ mb for $\pi^-$, compared with a DWIA calculation of 30 mb with very uncertain input parameters. If the $\gamma$-angular distribution can be measured, it will provide information complementary to the straight pion excitation curves.
Another type of inelastic pion scattering is charge exchange to discrete nucleon levels. Because of the difficulties associated with $\pi^0$ detection, the only place where this has been done clearly is in $^{13}\text{C}(\pi^+, \pi^0)^{13}\text{N}_{gs}$ with 180 MeV pions\(^\text{82}\). Even here only an over-all rate is measured by detecting just the positron emitted as the $^{13}\text{N}$ decays back to $^{13}\text{C}$. This gives a cross-section of $3.3 \pm 1.0$ mb. Such a process is interesting because in single scattering approximation it involves similar matrix elements to $\beta$-decay. People have even raised the possibility of studying such matrix elements systematically using pion charge exchange. In single scattering approximation, assuming an odd neutron outside of a $^{12}\text{C}$ core gets charge exchanged, we have

\[
F(q) = \int_{\pi^+ n \rightarrow \pi^- p} S(q) S(q),
\]

where $S(q)$ is the form factor for the distribution of the odd nucleon, which in practice was taken to be the same as the nuclear form factor. I have neglected the spin-flip term; this should have a small effect on the total rate since it is zero in the forward direction. Making slightly different assumptions, you get total rates of the order of 4.4-5.5 mb — quite satisfactory\(^{83,84}\).

We can now do "better" than this by using a DWIA

\[
F(q) = \int_{\pi^+ n \rightarrow \pi^- p} S(q) \int e^{i q \cdot r} \tilde{P}_T(r) \Gamma(b) \, d^3 r.
\]

The predicted cross-section\(^{57,85}\) is then about 0.15 mb, suggesting very strongly that the single scattering result is purely fortuitous. It has been suggested that spin flip will patch things up (it takes place at larger impact parameters and will thus be less influenced by the absorption), but I cannot see it making up this discrepancy. In principle we can do a more exact coupled channel formulation with $\pi^{+13}\text{C}$ and $\pi^{013}\text{N}$. Such calculations, within the framework of the Kisslinger equation, have been done up to 120 MeV for elements $^7\text{Li}$ and $^{27}\text{Al}$ (among others)\(^{86,87}\). Extrapolating in energy and interpolating in A, the prediction of 0.15 mb looks very reasonable (definitely not the 3.3 mb). It is a feature of this and the DWIA that the cross-section decreases as the resonance is approached.
How can things be improved? In the DWIA the only role of the multiple scattering is to provide a damping factor, which will in general reduce the cross-section. The coupled channel method will not differ greatly because the coupling is not very strong, i.e. multiple charge exchange is not very likely. When the basic \( \pi N \) interaction is very strong, one can think of many types of process which the DWIA has missed. For example, one can knock a proton out of the core and charge exchange the external neutron into the core, or charge exchange a neutron out of the core and knock the external neutron back in. If we assume a set of single particle wave functions as in our discussion of elastic scattering, then it is easy to write down what is essentially a second-order DWIA. That is, we calculate the profile function for the second-order processes described and then multiply by an absorption factor. The result does not look too good, an increase from 0.15 mb to 0.22 mb. However, the crude matrix elements put into the calculation are very similar to the ones involved in the \( 2^+ \) excitation of \( ^{12}\text{C} \). The simple model applied there underestimates the cross-section by a factor of six, and in the second order DWIA the factor enters squared! There is still room for hope therefore. A larger radius for the \( \frac{3}{2}^- \) orbital would also help. Perhaps a calculation building up the \( ^{13}\text{C} \) as a superposition of \( (0^+\text{n}) \) and \( (2^+\text{n}) \) would prove instructive.

It is my impression that these interesting reactions may end up telling us more about certain kinds of correlation functions in the nucleus, rather than about \( \beta \)-decay matrix elements.

12. **OTHER INELASTIC SCATTERINGS**

It is relatively easy to count the number of pions scattered through a given angle, without heeding their energy loss. After subtracting the elastic cross-section, we are left (at least at S1N energies) with a nuclear inelastic cross-section (very few pions produced). Providing the average energy loss is not too great, Glauber theory can provide an expression for this cross-section. We have to sum the transitions to all final states f
\[
\frac{d\sigma}{d\Omega} \left|_{\text{sum}} \right. = \sum_{f} | \langle f | F(\vec{q}, \vec{r}_1, \ldots, \vec{r}_A) | i \rangle |^2 \\
= \sum_{f} \langle i | F^+ | f \rangle \langle f | F | i \rangle \\
= \langle i | F^+ F | i \rangle ,
\]

where we have used closure over the final nucleon states. Hence the summed cross-section is just the expectation value of an operator in the ground state. There have been many applications of this formula to high-energy proton scattering\(^{71,88}\), but as yet no one has made a detailed comparison for low-energy pions. In Fig. 36, you can see Kofoed-Hansen's\(^{88}\) evaluation of the summed cross-section (elastic plus nuclear inelastic) for 20 GeV protons incident on \(^{12}\)C, neglecting the effects of correlations (which are particularly important for the small angle inelastic scattering). It would be nice to have a similar curve for the equivalent 260 MeV \(\pi^-\) data\(^2\). I would like to advance the proposition that the effects of pion absorption on the optical potential should show themselves in this sum rule for large \(q\), if they are at all important. Since pion absorption is mainly a two-nucleon affair, it takes place towards the centre of the nucleus (proportional to \(\rho^2\)) and we have to look at large \(q\) to investigate this region. If we accept the conventional model for absorption (one large angle scattering, followed by absorption on a second nucleon) then the large angle elastic scattering or excitation of low-lying levels may not be affected much by absorption. Finally, we have seen that the Glauber predictions of the elastic and total cross-sections are in reasonable rapport with experiment, so that it must be the breakup cross-section (probably the quasi-elastic) which is overestimated if absorption is really significant.

For a large nucleus, the sum rule for the inelastic scattering, away from the forward direction, can be expanded in a rapidly convergent series\(^{71,89}\)
\[ \frac{d\sigma}{d\Omega} |_{n=1} = \sum_{f \neq i} \frac{1}{F_{fi}^2} \]

\[ = N_s |f(q)|^2 + N_z \Lambda \left| \int |f(\vec{A})|^2 |f(\vec{A} - \vec{q})|^2 \frac{d^2 \Delta}{R^2} \right|^2 \]

where the number of effective nucleons is

\[ N_n = \frac{\Lambda}{n \cdot \sigma} \int e^{-\sigma A \mu(b)} \left[ \sigma A \mu(b) \right]^n d^3 b. \]

Fig. 36 Elastic and nuclear inelastic scattering of 19 GeV protons from Pb. The Glauber calculation is from Ref. 88.
This is called the shadowed multiple scattering expansion; it is basically a classical expression in that only cross-sections appear in it. To a good first approximation (not at too large angles) the inelastic scattering should look like the free πN scattering. However, the number of nucleons participating, \( N_1 \), is much smaller than \( A \) (for \(^{208}\)Pb, \( \sigma = 40 \) mb, \( N_1 \sim 10 \)) and often consists merely of a ring of nucleons around the periphery of the nucleus.

If we can neglect the size of the nucleus, then for a Gaussian nuclear density

\[
N_A^e = \frac{2 \pi}{\alpha^2} \left[ 1 - e^{-\frac{\sigma A^2}{2 2\pi}} \right] ,
\]

while for a uniform sphere distribution of radius \( R \)

\[
N_A^s = 3 A \frac{\sigma^2}{\pi R^2} \left( \frac{1 - e^{-\frac{x}{X}}}{x} \right) \bigg|_{x = \frac{3 \sigma A}{2 2\pi R^2}}.
\]

Of course for small \( \sigma \), \( N_1 \rightarrow A \), but a \( \sigma \rightarrow \infty \)

\[
N_A^e = \frac{2 \pi}{\sigma \alpha^2} ; \quad N_A^s = \frac{16}{9} \frac{R^6}{\sigma^3} \frac{n_3}{A^3}.
\]

These formulae have to be modified for small \( q^2 \), and also to take account of \( \beta^2 \). They are, however, already sufficient for you to understand one of the more puzzling aspects of the CERN-IPN data. As a by-product of their total cross-section measurements, they obtained a quantity which they identified as the 0° inelastic cross-section. As a function of energy, this was smallest at 150 MeV! Now the crude formula (157) predicts that

\[
\frac{d\sigma}{d\Omega}_{\text{inel}}^{0} = N_A \frac{R^2 \sigma^2}{16 \pi^2} \left( 1 + \varphi^2 \right) .
\]
In the uniform sphere model for large \( \sigma \)

\[
\frac{d\sigma}{d\Omega} \bigg|_{\text{inel}} \propto \frac{L^2 (1 + \rho^2)}{\sigma},
\]

while for a Gaussian

\[
\frac{d\sigma}{d\Omega} \bigg|_{\text{inel}} \propto \frac{L^2 \sigma (1 + \rho^2)}{L^2},
\]

The true nucleon distribution is intermediate between these two, and it is pleasing that \( k^2 (1 + \rho^2) \) is minimal around 180 MeV. These rough results will, I hope, stimulate somebody to evaluate Eq. (156) for low-energy pion scattering.

You can set up a similar formalism for pion charge-exchange reactions. However, there are no angular distributions available for \((\pi^\pm, \pi^0)\) scattering in the low-energy region. Integrating Eq. (157) with respect to \(q\), we expect for the total rates

\[
\sigma (\pi^+, \pi^0) \approx \frac{Z}{A} N_1 \sigma \pi^+ \eta \rightarrow \pi^0 p
\]

This has been tested\(^{91}\) for 70 MeV pions, and it is not inconsistent with the data. We cannot say any more until a better evaluation of the theory has been attempted.

The next process that springs to mind is double charge exchange. Although there are then no problems with neutral pions, the low cross-section has meant that no one has yet identified the excitation of a definite nucleon level in this way. There are many experiments which sum over all the final nuclear states\(^{92,93}\), but these are probably quite difficult to analyse. Typically the pion loses a large fraction of its energy in exciting the nucleus and there are probably many different mechanisms involved.
Let me finish by mentioning the quasi-elastic ($\pi$, $\pi N$) reactions. Since the total cross-section for $^{12}$C $^{\pi}_{\pi} \rightarrow ^{11}$C at 180 MeV is some 75 mb, and for $^{12}$C $^{\pi^+}_{\pi^+} \rightarrow ^{11}$C it is a similar number, we see that almost half of the inelastic cross-section is of the ($\pi$, $\pi N$) type. The first attempts to understand such reactions were based upon a single scattering approximation, but these clearly have to be greatly modified, since in the region of the $^{3/2}_{3/2}$ resonance they predict that the $\pi^-$ and $\pi^+$ reactions should be in the ratio of three to one, compared with an experimental one to one. Any attempts to interpret à la (p, 2p) must involve a very large charge-exchange term in the distorting potential. The phenomenon has still to be adequately explained.

13. Valediction

Since this is the last scheduled lecture, it is incumbent upon me to give a few words of advice to the boys who are leaving, and for those who will return, an exhortation to work before the start of the next school. What theoretical questions arising from my lectures should be answered first? I can only give those which spring to my own mind, and this is of course a very biased selection.

Do we need a non-local potential, or can we get away with a local one? If the Kisslinger equation is to be used, how important are the off-shell form factors? How strong are the effects of correlations on the optical potential for medium-energy pion scattering? For this we may have to study a multichannel problem. When we have a better feeling for pion absorption near the $^{3/2}_{3/2}$ resonance, can we calculate the feedback into the elastic optical potential?

Why is Glauber working as well as it does down to low energies? Could we improve it further by using the proper phase space, i.e. should the $q^2$ integrals be cut off at $4k^2$? We have tended in these lectures to think in terms of individual nucleons inside the nucleus, but the success of the deformed black sphere for 180 MeV pions on carbon should encourage one to study eikonal approximations to collective models. For example, Feshbach has shown that the Blair phase rules follow from the semi-classical approximation to a deformed square well potential with strong absorption, but of course the formalism can treat more realistic cases.
I suspect that the α-particle cluster model for $^{12}\text{C}$ and $^{16}\text{O}$ would be fruitful, in that it is a collective model, into which it is easy to feed the pion-nucleon information.

Of course we need more and better data, more nuclei and better energy resolution to measure excited states. We are promised data on helium and possibly deuterium in the not-too-distant future, but while the elements DOCH are interesting, some addition is required. I would predict now that $^{40}\text{Ca}$, $^{208}\text{Pb}$ will look like black discs to 180 MeV pions.

It is unfortunate that the excitation of $T = 1$ states and charge-exchange reactions have proved so hard, but that leaves scope for more powerful machines such as SIN. It would be nice to have coincidence experiments, detecting both the $\pi$ and $\gamma$ from $^{12}\text{C}(\pi, \pi')^{12}\text{C}^* \rightarrow ^{12}\text{C} + \gamma$, to measure the polarization of the excited states.

Fortunately the Bugg $\pi N$ amplitudes$^{70}$ in the energy range 70–300 MeV should be available later this year. If some of you consider measuring the scattering of pions from nuclei below this energy, then I would advise you also to consider measuring the scattering of pions from hydrogen as well, or the theorists may have an excuse for not analysing things accurately!

"There will be found, no doubt, when my history and tables of discovery are read, some things in the experiments themselves that are not quite certain, or perhaps that are quite false, which may make a man think that the foundations and principles upon which my discoveries rest are false and doubtful. But this is of no consequence, for such things must needs happen at first. It is only like the occurrence in a written or printed page of a letter or two mistaken or misplaced, which does not much hinder the reader, because such errors are easily corrected by the sense". [Aphorisms 1.CXVIII].
POSTSCRIPT

After these lectures were given, two papers appeared, independently suggesting the possible virtues of the local potential (66). Lee and McManus\textsuperscript{95}) found that above 150 MeV the exact solution for this potential agreed quite closely with the eikonal approximation; Fäldt\textsuperscript{96}) found a slightly less favourable result. Fäldt also stressed the importance of recoil effects in the Kisslinger equation, suggesting that this is the reason for the large size of the parameters found in Refs. 31 and 33. Some of these effects are already taken account of in certain papers (e.g. Ref. 38), but the fine points are a little lost in the ambiguities of the formalism, as exemplified by Eq. (35).

In its application to mesic atom data, the simple local potential (66) runs into difficulties if the nucleus has a sharp edge\textsuperscript{97}); the potential then involves derivatives of delta functions. The reason is quite clear in momentum space. If

\begin{equation}
\rho(r) = \rho_0 \Theta(r-R)
\end{equation}

then

\begin{equation}
S(q) \sim \frac{\cos qR}{q^2}
\end{equation}

for large \(q\). When this is multiplied by \(q^2\) from the p-wave amplitude, it is seen that the momentum space potential does not decrease with \(q\) for large \(q\), but merely oscillates. Any killing of \(f(q)\) for large \(q\) would remove the unphysical singularity. Now if one believes in Regge poles then \(f(s,q^2) \sim (q^2)^{\alpha(s)} \frac{1}{q^2} \) as \(q^2 \to \infty\). At threshold therefore, the amplitude should only grow like \((q^2)^{\alpha[\left(m_p+m_n\right)^2]} \sim q^{\frac{\alpha}{2}}\) and this is sufficient for the regularity of the potential.

Simple practical calculations by Scheck and myself thus far suggest that in light nuclei the p-wave amplitude is sufficiently strong to give effects which are at variance with the perturbation theory results [Eqs. (71) and (72)].
REFERENCES


   F. Binon et al., Coulomb-nuclear interference in π⁻¹²C elastic scattering, CERN preprint (submitted to Nuclear Phys.).


10) A fit to the data of Ref. 2 was carried out by R. Meunier, CERN Internal Report (1969), but he used spherical Bessel functions, rather than the more appropriate cylindrical ones.


12) See for example G. Fälndt, Pion-deuteron cross-sections in the resonance region, Harvard preprint 1971. This reference does not, however, explain all the ambiguities in the formalism.


15) J. Dowell, private communication.


26) Rutherford Laboratory proposal No. 79, by the Birmingham/Rutherford/Surrey group.
36) J.V. Lepore and R.J. Ridell, quoted in Ref. 34.

42) L.L. Foldy, Phys. Rev. 67, 107 (1945), and private communication.


46) This correlation is of quite a different type to the normal ones discussed. It is a rescattering effect on the nucleon itself.


62) E. Boschitz, private communication.

70) D. Bugg has intimated that this year he will publish accurate \( \pi^+ p \) angular distributions in the range 70-300 MeV/c^2.
75) M.P. Locher, private communication.
85) C. Wilkin, private communication.
87) A.K. Kerman and R.K. Logan, ANL 6848, p. 236.


94) H. Feshbach, Proceedings of the International School of Physics, Enrico Fermi, Course XXXVIII, p. 183.


97) L. Scherck, private communication.
\( \pi p \) ELASTIC SCATTERING, 0 to 300 MeV

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I. INTRODUCTION

Let me begin by reminding you of the main facts about this region.

i) The total cross-section is dominated by the \( P_{33} \) resonance:

\[
\sigma_T(\pi^+ p) = 4\pi \frac{\lambda^2}{\lambda_j^J} \left( J + \frac{1}{2} \right) \left| a_{\ell, J} \right|^2,
\]

where \( a_{\ell, J} \) is the partial wave amplitude, given by

\[
a_{\ell, J} = \frac{1}{2i} \left( \eta_{\ell, J} e^{2i \delta_{\ell, J}} - 1 \right).
\]

At the resonance, \( P_{33} \) contributes approximately 200 mb to \( \sigma_T(\pi^+ p) \), while all the other partial waves together amount to only 6 mb.

ii) The inelastic threshold is at 170 MeV, but the inelastic cross-section is very small below 300 MeV (it is 1.0 mb in \( \pi^- p \) and 0.2 mb in \( \pi^+ p \) at 300 MeV); hence for most partial waves we can set \( \eta = 1 \).

iii) \( \pi^+ p \) scattering is in a pure isospin state \( I = 3/2 \).

\( \cal A(\pi^- p \to \pi^- p) \) is given by \( \frac{1}{3} \cal A_{3/2} + \frac{2}{3} \cal A_{1/2} \)

\( \cal A(\pi^- p \to \pi^0 n) \) is given by \( \frac{\sqrt{2}}{3}(\cal A_{3/2} - \cal A_{1/2}) \)

where subscripts refer to isospin.

iv) If only S- and P-waves are significant, the angular distribution for \( \pi^+ p \) scattering is given by
where in the first approximation, we have kept only terms involving
\( a_{s} \); \( a_{p_{3}} \) refers to the S-wave, \( a_{p_{1}} \) and \( a_{p_{3}} \) to P-waves with \( J = 3/2 \)
and 1/2, respectively.

Thus, in the first approximation, \( \sigma_{T} \) determines \( |a_{p_{3}}|^{2} \), the \( \cos \theta \) term in \( d\sigma/d\Omega \) determines \( \text{Re} \ (a_{p_{3}}^{*} a_{s}) \) and the \( \cos^{2} \theta \) term in \( d\sigma/d\Omega \) determines
\( \text{Re} \ (a_{p_{3}}^{*} a_{p_{1}}) \).

In a full phase shift analysis, it is of course necessary to preserve
all the remaining small terms, plus contributions from D- and F-waves.
The formulae for \( \pi^{-} p \rightarrow \pi^{-} p \) are similar, except that the amplitude is a
linear combination of isospin 3/2 and 1/2 amplitudes. The only important
point to notice is that the first term becomes

\[
\frac{1}{q} \left( 1 + 3 \cos^{2} \theta \right) \left| a_{33} + 2a_{13} \right|^{2}
\]

Likewise in \( \pi^{-} p \rightarrow \pi^{0} n \), the first term becomes

\[
\frac{2}{q} \left( 1 + 3 \cos^{2} \theta \right) \left| a_{33} - a_{13} \right|^{2}
\]

Thus the branching ratio between \( \pi^{-} p \) and \( \pi^{0} n \) final states determines
\( \text{Re} \ (a_{33}^{*} a_{13}) \).
Table 1
Principal experimental data below 350 MeV

<table>
<thead>
<tr>
<th>$T_{\pi}$ (MeV)</th>
<th>Lab.</th>
<th>Date</th>
<th>Measurements</th>
<th>Accuracy</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>78-190</td>
<td>Chicago</td>
<td>1952-5</td>
<td>$\sigma_T^{+}$; $d\sigma/d\Omega^{+0}$</td>
<td>±3%, ±12%</td>
<td>1</td>
</tr>
<tr>
<td>150-750</td>
<td>Brookhaven</td>
<td>1958</td>
<td>$\sigma_T^{+}$</td>
<td>±2.5%</td>
<td></td>
</tr>
<tr>
<td>150,170,220</td>
<td>Carnegie</td>
<td>1956</td>
<td>$d\sigma/d\Omega^{+0}$</td>
<td>±5-10%</td>
<td>1</td>
</tr>
<tr>
<td>98</td>
<td>Liverpool</td>
<td>1958</td>
<td>$d\sigma/d\Omega^{+}$</td>
<td>±3-5%</td>
<td>1</td>
</tr>
<tr>
<td>30-41</td>
<td>Rochester</td>
<td>1960</td>
<td>$d\sigma/d\Omega^{+}$</td>
<td>±5-12%</td>
<td>1</td>
</tr>
<tr>
<td>247,310</td>
<td>Berkeley</td>
<td>1961-4</td>
<td>$d\sigma/d\Omega^{+0}$; $P$(backwards)$^{+0}$</td>
<td>±3%, 5%</td>
<td>1</td>
</tr>
<tr>
<td>70-295</td>
<td>CERN SC (Cambridge/Rutherford)</td>
<td>1970</td>
<td>$\sigma_T^{+}$; $d\sigma/d\Omega^{+}$</td>
<td>±0.5%, 1%</td>
<td>2</td>
</tr>
</tbody>
</table>

2. INTEREST OF MEASUREMENTS IN THIS REGION

High precision measurements in this region determine the long-range part of the $\pi$-nucleon interaction (small t and u). In P-waves the centrifugal barrier keeps the pion at large r unless the nuclear interaction is strongly attractive (as it is in the $P_{33}$ state), and one gets a precise measure of the long-range tail of the $\pi N$ interaction. The long-range interaction originates\(^3\) from N and $N^*$ exchange in the u-channel, and $\sigma$ and $\rho$ exchange in the t-channel. $N^*$ exchange may be derived with negligible error from $\pi N$ scattering data using crossing symmetry; $N$, $\sigma$, and $\rho$ exchange have completely different dependence to one another on I, J and $\ell$, and thus a complete determination of the S- and P-waves separates them clearly. It turns out that D- and F-waves are (with the exception of $D_{13}$ at high energy) so small that they can be derived more accurately from the theoretical fit to $N$, $\sigma$, and $\rho$ exchange than by experiment.

Figure 1 shows the present status of the phase shifts up to 300 MeV.
3. TOTAL CROSS-SECTIONS

They are measured by the transmission of $\pi^\pm$ beams through a liquid hydrogen target. Full details of the experimental technique are to be found in Ref. 2. Some of the more important points are:

a) it is essential to separate the $\pi$ beam cleanly from the contamination of $\mu$ and $e$ using a Cerenkov counter;

b) it is just as necessary to measure the momentum precisely as $\sigma_T$;

c) the ultimate precision is governed by understanding the shape of the extrapolation to zero solid angle; at small angles, corrections are necessary for Coulomb scattering, and for the effect of $\pi \rightarrow \mu\nu$ decays after the Cerenkov counter.

Results for $\pi^+ p$ are shown in Fig. 2. The agreement between the CERN measurements and earlier ones is fair, but the mass of the $N^*$ resonance is now found to be $1231 \pm 0.6$ MeV/c$^2$, rather than the earlier value of $1236 \pm 0.5$ MeV/c$^2$.

It is instructive to subtract from $\sigma_T$ the contributions of the small partial waves, so as to display the contribution from $P_{33}$ alone. This is done on Fig. 3; note that the $\pi^- p$ data have been corrected for the small inelasticity due to $\pi^- p \rightarrow n\gamma$. That is

$$
\sigma_T^{\pi^-p}(J^P=3^+, I=\frac{3}{2}) = \frac{8\pi\lambda^2}{3} \left\{ \sin^2 \delta_{33} - \frac{1}{2} (\eta - 1) \cos 2\delta - \frac{1}{4} (1 - \eta)(1 + \eta) \right\}
$$

a correction has been applied for the terms involving $(1 - \eta)$, and what is shown as the figure is $8\pi\lambda^2/3 \sin^2 \delta_{33}$. It is comforting that the experimental results touch the curve $8\pi\lambda^2/3$ within experimental error; this checks that both the normalization of $\sigma_T$ and the momentum scale are right (or they are wrong by compensating amounts, which seems unlikely).
Fig. 1 Phase shifts for πp scattering in the energy region 0 to 300 MeV.
Fig. 2 \( \pi^+ p \) total cross-sections from 150 to 400 MeV/c.
The total cross-section in the $P_{13}$ state. The full line shows $\frac{1}{2} \sigma_{\eta}(\pi^{+}p)$, the dashed line $\sigma_{\pi^{0}p}$, and the dash-dotted line $\sigma_{\eta}(\pi^{0}p)$. The latter two have been corrected for the small effect of inelasticity due to $\pi^{\pm} + \eta$ as explained in the text.
The interesting result which emerges immediately is that the N*0 is broader than N*++ by about 6 MeV. The origin of this effect is clear: the Coulomb potential pulls the π− towards the proton and enhances the probability of nuclear interaction, whereas the π+ is repelled, and the nuclear interaction is reduced.

4. **CALCULATION OF THE COULOMB BARRIER EFFECT**

One wants to calculate the correction due to diagrams of the form

![Diagram](image)

Fig. 4  Coulomb corrections to πp scattering.

One gets a qualitative feeling for the effect by treating the incident pion wave as distorted by the Coulomb potential; that is, instead of using spherical Bessel functions \( j_{\lambda}(kr) \) and \( n_{\lambda}(kr) \) in the scattering problem, one introduces Coulomb functions \( F_{\lambda}(kr) \) and \( C_{\lambda}(kr) \). Now expand the latter as a power series in \( j_{\lambda}(kr) \) and \( n_{\lambda}(kr) \). One finds

\[
\frac{1}{kr} F_{\lambda}(kr) = \left(1 \pm \frac{\pi \alpha}{2 \gamma^2}\right) j_{\lambda}(kr) + \ldots + \text{etc.}
\]

where \( \pm \) refer to \( \pi^+ p \) and \( \beta \) is the laboratory relativistic velocity of the incident pion; similarly for \( C_{\lambda}(kr) \). Then if one keeps only the first term in the series,

\[
\frac{\Gamma^{++}}{\Gamma^0} \approx 1 - \frac{2 \pi \alpha}{\beta^2} \approx 0.95.
\]
This fits the experimental data well; however, the derivation is not really valid because further terms in the series are not negligible.

The problem has been formulated more exactly by Auvil. He assumes that one can write down the Klein-Gordon equation with some potential (possibly energy-dependent) describing the πN interaction, and with the Coulomb interaction treated as a perturbation. Using Green's functions he deduces for π⁺p scattering the amplitude

\[ \mathcal{F}(\pi^+ p) = \frac{i}{2\gamma} \int e^{2\gamma(\delta - C_{33})} \left\{ e^{i\mathcal{R}_c^2} - 1 \right\} d\mathcal{R} \]

where \( \delta \) is the true nuclear phase shift (i.e. with the Coulomb potential switched off), and

\[ C_{33} = \int_{r=\infty}^{\infty} \mathcal{R}_c r^2 V_c(r) \left\{ e^{2\mathcal{R}_c} - \int e^{2\mathcal{R}_c} d\mathcal{R} \right\} dr \]

where \( V_c(r) \) is the Coulomb potential, and \( \mathcal{R}_c \) is the radial wave function for the nuclear scattering (with the Coulomb potential switched off).

The problem is now to choose a reasonable form for \( \mathcal{R}_c(kr) \). We have calculated \( C_{33} \) using a large variety of forms for \( \mathcal{R}_c(kr) \) corresponding to various potentials: square wells, Yukawa potentials, etc., which reproduce \( \delta_{33} \). The final results will be quoted for potentials of Yukawa form (or its gradient) derived from the best present estimates of \( \sigma, \rho, \) and N exchange; we find that varying the parameters of these potentials by reasonable amounts changes \( C_{33} \) by only ±15%, which is less than the experimental error.
Turning to \( \pi^- p \) scattering, the problem is somewhat more complicated since the Coulomb potential violates charge independence; i.e. it acts between \( \pi^- p \) and \( \pi^- p \) states, but does not couple to \( \pi^0 n \). Let us formulate this problem using the Schrödinger equation; the generalization is trivial. In the isospin representation, the nuclear interaction is described by a two-component equation (isospin states 3/2 and 1/2):

\[
\left( \nabla^2 + \ell^2 \right) \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix} = \frac{2u}{t^2} \begin{pmatrix} V_{3/2} & 0 \\ 0 & V_{1/2} \end{pmatrix} \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix}
\]

Now transform to the charge representation using Clebsch-Gordan coefficients:

\[
\left( \nabla^2 + \ell^2 \right) \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix} = \frac{2u}{t^2} \begin{pmatrix} \left( -\frac{1}{3} \right) \left( \frac{1}{3} \right) & V_{3/2} & 0 \\ \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) & 0 & \left( -\frac{1}{3} \right) \left( \frac{1}{3} \right) \end{pmatrix} \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix}
\]

We can now add the Coulomb potential to the right-hand side:

\[
\frac{2u}{t^2} \begin{pmatrix} V_c & 0 \\ 0 & V_c \end{pmatrix} \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix}
\]

Finally transforming back to the isospin representation:

\[
\left( \nabla^2 + \ell^2 \right) \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix} = \frac{2u}{t^2} \begin{pmatrix} \left( V_{3/2} + \frac{1}{3} V_c \right) & \left( -\frac{1}{3} \right) \left( -\frac{1}{3} \right) V_c \\ \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) V_c & \left( V_{1/2} + \frac{1}{3} V_c \right) \end{pmatrix} \begin{pmatrix} \psi_{3/2}^{\pi^-} \\ \psi_{1/2}^{\pi^-} \end{pmatrix}
\]

Physically what happens is that one starts with a plane wave at infinity; as the wave approaches the nucleon not only does the Coulomb potential distort the wave, but it changes the Clebsch-Gordan coefficient relating \((\pi^- p, \pi^0 n)\) to \((I = \frac{3}{2}, I = \frac{1}{2})\), i.e. a mixing amplitude \(f_{13}\) between the two isospin states appears. Finally, Auvil finds, writing subscript 3 for \(I = \frac{3}{2}\), and subscript 1 for \(I = \frac{1}{2}\).
\[ f_{\pi^{+}p \rightarrow p^{+}p} = \frac{1}{3} f_{3} + \frac{2}{3} f_{1} - \frac{4}{9} f_{13} \]

\[ f_{\pi^{-}p \rightarrow n^{0}n} = \frac{\sqrt{2}}{3} \left( f_{3} - f_{1} - \frac{1}{3} f_{13} \right) \]

\[ f_{n^{0}n \rightarrow n^{0}n} = \frac{2}{3} f_{3} + \frac{1}{3} f_{1} + \frac{4}{9} f_{13} \]

where

\[ f_{3} = \frac{1}{2i} \left( \eta_{3} e^{2\gamma} (\delta_{3} - \frac{1}{3} C_{33}) - 1 \right) \]

\[ f_{1} = \frac{1}{2i} \left( \eta_{1} e^{\beta} (\delta_{1} - \frac{2}{3} C_{11}) - 1 \right) \]

\[ f_{13} = C_{13} e^{\beta (\delta_{1} + \delta_{3})} \]

\[ C_{ij} = \int_{r=0}^{\infty} \rho_{r}^{2} V_{e}(r) \left( R_{i} R_{j} - \cos(\delta_{i} - \delta_{j}) \right) e^{2i(k_{r} r)} dr \]

Typically \( C_{33} \) is of magnitude 1.5\(^{o} \), \( C_{13} \) is 0.4\(^{o} \), and \( C_{11} \) is negligibly small. Finally one finds

\[ M^{++} - M^{0} = -1.3 \pm 0.5 \pm 1.8 \quad \text{before correction} \]

\[ = -2.9 \pm 0.5 \pm 1.8 \quad \text{after correction,} \]

\[ = -6.5 \pm 0.5 \pm 2.1 \quad \text{before correction} \]

\[ = -1.0 \pm 0.5 \pm 1.8 \quad \text{after correction.} \]

The first error is the statistical error, and the second is the present uncertainty in the subtraction of small partial waves.
5. THE TOTAL CHARGE EXCHANGE CROSS-SECTION

The Cambridge group measured the total cross-section for $\pi^- p \rightarrow$ neutrals at the CERN SC; the experiment is a very simple one, and it is surprising that it had never been attempted before. A box of scintillation counters surrounds a liquid hydrogen target on five sides out of six, and one measures the rate for $\pi^-$ in, nothing out. When a $\pi^-$ scatters backwards it can exit through the missing side of the box, but the recoil proton is detected instead. Small corrections are necessary for conversion of $\gamma$'s ($\sim 2\%$) and neutrons (1.5 to 4.0%) in the target or the scintillators; neutron conversion in the scintillators was calibrated directly at the Harwell cyclotron, and the remaining corrections were calculated.

Finally $\sigma_T(\pi^- p \rightarrow \pi^0 n)$ was obtained after applying small corrections for $\pi^- p \rightarrow \pi^0 \pi^0 n(\leq 1\%)$ and $\pi^- p \rightarrow \gamma n (\sim 1\%)$. Values are displayed in Fig. 5, and one immediately deduces values of the $\delta_{13}$ phase shift shown in Fig. 6. Prior to this experiment $\delta_{13}$ was the least well known of the phase shifts at low energy.

![Graph showing the total charge-exchange cross-section, $\sigma(\pi^- p \rightarrow \pi^0 n)$.](image)

Fig. 5  The total charge-exchange cross-section, $\sigma(\pi^- p \rightarrow \pi^0 n)$. 
6. $d\sigma/d\Omega$

The Cambridge group measured $d\sigma/d\Omega$ in two parts. The large $t$ region was measured in the conventional way by detecting $\pi$ and $p$ in coincidence; the small $t$ region and an overlap region was measured by identifying the scattered $\pi$ with a magnetic spectrometer arm of five spark chambers. The analysis of the counter data is complete, and the final accuracy is typically $\pm1\%$ or slightly better. As one would expect, agreement with present phase shift is only moderate; at several angles and energies, deviations of several standard deviations appears. The analysis of the spark chamber data is still in progress.

7. WHAT REMAINS TO BE DONE

The Cambridge-Rutherford experiment determines the elastic scattering amplitudes $\pi^+ p \rightarrow \pi^+ p$ and $\pi^- p \rightarrow \pi^- p$ from 70 to 295 MeV. It would be useful to check these amplitudes by measuring the polarization:
\[ P \frac{d\sigma}{d\Omega} = g e \cos \theta (D + \mathcal{E} \cos \theta) \]

where \( D \) depends largely on \( \text{Im} (a_{33}^* a_8) \), and \( \mathcal{E} \) depends largely on \( \text{Im} (a_{33}^* a_{P_1}) \). However, experimentally this is difficult. If one uses a polarized target, it is very difficult to get the low momentum \( \pi \) and \( p \) out of the magnetic field of the polarizing magnet; if one tries to measure the polarization of the recoil proton by scattering (in carbon, for example) the analysing power is poor at the low energy involved.

Measurement of \( \pi^- p \rightarrow \pi^0 n \) would check charge independence in the \( S^- \) and \( P_{3/2}^- \) waves. (The total cross-section has already been used to determine the \( P_{3/2}^- \) amplitude, and hence \( \delta_{13} \) assuming charge independence). However, it is again difficult to measure \( d\sigma/d\Omega \) with high precision; the Lausanne group are attempting this measurement at selected angles at the CERN SC, but estimate that it will be hard to achieve a precision better than \( \pm 7\% \).

Finally one is left with \( P(\pi^- p \rightarrow \pi^0 n) \) as a feasible experiment (using a polarized butanol target) to check charge independence or the accuracy of the elastic scattering amplitudes.

Experiments below 70 MeV would also be of interest; in this region \( P_{33} \) no longer dominates. Counter or spark chamber experiments of high accuracy are difficult, simply because of the problem of accounting for \( \pi \rightarrow \mu \) decays in flight. For example, at 70 MeV/c (20 MeV), the mean free path for decay is only 4 metres. The best way of doing this experiment would appear to be to use a liquid hydrogen bubble chamber, where both \( \pi \rightarrow \mu \) decays and small angle scatters (proton recoils a few mm long) are readily visible.
Finally it may be possible to determine the $\pi^- p$ scattering length directly by looking at $\pi$-mesic X-rays in hydrogen. The 2P to 1S transition has an energy of 2438 eV, of which 7 eV is due to the nuclear S-wave interaction.

These X-rays have been seen in an experiment at the CERN SC 4). The question is whether their energy can be measured with a precision of a fraction of an electron volt. It has been suggested it may be possible to find a photoelectric edge whose energy coincides sufficiently closely for an attenuation measurement to pinpoint the energy of the X-ray. It would also be possible to do this measurement in deuterium ($E_\gamma = 2596$ eV); however, in order to determine the $\pi^- n$ scattering length, it is necessary to understand the energy shift due to the process $\pi^- d \rightarrow mn$, and present information on this process is sufficiently imprecise to introduce an error of at least $\pm 1$ eV, and hence $\pm 0.01$ in the scattering length combination $a_1 + 2a_3$.

REFERENCES


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