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COHERENT INSTABILITY DUE TO ELECTRONS IN A COASTING PROTON BEAM

H.G. Hereward

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It was pointed out by Lapostolle \(^1\) that the 40 to 60 MHz betatron mode signals that are seen \(^2\) in the Intersecting Storage Rings are at about the frequency of electrons oscillating in the potential well of the beam. With a simple model one can estimate the thresholds and growth rates of instabilities arising from the interaction between the beam and these electrons.

**The Model**

Take a uniform density elliptic beam of dimensions 2a by 2b and suppose its barycentre is making small vertical oscillations described by

\[
z_1 = z_0 \exp i(\varphi_0 - \omega t)
\]

(1)

The rate of production of electrons per proton is \(\nu\), the electrons are produced at random within the beam ellipse and with small random transverse velocities. We assume almost all the electrons are within the proton beam and that they decay with a meanlife of \(\tau\); this is a reasonable model if the electrons drift longitudinally until they reach electrodes where they are quickly cleared. It is less satisfactory if they are lost by filling up the potential well until they overflow, or transversely as a result of proton heating or the instability itself; but still can be used for a rough estimate if we take \(\tau\) to be the life within the beam.

The dynamics is done in smooth approximation, linearised, with image and wall fields neglected. We calculate with a monochromatic beam, but use the known relation between Q-spread and Landau threshold \(^3\).

**The Electron Dynamics**

The vertical electric field, due to the proton beam, inside it, is

\[
E = \frac{Ne}{2\pi R c_0 \pi b(b + a)} (z - z_1)
\]

(2)

This gives the equation of motion for an electron

\[
z + k^2 z = k^2 z_1
\]

(3)

where

\[
k^2 = \frac{Ne^2}{2\pi R c_0 \pi b(b + a) m_e}
\]

(4)
I am neglecting the forces that the electrons exert on one another; the total force exerted by the electrons on themselves is zero.

Since (3) is linear I can calculate the motion of the electron barycentre by taking mean initial conditions. For the electrons born at time \( t_1 \) these are

\[
z = z_1(t_1) = z_0 \exp i(n\theta - \omega t_1), \quad \dot{z} = 0
\]

And the solution of (3) with those initial conditions is

\[
\frac{z}{z_0} = \frac{K^2}{K^2 - \omega^2 \left[ \exp i(n\theta - \omega t) - \exp i(n\theta - \omega t_1) \left[ \cos K(t - t_1) - \frac{i\omega}{K} \sin K(t - t_1) \right] \right] + \exp i(n\theta - \omega t_1) \cdot \cos K(t - t_1)}
\]

The mean displacement of all electrons at \( \theta, t \) is obtained by multiplying by \( \frac{1}{\tau} \exp(t_1 - t)/\tau \) and integrating \( dt \) from \( -\infty \) to \( t \). This gives, provided \( \text{Im } \omega > -1/\tau \),

\[
\langle z \rangle = z_1 \frac{K^2 - i\omega/\tau + 1/\tau^2}{K^2 - \omega^2 - 2i\omega/\tau + 1/\tau^2}
\]

If \( K \) and \( \omega \) are the same order of magnitude and \( 1/\tau \) is much less, this is to good approximation:

\[
\langle z \rangle = z_1 \frac{K^2}{K^2 - \omega^2 - 2i\omega/\tau}
\]

The fractional neutralisation is \( \mu \tau \) and the number of electrons per unit length in the beam is

\[
\mu \tau \frac{N}{2\pi R}
\]

For these slow coherent instabilities one may usually take the average \( \mu \tau \) round the circumference, if it is non-uniform.

* In a bunched beam electrons usually get lost between one bunch the the next, and this approximation is likely to fail.
The Proton Dynamics

The force exerted on an electron at \( z \) by the proton beam is, from (2)

\[
- e E = \frac{Ne^2}{2\pi R \epsilon_0 \gamma b(b + a)} (z - z_1)
\]

(10)

and the force exerted by the electron on the whole proton beam is just the opposite of this.

Without the electrons, the protons' equation of motion is

\[
\frac{d^2\zeta}{dt^2} + \alpha^2 Q^2 \zeta = 0
\]

(11)

where \( \alpha \) is the angular velocity, \( \dot{\zeta} \). For the mode (1) we have (these are total derivatives):

\[
\frac{d^2}{dt^2} \zeta = - (n\alpha - \omega)^2
\]

(12)

and (11) and (12) determine the unperturbed mode frequency

\[
\omega = \omega_0 \pm \alpha (n \pm Q)
\]

(13)

Now sum (10) over the electrons to get the perturbed proton equation

\[
\frac{d^2\zeta}{dt^2} + \alpha^2 Q^2 \zeta = \mu \tau \frac{Ne^2}{2\pi R \epsilon_0 \gamma b(b + a) \gamma m} (\langle z \rangle - z_1)
\]

(14)

where \( m \) is the proton mass. Using (7), the same approximations again, and making the abbreviation *

\[
\Delta = \frac{Ne^2}{2\pi R \epsilon_0 \gamma b(b + a) \gamma m 2Q \alpha^2}
\]

(15)

the perturbed frequency is given by

\[
(n\alpha - \omega)^2 = \Delta \left[ Q^2 + 2Q \Delta \mu \tau \left( \frac{\omega^2}{\omega^2 - R^2 + 2i\omega/\tau} \right) \right]
\]

(16)

* \( \Delta \) is the Q-shift that would be caused by 100% neutralisation.
Provided this Q-shift is small compared with Q, we can write (16) approximately:

$$
\omega = \mathcal{I} \left[ n + \left( Q \mu \tau \frac{\omega^2}{\omega^2 - K^2 + 2i\omega/\tau} \right) \right] \tag{17}
$$

For small enough $\Delta \mu \tau$ one can usually approximate by putting $\omega_0$ from (13) in place of $\omega$ in the right hand side of (17).

We see that the $2i\omega/\tau$ in the denominator produces an imaginary shift in $\omega$, and that this causes instability (i.e. Im $\omega$ positive) for the $n - Q$ modes*, or slow waves. This can be linked up with resistive wall theory by saying that the finite meanlife of the electrons makes then behave like a dissipative medium.

**Narrow Line Case, ISR Numbers**

As rough representative values I put $N = 10^{14}$ (5 Amperes), $b = 10$ mm and $a = 40$ mm. Then (4) gives me

$$
K = 4.6 \cdot 10^8 \text{ s}^{-1}
$$

so the electron frequency would be about 75 MHz. And (15) gives at $\gamma = 16$:

$$
\Delta = 0.11
$$

For successive $n$ values the $\omega_0$ are spaced by $\Omega = 2 \cdot 10^5 \text{ s}^{-1}$, so there will be some mode of the order of $n - Q = 230$ which is nearest the electron resonance, with

$$
\omega \approx K
$$

$$
|\omega - K| \leq \Omega/2 \quad \text{or less} \tag{18}
$$

Putting these into (17), the real part of the Q-shift is at least as big as

$$
\Delta \mu \tau \frac{\omega_0}{\Omega} = \Delta \mu \tau (n - Q) \tag{19}
$$

$$
= 25 \mu \tau
$$

We usually have $dQ/dp$ such that a Q-shift of the order of 0.04 is enough to overcome the Landau damping**, so a fractional neutralisation $\mu \tau$ like $2 \cdot 10^{-3}$ is enough to cross

* With $n > Q$

** For these $n - Q$ modes and $dQ/dp$ positive the $\Omega$-spread helps to increase the $\omega$-spread.
threshold if all electrons have the same frequency.

From the imaginary frequency shift one gets the growth rate (Landau damping neglected) of the unstable modes

$$\alpha = 2 \Delta \mu \ (n - Q)$$

and with our numbers this is 50 \(\mu\), so 6 s\(^{-1}\) at a pressure like \(10^{-10}\) torr nitrogen equivalent, since that pressure \(^4\) makes \(\mu = 0.12\) s\(^{-1}\).

For this process the ratio of the electron amplitude to the proton amplitude is of the order of at least \((n - Q)\), so at rather small proton amplitudes our assumption about the electrons being mostly within the proton beam will begin to fail, the electron motion will become rather nonlinear and go to lower frequencies. Presumably the proton amplitude will be limited, if not by the breakdown of the theory at least by the electrons hitting the wall.

**Broad Band Case**

Let us assume that the resonant frequencies \(K\) for the electrons present are spread out over a certain range. This can take account of the variation of beam dimensions with the \(\alpha\) and \(\beta\) functions around the ring and may in a crude way \(^5\) allow for the effect of transverse nonlinearity on the electron spectrum. Let \(P(K) \, dK\) be the fraction of electrons in the interval \(dK\) at \(K\). Now the averaging over the electrons that gave us (7) and (8) must be supplemented by an averaging over this spectrum, and in place of (8) we get

$$\langle z \rangle = z_1 \int_0^\infty \frac{K^2P(K) \, dK}{K^2 - \omega^2 - 2i\omega/\tau}$$

(21)

If \(\text{Im} \, \omega\) and \(1/\tau\) are both small the path of integration passes rather close to the zero of the denominator, and experience with such mappings \(^5\) indicates that we shall get about

$$\langle z \rangle / z_1 = \frac{\pi}{4} \frac{K_0}{\delta K}$$

(22)

for some \(\omega\) near \(K_0\), where \(K_0\) is the \(K\) at the peak of the spectrum and \(\delta K\) is its halfwidth at halfheight. Now using (14) and (15), that gives a \(Q\) - shift of

$$\Delta \mu \tau \left(1 - \frac{\pi}{4} \frac{K_0}{\delta K}\right)$$

(23)

The dense spectrum of electrons is acting as a dissipative environment. Since this is a kind of Landau damping process \(^6\) we should ask again whether we are justified in neglecting the forces that the electrons exert on one another. A reasonable criterion is that the associated electron frequency shift be much smaller than their spread:
\[ \frac{1}{2} \mu \tau K_0 \ll \delta K \]  

which limits us to fairly low levels of neutralisation.

For a fairly wide spectrum \( \delta K/k_0 = 10\% \) (\( \delta K \) is halfwidth at halfheight) and \( \Delta \) still 0.11, to reach a threshold like \( |\delta Q| = 0.04 \) needs, according to (23), a \( \mu \tau \) of \( 5\% \). The growth rate with proton Landau damping neglected is, for the slow waves,

\[ \alpha = \delta \mu \tau \frac{\pi}{4} K_0 \delta K \Omega \]  

(25)

and with those numbers this is \( 8 \cdot 10^4 \text{s}^{-1} \). This sort of electron spectrum simulates a rather potent dissipation.

Some electrons again have amplitudes much greater than the coherent proton motion. For the resonant ones we put \( K - \omega = -i \alpha \) into (8) and find

\[ \frac{\langle z^2 \rangle}{z_\Omega} = \frac{K}{2(\alpha + 1/\tau)} \]  

(26)

(here the triangular brackets indicate the average only over all ages present).

**Conclusions**

It seems possible that this mechanism explains the weak persistent \( n - Q \) spectral lines; it does not demand an unreasonably high electron population and it can limit at quite small coherent proton amplitudes.

To answer questions:

- does their amplitude fit the theory?
- how fast is this process diluting the proton beam?

it would be necessary to include nonlinearity and the aperture limit in the electron dynamics. The possibility of it causing rapid proton loss when the electron supply is more copious is theoretical speculation, but presumably big beam signals would be recognised if it occurred.

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References


