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TWO-LOOP β -FUNCTION FOR GENERALIZED NON-ABELIAN THIRRING MODELS

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ABSTRACT

By an explicit calculation, the second-loop coefficient of the β -function for the general non-Abelian Thirring model is shown to depend on the scheme-dependent contact term of the fermionic determinant and not to support the existence of a critical point of order $1/N_c$ (N_c = dimension of the fermion representation).

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The generalized non-Abelian Thirring models are fermionic quantum field theories in 1+1 space-time dimensions [Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1)$] with formal Lagrangian

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - \frac{1}{2} g J_\mu^\alpha J^\mu{}^\alpha \quad (1)$$

where

$$J_\mu^\alpha = \sum_{j=1}^{N_f} \bar{\psi}_j \gamma_\mu T^\alpha \psi_j, \quad \alpha = 1, \dots, \dim G \quad (2)$$

In Eqs. (1) and (2), $\psi = \{\psi_{aj}, a=1, \dots, N_c, j=1, \dots, N_f\}$ is a set of Dirac fields labelled by the "flavour" index j and the "colour" index a , which transforms under a unitary N_c -dimensional irrep. $R(G)$ of a compact, simple Lie group G . Hence $J_\mu^\alpha = J_\mu^\alpha T^\alpha$ is the total colour current, $\{T^\alpha, \alpha = 1, \dots, \dim G\}$ being a set of Hermitian generators of $R(G)$, normalized according to $\text{tr} T^\alpha T^\beta = \frac{1}{2} \delta^{\alpha\beta}$ [$\frac{1}{2}$ is the index of $R(G)$]. The overall internal symmetry of the Lagrangian (1) is $U(N_f) \times U(N_f) \times G$ which acts on $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi$ according to $\psi_{aj\pm} \rightarrow R_{ab}(\gamma) U_{jk}^{(\pm)} \psi_{bk\pm}$, $\gamma \in G$, $U^{(\pm)} \in U(N_f)$.

These models generalize to an arbitrary G and representation $R(G)$ and to $N_f > 1$, the well-known $G = SU(N_c)$ Thirring model [1]. This is often called the chiral Gross-Neveu model [2] owing to a Fierz identity relating $J_\mu^\alpha J^{\mu\alpha}$ to $(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2$ when $N_f = 1$, $G = SU(N_c)$ and $R = R_0 \equiv$ fundamental irrep. The $SU(N_c)$ Thirring model is known to be asymptotically free and to show dynamical mass generation. Its infra-red properties are also rather non-trivial. For $N_f = 1$, a Kosterlitz-Thouless phenomenon causes the decoupling of a massless excitation carrying the $U(1) \times U(1)$ symmetry and no colour [3]. For $N_f > 1$ the same decoupling takes place with the massless $U(N_f) \times U(N_f)$ excitations now described by a Wess-Zumino-Witten critical action [4].

Perhaps most importantly, the $SU(N_c)$ Thirring model, as well as its generalizations (1) and (2), for $N_f = 1$ and a certain class of irreps. $R(G)$, are exactly integrable quantum field theories [5] (as classically integrable theories with no restriction on $R(G)$, the generalized models (1) and (2) were first considered in Ref. [6]). Thus in many cases, their non-perturbative properties together with the exact spectrum and factorized S-matrix have been explicitly determined [7,8].

In this letter, I will address some more traditional problems concerning the generalized non-Abelian Thirring models using conventional perturbation theory. The motivation is two-fold. First of all, the two-loop β -function for these models is commonly known only for $G = SU(N_c)$, $R = R_0$, $N_f = 1$ (and in a particular

regularization scheme). Secondly, the connection of the Bethe ansatz solution of (1) (when possible) with conventional perturbation theory is very non-trivial [9] and it appears useful to gain as much information as possible using conventional means in order to make comparisons with the "exact" solutions.

The first question one can ask about the models (1) is whether they are strictly renormalizable theories for any G and R . Indeed the ultraviolet (UV) renormalization might in principle mix the non-Abelian current-current coupling $J_\mu^\alpha J^{\mu\alpha}$ with the Abelian one, $J_\mu J^\mu$, $J_\mu = \bar{\psi} \gamma_\mu \psi$ which is an allowed independent counterterm on account of symmetries and dimension (such a mixing exists, for instance, in the Bethe ansatz solution of $G = SU(N_c)$, $R = R_0$ and $N_f = 1$ [9]). I will show below that the pure non-Abelian coupling $J_\mu^\alpha J^{\mu\alpha}$ is stable up to second loop, i.e., the Abelian $J_\mu J^\mu$ is not induced by first- and second loop-infinities.

Having checked strict renormalizability up to second loop, one may look for the two-loop β -function $\beta(g)$. My result is:

$$\beta(g) = -\frac{1}{2\pi} \ell_{adj} g^2 \left(1 - \frac{2\lambda}{\pi} N_f \ell_R g\right) + \dots \quad (3)$$

where ℓ_{adj} is the index of the adjoint irrep. (identical to the quadratic Casimir of the adjoint) and λ is a parameter which depends on the regularization scheme. There are two "natural" values for λ in perturbation theory (see below), $\lambda = 1$ and $\lambda = \frac{1}{2}$. Hence the two corresponding regularization schemes are not in the same analyticity class. A third value for λ is provided by the Bethe ansatz β -function, $\lambda = 0$ (in the Bethe ansatz solution the β -function is given only by the first term in (3): $\beta(g) = \ell_{adj} g^2 / 2\pi$ [10]). All these three schemes lie, however, in the same universality class of the $g = 0$ UV fixed point which is characterized by the first term of $\beta(g)$. Notice that this term depends only on G and not on R or N_f . A few more comments on Eq. (3) will be made in the concluding remarks. Now the derivation of Eq. (3) will be outlined.

Let $\psi^{(0)}$ and g_0 be, respectively, the bare field and coupling constant. The quantum Lagrangian then reads:

$$\begin{aligned} \mathcal{L} &= i \bar{\psi}^{(0)} \not{\partial} \psi^{(0)} - \frac{1}{2} g_0 J_\mu^{(0)\alpha} J^{(0)\mu\alpha} \\ &= i \bar{\psi} \not{\partial} \psi - \frac{1}{2} g J_\mu^\alpha J^{\mu\alpha} + (Z-1) i \bar{\psi} \not{\partial} \psi + \frac{1}{2} (g - Z^2 g_0) J_\mu^\alpha J^{\mu\alpha} \quad (4) \end{aligned}$$

where $\psi = Z^{-\frac{1}{2}} \psi^{(0)}$ is the renormalized field, g is the renormalized coupling

constant and $J_\mu^\alpha = :\bar{\psi}\gamma_\mu T^\alpha\psi:$ is the perturbative colour current, the doublet dots representing normal ordering with respect to the perturbative vacuum defined by the free part $\bar{\psi}i\not{D}\psi$ of (4).

Rather than with (4), one can equivalently work with

$$\mathcal{L}' = i\bar{\psi}\not{D}\psi + (Z-1)\bar{\psi}i\not{\partial}\psi + \frac{1}{2Z^2g_0} A_\mu^\alpha A^{\mu\alpha} \quad (5)$$

where $D_\mu = \partial_\mu - iA_\mu$, $A_\mu = A_\mu^\alpha T^\alpha = A_\mu^\dagger$. Indeed a trivial Gaussian integration over A_μ reproduces $\mathcal{L}$ as in (4). This can be explicitly checked in all diagrams with no A-external legs. The net effect of the substitution of $\mathcal{L}$ with $\mathcal{L}'$ is to split (in all the allowed ways) the four fermion vertices of $\mathcal{L}$ into two $A\psi\psi$ -vertices connected by a trivial ultralocal A-propagator. The Feynman diagrams generated with (5) are then graphically identical to the diagrams of QED. The associated Feynman rules are given in Fig. 1.

According to the standard rules, the wave function renormalization constant Z and the bare coupling constant g_0 must be determined, as UV cut-off dependent power series in the renormalized coupling g , in such a way that the theories (4) or (5) can be expanded in powers of g with only finite contributions to any fixed order. Notice that the A-propagator (Fig. 1) should be expanded in powers of g , with each term of this expansion entering at different loop orders. The finite parts of Z and g_0 are fixed by two renormalization conditions which I choose to be:

$$S(p)^{-1} \Big|_{p^2=-\mu^2} = -i\not{p} \Big|_{p^2=-\mu^2} \quad (6)$$

$$\Gamma_4(\mu^2) \equiv \Gamma_4(p_1, \dots, p_4) \Big|_{p_j^2 = -\frac{1}{2}\mu^2, p_i p_i = 0} = -ig \gamma_\mu T^\alpha \otimes \gamma^\mu T^\alpha \quad (7)$$

where $S(p)$ is the full renormalized propagator and $\Gamma_4(p_1, \dots, p_4)$ is the renormalized four-point IPI fermion vertex function [in (7) the suitable antisymmetrization over ingoing and/or outgoing colour, flavour and chiral indices is implicitly understood]. μ^2 is the subtraction point.

The β -function $\beta(g)$ can be calculated from

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] g_0 = 0 \quad (8)$$

where $g_0 = g_0(g, \Lambda/\mu)$, with Λ the UV cut-off, is uniquely determined by Eqs. (6) and (7). To evaluate $\beta(g)$ up to two loops, one needs Z up to order g^2 and g_0 up to order g^3 . On general grounds one can write:

$$Z = 1 + z_1 g + z_2 g^2 + \dots \quad (9a)$$

$$g_0 Z^2 = g + c_1 g^2 + c_2 g^3 + \dots \quad (9b)$$

$$g_0 = g + (c_1 - 2z_1)g^2 + (c_2 + 3z_1^2 - 2z_2 - 2z_1 c_1)g^3 + \dots \quad (9c)$$

The calculation of the relevant Feynman diagrams is greatly simplified by a pair of crucial properties of Dirac γ matrices in two space-time dimensions. These are

$$\gamma_\mu \gamma_\nu \gamma^\mu = 0 \quad (10)$$

and

$$\gamma_\mu \gamma_\nu \gamma_\lambda = \gamma_\lambda \gamma_\nu \gamma_\mu \quad (11)$$

Together they entail

$$\gamma_\mu \gamma_{\nu_1} \dots \gamma_{\nu_n} \gamma^\mu = 0 \quad (12)$$

Equation (12) implies that all diagrams in which an A-particle is emitted and absorbed by the same fermion line vanish, no matter what happens everywhere else to the fermion line itself (Fig. 2). Of course, for this to happen, it is crucial that the fermion propagator be massless.

At this point a comment on the UV regularization is perhaps in order. Dimensional regularization cannot be recommended since it spoils Eq. (10), making much more work necessary. A more appropriate a priori regularization consists of multiplying the ultralocal A propagator by a momentum-dependent factor like $\Lambda^2(k^2 + \Lambda^2)^{-1}$ [it corresponds to replacing $A_\mu^{\alpha\mu\alpha}$ in Eq. (5) by $\Lambda^{-2} A_\mu^{\alpha} (-\square + \Lambda^2) A^{\mu\alpha}$]. This does not affect Eqs. (10)-(12) in any way and makes all relevant diagrams superficially convergent, except for the vacuum polarization $\Pi_{\mu\nu}^{\alpha\beta}(q)$ (Fig. 3) which remains logarithmic divergent as long as simple power-counting is considered. It is not difficult to see, however, that the identity (10) lowers the degree of divergence of $\Pi_{\mu\nu}^{\alpha\beta}(q)$, making it marginally convergent. As such, $\Pi_{\mu\nu}^{\alpha\beta}(q)$ cannot depend on the UV cut-off, whatever it is, but will in general depend on the regularization scheme adopted for fermion loops. In other words, $\Pi_{\mu\nu}^{\alpha\beta}(q)$ remains potentially ambiguous in its local part (i.e., in the part depending polynomially on q), which by

dimensional reasons and tensor analysis must be $\eta_{\mu\nu} \delta^{\alpha\beta}$ times a numerical, scheme-dependent constant. On the other hand, the non-local part of $\Pi_{\mu\nu}^{\alpha\beta}(q)$ is universal and can be calculated in any scheme. Pauli-Villars and dimensional regularization [N.B. Eq. (10) must then be replaced by $\gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\gamma_\nu$] give the transverse result (i.e., $q^\mu \Pi_{\mu\nu}^{\alpha\beta} = 0$)

$$\Pi_{\mu\nu}^{\alpha\beta}(q) = \frac{i}{\pi} N_f \ell_R \delta^{\alpha\beta} \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + i\varepsilon} \right) \quad (13)$$

corresponding to a gauge-invariant definition of the fermion determinant $\text{Det}\not{D}$. Hence, in general

$$\Pi_{\mu\nu}^{\alpha\beta}(q) = \frac{i}{\pi} N_f \ell_R \delta^{\alpha\beta} \left(\lambda \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + i\varepsilon} \right) \quad (19)$$

Besides the "natural" gauge-invariant choice $\lambda = 1$, there exists another "natural" choice: $\lambda = \frac{1}{2}$. It corresponds to a direct application of (9) in the integrand of $\Pi_{\mu\nu}^{\alpha\beta}$ and leads naturally to a traceless result: $\eta^{\mu\nu} \Pi_{\mu\nu}^{\alpha\beta} = 0$. This is proper to the two-dimensional decoupling of right- and left-movers in the formal Lagrangian (1) [it realizes at the renormalized level the formal decomposition $\text{Det}\not{D} = \text{Det}D_+ \text{Det}D_-$, $D_\pm = \frac{1}{2}D/(1 \pm \gamma_5)$].

Armed with the above considerations, let us first calculate the coefficients z_1 and z_2 in Eq. (9a). To second loop the fermion inverse propagator $S(p)^{-1}$ receives a non-vanishing contribution [cf. Eqs. (10)-(12) and Fig. 2] from the three diagrams of Fig. 4. In (4a) the A propagators are to lowest order in g [cf. Eq. (9b) and Fig. 1]; in (4b) the cross denotes the insertion of the counterterm $(Z-1)\bar{\psi}i\not{\partial}\psi$ up to order g^2 . One finds

$$\begin{aligned} (4a) &= i^2 (ig)^2 \int d^2\left(\frac{q}{2\pi}\right) \gamma^\mu T^\alpha \frac{i}{\not{p}-\not{q}+i\varepsilon} \gamma^\nu T^\beta \Pi_{\mu\nu}^{\alpha\beta}(q) \frac{\Lambda^4}{(q^2+\Lambda^2)^2} \\ &= \frac{-i}{\pi} g^2 N_f \ell_R C_R^{(2)} \int d^2\left(\frac{q}{2\pi}\right) \frac{\Lambda^4 \not{q}}{[(p-q)^2+i\varepsilon](q^2+\Lambda^2)^2} + \text{UV-finite terms} \\ &= i N_f \left(\frac{g \ell_R}{2\pi}\right)^2 \frac{\text{dim } G}{N_c} \not{p} \ln \frac{\Lambda^2}{-p^2} + \text{UV-finite terms} \end{aligned} \quad (15)$$

where $C_R^{(2)} = T^\alpha T^\alpha = \lambda_R \text{dim } G / N_c$ is the quadratic Casimir invariant of the irrep. R. Adding the value of (4b), $-i\not{p}(z_1 g + z_2 g^2)$, and inserting the result in Eq. (6) leads to

$$z_1 = 0, \quad z_2 = -N_f \left(\frac{\ell_R}{2\pi} \right)^2 \frac{\text{dim } G}{N_c} \ln \frac{\Lambda^2}{\mu^2} + \xi_1 \quad (16)$$

where ξ_1 is some finite numerical constant which originates from the UV finite terms of Eq. (15).

Let us consider now $\Gamma_4(\mu^2)$ [Eq. (7)]. The diagrams which give a non-zero contribution to the four-point function are listed in Fig. 5 (I have omitted crossed diagrams as well as leg exchanges, since these are irrelevant to my purposes). The two-loop diagrams have been grouped in a convenient way. All wavy lines are A propagators to all orders in g ; when expanded according to Eq. (9b), each propagator in a given diagram needs to be retained only up to terms which make the diagram of global order g^3 or less. This constitutes a quick way to take into account all possible counterterm insertions.

Diagram 5b is at least of order g^2 , but in view of the finiteness of $\Pi_{\mu\nu}^{\alpha\beta}(q)$ it is always finite and can be ignored. For (5c) one finds

$$\begin{aligned} (5c) &= g^2 \gamma_\mu \gamma^\lambda \gamma_\nu T^\alpha T^\beta \otimes \gamma^\mu \gamma^\rho \gamma^\nu T^\alpha T^\beta \gamma_{\lambda\rho}(p_1, p_2, p_3) \\ \gamma_{\mu\nu}(p_1, p_2, p_3) &= \int d^2\left(\frac{q}{2\pi}\right) \frac{\Lambda^4 (p_2+q)_\mu (p_1-q)_\nu}{(p_2+q)^2 (p_1-q)^2 (q^2+\Lambda^2) [(q+p_2-p_3)^2+\Lambda^2]} \\ &= - \int d^2\left(\frac{q}{2\pi}\right) \frac{q_\mu q_\nu}{(p_2+q)^2 (p_1-q)^2} \cdot \frac{\Lambda^4}{(q^2+\Lambda^2)^2} + \text{UV-finite terms} \quad (17) \\ &= \frac{i}{8\pi} \eta_{\mu\nu} \ln \frac{\Lambda^2}{-(p_1+p_2)^2} + \text{UV-finite terms} \end{aligned}$$

A similar calculation yields for (5d) [use is made of $(\gamma_\mu \otimes \gamma^\mu)^3 = 4\gamma_\mu \otimes \gamma^\mu$]

$$(5d) = -\frac{ig^2}{2\pi} \gamma_\mu T^\alpha T^\beta \otimes \gamma^\mu T^\beta T^\alpha \ln \frac{\Lambda^2}{-(p_1-p_2)^2} + \text{UV-finite terms} \quad (18)$$

Summing up (5c) and (5d) at the normalization point [cf. Eq. (7)] and using $[T^\alpha, T^\beta] = if^{\alpha\beta\gamma} T^\gamma$ and $f^{\alpha\gamma\delta} f^{\beta\gamma\delta} = \lambda_{\text{adj}} \delta^{\alpha\beta}$, I obtain:

$$(5c) + (5d) = -\frac{ig^2}{4\pi} \lambda_{\text{adj}} \gamma_\mu T^\alpha \otimes \gamma^\mu T^\alpha \ln \frac{\Lambda^2}{\mu^2} + \text{UV-finite terms} \quad (19)$$

Thus, expanding $g_0 Z^2$ in (5a) up to order g^2 , one finds from Eq. (7)

$$c_1 = -\frac{1}{4\pi} \ell_{adj} \ln \frac{\Lambda^2}{\mu^2} + \xi_2 \quad (20)$$

where ξ_2 is another finite numerical constant.

Let us now examine the two-loop diagrams. The overall UV divergence of the diagrams of group I (see fig. 5), expanded up to $O(g^3)$, are exactly cancelled by the two terms of order g^3 in diagram 5b, thanks to Eq. (20). Hence group I needs not be considered any further. The calculations for group II parallel very closely those of diagrams (5c) and (5d). The insertion of $\Pi_{\mu\nu}^{\alpha\beta}(q)$ [Eq. (14)] into an A-propagator does not introduce significant complications, and I will simply quote the final result for the divergent part of order g^3 :

$$(group II) = \frac{ig^3}{2\pi^2} (\lambda - \frac{1}{2}) N_f \ell_R \ell_{adj} \gamma_\mu T^\alpha \otimes \gamma^\mu T^\alpha \ln \frac{\Lambda^2}{\mu^2} \quad (21)$$

In the factor $\lambda - \frac{1}{2}$ one recognizes the contribution of the local piece $\propto \lambda \eta_{\mu\nu} \delta^{\alpha\beta}$ of $\Pi_{\mu\nu}^{\alpha\beta}(q)$ and that of the non-local one $\propto -q_\mu q_\nu / q^2$, down by $\frac{1}{2}$ due to the well-known rule $\int d^2 q q_\mu q_\nu f(q^2) = \eta_{\mu\nu} \int d^2 q q^2 f(q^2)$.

The first diagram of group III evaluates to:

$$\begin{aligned} & + g^3 \frac{N_f \ell_R}{\pi} \gamma_\mu T^\alpha \otimes \int d^2 \left(\frac{q_-}{2\pi} \right) \frac{\gamma_\nu T^\beta (p_3 - q) \gamma_\mu T^\alpha (p_1 - q) \gamma_\lambda T^\beta}{[(p_3 - q)^2 + i\epsilon] [(p_1 - q)^2 + i\epsilon]} \cdot \frac{q^\nu q^\lambda \Lambda^4}{q^2 (q^2 + \Lambda^2)^2} \\ & = \frac{-ig^3}{4\pi^2} N_f \ell_R \gamma_\mu T^\alpha \otimes \gamma^\mu T^\beta T^\alpha T^\beta \ln \frac{\Lambda^2}{-(p_1 - p_2)^2} + UV\text{-finite terms} \end{aligned} \quad (22)$$

Now

$$\begin{aligned} T^\beta T^\alpha T^\beta &= T^\beta T^\beta T^\alpha + T^\beta [T^\alpha, T^\beta] \\ &= C_R^{(2)} T^\alpha + i f^{\alpha\beta\gamma} T^\beta T^\gamma \\ &= C_R^{(2)} T^\alpha - \frac{1}{2} f^{\alpha\beta\gamma} f^{\alpha'\beta\gamma} T^{\alpha'} \end{aligned}$$

$$= \left(\frac{\dim G}{N_c} \ell_R - \frac{1}{2} \ell_{adj} \right) T^a \quad (23)$$

and $(p_1 - p_3)^2 = (p_2 - p_4)^2 = -\mu^2/4$ at the normalization point of Eq. (7). Hence one finds that group III contributes to $\Gamma_4(\mu^2)$ the divergent quantity

$$(\text{group III}) = \frac{-ig^3}{2\pi^2} N_f \ell_R \left(\frac{\dim G}{N_c} \ell_R - \frac{1}{2} \ell_{adj} \right) \gamma_\mu T^a \otimes \gamma^\mu T^a \ln \frac{\Lambda^2}{\mu^2} \quad (24)$$

There remain to calculate the UV divergences of order g^3 in the last two groups IV and V. Since, after all, we are interested only in the UV divergences, it is convenient to evaluate these six ladder diagrams at zero external momenta, rather than at the non-zero symmetric point of Eq. (7). This change has two effects. First of all, the theory being massless, at zero external momenta there will appear a host of IR divergences (some IR divergences already appeared in previous diagrams, e.g., in diagram 4a, but could be unambiguously buried in the UV-finite parts). Secondly, changing the normalization point with respect to other diagrams will introduce extra UV finite corrections. These can be safely ignored, provided the IR divergences are regulated in a consistent way in all the six ladder diagrams. Indeed the IR regulator will play the rôle of a new subtraction point, making IR divergent the corrections relative to the old subtraction point μ^2 . However, if the IR regularization is carried out consistently, no UV divergences get buried in the infinities arising as the IR cut-off, say m^2 , is sent to zero. With this in mind, one can proceed as follows.

By direct inspection and repeated use of Eq. (11), one can verify that there exists a unique space-spin factor for group IV and a unique one for group V, that is

$$(\text{group IV}) = I_1 F_1 \quad ; \quad (\text{group V}) = I_2 F_2 \quad (25)$$

with

$$I_1 = i g^3 \gamma_{\lambda_1} \gamma^{\mu} \gamma_{\lambda_2} \gamma^{\rho} \gamma_{\lambda_3} \otimes \gamma^{\lambda_1} \gamma^{\nu} \gamma^{\lambda_2} \gamma^{\sigma} \gamma^{\lambda_3} I_{\mu\nu} I_{\rho\sigma} \quad (26)$$

$$I_{\mu\nu} = \int d^2(q/2\pi) \frac{q_{\mu} q_{\nu}}{(q^2 + i\epsilon)^2} = \frac{1}{2} \eta_{\mu\nu} \int d^2(q/2\pi) \frac{1}{q^2 + i\epsilon} \quad (27)$$

$$I_2 = i g^3 \gamma_{\lambda_1} \gamma^{\mu} \gamma_{\lambda_2} \gamma^{\rho} \gamma_{\lambda_3} \otimes \gamma^{\lambda_1} \gamma^{\nu} \gamma^{\lambda_2} \gamma^{\sigma} \gamma^{\lambda_3} I_{\mu\nu\rho\sigma} \quad (28)$$

$$I_{\mu\nu\rho\sigma} = \int d^2\left(\frac{q}{2\pi}\right) \frac{q_{\mu} q_{\nu}}{(q^2 + i\epsilon)^2} \int d^2\left(\frac{k}{2\pi}\right) \frac{k_{\rho} (q - k)_{\sigma}}{(k^2 + i\epsilon) [(q - k)^2 + i\epsilon]} \quad (29)$$

Therefore F_1 and F_2 are the sums of the corresponding colour factors

$$F_1 = T^{\alpha} T^{\beta} T^{\gamma} \otimes (T^{\alpha} T^{\beta} T^{\gamma} + T^{\gamma} T^{\beta} T^{\alpha}) \quad (30)$$

$$F_2 = T^{\alpha} T^{\beta} T^{\gamma} \otimes (T^{\alpha} T^{\gamma} T^{\beta} + T^{\beta} T^{\alpha} T^{\gamma} + T^{\beta} T^{\gamma} T^{\alpha} + T^{\gamma} T^{\alpha} T^{\beta}) \quad (31)$$

The integrals I_1 and I_2 have actually been oversimplified by letting $\Lambda \rightarrow \infty$ in their integrands, thus returning the A-propagators to their original, momentum-independent, ultralocal form. The UV cut-off Λ will be restored later on when strictly necessary. This quicker procedure, although not fully rigorous, can be shown to give the correct results, at least in the case at hand.

In $I_{\mu\nu}$ the IR cut-off m^2 is now inserted, yielding

$$I_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu} \int d^2\left(\frac{q}{2\pi}\right) \frac{1}{q^2 - m^2 + i\epsilon} \simeq \frac{-i}{8\pi} \eta_{\mu\nu} \ln \frac{\Lambda^2}{m^2} \quad (32)$$

In $I_{\mu\nu\rho\sigma}$ the k-integration can be performed exactly, leading to

$$\begin{aligned} I_{\mu\nu\rho\sigma} &= \frac{i}{4\pi} \int d^2\left(\frac{q}{2\pi}\right) \frac{q_{\mu} q_{\nu}}{(q^2 + i\epsilon)} \left[\frac{1}{2} \eta_{\mu\nu} \left(1 + \ln \frac{\Lambda^2}{-q^2}\right) - \frac{q_{\mu} q_{\nu}}{q^2 + i\epsilon} \right] \\ &= \frac{1}{(8\pi)^2} \eta_{\mu\nu} \eta_{\rho\sigma} \int_0^{\Lambda^2} dy \frac{\ln(y^2/m^2) - \ln(1+y^2/m^2) + \ln(1+m^2/y^2)}{y^2 + m^2} + \tilde{I}_{\mu\nu\rho\sigma} \quad (33) \\ &= \frac{1}{2(8\pi)^2} \eta_{\mu\nu} \eta_{\rho\sigma} \left(\ln \frac{\Lambda^2}{m^2}\right)^2 + \tilde{I}_{\mu\nu\rho\sigma} + \text{UV-finite} \end{aligned}$$

where

$$\tilde{I}_{\mu\nu\rho\sigma} = \frac{i}{16\pi} (\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \int d^2\left(\frac{q}{2\pi}\right) \frac{1}{q^2 + i\varepsilon} \quad (34)$$

diverges as $\ln(\Lambda^2/m^2)$, but does not contribute to I_2 , because it gives a vanishing contraction with the γ -matrices in Eq. (28). Notice that the IR cut-off procedure used in deriving Eq. (33) is fully consistent with that used in Eq. (32). Inserting (32) and (33) in (26) and (28) and using $(\gamma_\mu \otimes \gamma^\mu)^5 = 16\gamma_\mu \otimes \gamma^\mu$, yields, up to UV finite terms

$$I_1 = -2I_2 = \frac{-1}{4\pi^2} \gamma_\mu \otimes \gamma^\mu \left(\ln \frac{\Lambda^2}{m^2} \right)^2 \quad (35)$$

The absence of subleading divergences in groups IV and V imply that they do not contribute to the β -functions, regardless of their final values taking F_1 and F_2 into account. It is, however, very instructive to carry the calculation to the end. We have:

$$(\text{group IV} + \text{V}) = \frac{1}{8\pi} \gamma_\mu \otimes \gamma^\mu \left(\ln \frac{\Lambda^2}{m^2} \right)^2 [F_2 - 2F_1] \quad (36)$$

with the overall colour factor

$$\begin{aligned} F_2 - 2F_1 &= T^\alpha T^\beta T^\gamma \otimes \{ T^\alpha [T^\gamma, T^\beta] + [T^\beta, T^\alpha] T^\gamma + [T^\beta, T^\gamma] T^\alpha + T^\gamma [T^\alpha, T^\beta] \} \\ &= -\frac{1}{2} [T^\alpha, [T^\beta, T^\gamma]] \otimes [T^\alpha, [T^\beta, T^\gamma]] \\ &= -\frac{1}{2} f_{\alpha\beta\gamma} T^\alpha \otimes T^\alpha \end{aligned} \quad (37)$$

Hence, shifting the terms proportional to $\ln m^2/\mu^2$ into the UV-finite parts:

$$\left[\Gamma_4(\mu^2) \right]_{\text{IV}+\text{V}} = -ig^3 c_1^2 \gamma_\mu T^\alpha \otimes \gamma^\mu T^\alpha + \text{UV-finite terms} \quad (38)$$

Notice that the UV divergences of the two-loop ladder diagrams are in exact geometrical sequence with those of the one-loop ladder diagrams (5c) and (5d). It seems plausible that this geometrization holds for all ladder diagrams.

Since all UV divergences up to second loop are proportional to the tree-level coupling $\gamma_\mu T^\alpha \otimes \gamma^\mu T^\alpha$, one sees that the Lagrangian (1) is strictly renormalizable, in the sense of the introductory remarks.

We are finally in a position to calculate the second coefficient c_2 in Eq. (9b). Adding (21), (24) and (38) into Eq. (7) yields a complete UV divergence cancellation provided

$$c_2 = c_1^2 + \frac{N_f l_R}{2\pi^2} \left(\lambda l_{adj} - \frac{\dim G}{N_c} l_R \right) \ln \frac{\Lambda^2}{\mu^2} \quad (39)$$

Collecting Eqs. (16) and (39) one obtains now from Eq. (9c):

$$g_0 = g + c_1 g^2 + \left[c_1^2 + \frac{\lambda}{2\pi^2} N_f l_R l_{adj} \ln \frac{\Lambda^2}{\mu^2} \right] g^3 + \dots \quad (40)$$

where I recall that $c_1 = -1/4\pi \lambda_{Adj} \ln \Lambda^2 / \mu^2$. Insertion of (40) in Eq. (8) gives the result (3) stated in the introduction.

The form (3) of the two-loop β -function may serve as a plausible check on the old claim [1] that the $SU(N_c)$ Thirring model possesses a non-trivial positive IR fixed point at $g = g_* = O(N_c^{-1})$ (see also Refs. [11]). From Eq. (3) with $N_f = 1$, $G = SU(N_c)$ and $R = R_0$ (which implies $\lambda_R = \lambda_{R_0} = \frac{1}{2}$ when the roots of the Lie algebra have unit length), one sees that the β -function has indeed a positive zero, but at $g = g_* = \pi/\lambda = O(1)$. Although approximate, this result, once combined with the form of $\beta(g)$ in the large N_c limit (suitably keeping gN_c fixed as $N_c \rightarrow \infty$), does not support the above-mentioned claim. In fact, the $N_c = \infty$ β -function reduces only to the first term in Eq. (3) [2] in agreement with a g_* of order 1 or larger (or maybe no g_* at all) and not with a g_* of order N_c^{-1} . The subtle question of the IR fixed points of the generalized non-Abelian Thirring model is further addressed in Ref. [12].

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$$\begin{aligned}
 \text{---}\overset{p}{\bullet}\text{---}\bullet &= i(\not{p} + i\varepsilon)^{-1} \\
 \text{---}\bullet\text{---}\bullet &= ig_0 Z^2 \eta_{\mu\nu} \delta^{\alpha\beta} \qquad \text{---}\overset{\mu\alpha}{\bullet}\text{---}\bullet = i\gamma^\mu T^\alpha
 \end{aligned}$$

Fig. 1

$$\begin{array}{c}
 \text{---}\bullet\text{---}\bullet \\
 \text{---}\bullet\text{---}\bullet \\
 \vdots \quad \vdots \quad \vdots \quad \vdots
 \end{array} = 0$$

Fig. 2

$$\pi_{\mu\nu}^{\alpha\beta}(q) = \mu\alpha \text{---}\bullet\text{---}\bullet \text{---}\bullet \text{---}\bullet \nu\beta$$

Fig. 3

$$\text{---}\bullet\text{---}\bullet \overset{p}{\text{---}} = \text{---}\bullet\text{---}\bullet + \text{---}\bullet\text{---}\bullet \text{---}\bullet \text{---}\bullet + \text{---}\bullet\text{---}\bullet \text{---}\bullet \text{---}\bullet + \dots$$

Fig. 4

$$\begin{aligned}
& \begin{array}{c} p_2 \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ p_4 \end{array} \\
& \begin{array}{c} p_1 \\ \hline \text{---} \end{array} \begin{array}{c} \text{---} \\ p_3 \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \\
& \quad (5a) \quad (5b) \quad (5c) \quad (5d) \\
& + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_I \\
& + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_{II} \\
& + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_{III} \\
& + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_{IV} \\
& + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_V \\
& + \dots
\end{aligned}$$

Fig. 5