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A NOTE ON THE FRADKIN-VILKOVISKY THEOREM

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In Memoriam Richard P. Feynman

ABSTRACT

It is argued that the usual statement of the Fradkin-Vilkovisky theorem, which determines the dependence of the path-integral for gauge-invariant systems on their gauge-fixing, cannot be correct. A more precise statement is given, by considering the structure of the modular space of gauge orbits. The discussion is illustrated in the case of the scalar particle with specific examples.

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I. INTRODUCTION

General procedures for the quantization of singular systems, i.e. systems described by an action invariant under some local gauge symmetry, have been developed in the recent past [1, 2, 3]. Popular examples of such systems are Yang-Mills theories, supergravity theories or reparametrization invariant theories such as general relativity or string theories.

In the Hamiltonian formulation, the most general setting is certainly provided by the Batalin-Fradkin-Vilkovisky (BFV) formalism [1] (for a review, see Ref.[4], whose notations are used here). Other approaches have been developed in the framework of a Lagrangian formulation [2, 3], but these can most probably also be understood from the BFV point of view [5, 6].

Within the BFV formalism, any physical (gauge-invariant) observable can be represented through a phase-space path-integral, taken with the Liouville measure and an effective action depending on a gauge-fixing function.

In this paper, we consider to what extent the BFV path-integral depends on this gauge-fixing function. Contrary to what is usually stated [1, 4], it will be argued that there is some dependence, so that the gauge-fixing of such systems has to be done appropriately.

A short review of the BFV formalism, following Ref.[4], is given in Section II. Section III discusses the structure of the space of orbits of the gauge group, and the correct statement for the Fradkin-Vilkovisky theorem, which determines the dependence of the BFV path-integral on the gauge-fixing function, is considered in Section IV. Section V illustrates our arguments in the case of a simple system: the relativistic scalar particle. Conclusions are given in Section VI.

II. THE BFV FORMALISM

Let us consider a gauge invariant system described by a collection of phase-space variables $z_A(t)$ (possibly including Grassmann variables), a canonical Hamiltonian $H_0(z_A)$ and a set of first-class constraints $\phi_\alpha(z_A) = 0$. (Second-class constraints are assumed to have been eliminated). We have the Poisson brackets [4]:

$$\left\{ z_A, z_B \right\} = C_{AB} \quad , \quad (1a)$$

$$\left\{ \phi_\alpha, \phi_\beta \right\} = C_{\alpha\beta}{}^\gamma \phi_\gamma, \quad (1b)$$

$$\left\{ H_0, \phi_\alpha \right\} = V_\alpha{}^\beta \phi_\beta, \quad (1c)$$

where generally the first-order structure coefficients $C_{\alpha\beta}{}^\gamma$ and $V_\alpha{}^\beta$ are functions of the phase-space variables. (The indices A and α, β, γ take their values in some set, which may be finite or infinite). Through their Poisson brackets, the constraints ϕ_α generate those local symmetries of the system which belong to the connected part of the gauge symmetry group.

The equations of motion which describe the time-evolution of such a system follow from the first-order action

$$S[z, \lambda] = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} (C^{-1})^{AB} \dot{z}_A \dot{z}_B - H_0 - \lambda^\alpha \phi_\alpha \right\}, \quad (2)$$

plus possibly boundary terms at t_1 and t_2 . Here, $\lambda^\alpha(t)$ are Lagrange multipliers for the constraints. This action is invariant under the transformations [4]:

$$\delta_\epsilon F(z_A) = \left\{ F(z_A), \epsilon^\alpha \phi_\alpha \right\}, \quad (3a)$$

$$\delta_\epsilon \lambda^\alpha = \dot{\epsilon}^\alpha + \lambda^\gamma \epsilon^\beta C_{\beta\gamma}{}^\alpha - \epsilon^\beta V_\beta{}^\alpha, \quad (3b)$$

with $\epsilon^\alpha(t)$ arbitrary infinitesimal functions of t .

In the BFV formalism [1], phase-space is enlarged in two steps. First, one introduces momenta $\pi_\alpha(t)$ conjugate to $\lambda^\alpha(t)$,

$$\left\{ \lambda^\alpha(t), \pi_\beta(t) \right\} = \delta_\beta^\alpha, \quad (4)$$

with the first-class constraints

$$\pi_\alpha = 0. \quad (5)$$

All first-class constraints can collectively be written as $G_a = (\pi_\alpha, \phi_\alpha)$. They satisfy the "algebra":

$$\left\{ G_a, G_b \right\} = C_{ab}{}^c G_c, \quad (6a)$$

$$\{ H_0, G_a \} = V_a^b G_b . \quad (6b)$$

Secondly, to each of these constraints G_a , one associates a pair of conjugate BFV ghosts $\eta^a(t)$ and $\mathcal{P}_a(t)$, with [4]:

$$\{ \eta^a, \mathcal{P}_b \} = -\delta_b^a . \quad (7)$$

By convention, η^a and $\mathcal{P}_a$ have ghost number (+1) and (-1) respectively. This defines the extended phase-space of the BFV formalism.

In this formalism, the local symmetry (3) of the action (2) is traded for a global symmetry generated by a Grassmannian quantity of ghost number (+1), the BRST charge Q_B [7], which satisfies the nilpotency condition

$$\{ Q_B, Q_B \} = 0 . \quad (8)$$

The construction of Q_B proceeds straightforwardly from the algebra of constraints (6) [4]. It is given as:

$$Q_B = \eta^a G_a - \frac{1}{2} (-1)^{\epsilon_b} \eta^b \eta^c C_{cb}^a \mathcal{P}_a + \text{"more"} \quad (9)$$

(where $\epsilon_b = 0$ or $\epsilon_b = 1$ depending on whether the index b is associated with a commuting or an anticommuting constraint G_b). Here, "more" stands for terms which involve more than five ghosts; they do not appear when the structure coefficients C_{ab}^c and V_a^b are simply constants.

The BRST charge is the basic quantity necessary in setting up a formulation of the system which explicitly embodies its local symmetries in the connected part of the gauge group. Physical observables are simply BRST invariant quantities, i.e. their Poisson bracket with Q_B vanishes. Among those, one has the BRST invariant Hamiltonian H_B , which can straightforwardly be obtained [4]. Typically:

$$H_B = H_0 + \eta^a V_a^b \mathcal{P}_b + \text{"more"} , \quad (10)$$

where "more" corresponds to terms containing at least four ghosts, which do not appear for constant structure coefficients C_{ab}^c and V_a^b .

Time-evolution of the system as described by the extended phase-space is then determined by an effective Hamiltonian

$$H_{\text{eff}} = H_B + \{ \Psi, Q_B \}, \quad (11)$$

where Ψ is, a priori, an arbitrary function on the extended phase-space of ghost-number (-1) . The equations of motion which follow from (11) have to be supplemented with BRST invariant boundary conditions at $t = t_1$ and $t = t_2$ [4].

The complete description of the system then also follows from some well determined first-order effective action.

For example, if all phase-space variables z_A are commuting ones, we have canonically conjugate pairs $z_A = (q^i, p_i)$, and (having in mind the path-integral to be discussed below) a convenient choice of BRST invariant boundary condition is [4]:

$$q^i(t_1) = q_1^i, \quad q^i(t_2) = q_2^i; \quad \pi_\alpha(t_1) = 0 = \pi_\alpha(t_2), \quad (12a)$$

$$\eta^\alpha(t_1) = 0 = \eta^\alpha(t_2); \quad \mathcal{P}_\alpha(t_1) = 0 = \mathcal{P}_\alpha(t_2), \quad (12b)$$

where the ghosts η^α (resp. $\mathcal{P}_\alpha$) are those associated with the constraints ϕ_α (resp. π_α). The corresponding effective action is:

$$S_{\text{eff}} = \int_{t_1}^{t_2} dt \left\{ \dot{q}^i p_i + \sum_\alpha \dot{\lambda}^\alpha \pi_\alpha + \sum_{\alpha_1} \dot{\eta}^{\alpha_1} \mathcal{P}_{\alpha_1} - \sum_{\alpha_2} \eta^{\alpha_2} \mathcal{P}_{\alpha_2} - H_{\text{eff}} \right\} \quad (13)$$

where α_1 (resp. α_2) are those indices corresponding to the constraints $G_{\alpha_1} = \pi_{\alpha_1}$ (resp. $G_{\alpha_2} = \phi_{\alpha_2}$).

The quantum BRST invariant system is then simply obtained by the correspondence principle; the quantum algebra of (anti)commutators of all degrees of freedom in the extended phase-space is determined from their Poisson brackets.

Physical states are obtained as states annihilated by the BRST charge (i.e. they correspond to cohomology classes of Q_B) and of zero ghost number, provided Q_B is nilpotent at the quantum level. For systems with an infinite number of degrees of freedom, these conditions may need modification and lead to constraints of consistency on the system due to normal ordering ambiguities. A well known example is that of string theories [8], where nilpotency of Q_B

(i.e. gauge invariance of the quantum string theory) restricts the number of allowed degrees of freedom [9] .

From this " Hilbert " space representation of the quantum system, it is possible to obtain a phase-space path-integral representation for any observable. This BFV path-integral over the extended phase-space is taken with the Liouville measure and the effective action which corresponds to the Hamiltonian (11). The BFV path-integral is BRST invariant (i. e. invariant for transformations in the connected part of the gauge group) provided the BRST charge is nilpotent in the canonically quantized system and the path-integral is evaluated with BRST invariant boundary conditions [4] .

For example, for a system with commuting degrees of freedom q^i only, the probability amplitude to "propagate" from a configuration $q^i(t_1)$ to a configuration $q^i(t_2)$ is given by

$$Z(q^i(t_2), q^i(t_1)) = \int [\mathfrak{D}\mu] e^{iS_{\text{eff}}} , \quad (14)$$

where $[\mathfrak{D}\mu]$ is the Liouville measure over the extended phase-space. The path-integral is taken with the boundary conditions (12) and S_{eff} is given in (13).

III. GAUGE FIXING

From the discussion above it should be clear that Ψ , which appears in H_{eff} and S_{eff} , effectively plays the role of a gauge-fixing function.

In the BFV formulation of gauge invariant systems, gauge fixing is tantamount to specifying a set of Lagrange multipliers $(\lambda^a(t))$. A good gauge-slice of this space of functions is obtained by selecting one and only one representative $(\lambda^a(t))$ of each orbit of the gauge symmetry group.

In order to discuss this question, we find it useful to introduce some concepts with a terminology borrowed from the theory of Riemann surfaces (as used in the context of string theories [8]) and generalized here to any gauge invariant theory.

Let us define " Teichmüller " space as the space of orbits, in the space of Lagrange multipliers λ^a , of the connected part of the gauge group. This space is determined by the infinitesimal transformations $\delta_\xi \lambda^a$ given in (3b). It may be of finite or infinite (not even countable) dimension.

The modular group (or mapping class group) is defined as the quotient of the complete gauge symmetry group by its connected component. The modular group has as many elements as there are disconnected components in the gauge group. It acts on "Teichmüller" space by permutations, with possibly fixed points for some modular transformations. The quotient of "Teichmüller" space by the modular group is thus well defined; let us refer to it as modular space.

All gauge inequivalent configurations of the system are hence characterized by all points in modular space. A complete gauge fixing of the system is thus determined by a gauge slice in the space of Lagrange multipliers which selects, for each point in modular space, one and only one representative $\{ \lambda^a(t) \}$ for one and only one gauge orbit in the equivalence class corresponding to that point in modular space. For a function Ψ which would achieve such a complete gauge-fixing, the path-integral (14) would reduce to an integral over modular space, giving a gauge invariant representation of the corresponding quantum observable.

However, as the BFV formalism explicitly incorporates only the gauge transformations connected to the identity, at best there may exist gauge-fixing functions Ψ such that the path-integral (14) reduces to an integral over "Teichmüller" space. We will refer to such functions Ψ as "good" gauge-fixing functions. In practice, this does not lead to a problem; the correct gauge invariant quantity is obtained by "moding out" the integral over "Teichmüller" space by the modular group, i.e. by restricting the integral to a fundamental domain of the modular group in "Teichmüller" space.

It may happen that no "good" gauge-fixing function Ψ exists. In this case the path-integral (14) reduces to an integral over "Teichmüller" space in which some regions of that space are either not covered at all or are covered more than once (possibly an infinite number of times). This corresponds to a Gribov problem [10]. As is well known, Yang-Mills theories [10, 11] and string theories [12, 6] suffer from such an ambiguity. As a practical consequence, generally, the integration measure over modular space is not determined from the path-integral [6]; another requirement is necessary.

Even if "good" gauge-fixing functions exist, it is conceivable that the same situation occurs for other choices of Ψ . Hence, it appears that the choice of Ψ is important [6] in obtaining a phase-space path-integral which reduces to an integral over a single cover of "Teichmüller" space, with a well determined integration measure.

A priori, one may thus expect the BFV path-integral to have some dependence on the choice of Ψ . The precise character of this dependence is determined by the Fradkin-Vilkovisky theorem [1].

IV. THE FRADKIN-VILKOVISKY THEOREM

The statement of this theorem is usually given as follows [1, 4]: there is no dependence whatsoever of the BFV path-integral on Ψ . As should be clear from the previous discussion, this cannot be correct. If those considerations were not convincing enough, explicit counter-examples are presented below in the case of a very simple system: the relativistic scalar particle.

In order to give the precise and correct statement of the theorem, let us introduce a last concept. We say that two gauge-fixing functions Ψ are equivalent if and only if they determine the same covering (including multicoverings and/or no coverings of some regions) of "Teichmüller" space. This defines gauge-equivalence classes of gauge-fixing functions [6].

Referring back to the proof of the theorem [1,4], one realizes that its correct and precise statement is rather [6]: the BFV path-integral does not specifically depend on a given gauge-fixing function Ψ , but only on its gauge-equivalence class.

In other words, for a given Ψ , the BFV path-integral indeed reduces (as was implicitly assumed in the previous section) to an integral over the covering of "Teichmüller" space determined by the gauge-equivalence class to which Ψ belongs. This is a very strong result indeed. The BFV path-integral does not depend on how the space of Lagrange multipliers is "sliced" by different gauge-fixing functions; it only depends on which orbits are intersected by the gauge-slice (including their multiplicities) but not on specific points on those orbits.

Putting it still differently, gauge-fixing functions Ψ which determine gauge-slices which can be transformed into one another by gauge transformations in the identity class of the modular group, lead to the same BFV path-integral.

In short, all these ideas are different aspects of the fact that the BFV path-integral is BRST invariant. Contrary to what is usually stated, this does not mean that it is independent of Ψ [6].

It also follows that the correct gauge invariant path-integral is given by the BFM path-integral only when evaluated for the gauge-equivalence class of "good" gauge-fixing function. This equivalence class does not exist when the description of the system suffers from a Gribov problem. In the latter case, one may still evaluate the BFM path-integral for each point in "Teichmüller" space separately, and then integrate those results over modular space with a measure to be determined from another requirement [6].

V. THE RELATIVISTIC PARTICLE

Let us illustrate the previous discussion by a specific example: the relativistic scalar particle [13, 14, 15, 6], with the non-linear action

$$S[x^\mu] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\dot{x}^2(\tau)} \quad (15)$$

Here τ parametrizes the world-line of the particle and $x^\mu(\tau)$ are D functions which define its space-time embedding. (Space-time is taken as D -dimensional with a Minkowski metric of signature $\eta^{\mu\nu} = (- + + \dots +)$, $\mu, \nu = 0, 1, \dots, D-1$). We consider a process where the initial and final space-time positions are fixed, i.e. $x^\mu(\tau_1) = x^\mu_1$ ($i = 1, 2$).

Clearly, the gauge symmetry of this system is the group of world-line diffeomorphisms which map the interval $[\tau_1, \tau_2]$ into itself.

In the Hamiltonian formulation, we have the phase-space variables x^μ and P_μ , with

$$\{x^\mu(\tau), P_\nu(\tau)\} = \delta^\mu_\nu, \quad (16)$$

the primary first-class constraint

$$\phi(\tau) \equiv \frac{1}{2} (P^2(\tau) + m^2) = 0, \quad (17)$$

and the canonical Hamiltonian $H_0 = 0$. The corresponding first-order action is

$$S[x^\mu, P_\mu, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu P_\mu - \frac{1}{2} \lambda (P^2 + m^2) \right], \quad (18)$$

where $\lambda(\tau)$ is the Lagrange multiplier for the constraint ϕ ($\lambda(\tau)$ actually corresponds to a world-line einbein). This action is invariant under:

$$\delta_\epsilon x^\mu = \epsilon p^\mu, \quad \delta_\epsilon p^\mu = 0, \quad \delta_\epsilon \lambda = \dot{\epsilon}, \quad (19)$$

where $\epsilon(\tau)$ is an infinitesimal function such that $\epsilon(\tau_i) = 0$ ($i=1, 2$).

It then follows that the quantity

$$c = \int_{\tau_1}^{\tau_2} d\tau \lambda(\tau) \quad (20)$$

(which is the total proper-time of the particle) is invariant under all transformations induced by the constraint [13, 14, 6]. Hence, c is the parameter which labels the orbits, in the space of functions $\lambda(\tau)$, of the connected part of the group of world-line diffeomorphisms. The one-dimensional real line $\mathbb{R}$ is thus the "Teichmüller" space for this system (and this process) [13, 16, 6].

However, the full group of gauge transformations which leave the action (15) invariant is doubly connected: it also contains those world-line diffeomorphisms which reverse the world-line orientation and exchange the two end-points τ_1 and τ_2 . The modular group is thus $\mathbb{Z}_2$. It acts on Teichmüller space as ($c \rightarrow -c$), so that modular space is simply $\mathbb{R}/\mathbb{Z}_2 \cong [0, +\infty[$ (with $c = 0$ being a singular point).

Therefore, the correct path-integral representation for the propagator, invariant under the complete gauge group of world-line diffeomorphisms, must be given by an integral over $c \in [0, +\infty[$ [13, 6]. This restriction to modular space has a physical meaning [13], as we will see.

To obtain the BFM extended phase-space, we first introduce a momentum $\pi(\tau)$ conjugate to $\lambda(\tau)$

$$\left\{ \lambda(\tau), \pi(\tau) \right\} = 1, \quad (21)$$

with the first-class constraints

$$G_1(\tau) = \pi(\tau) = 0, \quad G_2(\tau) = \phi(\tau) = 0. \quad (22)$$

Correspondingly, we then have two pairs of BFM ghosts $\eta^a(\tau)$, $\mathcal{P}_a(\tau)$ ($a = 1, 2$), with

$$\left\{ \eta^a(\tau), \mathcal{P}_b(\tau) \right\} = -\delta_b^a, \quad a, b = 1, 2. \quad (23)$$

The BRST charge is

$$Q_B = \eta^1 \pi + \frac{1}{2} \eta^2 (p^2 + m^2) , \quad (24)$$

with the BRST invariant Hamiltonian

$$H_B = 0 . \quad (25)$$

We impose the BRST invariant boundary conditions [6] :

$$x^\mu(\tau_i) = x_i^\mu , \quad \pi(\tau_i) = 0 , \quad \eta^2(\tau_i) = 0 = \mathcal{P}_1(\tau_i) , \quad i = 1, 2. \quad (26)$$

Let us consider the following set of gauge-fixing functions:

$$\Psi = \mathcal{P}_1 f(\lambda) + \mathcal{P}_2 \lambda , \quad (27)$$

where $f(\lambda)$ is any function of λ . We then have:

$$H_{eff} = \frac{1}{2} \lambda (p^2 + m^2) + f \pi - \mathcal{P}_2 \eta^1 - \frac{df}{d\lambda} \mathcal{P}_1 \eta^1 , \quad (28)$$

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu p_\mu + \dot{\lambda} \pi + \dot{\eta}^1 \mathcal{P}_1 - \eta^2 \dot{\mathcal{P}}_2 - H_{eff} \right] , \quad (29)$$

with the equations of motion:

$$\dot{x}^\mu = \lambda p^\mu , \quad \dot{p}^\mu = 0 ; \quad \dot{\lambda} = f(\lambda) , \quad \dot{\pi} = -\frac{1}{2} (p^2 + m^2) - \frac{df}{d\lambda} \pi + \frac{d^2 f}{d\lambda^2} \mathcal{P}_1 \eta^1 , \quad (30a)$$

$$\dot{\eta}^1 = \frac{df}{d\lambda} \eta^1 , \quad \dot{\mathcal{P}}_1 = -\mathcal{P}_2 - \frac{df}{d\lambda} \mathcal{P}_1 ; \quad \dot{\eta}^2 = \eta^1 , \quad \dot{\mathcal{P}}_2 = 0 . \quad (30b)$$

Given a solution for $\lambda(\tau)$, and the boundary conditions (26), one can solve in general for x^μ , p_μ , η^1 and η^2 , as:

$$x^\mu(\tau) = x_i^\mu + \Delta x^\mu \frac{1}{c} \int_{\tau_1}^{\tau} d\tau' \lambda(\tau') , \quad p^\mu(\tau) = \frac{\Delta x^\mu}{c} , \quad (31a)$$

$$\eta^1(\tau) = 0 = \eta^2(\tau) , \quad (31b)$$

where $\Delta x^\mu = x_2^\mu - x_1^\mu$, and c is defined in (20). The constraint ($P^2 + m^2 = 0$) must follow from the solution for $\pi(\tau)$ with the boundary conditions $\pi(\tau_i) = 0$ ($i=1, 2$).

Note that for $c > 0$ (resp. $c < 0$), positive-energy solutions propagate in the future (resp. past) and negative-energy ones in the past (resp. future). The restriction to $c \in [0, +\infty[$ thus corresponds to the description of, say, a particle. The same interpretation is valid for the propagator considered below (see (39)) [13, 6].

Canonical quantization of this system is straightforward [6]. Let us only note that nilpotency of the BRST charge is preserved through the quantization procedure.

The probability amplitude for the particle to propagate from x_1^μ to x_2^μ is represented by the BFV path-integral

$$Z = \int [dx^\mu dP_\mu d\lambda d\pi d\eta^1 d\mathcal{P}_1 d\eta^2 d\mathcal{P}_2] e^{iS_{\text{eff}}} \quad , \quad (32)$$

with the boundary conditions (26) and S_{eff} given in (29). Eq.(32) factorizes into:

$$Z = \int [d\lambda d\pi] e^{i \int_{\tau_1}^{\tau_2} d\tau (\dot{\lambda} - \frac{1}{2}) \pi} \int [d\eta^1 d\mathcal{P}_1 d\eta^2 d\mathcal{P}_2] e^{i \int_{\tau_1}^{\tau_2} d\tau [\mathcal{P}_2 \dot{\eta}^1 + \mathcal{P}_1 (-\dot{\eta}^2 + \frac{d}{d\lambda} \dot{\eta}^2)]} \cdot \int [dx^\mu dP_\mu] e^{i \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu P_\mu - \frac{1}{2} \lambda (P^2 + m^2)]} \quad . \quad (33)$$

These integrals can all be evaluated by time slices. Taking N time steps in the interval $[\tau_1, \tau_2]$, we define:

$$\epsilon = \frac{\Delta\tau}{N} \quad , \quad \Delta\tau = \tau_2 - \tau_1 \quad , \quad (34a)$$

and

$$g_i = g(\tau_i) \quad , \quad \text{with} \quad \tau_i = \tau_1 + i\epsilon \quad , \quad i = 0, 1, \dots, N \quad , \quad (34b)$$

for any function $g(\tau)$. One obtains (up to numerical factors), for the integral over x^μ and P_μ :

$$\left(\sum_{i=1}^N \lambda_i \epsilon \right)^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{\left(\sum_{i=1}^N \lambda_i \epsilon \right)} - m^2 \left(\sum_{i=1}^N \lambda_i \epsilon \right) \right] , \quad (35)$$

for the integral over the BfV ghosts:

$$\epsilon^N \det M , \quad (36a)$$

where M is a $(N-1) \times (N-1)$ matrix with the only non-vanishing entries

$$M_{i(i-1)} = \frac{-1}{\epsilon} - \frac{df(\lambda_i)}{d\lambda} , \quad M_{ii} = \frac{2}{\epsilon} - \frac{df(\lambda_i)}{d\lambda} , \quad M_{i(i+1)} = \frac{-1}{\epsilon} , \quad i=1, \dots, N-1 , \quad (36b)$$

and for the integral over λ and π :

$$\int_{-\infty}^{+\infty} d\lambda_N \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta(\lambda_{i+1} - \lambda_i - \epsilon f(\lambda_i)) . \quad (37)$$

The integral over π enforces the gauge-fixing condition $\dot{\lambda} = f(\lambda)$. As there is always one more integration over λ than over π , this will provide the final integration over Teichmüller space. The corresponding integration measure is determined from the integration over the BfV ghosts and the remaining Lagrange multipliers λ_i ($i=1, \dots, N-1$) [13,6]. That those factors should combine with the integration over λ_N into an integral over Teichmüller space (with no other dependence, as for example on $\Delta\tau$ or any other parameter which may appear in Ψ) is not obvious. This, however, is guaranteed by the Fradkin-Vilkovisky theorem.

For a given $f(\lambda)$, the problem thus reduces to finding how Teichmüller space is covered as λ_N varies from $(-\infty)$ to $(+\infty)$. Hence, for "good" gauge-fixing functions Ψ the path-integral (33) leads to:

$$Z = \int_{-\infty}^{+\infty} dc \quad c^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{c} - m^2 c \right] . \quad (38)$$

As was discussed before, the correct propagator is obtained by restricting the integration to modular space, which finally gives [13, 14, 6]:

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty dc \quad c^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{c} - (m^2 - i\epsilon)c \right] \quad (39)$$

(where a small imaginary part is added to m^2 in order to regulate the integral [13]). This is indeed the proper-time representation of the Feynman propagator for a scalar particle.

Let us now consider different choices of Ψ , i.e. of $f(\lambda)$, in order to show that indeed the BFM path-integral depends on the gauge-equivalence class of Ψ .

Example 1. The simplest example is :

$$f(\lambda) = \alpha \quad , \quad (40)$$

with α an arbitrary constant. The case $\alpha = 0$ corresponds to the proper-time gauge $\dot{\lambda} = 0$ [13,14,15,6] . The solution for $\lambda(\tau)$ is

$$\lambda(\tau) = \alpha (\tau - \tau_2) + \lambda_N \quad , \quad (41)$$

where we used $\lambda(\tau_2) = \lambda_N$ to fix the integration constant. The corresponding Teichmüller parameter is:

$$c(\lambda_N) = \frac{1}{2} \alpha (\Delta\tau)^2 + \lambda_N \Delta\tau \quad . \quad (42)$$

Therefore, as λ_N varies in the interval $]-\infty, +\infty[$, Teichmüller space is covered once and only once. The gauge-fixing functions Ψ given by (40) thus belong to the " good " gauge-equivalence class.

In the present case, the explicit evaluation of (36) and (37) can be completed [13, 14, 6] . One finds:

$$\epsilon^N \det M = \epsilon^N = \Delta\tau \quad , \quad (43a)$$

$$\int_{-\infty}^{+\infty} d\lambda_N \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta(\lambda_{i+1} - \lambda_i - \epsilon\alpha) = \int_{-\infty}^{+\infty} d\lambda_N \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta(\lambda_i - \lambda_N + \alpha\epsilon(N-i)) \quad . \quad (43b)$$

Hence, as expected, all factors combine into an integral over the Teichmüller parameter c (by the change of variables $\lambda_N \rightarrow c(\lambda_N)$) [13] . The correct result (39) is thus obtained for the propagator.

Example 2. Let us now consider [6] :

$$f(\lambda) = \alpha (\lambda - \lambda_0) \quad , \quad (44)$$

with α and λ_0 arbitrary constants. (For $\alpha = 0$, one recovers the proper-time gauge). We then have:

$$\lambda(\tau) = \lambda_0 + (\lambda_N - \lambda_0) e^{\alpha(\tau - \tau_2)}, \quad (45)$$

$$c(\lambda_N) = \lambda_0 \Delta\tau + \frac{1}{\alpha} (\lambda_N - \lambda_0) (1 - e^{-\alpha \Delta\tau}). \quad (46)$$

Here again, as λ_N varies from $(-\infty)$ to $(+\infty)$, one obtains a single cover of the entire Teichmüller space. Eq.(44) also defines "good" gauge-fixing functions.

Completing the evaluation of (36) and (37), we have:

$$\epsilon^N \det M = \frac{1}{\alpha} \left[\left(1 + \frac{\alpha \Delta\tau}{N} \right)^N - 1 \right], \quad (47a)$$

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta(\lambda_{i+1} - \lambda_i - \epsilon \alpha (\lambda_i - \lambda_0)) = \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta\left(\lambda_i - \frac{\lambda_{i0} + \alpha \epsilon \lambda_0}{1 + \alpha \epsilon}\right) \cdot \left(1 + \frac{\alpha \Delta\tau}{N}\right)^{-N+1}, \quad (47b)$$

so that

$$\lim_{N \rightarrow \infty} Z = \int_{-\infty}^{+\infty} d\lambda_N e^{-\alpha \Delta\tau} \frac{1}{\alpha} (e^{\alpha \Delta\tau} - 1) [c(\lambda_N)]^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta\tau)^2}{c(\lambda_N)} - m^2 c(\lambda_N) \right]. \quad (48)$$

As is guaranteed by the Fadkin-Vilkovisky theorem, all measure factors indeed combine into an integration over c , independent of α , λ_0 and $\Delta\tau$. The correct result (39) for the propagator is obtained again.

Example 3. As a first example of a "bad" gauge-fixing function Ψ , let us consider:

$$f(\lambda) = \alpha (1 - \beta \lambda^2), \quad (49)$$

with α, β strictly positive constants. (For $\beta \rightarrow 0$, one recovers the "good" gauge choice $f(\lambda) = \alpha$). We then have:

$$\lambda(\tau) = \frac{1}{\sqrt{\beta}} \frac{1 - \rho e^{-2\alpha\sqrt{\beta}\tau}}{1 + \rho e^{-2\alpha\sqrt{\beta}\tau}} \quad \text{with} \quad \rho = \frac{1 - \sqrt{\beta} \lambda_N e^{2\alpha\sqrt{\beta}\tau_2}}{1 + \sqrt{\beta} \lambda_N}, \quad (50)$$

and

$$c(\lambda_N) = \frac{-1}{\alpha\beta} \ln \left| \cosh(\alpha\sqrt{\beta}\Delta\tau) - \lambda_N \sqrt{\beta} \sinh(\alpha\sqrt{\beta}\Delta\tau) \right|. \quad (51)$$

The behaviour of the Teichmüller parameter is as follows. As λ_N varies from $(-\infty)$ to $\beta^{-1/2}(\text{tgh } \alpha\sqrt{\beta}\Delta\tau)^{-1}$, $c(\lambda_N)$ varies from $(-\infty)$ to $(+\infty)$ and vanishes for $\lambda_N = \beta^{-1/2}\text{tgh}(\alpha\sqrt{\beta}\Delta\tau/2)$. As λ_N varies from $\beta^{-1/2}(\text{tgh } \alpha\sqrt{\beta}\Delta\tau)^{-1}$ to $(+\infty)$, $c(\lambda_N)$ varies from $(+\infty)$ to $(-\infty)$ and vanishes for $\lambda_N = \beta^{-1/2}(\text{tgh } \alpha\sqrt{\beta}\Delta\tau/2)^{-1}$. Hence, we obtain in this case a double covering of Teichmüller space. Note that for $\beta \rightarrow 0$, Teichmüller space is covered only once, as it should.

Although the BFV path-integral is BRST invariant, it depends on the gauge-equivalence class of gauge-fixing functions. For the class of functions Ψ determined by (49), the BFV path-integral does not lead to the correct propagator.

Other gauge-fixing functions for which such a double covering of Teichmüller space is obtained are for example

$$f(\lambda) = \alpha (1 + \beta \lambda^2) , \quad (52)$$

with α, β strictly positive constants, or

$$f(\lambda) = \alpha \lambda (\lambda - \beta) \quad (53)$$

with α, β arbitrary constants. Note that (52) reduces to $f(\lambda) = \alpha$ for $\beta \rightarrow 0$ and that (53) reduces to $f(\lambda) = 0$ for $\alpha \rightarrow 0$. In both cases one thus recovers a "good" gauge-fixing function.

One may argue that, despite double covering of Teichmüller space, the correct result can nevertheless be obtained by a proper normalization of the BFV path-integral. However, in the non-singular limits $\beta \rightarrow 0$ or $\alpha \rightarrow 0$ in which a single covering is recovered, the normalization of the propagator would be off by a factor 2. This clearly shows that the BFV path-integral does depend on the gauge-equivalence class of gauge-fixing functions.

Example 4. If the previous examples were still not convincing enough, let us now consider

$$f(\lambda) = \alpha \lambda^3 , \quad (54)$$

with α a strictly positive constant. In the limit $\alpha \rightarrow 0$, one has the "good" gauge-fixing function $f(\lambda) = 0$. Solving for $\lambda(\tau)$, we obtain:

$$\lambda(\tau) = \frac{\lambda_N}{\sqrt{1 + 2\alpha\lambda_N^2(\tau_2 - \tau)}} , \text{ with } \tau < \tau_2 + \frac{1}{2\alpha\lambda_N^2} , \quad (55)$$

and

$$c(\lambda_N) = \frac{1}{\alpha\lambda_N} \left[\sqrt{1 + 2\alpha\Delta\tau\lambda_N^2} - 1 \right] . \quad (56)$$

As λ_N varies from $(-\infty)$ to $(+\infty)$, the Teichmüller parameter $c(\lambda_N)$ varies from $(-(2\Delta\tau/\alpha)^{1/2})$ to $(2\Delta\tau/\alpha)^{1/2}$ and vanishes at $\lambda_N = 0$. We thus obtain a single cover of a finite interval in Teichmüller space. Note that in the limit $\alpha \rightarrow 0$, a single covering of the complete Teichmüller space is nevertheless obtained, as it should.

This example shows with no ambiguity that the BFV path-integral indeed depends on the gauge-equivalence class of gauge-fixing functions, and that it cannot be independent of those functions.

As established by examples 1 and 2, the relativistic particle does not suffer from a Gribov problem [6]; there exists a gauge-equivalence class of "good" gauge-fixing functions. As said before, for systems with a Gribov problem one can nevertheless obtain the correct gauge invariant path-integral representation of a physical observable by evaluating the BFV path-integral for each point in Teichmüller space, and integrating those results over modular space, with a measure to be determined from some other requirement [6].

Let us illustrate this in the case of the particle [6]. In the space of Lagrange multipliers, any point on the orbit of Teichmüller parameter c can be represented by

$$\lambda(\tau) = \frac{c}{\Delta\tau} \frac{dh(\tau)}{d\tau} \quad (57)$$

for some function $h(\tau)$ such that $h(\tau_i) = \tau_i$ ($i=1,2$).

Consider then the gauge-fixing function

$$\Psi = \frac{1}{\beta} \left(\lambda - \frac{c}{\Delta\tau} \frac{dh}{d\tau} \right) \mathcal{P}_1 + \lambda \mathcal{P}_2 , \quad (58)$$

where β is an arbitrary constant parameter. For $h(\tau) = \tau$ (and $\beta \neq 0$), this choice corresponds to the second example above. After redefining π and $\mathcal{P}_1$ as $\beta\pi$ and $\beta\mathcal{P}_1$, which is a transformation leaving the Liouville measure invariant [4],

the limit $\beta \rightarrow 0$ is taken inside the BFM path-integral (32), which then reduces to:

$$Z = \int [d\lambda d\pi] e^{i \int_{\tau_1}^{\tau_2} d\tau (-\lambda + \frac{c}{\Delta\tau} \dot{h}) \pi} \int [d\eta^1 d\eta^2 d\eta^3 d\eta^4] e^{i \int_{\tau_1}^{\tau_2} d\tau (-\eta^2 \dot{\eta}_2 + \eta_2 \eta^1 + \eta_1 \eta^1)} \cdot \int [dx^\mu dP_\mu] e^{i \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu P_\mu - \frac{1}{2} \lambda (P^2 + m^2)]} \quad (59)$$

The integral over π clearly enforces the condition $\lambda = c(\dot{h}/\Delta\tau)$; taking the limit $\beta \rightarrow 0$ as explained above, the gauge-slice in the space of Lagrange multipliers has collapsed to a single point.

On the other hand, since for $\beta \neq 0$ the BFM path-integral leads to the correct result (38), the limit $\beta \rightarrow 0$ taken inside the integral must be singular in some manner. Indeed, evaluating (59) by time slices, one finds:

$$Z = \lim_{\substack{\epsilon \rightarrow 0, N \rightarrow \infty \\ \epsilon N = \Delta\tau}} \epsilon \int_{-\infty}^{+\infty} d\lambda_N \int_{-\infty}^{+\infty} \prod_{i=1}^{N-1} d\lambda_i \prod_{i=1}^{N-1} \delta(\lambda_i - \frac{c}{\Delta\tau} \dot{h}_i) \cdot \left(\sum_{i=1}^N \lambda_i \epsilon \right)^{-D/2} \cdot \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{\left(\sum_{i=1}^N \lambda_i \epsilon \right)} - m^2 \left(\sum_{i=1}^N \lambda_i \epsilon \right) \right] \quad (60)$$

$$= \left(\lim_{N \rightarrow \infty} \frac{\Delta\tau}{N} \int_{-\infty}^{+\infty} d\lambda_N \right) \cdot c^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{c} - m^2 c \right] \quad (61)$$

The singular character of the limit $\beta \rightarrow 0$ taken inside the path-integral thus simply appears through an ill-defined multiplicative factor. Absorbing it in the normalization of the path-integral (or simply ignoring it; for systems with an infinite number of degrees of freedom this amounts to the same), one concludes that the correct gauge invariant propagator is given by [6]:

$$Z(x_2^\mu, x_1^\mu) = \int_0^\infty dc \mu(c) c^{-D/2} \exp \frac{1}{2} i \left[\frac{(\Delta x)^2}{c} - (m^2 - i\epsilon) c \right] \quad (62)$$

where $\mu(c)$ is an undetermined integration measure. From the evaluation of the BFM path-integral in the "good" gauge-equivalence class, we know that $\mu(c) = 1$.

Note that even in the singular limit $\beta \rightarrow 0$ the BFM path-integral (61) is BRST invariant, as it does not depend on $h(\tau)$ but only on c , i.e. it is independent of the specific point on the gauge orbit at which the gauge-fixing is done. But this is indeed guaranteed by (the correct statement of) the Fradkin-

Vilkovisky theorem. By itself, the result (61) is already convincing enough that the usual statement of this theorem cannot be correct [6].

VI. CONCLUSION

In this paper, we have considered the dependence of the BFV phase-space path-integral [1,4] for constrained systems on the choice of a gauge-fixing function. The character of this dependence is determined by the Fradkin-Vilkovisky theorem [1].

Contrary however to the statement usually given for this theorem [1,4], we have argued that the BFV path-integral cannot be independent of the gauge-fixing function, but rather that it depends on gauge-equivalence classes of gauge-fixing functions (but not on a given function *per se*) [6].

This was done by characterizing the structure of the space of gauge orbits through concepts generalized from field theories defined on Riemann surfaces (e. g. string theories). It was shown how the correct gauge-invariant path-integral representation of a physical observable must reduce to an integral over modular space (the space of inequivalent gauge orbits), through the evaluation of the BFV path-integral for the gauge-equivalence class of "good" gauge-fixing functions. For systems which suffer from a Gribov problem [10], although such "good" gauge-fixing functions do not exist, we discussed how the correct quantities can nevertheless be obtained, up to an undetermined integration measure over modular space. For example, this was used in Ref.[6] in the evaluation of the free propagator for the closed and for the open bosonic string from the BFV path-integral.

All these aspects of gauge invariant systems were illustrated in the case of a very simple system: the relativistic scalar particle [13, 14, 15, 6]. In particular, explicit counter-examples to the usual statement [1,4] of the Fradkin-Vilkovisky theorem were presented; they all support the correct statement proposed in this paper.

The space of orbits of the complete gauge symmetry group is seen, once again, to play an important role in the quantization of constrained systems. The structure of that space, as characterized here, may well be of some help in unravelling the quantum behaviour of such systems. For example, an understanding of the precise structure of modular space in the case of

non-abelian gauge theories may turn out to be important in developing more insight into the dynamics of those theories.

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