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OFF-SHELL SIX-DIMENSIONAL SUPERGRAVITY IN HARMONIC SUPERSPACE

E. Sokatchev
CERN -- Geneva

ABSTRACT

A fully off-shell action for $N = 2$ $d = 6$ Poincaré supergravity in harmonic superspace is proposed. It consists of the off-shell actions for a scalar multiplet and for a self-dual tensor multiplet in conformal supergravity background. A new mechanism for describing self-dual tensor fields off shell in which conformal supergravity plays an active role is found.

*) On leave from the Institute for Nuclear Research and Nuclear Energy, Sofia, Bulgaria.

1. Introduction

The theory of N=2 d=6 supergravity has some rather unusual features compared to either N=1 or N=2 supergravity in d=4. It is sufficient to look at the content of the conformal (Weyl) multiplet [1,2] to see this:

$$e_a^m, \psi^{m\alpha i}, V^{mij}, T_{abc}^-, \chi^{\alpha i}, D. \quad (1.1)$$

Here e_a^m is the sexbein, $\psi^{m\alpha i}$ is the gravitino, v^{mij} is the SU(2) gauge field, $\chi^{\alpha i}$ and D are auxiliary spinor and scalar and T_{abc}^- is a **self-dual tensor**. It is just this tensor which causes problems when trying to couple the Weyl multiplet to matter multiplets. That can only be done with a matter multiplet which itself contains such a tensor. It is the so-called tensor multiplet [1] with a scalar σ , a spinor $\psi_{\alpha i}$ and an antisymmetric tensor B_{ab} subject to the self-duality constraint

$$F_{abc}^- = \partial_{[a} B_{bc]} - \varepsilon_{abcdef} \partial^d B^{ef} = 0. \quad (1.2)$$

Since (1.2) is in fact a dynamical equation, the whole tensor multiplet is on shell. As was discovered in [1], when the equation of motion (1.2) and those for the scalar and the spinor are put in a conformal supergravity background, they become algebraic relations allowing to express the Weyl multiplet components T , χ , D in terms of the tensor multiplet ones (e.g., $T_{abc}^- \sim F_{abc}^-$, etc.). This gives rise to a second version of the Weyl multiplet containing the unconstrained tensor field B_{ab} rather than the anti-self-dual one T_{abc}^- . Then it becomes possible to couple the new Weyl multiplet to other matter multiplets (e.g., to the scalar one). The latter serve as compensators for conformal invariance, and thus one obtains different versions of N=2 d=6 Poincaré supergravity.

In the construction of Ref.[1] there are two aspects one would like to develop further. First of all, the covariant equations of motion for the tensor multiplet (which allow to express part of the Weyl multiplet in terms of the tensor one) are given *ad hoc*, they do not follow from an action. It would be preferable to treat the tensor multiplet as an independent matter multiplet and to relate it to the Weyl multiplet only on shell. However, self-dual fields like B_{ab} (1.2) are known not to exist off shell on their own [3]. Indeed, the only way to derive the self-duality condition (1.2) from an action is to introduce a Lagrange multiplier for it. Then the latter itself satisfies a dynamical equation of motion, so one ends up with more on-shell degrees of freedom than initially planned. The second point about Ref.[1] we would like to emphasize is that the scalar multiplet compensator action used there is an on-shell one. Therefore the corresponding version of d=6 Poincaré supergravity (which for various reasons is the most interesting one) is also on shell.

In the present paper we solve both problems mentioned above. Using the methods of harmonic superspace [4-7] we are able to write down **off-shell** actions for both the tensor and the scalar multiplets in conformal supergravity background (the latter was first proposed in [7]). Surprisingly enough, the doubling of the tensor multiplet due to the introduction of a Lagrange multiplier multiplet is removed with help from the supergravity sector. Varying the fields T , χ , D from the Weyl multiplet one obtains equations which identify the components of the tensor multiplet with their Lagrange multipliers. Thus we find a new mechanism of constructing off-shell actions. We hope that it can have applications elsewhere, e.g. in the N=4 super-Yang-Mills theory.

The plan of the article is as follows. In Section 2 a harmonic superspace description of the tensor multiplet and its field-strength (or Lagrange multiplier) multiplet is developed. An off-shell action is proposed which, as explained above, has twice the number of degrees of freedom of the tensor multiplet. In Section 3 the formulation of $d=6$ conformal supergravity in harmonic superspace is given. It is essentially the same as that in $d=4$ [7]. The necessary information about the off-shell scalar multiplet in conformal supergravity background is also recalled. Section 4 is devoted to the covariantization of the tensor multiplet action from Section 2 in the presence of conformal supergravity. Finally, in Section 5 the mechanism of on-shell identification of the tensor multiplet with its Lagrange multiplier is illustrated by considering a simplified component action. The notation and conventions are explained in Appendix A. We refer the reader to Refs. [4-7] for specific information about the harmonic superspace approach. Some technical details of Section 4 are presented in Appendix B.

2. The tensor multiplet in harmonic superspace

The $d=6$ on-shell tensor multiplet [1] consists of the following fields:

$$\sigma, \quad B_{\alpha}^{\beta} \quad (B_{\alpha}^{\alpha} = 0), \quad \psi_{\alpha i}. \quad (2.1)$$

Here σ is a dimensionless scalar, B_{α}^{β} is a traceless bispinor representing an antisymmetric tensor B_{ab} , and $\psi_{\alpha i}$ is a pseudo-Majorana

spinor. The on-shell conditions for those fields needed to close the supersymmetry algebra are

$$\square \sigma = 0, \quad \partial^{(\alpha\beta} B_{\beta}^{\gamma)} = 0, \quad \partial^{\alpha\beta} \psi_{\beta i} = 0. \quad (2.2)$$

The equation for B_{α}^{β} is the spinor-notation form of the self-duality constraint

$$\partial^{[a} B^{bc]} = \varepsilon^{abcdef} \partial_d B_{ef}.$$

Obviously, the field B_{α}^{β} in (2.2) can undergo gauge transformations of the type

$$\delta B_{\alpha}^{\beta} = \partial_{\alpha\mu} \lambda^{\mu\beta} - \partial^{\beta\mu} \lambda_{\mu\alpha} \quad (2.3)$$

with an antisymmetric parameter $\lambda^{\alpha\beta} = -\lambda^{\beta\alpha}$ (or $\delta B^{ab} = \partial[a\lambda^b]$ in vector notation). This suggests another way of describing the tensor multiplet, this time using the self-dual field-strength

$$F_{(\alpha\beta)} = \partial_{(\alpha\mu} B_{\beta)}^{\mu} \quad (2.4)$$

instead of the gauge field B_{α}^{β} (note that the anti-self-dual field-strength $F^{(\alpha\beta)}$ vanishes according to (2.2)). Thus, the second form of the tensor multiplet is

$$\omega, F_{(\alpha\beta)}, \lambda_{\alpha i} \quad (2.5)$$

with on-shell conditions

$$\square \omega = 0, \quad \partial^{\alpha\beta} F_{\beta\mu} = 0, \quad \partial^{\alpha\beta} \lambda_{\beta i} = 0 \quad (2.6)$$

(the equation for $F_{\alpha\beta}$ follows from (2.2), (2.4) and the identity $\partial^{\alpha\beta}\partial_{\beta\gamma} = \partial_{\gamma\beta}\partial^{\beta\alpha} = \delta^{\alpha}_{\gamma}\square$). We stress that from now on we shall consider ω , $F_{\alpha\beta}$ and $\lambda_{\alpha i}$ as completely independent of σ , B_{α}^{β} and $\psi_{\alpha i}$ (i.e., the relationship (2.4) does not exist). Actually, later on the fields ω , F and λ will become the Lagrange multipliers for equations (2.2).

As stated above, the multiplets (2.1) and (2.5) are on-shell ones. The natural question arises whether it is possible to have them off shell and if there exists an action from which eqs. (2.2) and (2.6) follow as equations of motion. The answer is yes, but this is only possible if one uses harmonic superspace.

It is not difficult to show that the multiplet (2.1) is described by a harmonic superfield $v^{\alpha+}(x, \theta^+, \theta^-, u)$ subject to the following constraints:

$$D_{\alpha}^{+} V^{\beta+} - \frac{1}{4} \delta_{\alpha}^{\beta} D_{\gamma}^{+} V^{\gamma+} = 0, \quad (2.7a)$$

$$D^{++} V^{\alpha+} = 0. \quad (2.7b)$$

The first of them is very easy to solve in the analytic basis [6,7] in harmonic superspace, where $D_{\alpha}^{+} = \partial/\partial\theta^{\alpha-}$:

$$V^{\alpha+} = v^{\alpha+}(x, \theta^+, u) + \theta^{\alpha-} v^{(+2)}(x, \theta^+, u). \quad (2.8)$$

Here $v^{\alpha+}$ and $v^{(+2)}$ are arbitrary analytic superfunctions (not superfields, since $v^{\alpha+}$ transforms via $v^{(+2)}$ under supersymmetry transformations). The second constraint (2.7b) then becomes

$$D^{++} v^{(+2)} = 0, \quad (2.9a)$$

$$D^{++} v^{\alpha+} + \theta^{\alpha+} v^{(+2)} = 0. \quad (2.9b)$$

In order to find the component field content of these equations we expand the superfields $v^{(+2)}$ and $v^{\alpha+}$ in $\theta^{\alpha+}$:

$$\begin{aligned} v^{(+2)} &= f^{(+2)} + \theta^{\alpha+} \kappa_{\alpha}^{+} + (\theta^{+})^2{}^{\alpha\beta} A_{\alpha\beta} + (\theta^{+})_{\alpha}^3 \tau^{\alpha-} + (\theta^{+})^4 C^{(-2)}, \\ v^{\alpha+} &= \rho^{\alpha+} + \theta^{\beta+} (B_{\beta}^{\alpha} + \delta_{\beta}^{\alpha} \Sigma) + (\theta^{+})^2{}^{\beta\gamma} \omega_{\beta\gamma}^{-\alpha} + \\ &\quad + (\theta^{+})_{\beta}^3 E^{\beta\alpha(-2)} + (\theta^{+})^4 \varphi^{\alpha(-3)}. \end{aligned} \quad (2.10)$$

Putting this in eq. (2.9a) we find that $f^{(+2)}$ and κ_{α}^{+} are bilinear and linear in the harmonics,

$$\partial^{++} f^{(+2)} = 0 \rightarrow f^{(+2)} = f^{(ij)}(x) u_i^{+} u_j^{+},$$

$$\partial^{++} \kappa_{\alpha}^{+} = 0 \rightarrow \kappa_{\alpha}^{+} = K_{\alpha i}(x) u^{+i}.$$

Further, the equation for the vector $A_{\alpha\beta}$ ($= -A_{\beta\alpha}$)

$$\partial^{++} A_{\alpha\beta} + i \partial_{\alpha\beta} f^{(+2)} = 0$$

tells us that

$$A_{\alpha\beta} = a_{\alpha\beta}(x) - i \partial_{\alpha\beta} f^{(ij)}(x) u_i^{+} u_j^{-}.$$

Similarly, for the fields $\tau^{\alpha-}$ and $C^{(-2)}$ we find

$$\tau^{\alpha-} = \frac{i}{2} \partial^{\alpha\beta} \kappa_{\beta i} u^{-i},$$

$$C^{(-2)} = 2 \square f^{ij} u_i^- u_j^-,$$

as well as the constraint on the vector $a_{\alpha\beta}$:

$$\partial_{\alpha\beta} a^{\alpha\beta} = 0.$$

In a similar manner the second equation (2.9b) implies

$$\rho^{\alpha+} = \rho_i^{\alpha}(x) u^{+i}, \quad B_{\alpha}^{\beta} = B_{\alpha}^{\beta}(x), \quad \Sigma = \sigma(x) - f^{ij}(x) u_i^+ u_j^-,$$

$$\omega_{\beta\gamma}^{\alpha-} = -u^{-i} (i \partial_{\beta\gamma} \rho_i^{\alpha} + \delta_{\beta}^{\alpha} \kappa_{\gamma i} - \delta_{\gamma}^{\alpha} \kappa_{\beta i}),$$

$$a^{\alpha\beta} = -i \partial^{\alpha\gamma} B_{\gamma}^{\beta} - i \partial^{\alpha\beta} \sigma, \quad \varphi^{\alpha(-3)} = 0,$$

$$E^{\alpha\beta(-2)} = \frac{i}{2} \partial^{\alpha\beta} f^{ij} u_i^- u_j^-,$$

as well as the on-shell conditions

$$\square \sigma = 0, \quad \partial^{(\alpha\beta} B_{\beta}^{\gamma)} = 0, \quad \partial^{\alpha\beta} (\kappa_{\beta i} - i \partial_{\beta\gamma} \rho_i^{\gamma}) = 0$$

which coincide with (2.2) if $\kappa_{\beta i} - i \partial_{\beta\gamma} \rho_i^{\gamma}$ is identified with $\psi_{\beta i}$. One sees that the fields $f^{ij}(x)$ and $\rho^{\alpha}(x)$ remain arbitrary. Together with the gauge parameter $\lambda^{\alpha\beta}(x)$ from (2.3) they form a multiplet of gauge degrees of freedom. Another conclusion from the above analysis is that the constraint (2.7a) on the superfield $v^{\alpha+}$ is a kinematical one, whereas the dynamical equations follow from the second constraint (2.7b).

Going through similar steps one can verify that the field-strength tensor multiplet (2.4) can be described by the harmonic superfield $F(x, \theta^+, \theta^-, u)$ satisfying the constraints

$$D_{\alpha}^{+} D_{\beta}^{+} F = 0, \quad (2.11a)$$

$$D^{++} F = 0. \quad (2.11b)$$

Once again, the first of them just means that F is linear in θ^- :

$$F = \omega(x, \theta^+, u) + \theta^{\alpha-} \lambda_{\alpha}^{+}(x, \theta^+, u) \quad (2.12)$$

and the second one restricts the infinite harmonic series in (2.12) to the finite set (2.5) and produces the corresponding equations of motion (2.6).

From the above discussion one can conclude that the superfields $v^{\alpha+}$ or F subject to the kinematical constraints (2.7a) or (2.11a) describe off shell the tensor multiplet in its forms (2.1) or (2.5). Clearly, the price for having manifest supersymmetry is the infinite number of auxiliary fields (contained in the harmonic expansions of $v^{\alpha+}$ or F). The equations of motion (2.7b) or (2.11b) cut off the tails of auxiliary fields and put the remaining physical fields on shell. This situation is very similar to the off-shell description of the matter (hyper) multiplet [3].

The kinematical constraints (2.7a) and (2.11a) can be solved not only by expanding the superfields in θ^- as in (2.8) and (2.12), but also by introducing "prepotentials":

$$V^{\alpha+} = (D^+)^{3\alpha} V^{(-2)}, \quad (2.13a)$$

$$F = (D^+)^{3\alpha} F_{\alpha}^{(-3)}. \quad (2.13b)$$

Here $V^{(-2)}$ and $F^{(-3)}_{\alpha}$ are unconstrained superfields. Now it is very easy to write down an action from which the dynamical equations (2.7b) and (2.11b) will follow:

$$S_{TM} = \int d^6x d^8\theta du F_{\alpha}^{(-3)} D^{++} (D^+)^{3\alpha} V^{(-2)}. \quad (2.14)$$

It is obvious that the variation of $F^{(-3)}_{\alpha}$ in (2.14) produces (2.7a) with $V^{\alpha+}$ from (2.13a), and similarly for the variation of $V^{(-2)}$. In terms of component fields this means that the fields ω , F , λ (2.5) become Lagrange multipliers for the equations of motion (2.2) for the fields σ , B , ψ (2.1), and vice versa. However, the action (2.14) does in fact describe two tensor multiplets (one of the form (2.1) and the other of the form (2.5)). The basic reason for this is quite clear: the self-duality condition (2.2) on B_{α}^{β} can only be obtained from an action if a Lagrange multiplier ($F_{\alpha\beta}$) is introduced. The latter in turn propagates an equal number of degrees of freedom, so one inevitably doubles the number of on-shell degrees of freedom. The conclusion is that self-dual fields do not exist off shell on their own [2].

Surprisingly enough, this problem can be solved by introducing conformal supergravity into the game. As we shall see below, some of the fields of the $d=6$ Weyl multiplet act as additional Lagrange multipliers which identify the two tensor multiplets on shell.

3. The d=6 Weyl and scalar multiplets in harmonic superspace

The setup for describing the Weyl multiplet in d=6 harmonic superspace is exactly the same as for N=2 d=4 conformal supergravity [5,7]. Here we just recall some of the main points. Consider the following general coordinate transformation group in harmonic superspace:

$$\begin{aligned}\delta x^m &= \lambda^m(x, \theta^+, u), \\ \delta \theta^{\mu+} &= \lambda^{\mu+}(x, \theta^+, u), \\ \delta u^{+i} &= \lambda^{++}(x, \theta^+, u) u^{-i}, \quad \delta u^{-i} = 0, \\ \delta \theta^{\mu-} &= \lambda^{\mu-}(x, \theta^+, \theta^-, u).\end{aligned}\tag{3.1}$$

As one can see, the analytic subspace $(x^m, \theta^{\mu+}, u^{\pm i})$ is preserved by this choice of group. Further, introduce a covariant harmonic derivative $\mathcal{D}^{++}$ with the following transformation law

$$\delta \mathcal{D}^{++} = -\lambda^{++} \mathcal{D}^0\tag{3.2}$$

(the motivation for this definition has to do with the action of the conformal supergroup in flat harmonic superspace and is explained in [5]). This derivative involves a number of vielbeins:

$$\mathcal{D}^{++} = \partial^{++} + H^{(+4)} \partial^{--} + H^{++m} \partial_m + H^{++\mu+} \partial_\mu^- + H^{++\mu-} \partial_\mu^+.\tag{3.3}$$

In order to preserve the analyticity of the superfields on which $\mathcal{D}^{++}$ acts, one has to choose $H^{(+4)}$, H^{++m} and $H^{++\mu+}$ analytic as well:

$$\partial_\nu^+ H^{(+4)} = \partial_\nu^+ H^{++m} = \partial_\nu^+ H^{++\mu+} = 0,\tag{3.4}$$

whereas $H^{++\mu-}$ remains a general superfield. The transformation laws of the vielbeins above follow from (3.1) and (3.2):

$$\begin{aligned}\delta H^{(+4)} &= \mathcal{D}^{++} \lambda^{++}, \\ \delta H^{++m} &= \mathcal{D}^{++} \lambda^m, \\ \delta H^{++\mu+} &= \mathcal{D}^{++} \lambda^{\mu+} - \theta^{\mu+} \lambda^{++}, \\ \delta H^{++\mu-} &= \mathcal{D}^{++} \lambda^{\mu-} + \theta^{\mu-} \lambda^{++}.\end{aligned}\tag{3.5}$$

Now we can claim that the analytic vielbeins $H^{(+4)}$, H^{++m} and $H^{++\mu+}$ are the unconstrained prepotentials for d=6 conformal supergravity. To see this one goes to the Wess-Zumino gauge:

$$\begin{aligned}H^{(+4)} &= (\theta^+)^4 D(x), \\ H^{++m} &= (\theta^+)^2 e_{\mu\nu}^m(x) + (\theta^+)^3 \psi^{\mu m i}(x) u_i^- + \\ &\quad + (\theta^+)^4 V^{m i j}(x) u_i^- u_j^-, \\ H^{++\mu+} &= (\theta^+)^3 T^{(\nu\mu)}(x) + (\theta^+)^4 \chi^{\mu i}(x) u_i^-;\end{aligned}\tag{3.6a}$$

$$H^{++\mu-} = \theta^{\mu-} \rightarrow \mathcal{D}^{++} \lambda^{\mu-} = \lambda^{\mu+} - \theta^{\mu-} \lambda^{++},\tag{3.6b}$$

where one finds all the fields of the d=6 Weyl multiplet (type I) [1]. The remaining gauge freedom amounts to the standard set of coordinate transformations in x space: Weyl transformations, local $SU(2)$, local supersymmetry and local conformal supersymmetry. Restricting those gauge parameters one can fix the gauge even further. For example, both vielbeins $H^{(+4)}$ and $H^{++\mu+}$ can be entirely gauged away (this possibility follows from the superspin analysis made in [5]). The latter gauge choice

was advocated in [8] as the alternative to (3.5) which, as was claimed, should correspond to the type II Weyl multiplet of [1]. This, however, is not true because expressing, e.g., the field $D(x)$ in terms of the Weyl gauge parameter or in terms of the tensor multiplet component σ as in (3.22) does not lead to the same results at the level of the action.

The other harmonic covariant derivative $\mathcal{D}^{--}$ is defined by the relation

$$[\mathcal{D}^{++} - H^{(++)} \mathcal{D}^{--}, \mathcal{D}^{--}] = \mathcal{D}^0 \quad (3.7)$$

and has the transformation law

$$\delta \mathcal{D}^{--} = -(\mathcal{D}^{--} \lambda^{++}) \mathcal{D}^{--}. \quad (3.8)$$

Eq. (3.7) is a non-linear differential one allowing to express the vielbeins of $\mathcal{D}^{--}$,

$$\mathcal{D}^{--} = \partial^{--} + H^{--m} \partial_m + H^{--\mu+} \partial_\mu^- + H^{--\mu-} \partial_\mu^+, \quad (3.9)$$

in terms of those of $\mathcal{D}^{++}$. It can be solved as a power series in terms of superfields (for more detail see [6,7]) or in a closed form in terms of component fields in the Wess-Zumino gauge (3.6).

The $d=6$ scalar (or hyper) multiplet is described in harmonic superspace [9] in exactly the same way as in the $N=2$ $d=4$ case [4]. Namely, one considers an analytic superfield

$$q_a^+(x, \theta^+, u), \quad a = 1, \dots, 2n, \quad (3.10)$$

where n is the number of matter multiplets. The off-shell action for it is

$$S_{SM} = \int d^6x d^4\theta^+ du q^{+a} d_a{}^b D^{++} q^+_b, \quad (3.11)$$

where d_{ab} is an antisymmetric constant tensor with special properties (see [1]). In the simplest case of one such multiplet ($n=1$) considered below $d_{ab} = \varepsilon_{ab}$. Varying q^{+a} in (3.11) one obtains the on-shell equation

$$D^{++} q^+_a = 0. \quad (3.12)$$

It looks very much like the equations (2.7b) and (2.11b) for the tensor multiplet and, like them, reduces the content of q^+_a to just two fields,

$$A_a^i(x), \psi_{\alpha a}(x) \quad (3.13)$$

with the standard equations of motion

$$\square A_a^i = 0, \quad \partial^{\alpha\beta} \psi_{\beta a} = 0.$$

This is precisely the scalar multiplet with four scalars A_a^i and two spinors $\psi_{\alpha a}$.

The covariantization of the action (3.11) under the conformal supergravity group (3.1) is very simple [7]. It is sufficient to replace the harmonic derivative D^{++} by the covariant one $\mathcal{D}^{++}$ (3.3),

$$S_{SM}^{cov.} = \int d^6x d^4\theta^+ du q^{+a} \mathcal{D}^{++} q^+_a, \quad (3.14)$$

and to ascribe the following transformation law to q^+_a :

$$\delta q_a^+ = -\frac{1}{2} \Lambda q_a^+, \quad \Lambda = \partial_m \lambda^m - \partial_\mu^- \lambda^{\mu+} + \partial^{--} \lambda^{++}, \quad (3.15)$$

where Λ is the weight factor acquired by the volume element of analytic superspace $d^6x d^4\theta du$. (Note that the inhomogeneous terms $q^{+a}(\mathcal{D}^{++}\Lambda)q_a^+$ and $q^{+a}\Lambda^{++}q_a^+$ (see (3.2)) vanish because $q^{+a}q_a^+ = 0$) It is not hard to see that q_a^+ acts as a compensator for the Weyl, local conformal supersymmetry and local $SU(2)$ transformations which survive in the Wess-Zumino gauge (3.6). Indeed, the corresponding parameters a , $\eta_{\alpha i}$, λ_{ij} appear in Λ (3.15) (see [5]):

$$\Lambda = a + \theta^{\alpha+} \eta_{\alpha i} u^{-i} + \lambda_{ij} u^{+i} u^{-j} + \dots \quad (3.16)$$

Supposing that q_a^+ contains a non-vanishing constant term,

$$q_a^+ = u_a^+ + \tilde{q}_a^+,$$

one obtains from (3.15) and (3.16) in the linearized approximation

$$\delta \tilde{q}_a^+ = u_a^+ (a + \theta^{\alpha+} \eta_{\alpha i} u^{-i} + \lambda_{ij} u^{+i} u^{-j} + \dots) + \dots$$

Now it is clear that each of the parameters above is compensated by the corresponding components in $\tilde{q}_a^+$. In such a case one could conclude that (3.14) should be the action for d=6 Poincaré supergravity for this particular choice of the compensator.

Unfortunately, a closer look at the component form of (3.14) shows that this action is meaningless [1]. Indeed, consider the component expansion of q_a^+ (for simplicity we throw away the fermions):

$$q_a^+ = A_a^+ + (\theta^+)^2 \omega^\rho A_{\rho a}^- + (\theta^+)^4 A_a^{(-3)}. \quad (3.17)$$

Taking $\mathcal{D}^{++}$ in the Wess-Zumino gauge (3.5) and performing the θ integration one obtains

$$\begin{aligned} S_{SM}^{cov} = \int d^6x du \big(& 2A^{(-3)a} \partial^{++} A_a^+ - \frac{1}{4} A^{\alpha\beta a(-)} \partial_{\alpha\beta} A_a^+ - \\ & - \frac{1}{4} A^{+a} \partial^{\alpha\beta} A_{\alpha\beta a}^- - \frac{1}{4} A^{\alpha\beta a(-)} \partial^{++} A_{\alpha\beta a}^- + \\ & + D A^{+a} \partial^{--} A_a^+ + V^{m(-2)} A^{+a} \partial_m A_a^+ \big). \end{aligned} \quad (3.18)$$

Here $\partial_{\alpha\beta} = e^m_{\alpha\beta} \partial_m$, $A^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} A_{\gamma\delta}$, $V^{m(-2)} = v^{mij} u^-_i u^-_j$. Varying $A^{(-3)}$ one derives the equation

$$\partial^{++} A_a^+ = 0 \longrightarrow A_a^+ = A_{ai}(x) u^{+i}.$$

Further, using the gauge parameters a and λ_{ij} (Weyl and $SU(2)$) from (3.16) one can fix a gauge in which

$$A_{ai} = \varepsilon_{ai}. \quad (3.19)$$

Putting all this back into the action (3.18) one obtains its final form

$$S_{SM}^{cov} = \int d^6x \left[D - \frac{1}{16} (\partial_m e^{\alpha\beta m}) (\partial_n e^{\gamma\delta n}_{\alpha\beta}) \right] \quad (3.20)$$

(Note that both D and $e^m_{\alpha\beta}$ differ from the ones of [1] by field redefinitions, so (3.20) is covariant despite the unusual form of the second term.) It is clear now that the variation of D in (3.20) produces the meaningless equation $1 = 0$.

The way to rectify that has been found in [1]. The idea is to add to the scalar multiplet described by q^+_a another one, namely the tensor multiplet of Section 2. When its equations of motion (2.2) are properly

covariantized, they turn out to be algebraic relations allowing to express the Weyl multiplet components D , $\chi^{\alpha i}$, T^{ab} in terms of the tensor multiplet components σ , $\psi_{\alpha i}$, B_{α}^{β} as follows

$$\begin{aligned} T^{\alpha\beta} &= \sigma^{-1} \partial^{(\alpha} B^{\beta)} + \dots, \\ \chi^{\alpha i} &= \sigma^{-1} \partial^{\alpha\beta} \psi_{\beta}^i + \dots, \\ D &= \sigma^{-1} \square \sigma + \dots \end{aligned} \tag{3.22}$$

(the dots replace covariantizing terms). Plugging this into the scalar multiplet action (3.20) makes it meaningful.

We can go even further by carrying out that programme entirely **off shell**. First of all, we have a truly off-shell action (3.14) for the scalar multiplet as opposed to the on-shell one used in [1]. Second, instead of covariantizing the on-shell constraints on the tensor multiplet, we shall covariantize the off-shell action (2.14). Then we shall see that the "miracle" works both ways: not only D , χ , T are expressed in terms of σ , ψ , B on shell, but they in turn help to identify the components of the two copies (2.1) and (2.5) of the tensor multiplet on shell.

4. The covariant tensor multiplet action

The covariantization of the action (2.14) is complicated by the presence of the spinor derivatives $D^+_{\alpha} = \partial / \partial \theta^{\alpha-} = \partial^+_{\alpha}$. In a conformal supergravity background those transform in a non-trivial way (see (3.1)):

$$\delta \partial^+_{\alpha} = -\partial^+_{\alpha} \lambda^{\beta-} \partial^+_{\beta} = -\partial^+_{(\alpha} \lambda^{\beta)-} \partial^+_{\beta} - \frac{1}{4} (\partial^+_{\rho} \lambda^{\beta-}) \partial^+_{\alpha}, \quad (4.1)$$

where $\partial^+_{(\alpha} \lambda^{\beta)-}$ denotes the traceless part of $\partial^+_{\alpha} \lambda^{\beta-}$. This is a linear homogeneous transformation law, so the derivative of a weightless scalar superfield $\partial^+_{\alpha} \Phi$ remains covariant. However, the second order derivative of this superfield $\partial^+_{\alpha} \partial^+_{\beta} \Phi$ is not covariant since the inhomogeneous term $\partial^+_{\alpha} \partial^+_{\beta} \lambda^{\gamma-}$ appears in its transformation law. To cope with this one should introduce connection terms in the covariant derivatives. Because of the presence of weight factors like the one in (4.1), this is only possible for certain combinations of superfields and derivatives. As we shall see below, a proper covariantization of (2.14) does exist.

We shall first present the result and then explain how one can check the invariance of the action. The locally superconformal invariant version of (2.14) is

$$S_{TM}^{cov} = \int d^4x d^8\theta du \varepsilon^{\alpha\beta\gamma\delta} \left(\partial^{++} + \frac{2}{3} \Gamma^{++} \right) F_{\alpha}^{(-3)}. \quad (4.2)$$

$$\cdot \left[\partial^+_{\beta} \partial^+_{\gamma} \partial^+_{\delta} V^{(-2)} + 2 \Gamma^{(+2)}_{\beta\gamma} \partial^+_{\delta} V^{(-2)} + \partial^+_{\beta} \Gamma^{(+2)}_{\gamma\delta} V^{(-2)} \right].$$

Here Γ^{++} is defined as follows

$$\Gamma^{++} = \partial^- H^{(4)} + \partial_m H^{++m} - \partial_\mu H^{++\mu} \quad (4.3)$$

and has the transformation law

$$\delta \Gamma^{++} = \mathcal{D}^{++} \Lambda + 2 \lambda^{++} \quad (4.4)$$

(Λ is defined in (3.15)). The tensor multiplet prepotentials $F^{(-3)}_\alpha$ and $V^{(-2)}$ by definition transform in the following way

$$\delta F^{(-3)}_\alpha = -\partial_\alpha^+ \lambda^{\beta-} F_\beta^{(-3)} + \frac{4}{3} (\partial_\rho^+ \lambda^{\rho-}) F_\alpha^{(-3)} - \frac{2}{3} \Lambda F_\alpha^{(-3)}, \quad (4.5a)$$

$$\delta V^{(-2)} = \frac{2}{3} (\partial_\alpha^+ \lambda^{\alpha-}) V^{(-2)} - \frac{1}{3} \Lambda V^{(-2)}. \quad (4.5b)$$

The "semi" covariant derivative $\mathcal{D}^+_\alpha$ acting on a superfield with one or more spinor indices is given by

$$\mathcal{D}^+_\alpha \phi_{\beta\dots} = \partial_\alpha^+ \phi_{\beta\dots} + \Omega_{\alpha\beta}^{+\gamma} \phi_{\gamma\dots} + \dots \quad (4.6)$$

Here $\Omega^{+\gamma}_{\alpha\beta}$ is a connection with the following algebraic properties

$$\Omega_{\alpha\beta}^{+\gamma} = -\Omega_{\beta\alpha}^{+\gamma}, \quad \Omega_{\alpha\beta}^{+\beta} = 0 \quad (4.7)$$

and with the transformation law

$$\begin{aligned} \delta \Omega_{\alpha\beta}^{+\gamma} = & -\partial_\alpha^+ \lambda^{\rho-} \Omega_{\rho\beta}^{+\gamma} - \partial_\beta^+ \lambda^{\rho-} \Omega_{\alpha\rho}^{+\gamma} + \Omega_{\alpha\beta}^{+\rho} \partial_\rho^+ \lambda^{\gamma-} \\ & + \partial_\alpha^+ \partial_\beta^+ \lambda^{\gamma-} - \frac{1}{3} \partial_\alpha^+ (\partial_\rho^+ \lambda^{\rho-}) \delta_\beta^\gamma + \frac{1}{3} \partial_\beta^+ (\partial_\rho^+ \lambda^{\rho-}) \delta_\alpha^\gamma. \end{aligned} \quad (4.8)$$

This connection can be expressed in terms of the derivatives of the vielbeins H^{--m} and $H^{--\mu+}$ of $\mathcal{D}^{--}$ (3.9), as explained in Appendix B. Finally, the quantity $\Gamma^{(+2)}_{\alpha\beta}$ is defined by

$$\Gamma^{(+2)}_{\alpha\beta} = \frac{2}{5} \left(-\partial_{\mu}^{+} \Omega_{\alpha\beta}^{+\mu} + \Omega_{\alpha\mu}^{+\mu} \Omega_{\beta\mu}^{+\mu} \right) = -\Gamma_{\beta\alpha}^{(+2)}. \quad (4.9)$$

Now we give some hints how to check the invariance of the action integral (4.2). The transformation law of the harmonic connection Ω (4.4) follows from (3.1), (3.5), (3.15) and the identity $[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0$ (where $\mathcal{D}^0$ is the U(1) charge operator). Using the gauge (3.6b) for the parameter $\lambda^{\alpha-}$ and the analyticity property (3.4) one can show that

$$\mathcal{D}^{++} \partial_{\alpha}^{+} \lambda^{\beta-} = -\delta_{\alpha}^{\beta} \lambda^{++}. \quad (4.10)$$

Then it is easy to see that

$$\delta \left(\mathcal{D}^{++} + \frac{2}{3} \Gamma^{++} \right) F_{\alpha}^{(-3)} = \quad (4.11)$$

$$= \left[-\partial_{\alpha}^{+} \lambda^{\beta-} + \left(\frac{4}{3} (\partial_{\beta}^{+} \lambda^{\beta-}) - \frac{2}{3} \Lambda \right) \delta_{\alpha}^{\beta} \right] \left(\mathcal{D}^{++} + \frac{2}{3} \Gamma^{++} \right) F_{\alpha}^{(-3)}.$$

Further, using (4.5b), the analyticity of Λ ($\mathcal{D}^{+}_{\alpha} \Lambda = 0$), (4.8) and (4.9) and introducing the notation

$$V^{\alpha+} = \varepsilon^{\alpha\beta\gamma\delta} \left(\partial_{\beta}^{+} \partial_{\mu}^{+} \partial_{\delta}^{+} V^{(-2)} + 2 \Gamma_{\beta\mu}^{(+2)} \partial_{\delta}^{+} V^{(-2)} + \partial_{\beta}^{+} \Gamma_{\mu\delta}^{(+2)} V^{(-2)} \right) \quad (4.12)$$

one can verify that

$$\delta V^{\alpha+} = \left(\partial_{\beta}^{+} \lambda^{\alpha-} - \frac{1}{3} \partial_{\beta}^{+} \lambda^{\beta-} \delta_{\beta}^{\alpha} - \frac{1}{3} \Lambda \delta_{\beta}^{\alpha} \right) V^{\beta+} \quad (4.13)$$

(all the inhomogeneous terms cancel due to the presence of connection terms in (4.12)). Putting together (4.11) and (4.13) one sees that the integrand in (4.2) produces the overall weight factor $\partial^+_{\alpha} \lambda^{\alpha-} - \Lambda$ which cancels out the variation of the volume element.

It is worth remarking that $v^{\alpha+}$ defined in (4.12) is the covariantization of the flat superspace expression (2.13a). Indeed, using the fact that $\Gamma^{(+2)}_{\alpha\beta}$ is related to the anticommutator of two spinor derivatives,

$$\{\partial_{\alpha}^+, \partial_{\beta}^+\} \phi_{\mu} = R_{\alpha\beta\mu}^{(+2)\delta} \phi_{\delta}, \quad \Gamma_{\alpha\beta}^{(+2)} = \frac{2}{3} R_{\mu\alpha\beta}^{(+2)\mu}, \quad (4.14)$$

one can verify that $v^{\alpha+}$ (4.12) satisfies the covariant version of the constraint (2.7a):

$$\partial_{\alpha}^+ V^{\beta+} - \frac{1}{4} \delta_{\alpha}^{\beta} \partial_{\mu}^+ V^{\mu+} = 0, \quad (4.15)$$

$$\partial_{\alpha}^+ V^{\beta+} = \partial_{\alpha}^+ V^{\beta+} - \Omega_{\alpha\mu}^{+\beta} V^{\mu+}.$$

Further, the covariantization of the equation of motion (2.7b) is easily derived from the action (4.2) by variation of $F^{(-3)}_{\beta}$ (taking into account (4.3)):

$$\left(\partial^{++} + \frac{1}{3} \Gamma^{++} \right) V^{\alpha+} = 0. \quad (4.16)$$

Similar expressions can be obtained for the covariantization of the constraints (2.11) for the alternative description of the tensor multiplet on shell.

To summarize this section, we have succeeded in writing down the **off-shell** action for the tensor multiplet in the background of conformal

supergravity. Together with the **off-shell** action for the scalar multiplet (3.14) they constitute the action for **off-shell** Poincaré' supergravity plus a tensor multiplet. Of course, instead of the scalar multiplet it is possible to use other matter multiplets as compensators for conformal supergravity and thus to obtain different versions of Poincaré' supergravity. The main question now is: what happened with the doubling of the tensor multiplet in (2.14) discussed at the end of Section 2?

5. Conformal supergravity as a Lagrange multiplier for the tensor multiplet

In order to answer the above question one should carry out a detailed analysis of the component content of the combined action (3.14), (4.2). The hardest part of that exercise is to solve the equation for H^{--} following from the defining relation (3.7). In the Wess-Zumino gauge (3.6) it is a straightforward but rather tedious calculation. Once that has been achieved, the connection Ω can be computed and inserted in the action (4.2). Our aim here is to just indicate how the problem of doubling the tensor multiplet content is resolved. For this purpose it is sufficient to consider a simplified situation with only a few fields involved:

$$\begin{aligned} F_{\alpha}^{(-3)} &= (\theta^{-})_{\alpha}^3 \omega + (\theta^{-})^4 \theta^{\beta+} A_{\beta\alpha}, \\ V^{(-2)} &= (\theta^{-})_{\alpha}^3 \theta^{\alpha+} \sigma + (\theta^{-})^4 (\theta^{+})^{2\alpha\beta} a_{\alpha\beta}, \\ H^{(+4)} &= (\theta^{+})^4 D, \quad H^{++m} = (\theta^{+})^{2\alpha\beta} \gamma_{\alpha\beta}^m, \quad H^{++m+} = 0. \end{aligned} \quad (5.1)$$

Here all the fields are assumed to be functions of x only (the harmonic dependence has been eliminated using part of the equations of motion, see Section 2); the sexbein $e^m_{\alpha\beta}$ is reduced to its flat space value $\gamma^m_{\alpha\beta}$. Now it becomes quite easy to work out the connection

$$\Omega^{+\mu}_{\alpha\beta} = \theta^{\mu-} (\theta^+)^2_{\alpha\beta} D + (\theta^-)^2_{\alpha\beta} (\theta^+)^3_{\delta} \partial^{\delta\mu} D - \text{trace}. \quad (5.2)$$

The term with a derivative of D has too high a dimension to appear anywhere in the action (4.2), so we shall ignore it. Inserting (5.2) into (4.12) one finds (up to insignificant coefficients):

$$\begin{aligned} V^{\alpha+} = & \varepsilon^{\alpha\beta\mu\delta} [\varepsilon_{\beta\mu\delta\gamma} \theta^{\gamma+} \sigma + \varepsilon_{\beta\mu\delta\gamma} \theta^{\gamma-} (\theta^+)^2_{\alpha\beta} a_{\alpha\gamma} + \\ & + (\theta^-)^2_{\beta\mu} (\theta^+)^3_{\delta} D \sigma + (\theta^-)^3_{\beta} (\theta^+)^4_{\delta} D a_{\mu\delta}]. \end{aligned} \quad (5.3)$$

The other part of the Lagrangian is

$$\begin{aligned} (\partial^{++} + \frac{2}{3} \Gamma^{++}) F^{(-3)}_{\alpha} = & (\theta^-)^2_{\alpha\beta} \theta^{\beta+} \omega + (\theta^-)^3_{\beta} (\theta^+)^2_{\alpha\beta} A_{\mu\alpha} + \\ & + (\theta^-)^3_{\alpha} (\theta^+)^2_{\beta\mu} \partial_{\beta\mu} \omega + (\theta^-)^4_{\alpha} (\theta^+)^3_{\mu} \partial^{\mu\beta} A_{\beta\alpha}. \end{aligned} \quad (5.4)$$

Combining (5.3) and (5.4) and doing the harmonic (trivial in this case) and θ integration, and adding to all this the matter sector (3.20), one obtains the final form of the action:

$$S_{TM+SM}^{cov} = \int d^6 x [\sigma \partial^m A_m + a^m (A_m + \partial_m \omega) + D(\sigma \omega - 1)] \quad (5.5)$$

The corresponding equations of motion are very simple:

$$\partial^m A_m + D \omega = 0, \quad A_m + \partial_m \omega = 0, \quad (5.6a)$$

$$\partial_m a^m - D \sigma = 0, \quad a^m - \partial^m \sigma = 0, \quad (5.6b)$$

$$\sigma \omega = 1. \quad (5.6c)$$

From (5.6a) and (5.6b) one derives two expressions for D:

$$D = \omega^{-1} \square \omega = \sigma^{-1} \square \sigma, \quad (5.7)$$

whereas (5.6c) identifies the two scalar fields:

$$\omega = \sigma^{-1}. \quad (5.8)$$

We see that in addition to the mechanism discovered in [1], which makes it possible to express on shell part of the Weyl multiplet (namely the fields D , χ , T , see (5.7) and (3.22)) in terms of the tensor multiplet components, another "miracle" takes place. When the full off-shell action is considered, the same Weyl multiplet fields (D , χ , T) serve as Lagrange multipliers which allow to identify the fields from (2.5) with those from (2.1) on shell: ω with σ , $F_{\alpha\beta}$ with the curl of $B_{\alpha\beta}$, $\lambda_{\alpha i}$ with $\psi_{\alpha i}$. The conclusion is: there exists a **new mechanism for constructing off-shell theories**, in which supergravity plays an active role. Indeed, in flat space it is not possible to have an off-shell action for a single tensor multiplet, whereas in curved space this becomes a reality. One can imagine other possible applications of this mechanism, e.g., in 10-dimensional supergravity which in many respects resembles the 6-dimensional one, or in $N=4$ $d=4$ super-Yang-Mills theory, where the introduction of conformal supergravity may provide a natural

source for the specific self-duality constraint of that theory. Those possibilities are presently under investigation.

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Appendix A. Notation and conventions

We use six dimensional spinor notation similar to that of Ref. [10]. For example, the Grassmann coordinate $\theta^{\alpha i}$ has a left-handed spinor index (denoted by α upstairs) and an $SU(2)$ index. The corresponding spinor derivative $\partial_{\alpha i} = \partial / \partial \theta^{\alpha i}$ has a right-handed spinor index (α downstairs). Besides the Weyl condition, the spinors satisfy the pseudo-Majorana condition

$$\overline{\theta^{\alpha i}} \equiv \bar{\theta}^{\dot{\alpha}}_{\dot{i}} = C^{\dot{\alpha}}_{\beta} \varepsilon_{ij} \theta^{\beta j}, \quad (\text{A.1})$$

where C is unitary and satisfies $\bar{C}C = -1$. When the harmonic variables u^i are added, $\theta^{\alpha i}$ splits up into two $U(1)$ charged projections $\theta^{\alpha \pm} = \theta^{\alpha i} u^{\pm}_i$. The irreducible products of, e.g., $\theta^{\alpha +}$ are:

$$1, \theta^{\alpha+}, (\theta^+)^2{}^{[\alpha\beta]} = \frac{1}{2} \theta^{\alpha+} \theta^{\beta+}, \quad (\text{A.2})$$

$$(\theta^+)^3_{\alpha} = \frac{1}{3!} \theta^{\beta+} \theta^{\gamma+} \theta^{\delta+} \varepsilon_{\alpha\beta\gamma\delta}, \quad (\theta^+)^4 = \frac{1}{4!} \theta^{\alpha+} \theta^{\beta+} \theta^{\gamma+} \theta^{\delta+} \varepsilon_{\alpha\beta\gamma\delta}.$$

A couple of antisymmetric indices can be raised or lowered with the help of $\varepsilon^{\alpha\beta\gamma\delta}$:

$$(\theta^+)^2_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} (\theta^+)^2{}^{\gamma\delta}, \quad (\theta^+)^2{}^{\alpha\beta} = -\frac{1}{4} \varepsilon^{\alpha\beta\gamma\delta} (\theta^+)^2{}_{\gamma\delta}. \quad (\text{A.3})$$

Vectors and antisymmetric tensors are represented by bispinors:

$$\partial_a \rightarrow \partial_{[\alpha\beta]}, \quad B_{[ab]} \rightarrow B_{\alpha}{}^{\beta} \quad (B_{\alpha}{}^{\alpha} = 0), \quad (\text{A.4})$$

$$F_{[abc]} \pm \varepsilon^{abcdef} F_{def} = (F_{(\alpha\beta)}, F^{(\alpha\beta)}).$$

Appendix B. Constructing the connection $\Omega^+{}_{\alpha\beta}$

The construction makes use of two basic ingredients first introduced in [6]:

$$e_{\alpha}{}^{\mu} = \partial_{\alpha}^+ H^{-\mu+}, \quad (\text{B.1})$$

$$e_{\alpha\beta}^m = \partial_\alpha^+ \partial_\beta^+ H^{--m} - \partial_\alpha^+ e_\beta^\mu \cdot e_\mu^{-1\nu} \partial_\nu^+ H^{--m}, \quad (\text{B.2})$$

where H^{--m} and $H^{--\mu+}$ are vielbeins from the covariant derivative (3.9). They have rather regular transformation properties (see [6]):

$$\begin{aligned} \delta e_\alpha^\mu = & -\partial_\alpha^+ \lambda^{\nu-} e_\nu^\mu + e_\alpha^\nu \partial_\nu^- \lambda^{\mu+} + \partial_\alpha^+ H^{--m} \partial_m \lambda^{\mu+} - \\ & - (\partial^- \lambda^{++}) e_\alpha^\mu - (\partial_\alpha^+ H^{--m} \partial_m \lambda^{++} + e_\alpha^\beta \partial_\beta^- \lambda^{++}) H^{--\mu+}, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \delta e_{\alpha\beta}^m = & e_{\alpha\beta}^n (\partial_n \lambda^m - \partial_n \lambda^{\mu+} e_\mu^{-1\nu} \partial_\nu^+ H^{--m}) - \\ & - \partial_\alpha^+ \lambda^{\mu-} e_{\mu\beta}^m - \partial_\beta^+ \lambda^{\mu-} e_{\alpha\mu}^m - (\partial^- \lambda^{++}) e_{\alpha\beta}^m - \\ & - e_{\alpha\beta}^n [\partial_n \lambda^{++} (H^{--m} - H^{--\mu+} e_\mu^{-1\nu} \partial_\nu^+ H^{--m})]. \end{aligned} \quad (\text{B.4})$$

Out of them one can form the following combination:

$$\begin{aligned} \omega_{\alpha\beta}^{+\mu} = & \partial_\alpha^+ e_{\beta\gamma}^m \cdot e_m^{\beta\mu} - \partial_\alpha^+ e_\beta^\mu \cdot e_\mu^{-1\beta} + \partial_\beta^+ e_\alpha^\mu \cdot e_\mu^{-1\beta} \delta_\alpha^\mu = \\ = & (\partial_\alpha^+ \partial_\beta^+ \partial_\gamma^+ H^{--m} - \partial_\alpha^+ \partial_\beta^+ \partial_\gamma^+ H^{--\mu+} e_\mu^{-1\nu} \partial_\nu^+ H^{--m}) e_m^{\beta\mu}, \end{aligned} \quad (\text{B.5})$$

where $e_{\alpha\beta}^m e_m^{\gamma\delta} = \delta_{\alpha\beta}^{\gamma\delta}$. Using (B.3) and (B.4) one can check that

$$\begin{aligned} \delta \omega_{\alpha\beta}^{+\mu} = & \partial_\beta^+ \partial_\alpha^+ \lambda^{\mu-} - \partial_\alpha^+ (\partial_\beta^+ \lambda^{\mu-}) \delta_\beta^\mu + \partial_\beta^+ (\partial_\alpha^+ \lambda^{\mu-}) \delta_\alpha^\mu - \\ & - 2 \partial_\alpha^+ (\partial^- \lambda^{++}) \delta_\beta^\mu + 2 \partial_\beta^+ (\partial^- \lambda^{++}) \delta_\alpha^\mu + \dots, \end{aligned} \quad (\text{B.6})$$

where the dots denote homogeneous terms like in (4.8). Then it is clear that $\Omega^{+\mu}_{\alpha\beta}$ with the desired transformation law (4.8) is given by

$$\Omega_{\alpha\beta}^{+\mu} = -\omega_{\alpha\beta}^{+\mu} + \frac{1}{3} \omega_{\alpha\gamma}^{+\mu} \delta_\beta^\gamma - \frac{1}{3} \omega_{\beta\gamma}^{+\mu} \delta_\alpha^\gamma. \quad (\text{B.7})$$

The algebraic properties (4.7) follow from (B.5) and (B.7). The connection obtained has another important property in the gauge (3.6b) for $H^{++\mu-}$:

$$\mathcal{D}^{++} \Omega_{\alpha\beta}^{+\mu} = 0. \quad (\text{B.8})$$

The easiest way to see this is to realize that given the condition (3.6b) on $\lambda^{\mu-}$ the l.h.s. of (B.8) transforms homogeneously (see (4.10)), but there are no such components in the Wess-Zumino gauge (charge +3), so it must vanish. Another form of (B.8) is

$$[\mathcal{D}^{++}, \mathcal{D}_\alpha^+] = 0 \quad (\text{B.9})$$

for $\mathcal{D}_\alpha^+$ from (4.6).

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