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NEW REALIZATIONS OF THE VIRASORO ALGEBRA AS MEMBRANE SYMMETRIES

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ABSTRACT

We construct new realizations of the Virasoro algebra as subalgebras of area preserving diffeomorphisms of surfaces, which are the membrane symmetries. For the membrane torus, the $\text{SDiff}(T^2)$ algebra has a central extension of unique form and the corresponding Virasoro subalgebras are in general realized as non-trivial central extensions. These central extensions should be relevant to the question whether quantum string excitations may be contained in the membrane spectrum.

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Various motivations have recently raised interest in the possibility of constructing consistent relativistic quantum theories of extended objects other than strings [1,2]. In this work we study the quantum symmetries of two-dimensional extended objects, the closed bosonic membranes [3-9], focusing our attention to the reparametrization symmetries of the torus membrane.

If, as discussed e.g. in [10,11], after some membrane compactification the string excitations are contained in the torus membrane spectrum, it should be possible to find the quantum Virasoro algebra as a subalgebra of the torus symmetry, which in the light cone gauge is the group $\text{SDiff}(T^2)$ of the area preserving diffeomorphisms of the torus. The algebra of $\text{SDiff}(T^2)$ has been studied in [12] and the most general central extension of it has been determined. Also a particular Virasoro subalgebra has been found, but one with a trivial (i.e. absorbable in the redefinition of generators) central extension. A primary aim of the present work is to determine more general Virasoro subalgebras of $\text{SDiff}(T^2)$ with non-trivial central extensions.

The area preserving diffeomorphisms of the torus can be generated by infinitesimal transformations as well as by other elements not continuously connected to the identity, e.g. elements of the modular group $\text{SL}(2, \mathbb{Z})$ of 2×2 integer matrices of unit determinant. The identity component will be denoted by $\text{SDiff}_0(T^2)$. It consists of transformations of the local torus coordinates $(\sigma_1, \sigma_2) \in (-\pi, \pi) \times (-\pi, \pi)$

$$\sigma_1 \rightarrow \sigma_1(\sigma_1, \sigma_2) \quad , \quad \sigma_2 \rightarrow \sigma_2(\sigma_1, \sigma_2) \quad (1)$$

$$\det \frac{\partial(\sigma_1, \sigma_2)}{\partial(\sigma_1, \sigma_2)} = 1 \quad (2)$$

constructed from infinitesimal ones

$$\sigma_1 \rightarrow \sigma_1 + u_1(\sigma_1, \sigma_2) \quad , \quad \sigma_2 \rightarrow \sigma_2 + u_2(\sigma_1, \sigma_2) \quad (3)$$

$$\frac{\partial u_1}{\partial \sigma_1} + \frac{\partial u_2}{\partial \sigma_2} = 0 \quad (4)$$

The generators

$$L = u_1 \frac{\partial}{\partial \sigma_1} + u_2 \frac{\partial}{\partial \sigma_2}$$

are of the form

$$L_h = \frac{\partial h}{\partial \sigma_1} \frac{\partial}{\partial \sigma_2} - \frac{\partial h}{\partial \sigma_2} \frac{\partial}{\partial \sigma_1} \quad (5a)$$

$$P_{1,2} = -i \frac{\partial}{\partial \sigma_{1,2}} \quad (5b)$$

where h is a differentiable function on T^2 . If we choose as a basis on the torus

$$h_n = \exp i (n_1 \sigma_1 + n_2 \sigma_2) \quad , \quad L_n \equiv -L_{h_n}, \quad n = (n_1, n_2) \quad (6)$$

we find the explicit form of the $S\text{Diff}_0(T^2)$ algebra (the L_n part has been considered in [13])

$$[L_n, L_m] = n \times m L_{n+m} \quad (7a)$$

$$[P_{1,2}, L_n] = n_{1,2} L_n \quad (7b)$$

$$[P_1, P_2] = 0 \quad (7c)$$

where $n \times m = n_1 m_2 - n_2 m_1$.

It was shown in [12] that this algebra admits a unique central extension of the form :

$$[L_n, L_m] = n \times m L_{n+m} + a n \delta_{n,-m} \quad (8)$$

$a = (a_1, a_2) \in \mathbb{R}^2$, with the other relations (7b-c) unchanged. The last term in (8) signals a possibility of quantum anomalies. We know of no rigorous argument for the absence of diffeomorphism or global anomalies for the closed bosonic torus membrane. We note, in this context, that the "classical $a=0$ part" of (8) has modular invariant structure constants, i.e. for $A \in \text{SL}(2, \mathbb{Z})$

$$[L_{An}, L_{Am}] = n \times m L_{A(n+m)} + a \cdot (An) \delta_{An, -Am} \quad (9)$$

while the central extension coefficient is not invariant and it may indicate a particular role of the modular group.

Ref.[12] also raised the question of the existence of Virasoro subalgebras of the $a=0$ algebra (7a-c), i.e. linear combinations of the L_n 's

$$V_N = \sum_{n=0} c_N^n L_n, \quad N = 0, \pm 1, \pm 2, \dots \quad (10)$$

which satisfy the Virasoro algebra [14]

$$[V_N, V_{N'}] = (N - N') V_{N+N'} \quad N, N' \in \mathbb{Z} \quad (11)$$

(linear combinations containing P_1, P_2 are not admitted by (11)). This is equivalent to the following identity among the constants c_N^n

$$\sum_{n=0} c_N^n c_{N'}^{n-N} (n \times N) = (N - N') c_{N+N'}^N \quad (N \neq 0) \quad (12)$$

In [12] the particular form

$$c_N^{n_1 n_2} = \frac{(-1)^{n_1}}{n_1} \delta_{n_1, N} \quad n_1 \neq 0 \quad (13a)$$

$$c_N^{n_1 n_2} = 0 \quad n_1 = 0 \quad (13b)$$

was found to satisfy (12). (Notice that in ref. [12] the Fourier series (10) was employed for $\sigma_1 \in (0, 2\pi)$, and in order to 'transport' it to the range $\sigma_1 \in (-\pi, \pi)$ of the present work we have to multiply the coefficients c_N^n of [12] by a factor $\exp(in_1 \pi) = (-1)^{n_1}$).

Once a set of c_N^n 's satisfying (12) is available, it is very interesting to notice that, if the quantum algebra of $\text{SDiff}_0(T^2)$ is actually realized as a central extension (8) with $a \neq 0$, then the linear combinations (10) satisfy a central extension of the Virasoro algebra (11)

$$[V_N, V_{N'}] = (N - N') V_{N+N'} + C_{N, N'} \quad (14)$$

with

$$C_{N,N'} = \sum_{n \neq 0} c_N^n c_{N'}^{-n} a \cdot n \quad (15)$$

It is further easy to show that, since the central extension (8) is consistent with the Jacobi identity for the L_n 's

$$[L_n, [L_m, L_k]] + \text{cyclic perm.} = 0 \quad (16)$$

the $C_{N,N'}$ of (14) and (15) are consistent with the Jacobi identity for the V_N 's

$$[V_N, [V_L, V_M]] + \text{cyclic perm.} = 0 \quad (17)$$

This in turn implies that by an appropriate redefinition of V_N 's the $C_{N,N'}$ are necessarily of the form [15]

$$C_{N,N'} = (\gamma N + \beta N^3) \delta_{N,-N'} \quad (18)$$

The known Virasoro anomaly for the string is characterized by the appearance of an N^3 -term, while an N -term can be absorbed in a redefinition of the V_0 . It is somewhat disappointing that the c_N^n 's of (13) give only such a trivial central extension of (11), i.e. one with $\beta=0$ in (18).

So we now go back to the system (11) or (12) and attempt to determine its general solution, with the hope that there exist c_N^n 's which lead to $C_{N,N'}$'s of the form (18) with $\beta \neq 0$.

The problem of determining V_N is equivalent to that of finding a set of functions

$$f_N = \sum_{n \neq 0} c_N^n h_n = \sum_{n \neq 0} c_N^n \exp(in \cdot \sigma) \quad (19)$$

such that the corresponding V_N

$$V_N = \frac{\partial f_N}{\partial \sigma_2} \frac{\partial}{\partial \sigma_1} - \frac{\partial f_N}{\partial \sigma_1} \frac{\partial}{\partial \sigma_2} \equiv -L_f = L_{-f} \quad (20)$$

realize the requirement (11) in the form

$$[L_{-f_N}, L_{-f_{N'}}] = (N-N') L_{-f_{N+N'}} = L_{(N'-N)f_{N+N'}} \quad (21)$$

But in general

$$[L_{-f_N}, L_{-f_{N'}}] = L_{\{f_N, f_{N'}\}} \quad (22)$$

with the Poisson bracket defined as

$$\{f, g\} = \frac{\partial f}{\partial \sigma_1} \frac{\partial g}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_2} \frac{\partial g}{\partial \sigma_1} \quad (23)$$

so that (21) entails

$$\{f_N, f_{N'}\} = (N' - N)f_{N+N'} + A_{N, N'} \quad (24)$$

where the constants $A_{N, N'}$ must be antisymmetric in N, N' and consistent with the Jacobi identity for the Poisson bracket. These conditions imply the form

$$A_{N, N'} = (AN + BN^3) \delta_{N, -N'} \quad (25)$$

i.e. $A_{N, N'}$ vanishes if $N \neq -N'$ or $N=0$ or $N'=0$.

While relations (12) and (15) include the information that we are on the torus, relations (24) apply locally to all surfaces. The kind of the surface restricts of course the class of the allowed functions f_N .

We take $A_{N, N'} = 0$ in the subsequent analysis, simply in order to simplify the presentation. The same analysis done with a general $A_{N, N'}$ of the form (25) actually forces at some stage $A_{N, N'} = 0$ through consistency requirements.

The determination of c_N^n 's satisfying (12) has now been transformed to the problem of solving the infinite system of differential equations

$$\frac{\partial f_N}{\partial \sigma_1} \frac{\partial f_{N'}}{\partial \sigma_2} - \frac{\partial f_{N'}}{\partial \sigma_1} \frac{\partial f_N}{\partial \sigma_2} = (N' - N)f_{N+N'} \quad (26)$$

locally and then extending or interpreting the solutions as functions on the torus. Notice from (19) that f_N may be complex functions of the real arguments σ_1, σ_2 .

We start the construction with an arbitrary given function $f_0(\sigma_1, \sigma_2)$ and determine first the general form of the remaining f_N 's ($N \neq 0$) so that they satisfy the $N'=0$ equation

$$\frac{\partial f_0}{\partial \sigma_1} \frac{\partial f_N}{\partial \sigma_2} - \frac{\partial f_0}{\partial \sigma_2} \frac{\partial f_N}{\partial \sigma_1} = N f_N \quad (27)$$

This can be done by the method of characteristics. Introducing two functions $s(\sigma_1, \sigma_2)$, $t(\sigma_1, \sigma_2)$ which define a nonsingular coordinate change from σ_1, σ_2 to s, t and are such that the inverse functions $\sigma_1(s, t)$, $\sigma_2(s, t)$ satisfy

$$\frac{\partial \sigma_1}{\partial t} = - \frac{\partial f_0}{\partial \sigma_2}, \quad \frac{\partial \sigma_2}{\partial t} = \frac{\partial f_0}{\partial \sigma_1} \quad (28)$$

we find

$$f_N(\sigma_1, \sigma_2) = g_N(s(\sigma_1, \sigma_2)) e^{Nt(\sigma_1, \sigma_2)} \quad (N \neq 0) \quad (29)$$

where the functions g_N ($N \neq 0$) are still arbitrary. It is straightforward to show that (28) imply and are implied by

$$\{f_0, t\} = 1, \quad \frac{\partial f_0}{\partial t} = 0 \quad (30a, b)$$

so that f_0 is a function only of s , and f_0, t define locally canonical coordinates, i.e. the mapping $\sigma_1, \sigma_2 \rightarrow f_0, t$ is in $SDiff(T^2)$. It is clear that the freedom of choosing f_0 is equivalent to the freedom of choosing s , and from now on we identify

$$f_0(\sigma_1, \sigma_2) \equiv g_0(s) \equiv s \quad (31)$$

Notice also that conditions (28), (30a, b) remain satisfied if we replace simultaneously $s \rightarrow s \equiv \frac{s}{i}$, $t \rightarrow t \equiv it$.

As a next step we demand expressions (29) to satisfy eq.(26) for both integers

$N, N' \neq 0$. This gives $(g_N \equiv \frac{dg_N}{ds})$

$$Ng_N g_{N'} - N' g_{N'} g_N = (N - N') g_{N+N'} \quad (32)$$

Amazingly, the infinite system (32) admits an almost unique solution. We observe first that (32) are invariant under the transformation

$$g_N \rightarrow h^N g_N \quad (33)$$

where $h(s)$ is any differentiable function. This invariance permits to fix g_1

$$g_1 = 1 \quad (34)$$

We next notice that for $N' = -N$ (32) lead to the simple relation

$$g_N g_{-N} = s^2 + c_N, \quad N \neq 0, \quad c_N = c_{-N} \quad (35)$$

Combining this with (34) we have

$$g_{-1}(s) = s^2 + c_1 \quad (36)$$

We construct now the solution inductively. To determine g_2, g_{-2} we solve eq.(32) for $N=-2, N'=1$ and $N=2, N'=-1$, i.e.

$$g_{-2} = 3(s^2 + c_1)$$

$$(s^2 + c_1)g_2 + 4sg_2 = 3$$

to get

$$g_{-2} = s^3 + 3c_1 s + a_{-2} \quad (37a)$$

$$g_2 = \frac{s^3 + 3c_1 s + a_2}{(s^2 + c_1)^2} \quad (37b)$$

But g_2, g_{-2} must satisfy (35), which imposes relations among a_2, a_{-2}, c_1 and

c_2 . We find

$$a_2 + a_{-2} = 0 \quad , \quad a_2^2 = -4c_1^3 \quad (38a,b)$$

$$c_2 = 4c_1 \quad (38c)$$

Introducing the constant c by $c_1 = -c^2$, we find $c_2 = -(2c)^2$, $s^2 + c_1 = (s+c)(s-c)$ and

$$g_2 = \frac{s+2c}{(s+c)^2} \quad , \quad g_{-2} = (s-2c)(s+c)^2 \quad (39)$$

Relations (32) give now recursion relations for $N \geq 2$ and $N \leq -2$. Indeed, for $N' = 1$, $N \rightarrow N-1$ and $N' = -1$, $N \rightarrow -N$ we obtain respectively

$$g_N = \frac{1}{2-N} g_{N-1} \quad (40)$$

$$g_{-N-1} = \frac{1}{N-1} [2Nsg_{-N} - (s^2 - c^2) g_{-N}] \quad (41)$$

(These relations depend on the choice (34) for g_1). We notice that g_0 , $g_{\pm 1}$, $g_{\pm 2}$ are of the form

$$g_N = \frac{s+Nc}{(s+c)^N} \quad (42)$$

Proceeding inductively, we then assume that for some positive integer K the solutions $N = K-1$ and $N = -K$ are of the form (42) and we prove from relations (40) and (41) that this is also true for the $N = K$ and $N = -K-1$ solutions. Finally, by the invariance (33), we may choose

$$g_N = s+Nc = g_0+Nc \quad (43)$$

So the general local solution of (26) is

$$f_N(\sigma_1, \sigma_2) = [f_0(\sigma_1, \sigma_2) + Nc] e^{Nt(\sigma_1, \sigma_2)} \quad (44)$$

with f_0 and t arbitrary apart from the restriction

$$\{f_0, t\} = 1 \quad (45)$$

and there is a remaining arbitrariness (recall (33))

$$f_N(\sigma_1, \sigma_2) \rightarrow h^N(f_0(\sigma_1, \sigma_2)) f_N(\sigma_1, \sigma_2) \quad (46)$$

Recall also that we can everywhere replace simultaneously $f_0 \rightarrow -i f_0$, $t \rightarrow it$. The solution of ref. [12] corresponds to

$$c = 0 \quad (47a)$$

$$f_0 = \frac{1}{i} \quad \sigma_1 = \sum_{n_1 \neq 0} \frac{(-1)^{n_1}}{n_1} e^{in_1 \sigma_1} \quad (47b)$$

$$t = i\sigma_2 \quad (47c)$$

and satisfies the global eq.(12) on the torus with $c_N^{n'}$'s given in (13a,b), but leads to a trivial central extension without a N^3 -term.

To show how the local solutions (44), (45) can be extended on the torus, we choose the simplest example

$$f_N = \left(\frac{\sigma_1}{i} + Nc \right) e^{iN\sigma_1} \quad (48)$$

To proceed we have to define a periodic extension of σ_1 from say $(0, \pi)$ to $[-\pi, \pi]$ with period 2π . From experience of ref. [12], we adopt the straightforward odd extension [17], given by (47b). Differentiation of σ_1 will then induce δ -functions at $\sigma_1 = \pm\pi$, which have to be compensated so that the f_N 's are generalized solutions of (26) even at these points. This is consistently ensured by an odd extension of (48), defined by $g_N(-\sigma_1) = -g_N(\sigma_1)$ and $g_N(\sigma_1) = g_N(\sigma_1 + 2\pi)$ (so that $g_N(-\pi) = g_N(0) = g_N(\pi) = 0$), which requires the "constant" term Nc to be similarly extended from $(0, \pi) \rightarrow [-\pi, \pi]$ as an odd function [16]

$$(Nc)_{\text{odd}}(\sigma_1) = \begin{array}{ll} Nc & \sigma_1 \in (0, \pi) \\ -Nc & \sigma_1 \in (-\pi, 0) \end{array} \quad (49)$$

or

$$(Nc)_{\text{odd}} = - \frac{iNc}{\pi} \sum_{n_1 \neq 0} \frac{1-(-1)^{n_1}}{n_1} e^{in_1 \cdot \sigma_1} \quad (50)$$

So finally the extension of (48) on the torus is

$$f_N(\sigma_1, \sigma_2) = \sum_n c_N^{n_1, n_2} \exp(i(n_1 \sigma_1 + n_2 \sigma_2)) \quad (51)$$

with

$$\begin{aligned} c_N^{n_1, n_2} &= \left[\frac{(-1)^{n_1}}{n_1} - \frac{iNc}{\pi} \frac{1-(-1)^{n_1}}{n_1} \right] \delta_{n_1, N} \quad , \quad n_1 \neq 0 \\ &= 0 \quad , \quad n_1 = 0 \end{aligned} \quad (52)$$

It is not difficult to verify that (52) satisfy eq.(12). In checking the last statement one must use the regularization of logarithmically divergent sums applied in ref. [12], i.e. symmetric summation

$$\sum_{n \neq 0} \frac{a_n}{n} = 0 \quad \text{if} \quad a_n = a_{-n} \text{ and } a_n \rightarrow \text{const. as } n \rightarrow \pm\infty$$

as well as permit translations $n \rightarrow n+\kappa$ in logarithmically divergent sums over n .

We now calculate from (15) the anomalous term of the Virasoro subalgebra (14) whose generators V_N are defined by (1) + (52), and find that it is non-trivial

$$C_{N, N'} = -a_2 N \left[\frac{\pi^2}{3} + N^2 c^2 \right] \delta_{N, -N'} \quad (53)$$

$a_2 \in \mathbb{R}$. (If $a_2=0$, we can choose another solution f_N by replacing $\sigma_1 \rightarrow -\sigma_2$, $\sigma_2 \rightarrow \sigma_1$). This particular example shows that if $c \neq 0$ we expect to find non-trivial central extensions for the Virasoro subalgebras of $\text{SDiff}(T^2)$.

We can find more general solutions (44) by canonical or symplectic transformations $\sigma_1, \sigma_2 \rightarrow f_o, t$, and then by appropriate Fourier extensions on the torus we may determine the corresponding $c_N^{n_1, n_2}$'s. So we feel that up to these transformations and the invariance (46) the above example constitutes the

solution to our problem.

We conclude with some remarks and questions. The above realizations of the Virasoro algebra as subalgebras of $\text{SDiff}_0(T^2)$ are new ones, making use not of the conformal structure of surfaces but of their symplectic structure. Their very existence perhaps suggests new types of string theories. Another (but possibly related) point is the role of the central extension of the $\text{SDiff}_0(T^2)$ algebra. Under what circumstances does quantum mechanics of the torus membrane contain these anomalies? It is currently believed that there are no diffeomorphism anomalies for the membranes [1,2]. Does this imply that $\text{SDiff}(T^2)$, and in particular the modular subgroup of $\text{SDiff}(T^2)$, is free of anomalies? These questions are certainly worth of a careful examination. In case the $\text{SDiff}_0(T^2)$ algebra is realised quantum mechanically with anomalies, one must consider only half of its generators L_n, P_1, P_2 as constraints on the wavefunctions of the system

$$L_{n_1, n_2} \psi = 0 \quad , \quad n_1, n_2 \in \mathbb{Z}, \quad n_2 \geq 0 \quad (54a)$$

$$P_1 \psi = P_2 \psi = 0 \quad (54b)$$

similarly to the treatment of Virasoro generators in string theory. (In writing (54a) we used the fact that the central term $a \cdot n$ of (8) can be reduced to $a_2 n_2$ by a redefinition of the generators $L_n \rightarrow L_{An}$ with $A \in \text{SL}(2, \mathbb{R})$). In this case, the existence of the $c \neq 0$ Virasoro subalgebra corresponding to (52) and (53) gives an interesting result : because of the $\delta_{n_1, N}$ factor in (52), the membrane quantum constraints (54) imply the quantum constraints

$$V_N \psi = 0 \quad , \quad N \geq 0 \quad (55)$$

while the central term (53) ensures that there are no further constraints of the type $V_N \psi = 0$ ($N < 0$). So we have for the V_N the same Virasoro constraints as in the usual string theory. Thus, there is the possibility that the $a \neq 0$ torus membrane for appropriate values of a_2 and c contains the quantum string excitations.

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