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JACOBIAN SUBTLETIES AND THE COMPLETE PATH-INTEGRAL
BOSONIZATION OF THE NON-ABELIAN THIRRING MODEL

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ABSTRACT

The non-Abelian Thirring model invariant under an arbitrary simple Lie group G is written as a path integral over two bosonic fields taking values in G . The careful computation of the key functional Jacobian involved shows that it is a constant, in contrast with the value (a massless fermion determinant in the adjoint representation of G) previously assumed by several authors. We find two exact scale invariant critical points in the model. They are both described by a WZW sigma model of level $2N_F \ell_R / \phi^2$.

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The non-abelian Thirring model (also called multiflavor Chiral Gross-Neveu model) is defined for a generic, compact and simple Lie group G by the formal Lagrangian

$$\mathcal{L}(x) = i \sum_{j=1}^{N_F} \bar{\Psi}_j \not{\partial} \Psi_j - \frac{g}{2} \sum_{\alpha=1}^{\dim G} J_{\mu}^{\alpha} J^{\mu \alpha} \quad (1)$$

$$J_{\mu}^{\alpha} = \sum_{j=1}^{N_F} \bar{\Psi}_j \gamma_{\mu} T^{\alpha} \Psi_j, \quad T^{\alpha} = T^{\alpha \dagger}, \quad \text{tr}(T^{\alpha} T^{\beta}) = \ell_R \delta^{\alpha \beta}$$

Each fermionic field Ψ_j transforms under an irreducible unitary representation R of G . T^{α} are the generators of G in R and ℓ_R is the index of R .

It is convenient to express the vacuum-to-vacuum functional integral

$$\mathcal{Z} = \int d\psi d\bar{\psi} e^{i \int d^2x \mathcal{L}(x)} \quad (2)$$

as

$$\mathcal{Z} = \int dA d\psi d\bar{\psi} \exp i \int d^2x \left[i \bar{\psi} \not{D} \psi + \frac{1}{2\ell_R g} \text{tr}(A_{\mu} A^{\mu}) \right] \quad (3)$$

where the $A_{\mu} = A_{\mu}^{\alpha} T^{\alpha}$ are algebra-valued fields and $D_{\mu} = \partial_{\mu} - i A_{\mu}$. Integrating over the fermions yields

$$\int d\psi d\bar{\psi} e^{i \int d^2x \bar{\psi} \not{D} \psi} = [\text{Det}_R(\not{D})]^{N_F} \quad (4)$$

It is useful to parametrize A_{μ} in light-cone coordinates $x^{\pm} = \frac{1}{2}(x^0 \pm x^1) = x_{\mp}$

as

$$A_+ = i U^{-1} \partial_+ U, \quad A_- = i V^{-1} \partial_- V \quad (5)$$

where U and V are R -valued fields. $\text{Det}_R(\phi)$ is completely fixed by the Chiral anomaly up to a contact term^[1]

$$\text{Log} \frac{\text{Det}_R(\mathcal{D})}{\text{Det}_R(\mathcal{D})} = -i \Gamma_{WZ}[UV^{-1}] + \frac{i\lambda}{4\pi} \int d^2x \text{tr}(A_\mu A^\mu) \quad (6)$$

where for $\Omega = e^{i\omega}$,

$$\Gamma_{WZ}[\Omega] = \frac{1}{8\pi} \int d^2x \text{tr}(\partial_\mu \Omega \partial^\mu \Omega^{-1}) + \frac{1}{4\pi} \int_0^1 ds \int d^2x \varepsilon_{\mu\nu} \text{tr}[\omega(e^{-is\omega} \partial_\mu e^{is\omega})(e^{-is\omega} \partial_\nu e^{is\omega})] \quad (7)$$

is the Wess-Zumino functional and λ depends on the regularization chosen (e.g. $\lambda = 0$ in a gauge-invariant regularization).

Let us change the integration variables from (A_+, A_-) to (U, V) . We need then to compute the Jacobian $J(U, V)$ associated to this change

$$(A_+, A_-) \longrightarrow (U, V) \quad (8)$$

Eq. (5) defines A_+ and A_- in terms of U and V . It can be inverted with the result

$$\begin{aligned} U(x^+, x^-) &= U(0, x^-) \mathcal{P} \exp -i \int_0^{x^+} d\xi A_+(\xi, x^-) \\ V(x^+, x^-) &= V(x^+, 0) \mathcal{P} \exp -i \int_0^{x^-} d\xi A_-(x^+, \xi) \end{aligned} \quad (9)$$

$U(x)$ and $V(x)$ are uniquely defined as functionals of A_+ and A_- if we choose, for example

$$U(0, x^-) = 1 = V(x^+, 0) \quad (10)$$

We lose no generality with this choice since a transformation

$$U(x) \rightarrow L_1(x) U(x), \quad V(x) \rightarrow L_2(x) V(x) \quad (11)$$

leaves A_+ and A_- invariant for arbitrary $L_1, L_2 \in R$.

Infinitesimal variations in U, V

$$U \rightarrow U + \delta U, \quad V \rightarrow V + \delta V$$

are related to variations in A_+, A_- by

$$\delta A_+ = i \mathcal{D}_+ U^{-1} \delta U, \quad \delta A_- = i \mathcal{D}_- V^{-1} \delta V \quad (12)$$

where $\mathcal{D}^\pm$ stands for the covariant derivation in the adjoint representation of G

$$\mathcal{D}_\pm = \partial_\pm - i[A_\pm, \bullet]$$

It appears natural to assume that

$$J(U, V) = J_+(U) J_-(V) \quad (13)$$

Then, we can separately discuss the change $A_+ \rightarrow U$ yielding the Jacobian $J_+(U)$. $J_+(U)$ is the determinant of the map from the tangent space at U restricted by eq. (10), into the tangent space at $A_+ = i U^{-1} \partial_+ U$. By tangent

space at U we mean the linear space of Lie-algebra valued variations $U^{-1}\delta U$. These tangent spaces are not of identical nature since the δA_+ are chiral components of space-time 1-forms and $U^{-1}\delta U$ is a scalar function. However, thanks to eq. (10) these spaces are isomorphic. We find from eqs. (9)-(10)

$$U(x^+, x^-) = P \exp -i \int_0^{x^+} d\xi A_+(\xi, x^-) \quad (14)$$

This gives in a unique and explicit way the inverse mapping of $U \rightarrow A_+$ defined by eq. (5). Therefore the linear map

$$U^{-1} \delta U \xrightleftharpoons{\mathcal{D}_+} \delta A_+ \quad (15)$$

can be considered as an invertible matrix of infinite size whose determinant is precisely $J_+[U]$. Notice that the condition (10) excludes, as expected, zero modes of $\mathcal{D}_+$. The solutions of

$$\mathcal{D}_+ \psi = 0$$

are of the form

$$\psi = U^{-1}(x) \eta(x^-) U(x)$$

But $U^{-1}\delta U = \psi$ corresponds to a δU obtained from eq. (21) by a left multiplication by

$$L(x^-) = e^{\eta(x^-)} \simeq 1 + \eta(x^-)$$

This violates eq. (10) unless $\eta = 0$.

Keeping the above discussion in mind it is now straightforward to compute $J_+[U]$. Let Q_U denote the linear operator acting on scalars as

$$Q_U \omega = U \omega U^{-1} \quad (16)$$

Then

$$\mathcal{D}_+ = Q_U^{-1} \partial_+ Q_U \quad (17)$$

Here, Q_U^{-1} acts on + components of 1-forms and can be indeed identified with the inverse of Q_U since target and base space are isomorphic. It now follows that

$$\text{Det}(\mathcal{D}_+) = \text{Det}(\partial_+) = \text{constant} \quad (18)$$

A similar argument holds for $\text{Det}(\mathcal{D}_-)$. Therefore we can set after a trivial normalization

$$J(U, V) = 1 \quad (19)$$

Let us now discuss the flaws in the derivation of refs [1,2] incorrectly leading to a non-constant Jacobian $J(U, V)$. In refs. [1,2,3] $J(U, V)$ was identified with the fermion determinant (4) in the adjoint representation which is indeed non-constant [eq. (6) for $R = \text{Adjoint}$]. However this identification is not correct since the operators $\mathcal{D}_\pm$ appearing in the tangent mapping (15) and the covariant derivatives in the Dirac operator in the adjoint representation are actually not the same operator. The $D_\pm$ in the Dirac operator map fermions with opposite chirality

$$\begin{aligned} \mathcal{D}_+ &: V^+ \longrightarrow V^- \\ \mathcal{D}_- &: V^- \longrightarrow V^+ \end{aligned} \quad (20)$$

where $V^\pm$ is the space of $\pm$ chirality. V^+ and V^- are not isomorphic for generic fields $A_\pm$. Indeed for abelian configuration of (A_+, A_-) , after Wick rotation, the integrated chiral anomaly gives precisely $\dim(\text{Ker } D_+) - \dim(\text{Ker } D_-)$ while $\text{Ker}_\perp D_+$ and $\text{Ker}_\perp D_-$ are always isomorphic. Therefore, $D_\pm$ act on fermions as rectangular matrices of infinite size. On the other hand, the Dirac operator

$$\not{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \quad (21)$$

is a "square" matrix of infinite size thanks to eq. (20). Moreover we have

$$\text{Det } \not{D} = \text{Det}(D_- D_+) = \text{Det}(D_+ D_-)$$

This property can be easily justified replacing det by det' , the determinant over non-zero modes, if necessary. However, in general we have

$$\text{Det}(D_+ D_-) \neq \text{Det}(D_+) \text{Det}(D_-)$$

Actually the r.h.s., as it stands, requires a specific, non standard definition, since $D_\pm$ are not isomorphisms (they are not "square" matrices). In what concerns the Jacobian (13), in general, contrary to the implicit assumption of refs. [1,2]

$$\text{Det}(D_+) \text{Det}(D_-) \neq \text{Det}(D_+ D_-) \quad \text{or} \quad \text{Det}(D_- D_+)$$

The expressions $D_\pm D_\mp$ are meaningless since $D_\mp$ is not defined in the

whole image of $\mathcal{D}_+$.

It is easy to show that the non-constant Jacobian used in refs. [1,2] leads to unavoidable contradictions. Let us consider the δ -functional

$$\delta[J_-] = \int \mathcal{D}A_+ \exp i \int d^2x \text{Tr}(A_+ J_-) \quad (22)$$

and change variables from A_+ to U via $A_+ = i U^{-1} \partial_+ U$. The (wrong) identification of the Jacobian $J_+[U]$ with the fermion determinant in the adjoint representation yields

$$\begin{aligned} \int \mathcal{D}U \exp -i \left\{ \Gamma_{WZ}(U) + \frac{i}{4\pi} \int d^2x \text{Tr}[U^{-1} \partial_+ U \Omega^{-1} \partial_- \Omega] \right\} &= \\ = \int \mathcal{D}U \exp -i \left\{ \Gamma_{WZ}(U \Omega^{-1}) - \Gamma_{WZ}(\Omega^{-1}) \right\} &= \\ = (\text{constant}) \exp i \Gamma_{WZ}(\Omega^{-1}) \end{aligned} \quad (23)$$

where we parametrize J_- as

$$J_- = \frac{i}{4\pi} \Omega^{-1} \partial_- \Omega \quad (24)$$

and we use the fundamental property of the WZ functional

$$\Gamma_{WZ}[AB] = \Gamma_{WZ}[A] + \Gamma_{WZ}[B] + \int \frac{d^2x}{4\pi} \text{Tr}(A^{-1} \partial_+ A B \partial_- B^{-1}) \quad (25)$$

The δ -functional (22) is concentrated on $J_- = 0$ that is $\Omega = \Omega(x^+)$ by eq. (24). This is not the case for the expression (23). Hence $J_+[U] \neq e^{i\Gamma_{WZ}(U)}$. On

the other hand one does find the correct support $\Omega = \Omega(x^+)$ by taking $J_+[U] = 1$.

We can now come back to the vacuum-to-vacuum amplitude (3) and write it as a functional integral over U and V . Eqs. (3), (4), (6), (7) and (19) yield

$$Z = \int dU dV \delta_+[U] \delta_-[V] \exp \left\{ -i N_F \Gamma_{WZ}(UV^{-1}) + \frac{i}{2\ell_R g'} \int d^2x \text{Tr} A_\mu \hat{A}^\mu \right\} \quad (26)$$

where

$$\delta_-[U] = \prod_{x^-} \delta[U(x^-, x^+) - 1], \quad \delta_+[V] = \prod_{x^+} \delta[V(x^+, x^-) - 1]$$

and

$$g' = \frac{g}{1 + 2\lambda N_F g} \quad (27)$$

The original rigid G -symmetry of the purely fermionic formulation of eq. (2) manifests itself in eq. (26) as invariance of the whole integrand in (26) (including the gauge fixing δ 's!) under $U \rightarrow U_0 U U_0^{-1}$, $V \rightarrow U_0 V U_0^{-1}$ for any constant $U_0 \in R(G)$. Using eq. (25) and changing the integration variable $V \rightarrow V^{-1}$ in eq. (26)

$$Z = \int dU dV \delta_-[U] \delta_+[V] \exp \left\{ -i N_F (\Gamma_{WZ}[U] + \Gamma_{WZ}[V]) - \frac{i}{2} \left(\frac{1}{\ell_R g'} + \frac{N_F}{2\pi} \right) \int d^2x \text{Tr} [U^{-1} \partial_+ U V \partial_- V^{-1}] \right\} \quad (28)$$

Since $\Gamma_{WZ}[U]$ is precisely the action of a critical WZW model, eq. (25) describes two such models coupled by a current-current interaction. At

$$g = g_* = - \frac{2\pi}{\ell_R N_F (\lambda + 1)} \quad (29)$$

the interaction vanishes and we obtain a conformal invariant theory.

In terms of the product field $\Omega = U V^{-1}$, eq. (22) produces a critical WZW model when $g' = \infty$. That is when

$$g = g'_* = - \frac{2\pi}{\ell_R N_F \lambda} \quad (30)$$

That is, we find two distinct IR fixed points in the non-abelian Thirring model, for generic values of λ . Fixed points can be also investigated from the zeros of the β -function in perturbation theory. At two-loops one has^[5]

$$\beta(g) = - \frac{\ell_{Adj}}{2\pi} g^2 \left[1 - \frac{\ell_R N_F}{\pi} (\lambda + 2) g \right] + O(g^4) \quad (31)$$

which shows only one perturbative critical point for $g = g_*^{(pert)} = \pi[N_f \ell_R (\lambda + 2)]^{-1}$. Let us recall that besides the "natural" gauge-invariant choice $\lambda = 0$, there exists another "natural" choice : $\lambda = -1$. This corresponds to the exact factorization of $\det \not{D}$ into $\exp[-i\Gamma_{WZ}(U)]$, the contribution of right-handed fermions, times $\exp[-i\Gamma_{WZ}(V^{-1})]$, the contribution of left-handed ones. In this "decoupling" scheme, only $g_*^{(pert)} = 2\pi/[\ell_R N_f]$ stays finite and is qualitatively well approximated by $g_*^{(pert)} = \pi/[\ell_R N_f]$. On the other hand, in the gauge-invariant scheme $\lambda = 0$, the finite critical point is $g_* = -2\pi/[\ell_R N_f]$, while $g_*^{(pert)} = \pi/[2\ell_R N_f]$ is again positive.

At any rate, both the two-loop β -function and our exact functional

derivation completely exclude the possibility of a critical point of order ℓ_{Adj}^{-1} . In the case $G = \text{SU}(N)$, $R = \text{fundamental irrep}$ (so that $\ell_R = 1/2$, $\ell_{\text{Adj}} = N$ in a conventional normalization), the existence of such a critical point of order N^{-1} was claimed in ref. [4].

Another observation is in order concerning the limit $N_f \rightarrow \infty$. In this limit, the non-abelian Thirring model should reduce to the principal Chiral model over G [1,3] (more precisely to a relevant subsector of it [3]). Since the principal Chiral model is itself asymptotically free, it is clear that it will arise in the $N_f \rightarrow \infty$ limit only if the non-abelian Thirring model is regularized in such a way that $\lambda > 0$, to avoid the presence of a positive critical point g_*' vanishing as $N_f \rightarrow \infty$ (see eq. (30)).

We shall now give a better characterization of the critical points g_* and g'_* . To this end let us introduce various source terms in the vacuum functional Z of eq. (2), via

$$\mathcal{Z} \rightarrow \mathcal{Z} + h_\mu^\alpha J^{\mu\alpha} + \sum_{j=1}^{N_f} (\bar{\Psi}_j \eta_j + \bar{\eta}_j \Psi_j) \quad (32)$$

The various steps leading to expressions (26), (28) can be repeated with minor modifications, and we find

$$\hat{\mathcal{Z}}[h, \eta, \bar{\eta}] = \hat{\mathcal{Z}}[h, \eta, \bar{\eta}] \exp\left(\frac{i}{2g} \int d^2x h_\mu^\alpha h^{\mu\alpha}\right) \quad (33)$$

$$\begin{aligned} \hat{\mathcal{Z}}[h, \eta, \bar{\eta}] = & \int \mathcal{D}U \mathcal{D}V \delta_+[U] \delta_+[V] \exp\{-i\Gamma[U, V] + \\ & + \frac{i}{g} \int d^2x h_\mu^\alpha A^{\mu\alpha} - i \sum_{j=1}^{N_f} \int d^2x d^2y \bar{\eta}_j(x) S(x, y) \eta_j(y)\} \end{aligned} \quad (34)$$

where $\Gamma[U, V]$ is the same action functional appearing in e.g., eq. (28)

$$\Gamma[U, V] = -N_F \Gamma_{WZ}[UV^{-1}] - \frac{1}{2\ell_R g'} \int d^2x \text{tr}[U^{-1} \partial_+ U V^{-1} \partial_- V] \quad (35)$$

and $S(x, y)$ is the exact fermion propagator in the A field:

$$S(x, y) = (i\not{\partial} + i0)^{-1}(x, y) = \begin{pmatrix} U(x)^{-1} & 0 \\ 0 & V(x)^{-1} \end{pmatrix} S_0(x-y) \begin{pmatrix} V(y) & 0 \\ 0 & U(y) \end{pmatrix}. \quad (36)$$

In eq. (33) $S_0 = (i\not{\partial} + i0)^{-1}$ is the free fermion propagator (trivial in color indices) and we have implicitly chosen the chiral γ -representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From eqs. (33) and (34) one sees that the fermionic current correlators can be calculated in the bosonic theory of the fields U and V with the correspondence

$$J_{\pm}^{\alpha} = J_0^{\alpha} \pm J_1^{\alpha} \longleftrightarrow \frac{1}{g} A_{\pm}^{\alpha} = \frac{i}{g\ell_R} \begin{pmatrix} \text{tr } T^{\alpha} U^{-1} \partial_+ U \\ \text{tr } T^{\alpha} V^{-1} \partial_- V \end{pmatrix} \quad (37)$$

Strictly speaking, this correspondence holds exactly for correlators involving only J_+^{α} [J_-^{α}] and no J_-^{α} [J_+^{α}]. In the case of mixed correlators, the contact term $\sim h_{\mu}^{\alpha} h^{\mu\alpha} - h_{+}^{\alpha} h_{-}^{\alpha}$ in eq. (33) plays a role and a minor change to the correspondence (37) is needed. For instance

$$\langle J_+^{\alpha}(x) J_-^{\beta}(y) \rangle = \frac{1}{g^2} \langle A_+^{\alpha}(x) A_-^{\beta}(y) \rangle + \frac{\delta^{\alpha\beta}}{2g} \delta^{(2)}(x-y) \quad (38)$$

As for the fermion fields correlators, eq. (34) shows that they can be calculated via the correspondence

$$\psi_j \longleftrightarrow \begin{pmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \chi_j, \quad \bar{\psi}_j \longleftrightarrow \bar{\chi}_j \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}, \quad 1 \leq j \leq N_F \quad (39)$$

where x_j is a free field transforming as ψ_j under G . Of course the correspondence (39) is not to be regarded as an operator identity (we are actually enlarging the Hilbert space by introducing the fields x_j), but as a practical rule for translating ψ -correlators in the U-V language.

Let us now consider the critical point g_* (eq. 29), at which the theory decouples into two separate WZW models :

$$\Gamma[U, V] = -N_F \Gamma_{WZ}[U] - N_F \Gamma_{WZ}[V^{-1}] \quad (40)$$

At this point we have conformal invariance and the global color invariance under G is promoted to a pair of Kac-Moody symmetries, with current generators $b_+^\alpha(x^+)$ and $b_-^\alpha(x^-)$. At first sight one could expect four Kac-Moody algebras from the action (40) (two for each WZW model). However we still have the constraints $U(0, x^-) = 1$, $V(x^+, 0) = 1$. They imply that only the right Kac-Moody symmetry $U \rightarrow U R(x^+)$ of $\Gamma_{WZ}[U]$, generated by $U^{-1} \partial_+ U$, and the left Kac-Moody symmetry $V \rightarrow L(x^-) V$ of $\Gamma_{WZ}[V^{-1}]$, generated by $(\partial_- V^{-1}) V = -V^{-1} \partial_- V$ are explicitly realised in the Hilbert space. Since $R(0) = 1 = L(0)$ to respect the gauge fixing, one sees that the rigid $G_L \times G_R$ invariance is realised on (40), at the decoupling point g_* , as $U \rightarrow U_0 U U_0^{-1}$, $V \rightarrow U_0' U U_0'^{-1}$, with independent constant U_0 and U_0' . From (40) we can read off the level of these Kac-Moody algebras ; it is given by $n = 2N_F \ell_R / \phi^2$, where ϕ^2 is the squared length of the longest root of the Lie algebra of G . Hence we must identify :

$$b_+(x) = \frac{\ell_R N_F}{2\pi} U^{-1} \partial_+ U, \quad b_-(x) = \frac{\ell_R N_F}{2\pi} V^{-1} \partial_- V, \quad (41)$$

It is evident that, as far as current correlators are concerned, we can

replace the two decoupled but constrained WZW models with one single unconstrained WZW model $\Gamma_{\text{WZ}}[\Omega]$, upon identifying $U = \Omega$, $V = \Omega^{-1}$. From eq. (37) we now see that, apart from the trivial contact term in (38).

$$J_+ = (\lambda + 1) b_+ \quad , \quad J_- = (\lambda + 1) b_- \quad (42)$$

At the critical point g_* , the single WZW theory $\Gamma_{\text{WZ}}[\Omega]$ can be used to calculate also the ψ -correlators. For instance we have $(\psi_{\pm} = \frac{1}{2}[1 \mp \gamma_5]\gamma)$

$$\langle \psi_{a, j_+}(x) \psi_{b, k_+}^{\dagger}(y) \rangle = \frac{\delta_{jk}}{2\pi i} \frac{1}{x^+ - y^+} \langle \Omega_{ac}^{-1}(x) \Omega_{cb}(y) \rangle \quad (43)$$

$$\text{and} \quad \langle \psi_{+(1)} \psi_{+(2)}^{\dagger} \psi_{+(3)} \psi_{+(4)}^{\dagger} \rangle = \delta_{j_1 j_2} \delta_{j_3 j_4} [2\pi i (x_1^+ - x_2^+)]^{-1} \\ [2\pi i (x_3^+ - x_4^+)]^{-1} \langle \Omega_{a_1 b}^{-1}(x_1) \Omega_{b a_2}(x_2) \Omega_{c_3 c}^{-1}(x_3) \Omega_{c a_4}(x_4) \rangle \quad (44)$$

$$\text{while} \quad \langle \psi_{+(1)} \psi_{+(2)}^{\dagger} \psi_{-(3)} \psi_{-(4)}^{\dagger} \rangle = \delta_{j_1 j_2} \delta_{j_3 j_4} [2\pi i (x_1^+ - x_2^+)]^{-1} \\ [-2\pi i (x_3^- - x_4^-)]^{-1} \langle \Omega_{a_1 b}^{-1}(x_1) \Omega_{b a_2}(x_2) \Omega_{c_3 c}^{-1}(x_3) \Omega_{c a_4}^{-1}(x_4) \rangle \quad (45)$$

In these formulae the Ω -correlators can be calculated using the various methods of conformal field theory applied to the WZW model[6]. Notice that, whenever the level of the Kac-Moody algebra exceeds one, i.e. $2N_f \ell_R / \phi^2 > 1$, the four point function in (44) will not factorize according to Wick rules. In such case the conformal theory at $g = g_*$ cannot be mapped by non-abelian bosonisation rules into a free field theory.

The behavior of the theory at the other critical point g^* (eq. 30) is more singular. At $g = g^*$ we are left with a single WZW action

$$\Gamma[U, V] = - N_F \Gamma_{WZ} [UV^{-1}] \quad (46)$$

but we still have to integrate over both U and V . By a change $U = \Omega V$ we can extract the V -integration, in the sense that the action will not depend on V . On the other hand the constraint $U(0, x^-) = 1$ becomes $\Omega(0, x^-) = V(0, x^-)^{-1}$, with V to be integrated over (keeping $V(x^+, 0) = 1$). Such a constraint is therefore empty. Again the theory reduces to a single, unconstrained WZW model in terms of Ω . However the integration over V acts as an integration over a new gauge freedom, which kills the expectation values of gauge-variant quantities like J_μ or ψ (the change $U = \Omega V$ does not eliminate V from the source terms in $Z[h, \eta, \bar{\eta}]$). Properly normalized expectation values of gauge-invariant quantities (that is functions of $\Omega = UV^{-1}$ alone) exist and are non trivial also at $g = g^*$. They coincide with those of a WZW model over G again with level $2\ell_F N_F / \phi^2$.

In conclusion, let us briefly examine how the results of ref. (2b) and (2c) are modified by the use of the correct Jacobian $J = 1$ [eq. (19)]. First, the level of the Kac-Moody algebras is actually N_F as follows from eq. (41) for the $SU(N)$ group in the fundamental representation. Second, the mass spectrum in $SU(N)$ QCD₂ depends on this Jacobian. Repeating the analysis of ref. (2c) the mass of the fundamental meson is now $m^2 = N_F g^2 / (2\pi) + O(g^4)$. When $N \rightarrow \infty$ with $g^2 N = \text{fixed}$, this meson becomes massless in perfect

agreement with the results of ref. [7] for the lightest bound-state.

As far as the $N_f \rightarrow \infty$ limit of the non-abelian Thirring model which leads to the principal chiral sigma model^[1,3], the use of the correct Jacobian (19) does not affect the results.

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