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STRING VERTICES AND INDUCED REPRESENTATIONS

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ABSTRACT

It is shown that string vertices carry an induced representation of a graded extension of the Virasoro group, to incorporate the ghosts with conformal dimension two and the BRST charge. Decoupling of zero norm physical states, which determines the measure, is a consequence of the behaviour of the vertex under this group.

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It has been clear for many years that our understanding of string theory is far from complete. Indeed, many more recent developments in string theories have served to underline this lack of understanding. In an attempt to rectify these defects, and in a desire to find a complete and transparent description of string theory, two new formulations of string theory have been given. One goes under the name of the "group theoretic approach", while the other is based on the mathematics of Grassmannians.

The group theoretic approach has proved to be a powerful method of computing string perturbation theory, and this may bode well for its extension to include non-perturbative techniques and define new string theories. It is an extension of this method which is the subject of this paper. This method had its origins in the development of gauge-covariant string theory, but it possessed many of the advantages inherent in the old dual model [1]. In this model, one found oscillator vertices which, when integrated over their dependence on Koba-Nielsen co-ordinates [2] and moduli [3] gave the scattering of arbitrary string states. At tree level, such vertices were deduced [4,5] by factorizing the well-known formula for the scattering of N tachyons [6]. By sewing such tree vertices one was able to deduce vertices and their integration measure for multiloop string scattering [7]. Although not technically correct above one loop, due to the presence of negative norm propagating in the loops these scattering amplitudes had many of the correct generic features. The one-loop exception was due to the ability to insert the Brink-Olive projector [8].

However, what was not shown for these vertices was that they decoupled zero norm physical states. In QED, these states are of the form $\epsilon^\mu = k^\mu \epsilon(k)$, while in string theory they are $L_{-1}|\Omega\rangle$, $(L_{-2} + 3/2 L_{-1}^2)|\Omega\rangle$ [9], etc., or, in BRST language, $Q|\Lambda\rangle$ [10]. The decoupling of these states requires a knowledge of how string vertices behaved under the conformal group. Of course, it was known almost from the beginning of string theory that the operators, say in the bosonic string, corresponding to the scattering of two arbitrary states and one fixed string state, were operators of conformal weight 1, for example, the Fubini-Veneziano operator e^{ikQ} for tachyon emission [11]. (The corresponding fermion emission vertex in string theory had its superconformal properties computed in Ref. [12].) The three-string vertex, however, summarizes in one expression all such vertices and only in the paper of Ref. [13] were its conformal identities for the special case of $L_0, L_{\pm 1}$ discussed.

The general understanding of the conformal properties of string vertices came from the group theoretic approach to string theory [14-19]. The origin of this development was within gauge-covariant string theory where, in an intermediate step to establish gauge invariance, the conformal properties of the gauge-covariant three-string vertex, that is, the way it transmitted and reflected L_n 's, were

derived [20,21] using an overlap relation. Overlap relations, which are discussed below, are identities which relate the action of any conformal operator of given dimension acting on one leg of a vertex to its action on another leg or legs. These relations were first considered within the context of gauge-covariant string theory, in Refs. [20] and [21]. It was realized that such relations were very powerful and held for conformal operators and vertices in general [22,23]. Indeed, the works in Refs. [22] and [23] are not concerned with string field theory. Taking the conformal operator to be $L(z) = \sum_n L_n z^{-n}$, one therefore obtained [22,21] the general conformal properties of the old dual model [24] three-string vertex, its BRST invariant extension to include ghost oscillators [21] and the three-string Neveu-Schwarz vertex [22] which was constructed using this method of overlap relations.

Indeed, the overlap equations became the central tool in the formulation of the group theoretic approach to string theory [14-19], and were extensively developed there. They allowed the efficient evaluation of string, tree and multi-loop vertices, automatically encoding their behaviour on Koba-Nielsen co-ordinates and moduli, as well as their conformal properties. In particular, they yield the set of conformal transformations that preserved a vertex and those which changed its moduli and Koba-Nielsen co-ordinates. The latter were essential for the determination of the measure over which the vertex is integrated. In fact, the measure is deduced by demanding that zero norm physical states decouple.

In an independent development, it was realized by Belinson [25] as a result of a generalization of a theorem of Kodaira and Spencer, that the moduli space of Riemann surfaces could be viewed as coset manifolds. This approach has been implemented in the Grassmannian approach [26-29] and used in the works of Refs. [30] and [31].

In Ref. [32] the relation was given between the vertices and their conformal properties found by the group theoretic method, i.e., the existence of moduli-changing and preserving conformal transformations, and the moduli space as a coset space viewpoint. The purpose of this paper is to extend this development and to show precisely how string vertices are no more than an example of induced representations where the isotropy group is trivially realized. This fact was alluded to in earlier papers on the group theoretic method, and indeed this inspired the name "group theoretic method". We begin by summarizing the relevant features of the group theoretic approach since it is somewhat scattered in the literature. We then demonstrate how, taking the relevant isotropy group and the vertex corresponding to this group, the vertex is given at any other point of Koba-Nielsen and moduli space by a boost as implemented by the theory of induced representations. This discussion can be carried out with and without ghosts. In the former case one has the Virasoro

group, but in the latter case one uses a graded extension of the Virasoro group to include the ghost oscillators of conformal dimension two ($\bar{\beta}_n$'s) and the BRST charge. The integration over the "relevant" part of the coset space explains in a natural way the integration over the Koba-Nielsen co-ordinates and the loop moduli, and automatically provides the correct ghost $\bar{\beta}$ insertions. The decoupling of zero norm physical states is then seen to be a consequence of the fact that the vertex is an induced representation for that supersymmetry subgroup which includes the BRST charge.

We now review the relevant aspects of the group theoretic approach [14-19,32], which assumes that the scattering of N physical states is given by

$$W = \int \prod_i' dz_i \prod_r' d\sigma_r \bar{f}(z_i, \sigma_r) V(z_i, \sigma_r) |x\rangle_1 \dots |x\rangle_N \quad (1)$$

where $|x\rangle$ are physical states and z_i ($i = 1, \dots, N$) and v_r ($r = 1, \dots, 3(g-1) \equiv g_1$, $g > 2$) are the Koba-Nielsen co-ordinates and v_r are the moduli. The vertex V is determined by requiring that it satisfy overlap equations; to be precise, if $R(z)$ is a conformal operator of dimension d , then V obeys the equation

$$V \frac{R^{(i)}(\xi^i)}{(\xi^i)^d} = V \frac{R^{(j)}(\xi^j)}{(\xi^j)^d} \left(\frac{d\xi^j}{d\xi^i} \right)^d \quad (2)$$

where ξ_i is any analytic function of co-ordinate z which vanishes at z_i , and similarly for ξ_j . We can therefore write

$$\xi^i = (v^i)^{-1} z = \sum_{n=1}^{\infty} \hat{a}_n^i (z - z_i)^n \quad ; \quad \hat{a}_1 \neq 0 \quad (3)$$

In terms of the conformal transformations L_n , we may write

$$(v_i) = \exp\{z_i L_{-1}^{(i)}\} \exp\left\{-\sum_{n=0}^{\infty} a_n^{(j)} L_n^{(j)}\right\} \quad (4)$$

The ξ^i and ξ^j are co-ordinate patches around z_i and z_j and the co-ordinate z is a co-ordinate patch for which both sides of Eq. (3) are defined. This equation then states that when the action of two conformal operators on V are defined at a common point, they are equal. For the active point of view, see Ref. [17]. The superscript indices i and j tell us on which leg the conformal operator acts.

For each loop we have the overlap conditions

$$V \frac{R^i(\bar{z}^i)}{(\bar{z}^i)^d} = V \frac{R^i(P_n^i \bar{z}^i)}{(P_n^i \bar{z}^i)^d} \left(\frac{d(P_n^i \bar{z}^i)}{d \bar{z}^i} \right)^d \quad (5)$$

Using Eq. (2) to change the indices in Eq. (5), we find that $P_n^i = (V^i)^{-1} P_n (V^i)$. We take P_n , $n = 1, \dots, g$ to be an arbitrary but given element of $SU(1,1)$ for reasons that will become apparent below. The moduli and Koba-Nielsen dependence of V is defined through Eqs. (2) and (5), there being three parameters in each P_n and one z_j in each $(V_j)^{-1}$. The vertex also depends on a_n^j , $n = 0, 1, \dots, \infty$, $j = 1, \dots, N$. However, as we shall see, the physical result is independent of these later parameters.

The above equations are defined on the Riemann sphere, but for loops Eq. (5) tells us that one may work on only the fundamental domain corresponding to the group generated by the P_n , $n = 1, \dots, g$, that is, the region outside all the isometric circles of P_n . We recognize here the Schottky representation of Riemann surfaces. One can further cut the Schottky picture and consider the Fuchsian representation. Corresponding to this further dissection, one inherits one extra overlap condition for each loop.

Equations (2) and (5) hold for any conformal operator. For example, for the bosonic string we may use $Q^\mu(z)$, $P^\mu(z)$, $L(z)$ and the ghosts $\beta(z)$, $\bar{\beta}(z)$ which have conformal weights -1 and $+2$ respectively, and are more usually denoted by $c(z)$ and $b(z)$ in the literature. In the case of loops, some care must be taken where Q^μ and $L(z)$ are concerned, and the reader may consult Refs. [14]-[19]. These equations uniquely determine V up to a multiplicative function of the moduli. For explicit expressions for V , see Refs. [14]-[19]. The decoupling of zero norm physical states determines the measure $\bar{f}(z_i, v_r)$, given a choice of the mentioned multiplicative function, the final result for string scattering being independent of the choice.

One can integrate Eq. (2) to obtain what is called an integrated overlap equation of the form [14-23]

$$V \left\{ \sum_{j=1}^N \oint_{\bar{z}=0} d\bar{z}_j \, \frac{R^j(\bar{z}^j)}{(\bar{z}^j)^d} \left(\frac{d\bar{z}_j}{d\bar{z}^j} \right)^{d-1} \right\} = 0 \quad (6)$$

An alternative, although subsequent, derivation of certain integrated overlap equations from the Polyakov path integral is given in Ref. [28], and they were also discussed in Ref. [29]. We may identify ϕ as a tensor of rank $(1-d)$. The above

equation is valid provided ϕ is defined on the Riemann sphere for trees and also satisfies the relevant automorphic conditions for loops, i.e.,

$$\phi(\bar{z}_i) = \phi(P_n^i \bar{z}_i) \left[\frac{d(P_n^i \bar{z}_i)}{d\bar{z}_i} \right]^{1-d} \quad (7)$$

In addition, ϕ can have poles only at Koba-Nielsen points, but is analytic elsewhere. These requirements are an obvious consequence of this integration procedure, which is discussed further in Refs. [14]-[20] and [22]. In certain cases, such as $R = P^\mu(z) = z(d/dz)Q^\mu$, one can use functions more general than single-valued automorphic functions [15].

As the vertex V is determined, up to the above-mentioned function, by the overlaps, so consequently are all of its properties. In particular, it is instructive to write down explicitly some of its conformal properties. One can easily show that

$$V(z_j, \sigma_r) = V(z_j, \sigma_r) P_n^i \quad \forall n \text{ and } i \quad (8)$$

and

$$V(z_1, \dots, z_N, \sigma_r) \prod_{j=1}^N [T_{j+1}^{(i)}]^{-1} = V(z_2, \dots, z_N, z_1, \sigma_r) \quad (9)$$

where $T_j = (V_j)^{-1}(V_{j-1})$. One simply shows that the left- and right-hand sides of these equations satisfy the same overlap equations.

Let us now consider the action of a conformal transformation $\mathcal{X}^i$, where i refers to the leg of V upon which it acts:

$$V \rightarrow \hat{V} = V \prod_{i=1}^N \mathcal{X}^i \quad (10)$$

Using Eqs. (2) and (5) or Eqs. (8) and (9), one can easily see that one can change the moduli v_r , the Koba-Nielsen variables z_j and the a_n^j by

$$V(\hat{\sigma}_r, \hat{z}_j, \hat{a}_n^j) = V(\sigma_r, z_j, a_n^j) \prod_{i=1}^N \mathcal{X}^i \quad (11)$$

provided [14-19]

$$(T_{i+1})^{-1} \mathcal{X}^{i+1} \hat{T}_{i+1} = \mathcal{X}^i \quad \forall i \quad (12)$$

$$(P_n^i)^{-1} \mathcal{U}^i(\hat{P}_n^i) = \mathcal{U}^i \quad \forall i, n \quad (13)$$

The first of these equations is solved by taking

$$\mathcal{U}^i = (v^i)^{-1} \mathcal{U} \hat{V}^i \quad (14)$$

where

$$\hat{V}^i = v^i (\hat{z}_j, \hat{a}_n^j) \quad (15)$$

while the second equation now reads

$$P_n^{-1} \mathcal{U} \hat{P}_n = \mathcal{U} \quad (16)$$

where $\hat{P}_n = P_n(\hat{v}_r)$.

One can in particular consider such conformal transformations which are infinitesimal. Those which leave the moduli v_r inert but make an infinitesimal change $\hat{z}_j = z_j + \epsilon f_j$, $\hat{a}_n^j = a_n^j + \epsilon k_j$ are given by

$$V(\sigma_r, \hat{z}_j, \hat{a}_n^j) = V(\sigma_r, z_j, a_n^j) \prod_{i=1}^N (v^i)^{-1} \hat{V}^i \quad (17)$$

On the other hand, if we do not change z_j and a_n^j , then $v_j = \hat{v}_j$, but we change the moduli $v_r \rightarrow v_r = v_r + \epsilon h_r(v)$. Then Eq. (16) reads

$$P_n(z) + \epsilon f(z) \frac{dP_n(z)}{dz} = \hat{P}_n(z) + \epsilon f(P_n(z)) \quad (18)$$

Implemented on the vertex, this equation becomes

$$\sum_{r=1}^{g_1} h_r(v) \frac{\partial V}{\partial v_r} = V \left\{ \sum_{j=1}^N \oint_{\gamma_j=0} d\bar{z}_j \frac{L^j(\bar{z}_j) f^j(\bar{z}_j) + c_f}{(\bar{z}_j)^2} \right\} \equiv V(L_f + c_f) \quad (19)$$

where c_f is a constant which is easily determined by taking a suitable state on either side of this equation. It corresponds to the presence of a multiplier which has not been explicitly written in Eq. (11). Clearly, the moduli-changing equations are sensitive to the multiplicative function, discussed above, which is not determined by the overlap equations. The value of c_f is fixed relative to the choice of this function. We recall, however, that to find the actual scattering amplitude, we multiply by the measure $\bar{f}$ and integrate over all the Koba-Nielsen variables and

moduli as in Eq. (11). However, the measure $\bar{f}$ is closely related to the constants c_f and the final result is independent of what choice of multiplicative factor for V one takes. We will discuss this point further below.

Also of importance to us in this paper are those transformations which leave the vertex inert, i.e., which change neither v_r , z_j , nor a_n^j . For the case of a tree graph with N external legs, these are given by $\mathcal{H}^{ij}$'s which satisfy

$$\mathcal{H}^{i+1} = (V^{i+1})^{-1} V^i \mathcal{H}^i (V^i)^{-1} (V^{i+1}) \quad (20)$$

while, if the contribution has loops, they should also obey

$$P_n^{-1} \mathcal{H} P_n = \mathcal{H}. \quad (21)$$

For infinitesimal transformations, we take Eq. (18) with $P_n = \hat{P}_n$, which means that $f(z)$ transforms as a vector field (i.e., a tensor of rank -1) and is defined on the Riemann surface itself. In fact, we have already met this group since we may consider taking $R(z) = L(z)$, which has conformal weight 2 in Eq. (6), and we find

$$V \left\{ \sum_{j=1}^N \oint_{\bar{z}=0} d\bar{z}_j \left[\frac{L^j(\bar{z}^j)}{(\bar{z}^j)^2} \left(\frac{d\bar{z}^j}{d\bar{z}^i} \right) + c \right] \right\} = 0 \quad (22)$$

that is, an equation of the same type. Examining the previous equations which define conformal transformations that preserve the vertex, we find that they are the same as those given by Eq. (22). Clearly, the set of such transformations forms a group which, for a vertex with g loops and N external lines, we denote by H_N^g .

The transformation properties of the vertex can be summarized by writing the dependence of the vertex V on v_r and z_j , a_n^j through P_n and V^j , i.e., $V(v_r, z_j, a_n^j) = V(P_n, V^j)$. We recall that the vertex $V(P_n, V^j)$ is determined, up to a multiplicative function, by the overlaps which involve V^j and P_n . As such, V depends not on N V^j 's, but only on $(V^j)^{-1} V^i$ for all i, j ; that is, only $N-1$ conformal transformations. [We note that $(V^j)^{-1} V^k = (V^j)^{-1} V^i (V^i)^{-1} V^k$.] We also observe that it is $SU(1,1)$ -invariant in the sense that if $S \in SU(1,1)$, then

$$V(S P_n S^{-1}, S V^j) = V(P_n, V^j) \quad (23)$$

This explains the 3 in the count of the number of moduli.

The conformal transformation property of the vertex embodied in Eqs. (12) and (13) may be written as

$$V(P_n, V_j) \prod_{i=1}^n \mathcal{X}^i = V(\mathcal{X}^{-1} P_n \mathcal{X}, \hat{V}_j) \quad (24)$$

where $\mathcal{X}^j$ is split up according to Eq. (14), i.e., $\hat{V}_j$ is of the required form of Eq. (3) and $\mathcal{X}^{-1} P_n \mathcal{X} \in \text{SU}(1,1)$. We have again omitted the possible scaling of V by a function of v_r and z_j .

Applying two such transformations successively, we find

$$V(P_n, V_j) \prod_i \mathcal{X}_1^i \mathcal{X}_2^i = V(\mathcal{X}_2^{-1} \mathcal{X}_1^{-1} P_n \mathcal{X}_1 \mathcal{X}_2, V_j) \quad (25)$$

where

$$\begin{aligned} \mathcal{X}_1^j &= (V_j)^{-1} \mathcal{X}_1 (V_1^j) \\ \mathcal{X}_2^j &= (V_1^j)^{-1} \mathcal{X}_2 (V_2^j) \end{aligned} \quad (26)$$

One can verify that the order of the factors is correct by carrying out these transformations on the overlap equations.

To get a concrete feeling for what is going on, let us compute H_N^g for the case where $(V_j)^{-1} z = z - z_j$ (i.e., $a_n^j = 0$). To find all generators of this group, we need only produce a basis of vector fields ϕ which satisfy the constraints discussed above. Indeed, conversely, given all the L_n -preserving transformations, we can read off all vector fields with these properties. Consider first the case of a tree graph with N legs. The vector field ϕ must be defined in both co-ordinate patches on the Riemann sphere [14-19]. There are three analytic vector fields which we may take to be given by $\phi = 1$, $\phi = z_j + \xi_j$, $\phi = (\xi_j + z_j)^2$. We therefore find the corresponding three identities

$$\begin{aligned} V \sum_{j=1}^N L_{-1}^j &\equiv V \ell_{-1} = 0 \quad ; \quad V \sum_{j=1}^N (z_j L_{-1}^j + L_0^j) \equiv V \ell_0 = 0 \\ V \sum_{j=1}^N \{ z_j^2 L_{-1}^j + 2 z_j L_0^j + L_1^j \} &\equiv V \ell_1 = 0 \end{aligned} \quad (27)$$

A basis of functions ϕ with poles only at one Koba-Nielsen point is given by

$$\phi_j^n = \left(\frac{1}{z_j}\right)^n = \frac{1}{(z_i + z_c - z_j)^n} \quad n \geq 1, \forall j \quad (28)$$

From these we find the identity

$$\forall \left\{ L_{-n-1}^j + \sum_{\substack{k=1 \\ k \neq j}}^N \sum_{p=0}^{\infty} \frac{L_{p-1}^k}{(z_k - z_j)^{n+p}} \binom{-n}{p} \right\} \equiv \forall e_{-n-1}^j = 0 \quad n \geq 1 \quad \forall j \quad (29)$$

Clearly, the above set of functions provide a complete basis for all allowed ϕ 's and so

$$H_N^0 = \left\{ e_r, r = -1, 0, 1; e_{-m}^j; j = 1, \dots, N, m \geq 2 \right\} \quad (30)$$

Consider now the case of a g -loop vertex which, for simplicity, has only one external string. We are now interested in vector fields defined on the relevant Riemann surface that have poles at only one point. The Riemann-Roch theorem tells us that at that point they must have a pole of order $g_1+1 \equiv 3(g-1)+1$ ($g > 2$), that is, a suitable basis is of the form

$$\phi_m(z) = \frac{1}{z^{g_1+1+m}} + \sum_{n=-g_1}^{\infty} a_{mn} z^n \quad m \geq 0 \quad (31)$$

The corresponding L_n identities are

$$\forall \left\{ L_{-g_1-2-m}^1 + \sum_{n=-g_1}^{\infty} a_{mn}'' L_{n-1}^1 + c \right\} = \forall e_{-g_1-2-m}^1 = 0 \quad m \geq 0. \quad (32)$$

and so

$$H_1^g = \left\{ e_p^1, p \geq g_1+2 \right\}. \quad (33)$$

For more than one point, the discussion is as above, the basis being given by $f_m^j(z) = f_m(z+z^j-z^k)$ for each z^k . However, since one can have poles at more than one point, the Riemann-Roch theorem is a little different, and one can have a pole of a lower order than g_1+1 , provided one has a pole at another point.

This implies additional L identities. In fact, one can arrive at these identities in another way. Examining Eq. (19), one sees that one can change the moduli v_r by having a pole at any of the Koba-Nielsen points. Hence, if we change v_r in the same way at two points, we may subtract the two equations to obtain an infinitesimal transformation which has poles at both points and leaves the vertex inert.

The above discussion holds whether one has a vertex which incorporates ghosts or not. However, if one does incorporate ghosts, one can show [19] that there are no constants in the integrated L_n identities, and that it is possible to adjust the vertex V by a multiplicative factor in such a way that the constants in Eqs. (19) and (22) are removed. In fact, one scales V by the measure $\bar{f}$ itself, and one can show that c_r is essentially the derivative of $\ln \bar{f}$ with respect to v_r . To be precise, in the case for which the moduli change is $v_r \rightarrow v_r + \delta_{rs} \epsilon$, then $c_s = -(\partial/\partial v_s) \ln \bar{f}$ [19]. This is to be expected, as the L_n 's without ghosts have a central term in their algebra which means that they do not quite form a conformal operator of weight 2. However, with ghosts in $D = 26$, there is no central term and so they really do have conformal weight 2.

Let us now take the vertex to include ghosts and then, in addition to the L_n identities above, the vertex is also preserved by identities involving $\bar{\beta}(z)$. Since this operator has the same conformal weight (i.e., 2) as L_n , these identities are the same except for the replacement $L_n \rightarrow \bar{\beta}_n$. In this case, however, as the $\bar{\beta}_n$ are odd, there are not constants possible. We use the same notation as before for the preserving transformations, i.e., $\bar{b}_r$, $r = -1, 0, 1$ and $\bar{b}_{-n}^j$ preserve a tree graph, while $\bar{b}_{-g-2-m}^1$ preserves the g -loop tadpole graph. The vertex is also preserved by the integrated identities for α_n^μ and β_n discussed in Refs. [14]-[19]. However, we will not discuss these types of identity for the majority of this paper.

The vertex V will also be annihilated by the action of Q :

$$V \left\{ \sum_{j=1}^N \varphi^j \right\} = 0. \quad (34)$$

We observe that $Q \equiv \sum_{j=1}^N Q^j$ is an integral of a current j of conformal weight one [33]. Although strictly one should verify this explicitly by manipulating the β_n , $\bar{\beta}_n$ and α_n^μ identities to establish this result, as j is a conformal object of weight 1, we may apply the integrated overlap directly to it to find the above result. More generally, we note that

$$\forall \sum_{j=1}^N \left\{ \oint d\vec{z}_j \exp \left(j \frac{(\vec{z}_j)}{(\vec{z}_j)} \right) \right\} = 0 \quad (35)$$

for any function ϕ which satisfies the requirements given above. Obviously, all these symmetry transformations form a group; however, in what follows we will mainly consider the group which we denote by I_N^g , formed by the L 's, the $\bar{\beta}$'s and Q . Hence, for trees,

$$I_N^0 = \left\{ \ell_r, \bar{\ell}_r ; r = -1, 0, 1 ; \ell_{-n}^j, \bar{\ell}_{-n}^j \right. \\ \left. j = 1, \dots, N \quad n = -2, -3, \dots \quad \text{and } Q \right\} \quad (36)$$

while for the tadpole

$$I_1^g = \left\{ \ell_{-p}' , \bar{\ell}_{-p}' \quad p = g_1 + 2, g_1 + 3, \dots \quad \text{and } Q \right\} \quad (37)$$

and similarly for an arbitrary leg multiloop vertex once we include the additional identities discussed above.

Examining Eq. (14), we see that the action of any conformal transformation

$$\mathcal{U}^j = \exp c_n^j L_n^j \quad (38)$$

on the vertex can be taken to not change the loop cycling transformations P_n , but does change the external leg cycling transformations V^j from

$$V^j \rightarrow \hat{V}^j \equiv V^j \mathcal{U}^j \quad (39)$$

If one has used no ghosts, then the action of L_n , $n \geq 1$ annihilates physical states while L_0 is 1 on these states. Going through the procedure of decoupling of zero norm physical states, one finds a corresponding change in the measure $\bar{f}$, such that the final result W of Eq. (1) for string scattering is the same on physical states. An equivalent mechanism operates when we work with vertices which include ghosts (see Refs. [18] and [19]). Hence the values of a_n^j which we take in the cycling transformations V^j are irrelevant.

The transformations which change the Koba-Nielsen points are computed from Eq. (12), and clearly involve L_{-1}^j if we change z_j , and in general also L_n^j , $n \geq 0$.

The moduli-changing transformations are those that remain, and so these must involve functions f having poles of order less than g_1 , since higher poles may be removed as they correspond to the preserving transformations discussed above.

Once one has chosen a moduli change, the functions which implement this change according to Eq. (19) are found by solving Eq. (18). However, they are of the generic form

$$f_r = \sum_{m=1}^{g_1} \frac{a_m^{(r)}}{z^m} + \sum_{p=0}^{\infty} a_m^{(r)} z^p; r=1, \dots, g_1 \quad (40)$$

For examples of this specific procedure, see Refs. [14]-[19].

This completes our summary of the aspects of the group theoretic method [14-19,32] which are relevant to us in this paper. Our purpose now is to show that vertices are induced representations of $U_N = U \times \dots \times U$ (N factors). U is the Virasoro group plus the $\bar{\beta}_n$ and Q , i.e.,

$$U_N = \{ \bar{\beta}_n^i, L_n^i; n = 0, \pm 1, \pm 2, \dots, i=1, \dots, N \} \quad (41)$$

and $Q = \sum_{j=1}^N \phi_j^i$
with isotropy group I_N^Q .

To begin at the beginning, we recall the definition of an induced representation and some related material [34]. Consider a group G with a subgroup H ; let us denote the generators of H by H_i , and the remaining generators of G by K_a . We parametrize the coset space G/H by co-ordinates denoted by ξ^a , and define the action of any element g of G on G/H by $g:\xi^a \rightarrow \xi'^a = \tau(g)\xi^a$, where $g(\xi^a)g = h(\xi, g)g(\xi'^a)$, $h(\xi, g)$ is a member of H and $g(\xi^a)$ is any element of G belonging to the coset of G/H labelled by ξ^a . In particular, in the neighbourhood of the origin of the group, we may choose the coset representative

$$g(\xi^a) = \exp \sum_a \xi^a K_a = e^{\xi \cdot K} \quad (42)$$

and the above equation reads

$$\exp(\xi \cdot K) \cdot g = \exp \omega \cdot H \exp \xi' \cdot K. \quad (43)$$

Let $\{g_0\}$ be a set of vectors in a Hilbert space $\mathcal{H}$ defined for all g_0 belonging to the group G and which, for any g and g_0 in G and h in H , satisfy the relations

$$\langle_n g_0 | U(g) = \langle_n g_0 g | \quad (44)$$

$$\langle_n h g_0 | = \sum_m \langle_m g_0 | D^m_n(h^{-1}) \quad (45)$$

where $U(g)$ are operators and $D^m_n(h)$ is a matrix which carries a representation of H . The above definition of an induced representation is equivalent to the following definition, which is probably more palatable to most physicists.

We consider a set of vectors $\langle_n \xi |$ defined on G/H and carrying a representation of G such that

$$\langle_n 0 | U(h) = \sum_m \langle_m 0 | D^m_n(h^{-1}) \quad (46)$$

$$\langle_n 0 | U(e^{\xi \cdot K}) = \langle_n \xi | \quad (47)$$

The latter definition illustrates the fact that, given the representation of $\langle_n 0 |$ under H at one point of the coset, we can define $\langle_n \xi |$ by the action of $U(e^{\xi \cdot K})$ and then deduce the action of the whole group on $\langle_n \xi |$. This procedure assumes that the action of the generators K on $\langle_n 0 |$ is known. The relation between these two definitions is found by defining, for $g_0 = h \exp x \cdot K$,

$$\langle_n g_0 | \equiv \sum_m \langle_m x | D^m_n(h^{-1}) \quad (48)$$

For the general transformation we have

$$\langle_n \xi | U(g) = \sum_m \langle_m \tau(g) \xi | D^m_n(h^{-1}(\xi, g)) \quad (49)$$

Before considering the graded extension of the Virasoro group discussed above, it is illuminating to consider just the Virasoro case. Here, for a multiloop graph with N external strings, we have the isotropy group H_N^g . At the tree level, the coset generators are L_{-1}^j for all j except any three chosen values and L_n^j , $n \geq 0$, $j = 1, \dots, N$, the former corresponding to the presence of the three identities ℓ_r , $r = -1, 0, 1$, while for a multiloop graph with one external leg the coset generators are the g_1 L_f 's of Eqs. (19) and (40), as well as L_n 's $n \geq -1$. In other words, the coset contains L_p ; $-g_1 - 1 \leq p < \infty$. For a graph with N external legs and g loops, the coset contains $L_p^{i_1}$; $-g_1 - 1 \leq p < \infty$ for some i_1 and L_p^j , $p \geq -1$ for $j = 1, 2, \dots, N$. An exception is for one loop, where $\sum_j L_{-1}^{(j)}$ is in the isotropy group

and so only $N-1$ L_{-1} 's are in the coset.

The reader will have observed that there is some ambiguity in how one chooses the coset generators. In fact, one choice is as good as any other, apart from possible boundary terms. From the method of choosing the coset generators, it follows that one can go from one choice to another by using a transformation from the isotropy group. One can also consider choices which maintain the cyclicity of the vertex by using a Faddeev-Popov procedure. These two points will be discussed elsewhere .

We now consider Eq. (44) and define

$$\langle g_o^j | = V(g_o^{-1} P_{no} g_o, V_1^j) \quad (50)$$

where $g_o^j = (V_o^j)^{-1} g_o (V_1^j)$, and P_{no} and V_o^j are arbitrary, but fixed, loop and leg cycling transformations. The group H_N^g corresponds to the vertex with overlaps which involve P_{no} and V_o^j . In particular, a preserving transformation can be read off from Eqs. (13), (34) and (45) to be of the form

$$h^j = (V_o^j)^{-1} \mathcal{H} V_o^j s^j \quad (51)$$

where

$$\begin{aligned} \mathcal{H}^{-1} P_{no} \mathcal{H} &= P_{no} \\ \text{and } V_o^j s^j (V_o^j)^{-1} &= V_o^i s^i (V_o^i)^{-1}. \end{aligned} \quad (52)$$

as then the action of s^j preserves $(V_o^j)^{-1} V_o^i$ for all i, j .

Clearly, we have Eq. (45), since

$$\begin{aligned} \langle h^j g_o^j | &= V(g_o^{-1} \mathcal{H}^{-1} P_{no} \mathcal{H} g_o, V_1^j) \\ &= V(g_o^{-1} P_{no} g_o, V_1^j) = \langle g_o^j | \end{aligned} \quad (53)$$

Also we find Eq. (44), as

$$\begin{aligned}
 \langle g_0^j | u^{(i)}(\mathcal{X}^i) &= V(\bar{g}_0^j P_{0n} g_0, V_1^j) \pi \mathcal{X}^i \\
 &= V(\hat{g}^{-1} \bar{g}_0^{-1} P_{0n} g_0 \hat{g}, \hat{V}_1^j) = \langle (g_0 \bar{g})^j |
 \end{aligned}
 \tag{54}$$

where

$$\mathcal{X}^i = (V_1^j)^{-1} \hat{g}(\hat{V}_1^j).$$

Consequently, we see that the vertex is indeed an induced representation of the Virasoro group. We now consider the graded extension of the Virasoro group, first at the tree level. We are to consider states which are to transform under I_N and have I_N^0 as isotropy group. This particular isotropy group is associated with a specific cycling $(\bar{V}_j)$, i.e., a particular set of $\bar{a}_n^j$ and $\bar{b}_j$ Koba-Nielsen co-ordinates which is the origin for our coset space. We denote this vertex, which satisfies the corresponding integrated and unintegrated $P^\mu(z)$, $\beta(z)$ and $\bar{\beta}(\bar{z})$ identities of Eqs. (2), (5) and (6), by $V(\bar{b}_j, 0, \bar{a}_n^j, 0)$. For an explicit expression, see Refs. [14]-[19]. We follow the general method of Eqs. (44) and (45), or (46) and (47), but taking the representation of the isotropy group I_N^0 to be trivial. The coset contains the same L_n 's discussed for this case above, the bosonic generators L_{-1}^j , $j = 1, \dots, N$ except for i_1, i_2, i_3 indices and L_n^j , $n > 0$, $j = 1, \dots, N$. We also have the anticommuting generators $\bar{\beta}_{-1}^j$, $j = 1, \dots, N$ except i_1, i_2, i_3 and $\bar{\beta}_n^j$, $n > 0$, $j = 1, \dots, N$. Correspondingly we may label the coset by z^j , $j = 1, \dots, N$; except $j = i_1, i_2, i_3$, a_n^j and ϕ^j , $j = 1, \dots, N$; except $j = i_1, i_2, i_3$ and ξ_n^j , and we may write the representative group element as

$$\begin{aligned}
 g &= \exp \sum_j' z^j L_{-1}^j \exp \sum_j' \psi^j \bar{\beta}_{-1}^j \prod_{k=1}^N \exp \sum_{n=0}^{\infty} a_n^k L_n^k \\
 &\cdot \prod_{k=1}^N \exp \sum_{n=0}^{\infty} \bar{z}_n^k \bar{\beta}_n^k
 \end{aligned}
 \tag{55}$$

where $\sum_j'$ means the sum over j except for i_1, i_2, i_3 . The above notation for the vertex $V(\bar{b}_j, 0, \bar{a}_n^j, 0)$ means the general vertex $V(z_j, \psi_j, a_n^j, \xi_n^j)$ evaluated at $\psi_j = \xi_n^j = 0$, $z_j = \bar{b}_j$, $a_n^j = \bar{a}_n^j$.

Following the general procedure for boosting of Eq. (47), we have that

$$V(z_j, \psi_j, a_n^j, \bar{z}_n^j) = V(\bar{z}_j^0, 0, \bar{a}_n^j, 0) g \tag{56}$$

If we restrict ourselves to a g with $\phi_j = \xi_n^j = 0$, the reader will immediately recognize that we have here Eq. (14), which changes the cycling transformation and shifts the Koba-Nielsen points. The action of an arbitrary group element can be computed, as usual, from Eq. (56), and we observe that the isotropy group for $V(z_j, \phi_j, a_n^j, \xi_n^j)$ is given by $g^{-1} I_N^0 g$. The unintegrated overlap equations, i.e., Eqs. (2), are satisfied by $V(z_j, 0, a_n^j, 0)$ and we can find those for $V(z_j, \phi_j, a_n^j, \xi_n^j)$ by boosting this equation. For example, one finds that

$$V(z_j, \psi_j, 0, 0) \left\{ L_\phi + \sum_{j=1}^N \psi_j \frac{\partial}{\partial z_j} \bar{\beta} \phi \right\} = 0$$

$$V(z_j, \psi_j, 0, 0) \left\{ [(z-z_i) \beta^i(z-z_i) - \psi^i] \right. \\ \left. - [(z-z_j) \beta^j(z-z_j) - \psi^j] \right\} = 0 \quad (57)$$

while the $\bar{\beta}$ identities have no extra terms. For $\phi_j = 0, \xi_n^j = 0$, all the identities are given by Eq. (2) as usual, the boost just changing the argument and providing the correct factors associated with the conformal weight of the operator R involved. The transformation properties of V are given by Eq. (49) and in particular we find that

$$V(z_j, \psi_j, a_n^j, \xi_n^j) e^{\bar{\epsilon} \phi} \\ = V(z_j + \bar{\epsilon} \psi_j, \psi_j, a_n^j + \bar{\epsilon} J_n^j k(a), \xi_n^j - \frac{1}{2} \bar{\epsilon} \sum_{m,p} J_m^j J_p^j (m-p) \delta_{m+p,n}) \quad (58)$$

In a usual coset space approach, one now integrates the functions which carry the representation over the coset with a group-invariant measure. In this way, one obtains a unitary representation. However, in string theory we are asked to compute the scattering amplitude applying the physical states $|\chi\rangle_i, i = 1, \dots, N$ as follows:

$$W = \int \prod_i' d z_i \prod_i' d \psi_i V(z_j, \psi_j, a_n^j, \xi_n^j) |\chi\rangle_1 \dots |\chi\rangle_N \quad (59)$$

We only integrate over $N-3$ z_i 's and $N-3$ ψ_i 's and not the whole coset. This is because W is independent of a_n^j and ξ_n^j , a fact we now demonstrate. We note that the physical states $|\chi\rangle$ are defined by $Q|\chi\rangle = 0$ and satisfy $\bar{\beta}_0 |\chi\rangle = 0$ and have ghost number $\frac{1}{2}$. We recall that the fields in gauge-covariant string theory have ghost number $\frac{1}{2}$ and $\bar{\beta}_0 |\chi\rangle = 0$ is a gauge-fixing condition [36]. They can be shown to be of the form [37]

$$|x\rangle = \psi|-\rangle + \phi|1\rangle \quad (60)$$

where $\bar{\beta}_0|-\rangle = \alpha_n^\mu|-\rangle = \beta_n|-\rangle = \bar{\beta}_n|-\rangle = 0$ $n \geq 1$, ψ is independent of $\beta, \bar{\beta}$ oscillators, and $L_n \psi|-\rangle = 0$, $n \geq 0$. The ξ_n^j dependence of the vertex is given by

$$V(z_j, \psi_j, a_n^j, \xi_n^j) = V(z_j, \psi_j, a_n^j, 0) \prod_k \exp \sum_{n=0}^{\infty} \xi_n^j \bar{\beta}_n^j \quad (61)$$

However, the states $|x\rangle$ carry ghost number $\frac{1}{2}$ and the vertex $\int \prod_j' dz_j d\psi_j V(z_j, \psi_j, a_n^j, 0)$ has already the correct ghost number to give a non-zero result; therefore any ξ_n^j dependence has the wrong ghost number, and so gives no contribution. Consider first then only external states of positive norm

$$V(z_j, \psi_j, a_n^j, \xi_n^j) \prod_{k=1}^N \psi_k |-\rangle_k = V(z_j, \psi_j, a_n^j, 0, 0) \prod_{k=1}^N \psi_k |-\rangle_k \quad (62)$$

Clearly, on these states the scattering does not depend on a_n^j . This leaves the presence of the zero norm physical states. However, as we will shortly show, these states decouple and so W does not depend on ξ_n^j or a_n^j .

We recall that zero norm physical states should decouple. We observe that Eq. (58) reads infinitesimally, for $\xi_n^j = 0$,

$$V(z_j, \psi_j, a_n^j, 0) \phi = \sum_j' \psi_j \frac{\partial}{\partial z_j} V(z_j, \psi_j, a_n^j, 0) \quad (63)$$

and hence

$$\begin{aligned} W \{ |x\rangle_1 \dots \phi^{(i)} |1\rangle_i \dots |x\rangle_N \} \\ = \int \prod_i' dz_i d\psi_i \sum_j' \psi_j \frac{\partial}{\partial z_j} V(z_j, \psi_j, a_n^j, 0) \{ |x\rangle_1 \dots |1\rangle_i \dots |x\rangle_N \} \\ = 0 \end{aligned} \quad (64)$$

up to a boundary term which vanishes at tree level. At one loop, however, such terms were found [15,38] to be non-zero for all open bosonic strings except for the gauge group $SO(2^{13})$. Consequently, the decoupling of zero norm physical states can be seen to be a result of the fact that the vertex is an induced representation of the graded extension of the Virasoro group. We observe that the part of the group being used here involves $\bar{\beta}_{-1}$, L_{-1} and Q which form an $N = 2$ superalgebra.

It is useful to make contact with previous work on the computation of ghost vertices. One immediately sees that $\int \prod_j' d\psi_j V(z_j, \psi_j, a_n^j, 0)$ is what was called U in Ref. [18], and it was shown that this was the correct vertex to use with only the z_j Koba-Nielsen integration. In particular, we realize that for $a_n^j = 0$

$$\int \prod_j' d\psi_j V(z_j, \psi_j, a_n^j, 0) = V(z_j, 0, 0, 0) \prod_j' \bar{\beta}_1^j \quad (65)$$

which was the vertex for the $(v^j)^{-1} \mathbf{z} = \mathbf{z} - \mathbf{z}_j$ cycling, and whose explicit form is given in Ref. [18]. By sewing the BRST-invariant three-string vertex of Ref. [21], the authors of Ref. [39] have also constructed a vertex with N-3 ghost insertions.

For some of the cycling transformations which are required to make agreement with some of the old literature, say the α_n^μ vertex for N string scattering of Lovelace and Olive [5], the above discussion needs modifying. The problem is that one could take a cycling transformation for which the a_n^j actually depended on z^j . In this perhaps unnatural case, the decoupling does not take place in the simple way discussed above, since the $\partial/\partial z^j$ derivative snags on the a_n^j . The solution to this problem is to consider the vertex

$$V(z_j, 0, 0, 0) \left\{ \exp \sum_j' \psi_j \bar{\beta}_1^j \right\} \left\{ \exp \sum_j \sum_{n=0}^{\infty} a_n^j L_n^j \right\} \\ \cdot \left\{ \exp \sum_j \psi_j \bar{\beta}_{\chi_j} \right\} \left\{ \exp \sum_j \sum_{n \geq 0} \gamma_n^j \bar{\beta}_n^j \right\} \quad (66)$$

where $\bar{\beta}_{\chi_j}$ is the expression $e^{-(a \cdot L)} \partial/\partial z^j e^{a \cdot L}$, with L_n replaced by $\bar{\beta}_n$. It is then straightforward to show that decoupling occurs.

Let us now turn to the multiloop case which, for simplicity, we take to have only one external string. The generators outside the isotropy group are the L 's discussed above and $\bar{\beta}_{fr}^{(1)}$, $r = 1, \dots, g_1$ as well as $\bar{\beta}_{-1}^{(1)}$ and $\bar{\beta}_n^{(1)}$, $n > 0$. An exception is at one loop, where $\bar{\beta}_{-1}^{(1)}$ is actually in $I_1^\dagger$. The coset can therefore be parametrized by v_r, ψ_r , $r = 1, \dots, g$, which correspond to the moduli, that is, to the generators $L_{fr}^{(1)}$, $\bar{\beta}_{fr}^{(1)}$ as well as the co-ordinates z_j, ψ_j corresponding to $L_{-1}^{(1)}$ and $\bar{\beta}_{-1}^{(1)}$ and $a_n^{(1)}$, $\xi_n^{(1)}$ corresponding to $L_n^{(1)}$, $\bar{\beta}_n^{(1)}$, $n > 0$. We may write a general vertex as

$$\begin{aligned}
 V(\sigma_r, \bar{\Psi}_r, z_1, \psi_1, a_n^1, \tau_n^1) &= V(\bar{\sigma}_r, 0, \bar{z}_1, 0, 0, 0) \\
 &\cdot g(\sigma_r, \bar{\sigma}_r) \exp \left\{ \sum_{r=1}^{g_1} \bar{\Psi}_r \bar{\beta}_{f_r}^1 \right\} \exp(z_1 L_{-1}^1 + \psi_1 \bar{\beta}_{-1}^1) \\
 &\exp \left\{ \sum_{n=0}^{\infty} a_n^1 L_n^1 \right\} \exp \left\{ \sum_{n=0}^{\infty} \tau_n^1 \bar{\beta}_n^1 \right\}
 \end{aligned} \tag{67}$$

where $g(v_r, \bar{v}_r)$ is composed of coset generators $L_{\bar{f}_r}$ and boosts the loop moduli from $\bar{v}_r$ to v_r . The $\bar{\beta}_{\bar{f}_r}$ correspond to the moduli change at the point v_r . The scattering amplitude is given by putting physical states on

$$W = \int \prod_{r=1}^{g_1} d\sigma_r \prod_{r=1}^{g_1} d\bar{\Psi}_r dz_1 d\psi_1 \bar{f} V \tag{68}$$

except at one loop, where dz_1 and $d\psi_1$ are not present. The result is independent of $\xi_n^{(1)}$ and $a_n^{(1)}$ for the same reasons as in the tree case given above. We realize that after performing the $\bar{\Psi}_r$ and ψ_1 integrations, one finds the ghost prefactors which were introduced in Ref. [19] and in the last paper of Ref. [27].

The determination of the measure $\bar{f}$ by demanding decoupling of zero norm physical states is given in Ref. [19]. Its determination in this new formalism (i.e., before integration of the anticommuting variables) proceeds as for the tree case, and results from the fact that V carries an induced representation which includes Q . One uses the formula

$$\begin{aligned}
 &V(\sigma_r, \bar{\Psi}_r, z_1, \psi_1) \exp \bar{E} \bar{\Phi} \\
 &= V(\sigma_r, 0, z_1, 0) \exp \left\{ \sum_r \bar{\Psi}_r \bar{\beta}_{f_r} + \bar{E} \sum_s \bar{\Psi}_s L_{f_s} \right\} \\
 &\cdot \exp \left\{ \psi_1 \bar{\beta}_{-1}^1 + \bar{E} \psi_1 L_{-1}^1 \right\} \\
 &= V(\sigma_r, 0, z_1, 0) \exp \left\{ \sum_s \bar{E} \bar{\Psi}_s L_{f_s} \right\} \exp \left\{ \sum_r \bar{\Psi}_r \bar{\beta}_{f_r} \right\} \\
 &\exp \left\{ \frac{1}{2} \sum_{r,s} \bar{\Psi}_s \bar{\Psi}_r [L_{f_s}, \bar{\beta}_{f_r}] \right\} \exp(\psi_1 \bar{\beta}_{-1}^1 + \bar{E} \psi_1 L_{-1}^1)
 \end{aligned}$$

$$\begin{aligned}
&= \left[\exp \left\{ \sum_s \bar{\epsilon} \Phi_s \left(\frac{\partial}{\partial \sigma_s} - c_s \right) \right\} V(\sigma_r, 0, z_1, 0) \right] \\
&\cdot \exp \left\{ \sum_r \Phi_r \bar{\beta}_{fr} \right\} \exp \frac{1}{2} \sum_{r,s} \bar{\epsilon} \Phi_s \Phi_r [L_{fs}, \bar{\beta}_{fr}] \\
&\exp(\psi_1 \bar{\beta}'_{-1} + \bar{\epsilon} \psi_1 L'_1) \\
&= \left[\exp \left\{ \sum_s \bar{\epsilon} \Phi_s \left(\frac{\partial}{\partial \sigma_s} - c_s \right) \right\} V(\sigma_r, \Phi_r, z_1, \psi_1) \right] \\
&\cdot \exp(\psi_1 \bar{\beta}'_{-1} + \bar{\epsilon} \psi_1 L'_1) - V(\sigma_r, 0, z_1, 0) \exp \left\{ \sum_r \Phi_r \bar{\beta}_{fr} \right\} \\
&\cdot \sum_{r,s} \bar{\epsilon} \Phi_s \Phi_r \left\{ \frac{\partial \bar{\beta}_{fr}}{\partial \sigma_s} + \frac{1}{2} [L_{fs}, \bar{\beta}_{fr}] \right\} \\
&\cdot \exp(\psi_1 \bar{\beta}'_{-1} + \bar{\epsilon} \psi_1 L'_1)
\end{aligned}$$

(69)

$$= V(\sigma_r + \bar{\epsilon} \Phi_r, \Phi_r, z_1 + \bar{\epsilon} \psi_1, \psi_1) \exp \sum_s \bar{\epsilon} \Phi_s c_s$$

where we have used the identities discussed in Ref. [19] [i.e., Eqs. (30) and (31) and the discussion below them]. Using Eqs. (69) and (68) and demanding decoupling implies that

$$\frac{\partial \bar{f}}{\partial \sigma_r} + c_r = 0 \quad ; \quad \frac{\partial \bar{f}}{\partial z_1} = 0 \quad (70)$$

As discussed above, we may scale the vertex carefully in such a way that $c_r = 0$ in the identity of Eq. (19), and we conclude that $\bar{f} = 1$ in this case.

For the multiloop vertex with many external legs, the discussion is almost identical. The only difference is that one has z_j , $j = 1, \dots, N$ and ϕ_j , $j = 1, \dots, N$ except for one loop, where we have one less commuting and anticommuting co-ordinate, due to the one analytic vector field. The number of "loop" coset parameters is still $g_1 v_r$'s and $g_1 \bar{\psi}_r$'s, due to the presence of the additional elements in the isotropy group discussed above.

One of the advantages of viewing vertices as induced representations is that one can reverse the above procedure. Namely, suppose we assume that the vertices are induced representations; then, if we can compute them for a particular point

and find their isotropy group at that point, we can then determine the vertices at any point by boosting with coset generators whose action on the vertex is in terms of L_n 's, $\bar{\beta}_n$'s, etc., and so is known.

To give a concrete example, let us consider the case of Neveu-Schwarz tree graphs. The vertices will depend on $N-3$ z_j 's and $N-2$ θ_j 's, and were found in Refs. [16] and [22], using a superspace description of cycling transformations, superconformal operators, etc. However, we could also proceed as follows: let us first compute the vertex for the particular point $z_j, \theta_j = 0$, i.e. $V(z_j, 0)$. This is given by the straightforward overlaps of Eqs. (2) and (5) for R , taken to be P^μ and $H^\mu(z) = \sum_r b_{-r}^\mu z^r$, and for the ghost contribution with R , taken to be $\bar{\beta}, \beta, \bar{g}$ and g in turn. The $\alpha_r^\mu, \beta_n, \bar{\beta}_n$ part of the vertex is as for the bosonic case [14,17,18], while the b_r^μ dependence is given in Ref. [17]. We may also compute the isotropy group of this vertex at the point $z_j, \theta_j = 0$ by using the overlap with $R = L(z), \bar{\beta}(z), G(z) = \sum_r G_{-r} z^r$ and $\bar{g}(z) = \sum_r \bar{g}_{-r} z^r$ respectively. One finds that the isotropy group contains the same L_n 's, $\bar{\beta}(z)$ and Q as for the bosonic vertex discussed earlier, and also contains, for the simplest choice of cycling

$$G_{-n-\frac{1}{2}}^i \equiv G_{-n-\frac{1}{2}}^i + \sum_{j=1}^N \sum_{p=0}^{\infty} \binom{n}{p} \frac{G_{p-\frac{1}{2}}^j}{(z_j - z_i)^{n+p}}, \quad \sum_{j=1}^N G_{-\frac{1}{2}}^j, \quad \sum_{j=1}^N (z_j G_{-\frac{1}{2}}^j + G_j^j) \quad (71)$$

The coset therefore contains $G_{-\frac{1}{2}}^j$ for all j except for two values i_1 and i_2 , as well as $G_r^j, r > \frac{1}{2}, j = 1, \dots, N$.

The ghosts $\bar{\beta}_n$ and $\bar{g}_r$ are also in the isotropy group and coset in a one-to-one correspondence with the way the L_n 's and G_r 's appear in these two pieces. The vertex for any θ_j and k_j is then given by

$$V(z_j, \psi_j, \omega_j, k_j) = V(z_j, 0, 0, 0) \exp \sum_j \psi_j \bar{\beta}_{-1}^j \exp \left\{ \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^N \omega_j G_{-\frac{1}{2}}^j + \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^N k_j \bar{g}_{-\frac{1}{2}}^j \right\} \quad (72)$$

and the scattering amplitude is given by

$$\int \prod_j d z_j d \psi_j \prod_{\substack{i=1 \\ i \neq i_1, i_2}} \pi d \omega_i d k_i V(z_j, \psi_j, \omega_j, k_j). \quad (73)$$

which yields a result in agreement with Refs. [19] and [20] when one performs all the ghost algebra. The integrals over k_j should be taken to yield δ functions of

$\bar{g}_{-1/2}$. The ghost extension of the three-vertex has been given in Ref. [40] for a different choice of cycling transformation. The extension of the discussion to include multiloop Neveu-Schwarz graphs is straightforward.

It is clear that one can apply the strategy given above to the Ramond sector since, as has already been noted [14,17], the part of the vertex which is independent of the anticommuting parameter is determined by taking suitable conformal operators in Eqs. (2) and (5). This is the subject of the paper of Ref. [35], where the group theoretic method is extended to the Ramond sector.

It is instructive to note the relation between sewing and the vertices with anticommuting Koba-Nielsen variables. We note when we sew the BRST-invariant three-vertex [21] with an appropriate propagator to yield a tree or multiloop vertex, that the loop moduli and Koba-Nielsen variables arise from the propagators. The propagator depends upon the cycling chosen. For example, the propagator for the cycling choice that leads to the Lovelace-Olive vertex and particular Koba-Nielsen identification has the simple form

$$\int_0^1 \frac{dx}{x} x^{L_0} \quad (74)$$

We note [17] that sewing with the propagator produces a composite vertex which has the required Q^μ overlaps, and a little thought shows that, by the same argument, one will also find that the β and $\bar{\beta}$ overlaps of Eqs. (2) and (5) are also obeyed by the composite vertex, since they are satisfied by the three-vertex and go through the propagator according to their conformal weight. As such, the composite vertex which is determined by these overlaps must be the vertex with the anticommuting parameters set to zero. We also see that this vertex is annihilated by Q , since Q passes through the BRST-invariant three-vertex and goes through the propagator trivially, as the latter is just a conformal transformation.

One knows, however, that the propagator above is not satisfactory, as it must have non-zero ghost number; essentially, this is because any action must be non-zero between physical states with positive norm which are built by α_n^μ 's on the $|->$ vacuum. In fact, the correct propagator to use is found by including antighost factors in such a way that Q commutes with the propagator up to a total divergence. For example, we should modify the propagator of Eq. (74) by

$$\int_0^1 \frac{dx}{x} x^{L_0} \bar{\beta}_0$$

which we may write as

$$\int \frac{dz}{z} d\psi \exp(\ln z L_0 + \psi \bar{\beta}_0)$$

Sewing with this vertex, we find a vertex with the correct ghost number. Indeed, an N-string tree-level vertex and the loop measure [39,41] have been constructed by precisely this method, by sewing the BRST-invariant three-vertex of Ref. [21] with the appropriate propagator [39] for the Lovelace cycling choice. Indeed, writing the propagator in the second form, it is clear that one obtains a vertex before the z and ψ integration in the propagation which depends on $N+g_1$ bosonic and $N+g_1$ anti-commuting variables. This is of course the vertex discussed in this paper. We note that by making repeated use of the overlap equations for the three-vertex, we may move the $\exp \psi \bar{\beta}_0$ factor to the outside of the graph, to produce the required additional $\bar{\beta}$ factors. We recall that although this vertex has the usual Q^μ , P^μ and $\bar{\beta}$ overlaps, its β overlaps are not of the straightforward type of Eqs. (2) and (5). From the coset point of view, this is because one is no longer at the point where the anticommuting variables vanish, while from the sewing point of view, it is because the β identities do not pass simply through the propagator.

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A P P E N D I X

In this section, we derive the ghost oscillator extension of the vertex which describes the scattering of Neveu-Schwarz strings at tree level, as well as its superconformal and ghost prefactors. The latter are discussed along the lines of Ref. [18], but it is shown that the result is in agreement with the derivation using the strategy set out in this paper [i.e., Eq. (73)].

The α_n^μ and b_r^μ parts of the vertex were derived in Refs. [16] and [17]. We now derive the ghost part where $\bar{g}(z) = \sum_r \bar{g}_r z^{-r}$ and $g(z) = \sum_r g_r z^{-r}$ are the ghosts of conformal weight $3/2$ and $-1/2$ respectively. We note the identities

$$\begin{aligned} [g_s, \bar{g}_r] &= \delta_{r+s, 0} \quad , \quad G_s = [\varphi, \bar{g}_s] \\ [g_s, N] &= -g_s \quad , \quad [\bar{g}_s, N] = +\bar{g}_s. \end{aligned} \quad (A1)$$

where the BRST charge Q is given by

$$\begin{aligned} Q &= \sum_n \alpha_n^{(a,b)} \beta_n + \sum_s G_{-s}^{(a,b)} g_s - \frac{1}{2} : \sum_{m,n} (m-n) \beta_{-m} \beta_{-n} \bar{\beta}_{m+n} : \\ &+ \sum_{n,s} \left(\frac{3n}{2} + s \right) : \bar{g}_{-s} g_{n+s} \bar{\beta}_{-n} : - \sum_{r,s} g_r g_s \bar{\beta}_{-r-s} - \frac{1}{2} \bar{\beta}_0 \end{aligned} \quad (A2)$$

and N is the ghost number operator

$$N = \left(\sum_{n=-1}^{\infty} \beta_{-n} \bar{\beta}_n - \sum_{n=2}^{\infty} \bar{\beta}_{-n} \beta_n \right) - \left(\sum_{r=\frac{1}{2}}^{\infty} g_{-r} \bar{g}_r + \bar{g}_r g_r \right) \quad (A3)$$

Equation (6) for the conformal operator $\bar{g}(z)$ becomes, for Neveu-Schwarz lines,

$$V_0 \left\{ \sum_{j=1}^M \oint d\bar{z}_j \phi \frac{\bar{g}^j(z_j)}{(\bar{z}_j)^{3/2}} \right\} = 0 \quad (A4)$$

where we recall that we are using the simplest cycling choice $\xi_j^j = z - z^j$. In order for ϕ to have poles at Koba-Nielsen points only, it must go like at most $(\xi_j)^1$ for ξ_j large. Consequently, we have two analytic tensors which we may choose to be $\phi = 1$ and $\phi = \xi_j + z_j$. These lead to the two identities

$$V_0 \sum_{j=1}^M \bar{g}_{-\frac{1}{2}}^j = 0 \quad ; \quad V_0 \sum_{j=1}^M \left(z_j \bar{g}_{-\frac{1}{2}}^j + \bar{g}_{\frac{1}{2}}^j \right) = 0 \quad (A5)$$

A complete basis for all the remaining ϕ 's is given by

$$\phi = \frac{1}{(\bar{z}_i)^n} = \frac{1}{(\bar{z}_j + z_j - z_i)^n} = \sum_{p=0}^{\infty} \binom{-n}{p} \frac{(\bar{z}_j)^p}{(\bar{z}_j - z_i)^{n+p}} \quad n \geq 1 \quad (A6)$$

which lead to the identities

$$V_0 \left\{ \bar{g}_{-n-\frac{1}{2}}^j + \sum_{\substack{j=1 \\ j \neq i}}^m \sum_{p=0}^{\infty} \binom{-n}{p} \frac{1}{(\bar{z}_j - z_i)^{n+p}} \bar{g}_{p-\frac{1}{2}}^j \right\} = 0, n \geq 1 \quad (A7)$$

We can now solve for V . We observe that we have identities above which contain $\bar{g}_{-r}^i$, $r \geq 3/2$, and $\bar{g}_s^j$, $s \geq -1/2$. As such, they completely determine the dependence of the vertex on g_r , $r \geq 3/2$. With this in mind, we choose to work with a vacuum

$$\langle 0 | \bar{g}_r = 0 \quad r \leq -3/2, \quad \langle 0 | g_s = 0, s \leq \frac{1}{2} \quad (A8)$$

The vertex is given by

$$V_0 = \left\{ \prod_{i=1}^m \langle 0 | \right\} \exp - \left\{ \sum_{\substack{i,j=1 \\ i \neq j}}^m \sum_{n=1}^{\infty} g_{n+\frac{1}{2}}^i \binom{-n}{p} \bar{g}_{p-\frac{1}{2}}^j \right. \\ \left. \delta \left(\sum_{j=1}^m \bar{g}_{-\frac{1}{2}}^j \right) \delta \left(z^j \bar{g}_{-\frac{1}{2}}^j + \bar{g}_{\frac{1}{2}}^j \right) \right\} \quad (A9)$$

which is easily seen to solve the above identities.

This solution for the vertex is not unique since we can multiply it by any function of $\bar{g}_r$ and it clearly will still satisfy the identities. We must however also consider the identity for the ghosts, namely

$$V_0 \left\{ \sum_{j=1}^m \oint d\bar{z}_j \phi g^j(\bar{z}_j) (\bar{z}_j)^{\frac{1}{2}} \right\} = 0 \quad (A10)$$

where ϕ goes as $(\bar{z}_j)^{-3}$ as $\bar{z}_j \rightarrow \infty$. If we have poles at only one Koba-Nielsen point,

$$\phi = \frac{1}{(\bar{z}_i)^n} = \frac{1}{(\bar{z}_j + z_j - z_i)^n} = \sum_{p=0}^{\infty} \binom{-n}{p} \frac{(\bar{z}_j)^p}{(\bar{z}_j - z_i)^{n+p}} \quad n \geq 3 \quad (A11)$$

and we find the identity

$$V_0 \left\{ g_{-n+3/2}^i + \sum_{\substack{j=1 \\ j \neq i}}^m g_{p+3/2}^j \binom{-n}{p} \left(\frac{1}{z_j - z_i} \right)^{n+p} \right\} = 0. \quad (A12)$$

These are automatically satisfied by the vertex of Eq. (A9). However, it is still not uniquely determined, as we may multiply by any function of $\bar{g}_{\pm 1/2}$. The ϕ 's of Eq. (A11) do not constitute a complete set of allowed ϕ 's, since we can provide functions which have poles at more than one point and which are not expressible in terms of those of Eq. (A11). The functions that complete the set are

$$\phi = \left(\frac{1}{z_j} \right)^2 - \left(\frac{1}{z_i} \right)^2 \quad \text{and} \quad \left(\frac{1}{z_i} \right)^2 - \frac{1}{z_i z_j} \quad \forall i, j \text{ except } i=j \quad (A13)$$

They lead to the identities

$$V_0 \left\{ g_{-1/2}^i - g_{-1/2}^j + \dots \right\} = 0 \quad (A14)$$

and

$$V_0 \left\{ g_{-1/2}^i - \frac{g_{-1/2}^i}{(z_i - z_j)} + \frac{g_{-1/2}^j + \dots}{(z_i - z_j)} \right\} = 0 \quad (A15)$$

where $+\dots$ denotes terms involving g_s , $s \geq 3/2$. Any additional factor by which we may multiply the vertex of Eq. (A9) is of the form

$$\sum_j c^j \bar{g}_{-1/2}^j + \sum_j d^j \bar{g}_{1/2}^j \equiv \sum_j \oint d\bar{z}_j \psi \frac{\bar{g}^j(z_j)}{(z_j)^{3/2}} \quad (A16)$$

Such a factor must commute with the identities of Eqs. (A14) and (A15), which imply that

$$d^i = d^j \quad \text{and} \quad (z_i - z_j) d^i = c^i - c^j \quad (A17)$$

If $d^i = 0$, then $c^i = 1$ and we recover $\sum_j \bar{g}_{-1/2}^j$. If $d^i \neq 0$, we may set $d^i = 1$ and so $c^i - c^j = z_i - z_j$. The homogeneous solution to this equation is the possibility which we already have, while the inhomogeneous possibility is $(\sum_j z_j \bar{g}_{-1/2}^j + \bar{g}_{1/2}^j)$.

Consequently, we only have the two solutions we already know, and so we find that the vertex of Eq. (A9) is unique. The reader who wishes an exercise with overlap equations may repeat the above for the case of the $\beta, \bar{\beta}$ overlaps for the

tree-level open bosonic string. Indeed, the above illustration of how the ghost overlap identities for fields of conformal weight d and $1-d$ determine the tree-level vertex is quite general. Essentially, one solves the antighost identities. These involve analytic tensors which are realized by δ functions on the appropriate vacuum. One then looks at the ghost identities, which demonstrate that the vertex found by multiplying by the above δ functions and no more is the unique solution. Crucial to this is the use of ϕ 's with poles at more than one point. The additional factors by which we may multiply the vertex can be written in the form $\sum_j \int d\xi_j \phi R^j(\xi^j)$, and must satisfy

$$\sum_j \oint d\bar{z}_j \psi e^{(1-d)} = 0. \quad (A18)$$

where ϕ is any ϕ from the ghost overlap. From consistency, the analytic antighost factors obey this equation, but one must show that there are no more. This is to be expected, as if there existed another, we could multiply the vertex by it. However, this vertex would satisfy an antighost identity which could be seen to arise from an antighost ϕ which is not of the allowed type. In fact, this does not occur.

The situation for loops is a little different, as it is the ϕ 's for the ghost identities which can be analytic. However, one finds that the ϕ 's for the antighosts with poles at only one Koba-Nielsen point do not form a complete set, and those with poles at more than one point are required.

Since $G(z)$ and $\bar{g}(z)$ have the same conformal weight, they have the same integrated overlap identities, and so occur in the isotropy group of V_0 in the same way. Examining Eqs. (A5) and (A7), we find the isotropy group for $\bar{g}(z)$ which is in one-to-one correspondence with that for $G(z)$ in Eq. (71).

We observe that the vertex of Eq. (A9) involves δ functions of antighost oscillators which are bosonic. In fact, the same is true for the anticommuting oscillators in the open bosonic string as $\bar{\beta}_{-1} = \delta(\bar{\beta}_{-1})$, but their meaning in the case under consideration in this section requires further classification. The relation between $g_{\frac{1}{2}}$ and $\bar{g}_{-\frac{1}{2}}$ is $[g_{\frac{1}{2}}, \bar{g}_{-\frac{1}{2}}] = 1$, and they may be viewed in the same way as x and p ; that is, in the "co-ordinate representation", we may take $\bar{g}_{-\frac{1}{2}} = \bar{g}_{-\frac{1}{2}}$, $g_{\frac{1}{2}} = -(\partial/\partial \bar{g}_{-\frac{1}{2}})$. In this representation, $\langle 0|$ is a constant independent of $g_{-\frac{1}{2}}$, and $\delta(\bar{g}_{-\frac{1}{2}})$ is the usual type of δ function. We note that the more usual vacuum $\langle +|$, defined by

$$\langle +| \bar{g}_{-r} = 0 = \langle +| g_{-r} \quad r \geq \frac{1}{2} \quad (A19)$$

is related to the $\langle 0|$ by

$$\langle 0 | \delta(\bar{g}_{-\frac{1}{2}}^j) = \langle +1 \quad (A20)$$

It follows from Eq. (A1) that

$$[N, \delta(g_s)] = -\delta(g_s) ; [N, \delta(\bar{g}_s)] = +\delta(\bar{g}_s) \quad (A21)$$

To find the full vertex V , one must, as discussed in this paper and in Refs. [16] and [17], introduce a θ dependence or, equivalently, $G_{-\frac{1}{2}}$ prefactors. We simply recall the result

$$\begin{aligned} V(z_j) &= V_0(z_j) \prod_{j=1}^{m-2} G_{-\frac{1}{2}}^j \equiv \int \prod_{j=1}^{m-2} d\omega_j V(z_j, \omega_j) \\ &= \int \prod_{j=1}^{m-2} d\omega_j V_0(z_j) \exp\left(\sum_{j=1}^{m-2} \omega_j G_{-\frac{1}{2}}^j\right) \quad (A22) \end{aligned}$$

It finally remains to comment on the ghost prefactors by which one must multiply the vertex in order to find a vertex which gives non-zero on physical states. The latter have ghost number 1, and so, in order to give a non-zero result, the vertex must have ghost number M . The vertex of Eq. (A9), when supplemented by its α_n^μ , b_n^μ , β_n and $\bar{\beta}_n$ pieces, has ghost number $M+1$. It is also annihilated by the sum of the Q 's. We are in the same position as in the bosonic theory, and the resolution is the same as in that case [18]. We must multiply by antighost ghost factors that lead to the vertex having the correct decoupling properties and ghost number [18]. Such a vertex is given by

$$\begin{aligned} U &= \int \prod_{j=1}^{m-2} d\omega_j V_0(z_j) \exp\left\{\sum_{j=1}^{m-2} \omega_j G_{-\frac{1}{2}}^j\right\} \\ &\quad \cdot \prod_{j=1}^{m-2} \delta(\bar{g}_{-\frac{1}{2}}^j + \omega_j \bar{\beta}_{-1}^j) \left\{ \prod_{j=1}^{m-3} \bar{\beta}_{-1}^j \right\} \\ &= V_0(z_j) \left\{ \prod_{j=1}^{m-2} G_{-\frac{1}{2}}^j \prod_{j=1}^{m-2} \delta(\bar{g}_{-\frac{1}{2}}^j) \right\} \left\{ G_{-\frac{1}{2}}^{m-2} \delta(\bar{g}_{-\frac{1}{2}}^{m-2}) + \bar{\beta}_{-1}^{m-2} \delta(\bar{g}_{-\frac{1}{2}}^{m-1}) \prod_{j=1}^{m-3} \bar{\beta}_{-1}^j \right\} \quad (A23) \end{aligned}$$

The specific form of the δ function is determined by requiring that decoupling of zero norm physical states occur. We observe that

$$\begin{aligned} U \varphi &= V_0(z_j) \left\{ \prod_{j=1}^{m-3} G_{-\frac{1}{2}}^j \right\} \left\{ -\sum_{k=1}^{m-3} G_{-\frac{1}{2}}^k \delta'(\bar{g}_{-\frac{1}{2}}^k) + \bar{\beta}_{-1}^k \delta''(\bar{g}_{-\frac{1}{2}}^k) \right\} \\ &\quad \cdot \prod_{\substack{j=1 \\ j \neq k}}^{m-3} \delta(\bar{g}_{-\frac{1}{2}}^j) \left(G_{-\frac{1}{2}}^{m-2} \delta \bar{g}_{-\frac{1}{2}}^{m-2} + \bar{\beta}_{-1}^{m-2} \delta'(\bar{g}_{-\frac{1}{2}}^{m-1}) \right) \prod_{j=1}^{m-3} \bar{\beta}_{-1}^j \cdot (-1)^{m-4}. \end{aligned}$$

$$\begin{aligned}
 & + V_0(z) \prod_{j=1}^{m-3} G_{-\frac{1}{2}}^j \left\{ G_{-\frac{1}{2}}^{m-2} \delta(\bar{g}_{-\frac{1}{2}}^{m-2}) + \bar{P}_{-1}^{m-2} \delta'(\bar{g}_{-\frac{1}{2}}^{m-2}) \right\} \\
 & \cdot \left\{ \sum_{k=1}^{m-3} \bar{P}_{-1}^1 \dots L_{-1}^k \dots \bar{P}_{-1}^{m-3} (-1)^{m-3-k} \right.
 \end{aligned}
 \tag{A24}$$

where we have used the formula

$$[\delta(\bar{g}_{-\frac{1}{2}}), Q] = -G_{-\frac{1}{2}} \delta'(\bar{g}_{-\frac{1}{2}}) - \bar{P}_{-1} \delta''(\bar{g}_{-\frac{1}{2}})
 \tag{A25}$$

We then find that $WQ = 0$, where the scattering amplitude is given by

$$W = \int \prod_{i=1}^{m-3} dz_i U \bar{f}
 \tag{A26}$$

One easily verifies that the measure $\bar{f}$ is 1. As a result of Eq. (A21), we find that U has the correct ghost number. Indeed, one may write U as an exponential of oscillators acting on the $\langle + |$ vacua.

One recovers the result of Eq. (73) found by taking from the beginning the vertex to be an induced representation of the extended super Virasoro group. The latter method, however, automatically provides a vertex with the correct ghost number and decoupling properties in a rather efficient way. The one subtlety arises from making in Eq. (73) a choice of coset which is preserved by the action of Q . This point and the general theory behind the coset choice will be given elsewhere.