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PHENOMENOLOGY OF PERIODIC WINDOWS IN ONE DIMENSIONAL MAPS

J. Dias de Deus^{a)}, R. Dilão^{a),b)} and A. Noronha da Costa^{a)}

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It is well known that dissipative dynamical systems present a highly complex behaviour when a certain control parameter is varied. This complexity is associated to the presence of a large number of unstable periodic orbits. In fact, a commonly observed phenomenon in these systems is the appearance of stable periodic orbits by tangent bifurcation as the parameter is increased. Further increase in the parameter leads to the destabilization by period doubling and to the generation of chaotic bands that finally collapse onto an unstable periodic orbit with the same period (crisis) ¹⁾. The parameter region between tangent bifurcation and crisis is usually called a periodic window.

Maps of the interval have been a valuable tool in the analysis of dissipative systems, namely by the prediction of universal constants that can be readily measured in experiments. In particular, universality in the structure of periodic windows is expressed by the fact that the ratio, $m_c = (\mu_c - \mu_o) / (\mu_d - \mu_o)$, where μ_o , μ_d and μ_c are respectively the parameter values for tangent bifurcation, first period doubling and crisis, approaches the value $9/4$ when the period increases to infinity ²⁾.

The size of windows varies by orders of magnitude in the same bifurcation diagram, according to period and position in parameter space. For example, period 3 corresponds to the largest window for the logistic map and is clearly visible, while a certain detail is needed to observe longer periods. In fact, the period 3 is generally the easiest to observe experimentally ³⁾, while small windows corresponding to long periods tend to be washed out by noise.

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It is well known that dissipative dynamical systems present a highly complex behaviour when a certain control parameter is varied. This complexity is associated to the presence of a large number of unstable periodic orbits. In fact, a commonly observed phenomenon in these systems is the appearance of stable periodic orbits by tangent bifurcation as the parameter is increased. Further increase in the parameter leads to the destabilization by period doubling and to the generation of chaotic bands that finally collapse onto an unstable periodic orbit with the same period (crisis) ¹⁾. The parameter region between tangent bifurcation and crisis is usually called a periodic window.

Maps of the interval have been a valuable tool in the analysis of dissipative systems, namely by the prediction of universal constants that can be readily measured in experiments. In particular, universality in the structure of periodic windows is expressed by the fact that the ratio, $m_c = (\mu_c - \mu_0) / (\mu_d - \mu_0)$, where μ_0 , μ_d and μ_c are respectively the parameter values for tangent bifurcation, first period doubling and crisis, approaches the value $9/4$ when the period increases to infinity ²⁾.

The size of windows varies by orders of magnitude in the same bifurcation diagram, according to period and position in parameter space. For example, period 3 corresponds to the largest window for the logistic map and is clearly visible, while a certain detail is needed to observe longer periods. In fact, the period 3 is generally the easiest to observe experimentally ³⁾, while small windows corresponding to long periods tend to be washed out by noise.

Asymptotic estimates were given ⁴⁾ for the size of period p windows in parameter space, $\delta\mu(p)$, and for the width of the chaotic bands at crisis, $\delta x(p)$, in terms of parameters of the p -iterated map,

$$\delta\mu(p) \underset{p \rightarrow \infty}{=} \frac{9}{4} \frac{2}{|a^{(p)} \cdot b^{(p)}|}, \quad \delta x(p) \underset{p \rightarrow \infty}{=} \frac{8}{|b^{(p)}|} \quad (1)$$

and numerical evidence was presented for the asymptotic relation,

$$\frac{(\delta x(p))^2}{\delta\mu(p)} \propto \frac{a^{(p)}}{b^{(p)}} \underset{p \rightarrow \infty}{=} \text{const.} \quad (2)$$

where,

$$a^{(p)} = \frac{\partial f^p}{\partial \mu}(x_0, \mu_0), \quad b^{(p)} = \frac{\partial^2 f^p}{\partial x^2}(x_0, \mu_0) \quad (3)$$

and x_0 is the period p point nearest to the critical point at tangent bifurcation, $\mu = \mu_0$. However, this approach does not permit to compare the size of different windows.

We here show that the size of the period p windows converging to the final aperiodic regime decreases geometrically with the period. Asymptotic estimates are obtained for $\delta\mu(p)$ and $\delta x(p)$ when $p \rightarrow \infty$, permitting to compare the widths of windows of different periods. These estimates are given in terms of the derivatives of the map $f_\mu(x)$ and derived extending the scaling properties of the iterates in the approach to the final aperiodic regime.

We consider analytic and even maps $f_\mu(x)$, $0 \leq \mu \leq 1$, $-1 \leq x \leq 1$, with a single and non degenerated critical point at $x=0$, such that $f_\mu(-1) = f_\mu(1) = -1$ for all μ and $f_\mu(0)$ is a monotonic increasing function of μ . In the final regime, $\mu = 1$, the trajectory

of the critical point falls onto the unstable fixed point at $x^* = -1$ after two iterations (final crisis), $f^2_1(0) = x^*$. In ref. 5 it was proved that the asymptotic behaviour in μ of the iterates $f^p_\mu(0)$ is described by a scaling function,

$$f^p_\mu(0) \underset{p \rightarrow \infty}{=} F(\Delta^p(1-\mu))$$

where $\Delta = f'_1(x^*)$. The zeros of the scaling function F are associated to a family of period p orbits containing the critical point, $f^p_{\mu_p}(0) = F(\Delta^p(1-\mu_p)) = 0$ with $3 \leq p < \infty$, converging geometrically to the final regime when $p \rightarrow \infty$, $1 - \mu_p \propto \Delta^{-p}$. In fact, these μ_p are the largest parameters for which a period p superstable orbit exists 5).

We now generalize these results, showing that both the dependence in the parameter μ and in x are described by a single scaling function when $p \rightarrow \infty$. Expanding $f^p_\mu(x)$ in μ and x around $\mu = 1$ and $x = 0$ and taking the leading order terms of all the derivatives when $p \rightarrow \infty$, we obtain 6),

$$f^p_{\mu(x)} \underset{p \rightarrow \infty}{=} H(Z) \quad (4)$$

where $H(Z)$ is an analytic scaling function in the variable

$$Z = \Delta^p[\alpha(\mu - 1) + \beta x^2/2] \quad (5)$$

with,

$$\alpha = \frac{\partial f}{\partial \mu}(0, 1) \quad \text{and} \quad \beta = \frac{\partial^2 f}{\partial x^2}(0, 1) \quad (6)$$

The parameters $a(p)$ and $b(p)$ in (3) are then easily estimated from the scaling function. For $x=0$ and $0 \leq \mu \leq 1$

$$a(p) = \frac{\partial f^p}{\partial \mu} \Big|_{p \rightarrow \infty} \propto \Delta^p \frac{\partial H}{\partial z}, \quad b(p) = \frac{\partial^2 f}{\partial x^2} \Big|_{p \rightarrow \infty} \propto \beta \Delta^p \frac{\partial H}{\partial z} \quad (7)$$

and we obtain, for $b(p) \neq 0$,

$$a(p)/b(p) = \alpha/\beta. \quad (8)$$

in agreement with (2).

Relation (8) fails for $b(p)=0$ (we suppose $a(p) \neq 0$). From the definition of $b(p)$, (3), and the fact that $x=0$ is the critical point, $b(p) = f'_\mu(f^{p-1}_\mu(0)) \dots f'_\mu(f_\mu(0))f''_\mu(0)$ and, $b(p)=0$ if, and only if, for some $1 \leq k < p$ and $0 \leq \mu' \leq 1$, $f^k_{\mu'}(0)=0$, which happens at superstable orbits of period $k < p$. As the sequence of parameters μ_p for which there is a last period p is increasing, for $\mu > \mu_p$ there are no superstable periods $k < p$ and relation (8) is verified. In Fig. 1 we test the scaling relation (8) in the approach to the final aperiodic regime and we conclude that (8) holds for each p except for small parameter regions around superstable orbits with periods $j < p$.

Estimates (1) for $\delta\mu(p)$ and $\delta x(p)$ can now be expressed in terms of parameters α and β , (6), characterizing the map f_μ . After substitution of (7) into (1) we obtain, for the size of the period p window and the width of the period p chaotic band,

$$\begin{aligned} \delta\mu(p) & \Big|_{p \rightarrow \infty} = \frac{9}{4} \cdot \frac{8}{|\alpha\beta|} \cdot \frac{\Delta^{-2p}}{|H'(z_p)|^2} \\ \delta x(p) & \Big|_{p \rightarrow \infty} = \frac{16}{|\beta|} \cdot \frac{\Delta^{-p}}{|H'(z_p)|} \end{aligned} \quad (9)$$

where $Z_p = \alpha \Delta^p (\mu_p - 1)$ is the value of the scaling variable at the last superstable period p orbit and $x=0$. Equations (9) are tested in Fig.2 for the logistic map. The size of the windows in parameter space and the width of the chaotic bands decrease geometrically with the period, satisfying asymptotically the universal relation,

$$\delta\mu \underset{p \rightarrow \infty}{=} c(\delta x)^2 \quad (10)$$

where $c = |(9\beta)/(\alpha 2^7)|$ is a map dependent constant. Equations (9) and (10) hold for any analytic map of an interval with one quadratic critical point.

In the case of a quadratic map the scaling function is easily computed and we obtain 6),

$$f_{\mu}^p(x) \underset{p \rightarrow \infty}{=} H(Z) = -\cos\sqrt{Z/2} \quad (11)$$

with $Z = \Delta^p[|\alpha|(1-\mu) + |\beta|x^2/2]$ and $\Delta=4$. Asymptotic estimates for the position in the parameter space of superstable periods are readily obtained. As $f_{\mu_p}^p(0) = H(Z_p) = 0$, from (11) we have,

$$\mu_{p(j)} = 1 - \frac{\pi^2 (2j+1)^2 \Delta^{-p}}{2|\alpha|}, \quad j = 0, 1, \dots, \mathcal{N}_p \quad (12)$$

where $j=0$ corresponds to the last superstable of period p , $\mu_{p(0)} = \mu_p$ and $\mathcal{N}_p$ is the number of period p windows. A test of equation (12) is presented in Table 1 for the last superstables of periods $3 \leq p \leq 7$ of the map $f_{\mu}(x) = 2\mu - 1 - 2\mu x^2$. The formula is still reasonably accurate for $\mu_{p(j)}$ with $j \neq 0$ small ($j \ll \mathcal{N}_p$). For example, for

the two period 7 superstable points preceding the last one (i.e., $j=1$ and $j=2$), by (12), $\mu_{7(1)}=0.998644$ and $\mu_{7(2)}=0.996235$, which corresponds to errors of 0.001% and 0.004%, respectively.

Estimates (9) can also be explicitly computed for the quadratic map. By (9) and (11),

$$\begin{aligned}\delta\mu_{p(j)} &= \frac{18 \pi^2 (2j+1)^2 \Delta^{-2p}}{|\alpha\beta|}, \quad j=0, 1, \dots, \mathcal{N}_p \\ \delta x_{p(j)} &= \frac{16 \pi (2j+1) \Delta^{-p}}{|\beta|}, \quad j=0, 1, \dots, \mathcal{N}_p\end{aligned}\tag{13}$$

and these expressions are tested in Table 2 and Fig. 2 for the last windows of a quadratic map. These estimates are still reasonably accurate for $j \neq 0$ small, Fig. 3.

It is interesting to note that from (13) and taking in account the fact that for fixed p ⁷⁾, $j_{\max} \propto 2^{p-1}/p$, it follows that, excluding the special case of period 1, period 3 has the largest window.

In conclusion, a unified framework is presented for the scaling behaviour of the iterates $f^p_\mu(x)$ when $p \rightarrow \infty$, both with respect to the μ and x dependence. It is shown that $f^p_\mu(x) = f_\mu(f^{p-1}_\mu(x)) = H(Z)$ when $p \rightarrow \infty$ and H is a scaling function in the variable Z (expressions (4) and (5)). This leads to estimates for the size of the period p windows.

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Table Captions:

TABLE 1: Test of formula (12) for the last superstable parameters of period p ($j=0$) of the quadratic map $f_\mu(x) = 2\mu - 1 - 2\mu x^2$, $0 \leq \mu \leq 1$ and $-1 \leq x \leq 1$.

TABLE 2: Test of formula (13) for the size of the last period p window in the bifurcation diagram of the quadratic map.

Figure Captions:

FIGURE 1: Test of (8) for the quadratic map. With $f_\mu(x) = 2\mu - 1 - 2\mu x^2$, we have $\alpha = 2$, $\beta = -4$ and so $|a^{(p)}/b^{(p)}| = 1/2$. We indicate the values of the parameter for which $f_\mu(x)$ has superstable orbits of periods 5, 6 and 7.

FIGURE 2: Test of relations (13) for the size of the last period p windows and the width of the chaotic bands at window crisis in the quadratic map.

FIGURE 3: Test of equation (13) for the size of the period 7 windows in the quadratic map as a function of j ($0 \leq j \leq 8$). The numerical values are represented as black circles. For the two period 7 windows preceding the last one (i.e., $j=1$ and $j=2$), our estimates give, $\delta\mu_{7(1)} = 7.44535 \times 10^{-7}$ and $\delta\mu_{7(2)} = 2.06815 \times 10^{-6}$, which correspond to relative errors of 3.3% and 8.2%, respectively.

TABLE 1:

p	estimated μ_p	μ_p	Relative error (%)
3	0.96144685	0.95796851	0.36309
4	0.99036171	0.99006753	0.02971
5	0.99759043	0.99756676	0.00237
6	0.99939761	0.99939578	0.00018
7	0.99984940	0.99984926	0.00001

TABLE 2:

p	estimated $\delta\mu_p$	$\delta\mu_p$	Relative error (%)
3	$5.4215 \cdot 10^{-3}$	$7.0934 \cdot 10^{-3}$	23.6
4	$3.3884 \cdot 10^{-4}$	$3.7341 \cdot 10^{-4}$	9.3
5	$2.1178 \cdot 10^{-5}$	$2.1852 \cdot 10^{-5}$	3.1
6	$1.3236 \cdot 10^{-6}$	$1.3364 \cdot 10^{-6}$	1.0
7	$8.2726 \cdot 10^{-8}$	$8.2962 \cdot 10^{-8}$	0.3

Fig. 1

Fig. 2

Fig. 3