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QUANTUM CONSISTENCY OF OPEN STRING THEORIES

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ABSTRACT

Anomaly cancellation in open string theories is shown to select the gauge group $SO(2^{D/2})$ uniquely, although finiteness of (1-loop) amplitudes can be achieved for any $SO(n)$ or $USp(n)$ gauge group. The discussion is mainly restricted to open bosonic strings, but it also applies to Type I superstrings. These results open new perspectives for the recent suggestion made in Ref.[1], the generalisations of which allow for the construction of new open string theories in less than 10 dimensions.

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1. INTRODUCTION

The Type I open superstring theory is free of any 1-loop gauge anomalies for only one gauge group, namely $SO(32)$ [2-6]. For that theory, any amplitude with massless external states can be made finite, provided the contributions from oriented and unoriented world-sheets are appropriately summed [6-12]. However, divergences in each of those classes of diagrams can be made to cancel against each other for other gauge groups as well, as for example $SO(8)$ [8,9].

These results are not sufficient to establish the $SO(32)$ superstring as a consistent quantum theory. All zero-norm states, and not only the longitudinal gauge bosons [2] and gravitons [13], must decouple from amplitudes, and any amplitude with physical external states should be finite (up to the problem of mass renormalisation [14]).

The same questions can of course be raised in the context of open bosonic string theories. Using the same summation of oriented and unoriented diagrams which makes Type I superstring amplitudes finite for $SO(2^5)$, it has been shown [15] that the M-tachyon amplitude in open bosonic strings is finite for $SO(2^{13})$, and that the closed string dilation tadpole vanishes for this gauge group as well [16,17]. Moreover, it is believed that all zero-norm states decouple from any amplitude [18,19]. This is indeed to be expected for the massless sector, since the corresponding effective field theory of bosonic fields is free of any gauge and gravitational anomalies. However, this is not so clear for the massive sector and it has indeed been suggested that decoupling of massive zero-norm states requires the gauge group $SO(2^{13})$ [20].

The appearance of the gauge groups $SO(2^{D/2})$ for open string theories must have a more profound origin than that of merely numerical coincidences: the almost identical contributions from orientable and non-orientable diagrams is indeed very suggestive [21].

The underlying reason for the quantum consistency of open string theories with gauge group $SO(2^{D/2})$ is certainly to be found in the recent proposal [1] that these can be obtained from closed string theories by a $\mathbb{Z}_2$ -orbifold construction in the world-sheet (and not in the space in which the string degrees of freedom take their values [22]). After all, this is the intuitive idea that an open string is "half" a closed string, with left- and right-moving fields identified.

Indeed, it has been claimed [1] that such a construction leads uniquely to the gauge group $SO(2^{D/2})$ [23]. As the concept of modular invariance gains its full applicability again in such an approach to open strings, it is to be expected

that quantum consistency (i.e. finiteness and decoupling of zero-norm states) would simply follow from modular invariance, as it does for theories of closed strings [24,25]. In particular, it would determine [1] how oriented and unoriented diagrams have to be summed, thereby solving the problem of regularisation of these theories. We are thus led to conjecture that all consistent quantum theories of open strings are to be obtained through such a $\mathbb{Z}_2$ -orbifold construction in the world-sheet.

In this paper, we show how quantum consistency of open bosonic strings selects the gauge group $SO(2^{13})$ uniquely. Together with the corresponding results for Type I superstrings, this proves the conjecture for all known open string theories.

The approach taken in Ref.[1] thus opens the way to many new developments. One obvious direction is in the construction of new open string theories (with or without supersymmetry) in space-time dimensions ranging from 10 to 4 (or less) [26], with possibly large gauge groups and chiral spectra of fermions. Open string model building becomes possible. Another direction is in the generalization of the approach to other orbifold constructions in the world-sheet (such as $\mathbb{Z}_n$). This clearly leads to the conjecture that all consistent quantum string theories can be obtained through such constructions. All string theories would then be theories of closed strings only, characterized by different orbifold structures in their respective world-sheets.

In this paper, the discussion is restricted to open bosonic strings (in 26 dimensions), although our considerations also apply to Type I superstrings. The question of finiteness of (1-loop) amplitudes (with external open string states only) is first considered, with the conclusion that they can be made finite for any $SO(n)$ or $USp(n)$ gauge group by appropriate regularisations of oriented and unoriented diagrams. Moving on to the question of (1-loop) anomaly cancellation, we show how this uniquely selects the gauge group $SO(2^{13})$ independently of any regularisation ambiguities, although the open string tachyon makes the theory inconsistent. Details of these results will be presented separately [27].

2. FINITENESS

From Ref.[28], we know that for every state

$$|\phi; k\rangle = \sum C_{\mu_1 \dots \mu_m} \alpha_{-n_1}^{\mu_1} \dots \alpha_{-n_m}^{\mu_m} |0; k\rangle \quad (1)$$

in the Hilbert space of the open bosonic string (we use the notations of Ref.[29] with $2\alpha' = 1$), which satisfies the conditions

$$(L_0 = \frac{1}{2} p^2 + N) |\phi; k\rangle = \Delta |\phi; k\rangle, \quad L_n |\phi; k\rangle = 0, \quad n \geq 1, \quad (2)$$

there exists, in 1-to-1 correspondence, a primary conformal field $\phi(z)$ of weight Δ , such that

$$[L_n, \phi(z)] = z^{n+1} \partial_z \phi(z) + \Delta(n+1) z^n \phi(z) \quad (3a)$$

$$|\phi; k\rangle = \lim_{z \rightarrow 0} \phi(z) |0; 0\rangle \quad (3b)$$

and given by

$$\phi(z) = \sum c_{\mu_1 \dots \mu_m} : \prod_{j=1}^m \frac{i}{(n_j-1)!} \partial_z^{n_j} X^{\mu_j}(z) e^{ikX(z)} : , \quad (4)$$

where

$$X^\mu(z) = q^\mu - i p^\mu \ln z + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{n} \alpha_n^\mu z^{-n}. \quad (5)$$

Amplitudes of physical states (i.e. states which satisfy (2) with $\Delta = 1$) are given by the correlation function of the corresponding fields, integrated over the positions of these vertices $\phi(z)$ on the boundary of the appropriate Riemann surface. The factors $\partial_z^{n_j} X^{\mu_j}(z)$ in (4) can be included in the vertex $\exp(ikX(z))$ as $\exp(ik_\mu X^\mu(z) + i\eta_\mu \partial_z^{n_j} X^\mu(z))$; by taking the appropriate terms in an expansion in the auxiliary vectors η_μ , the corresponding amplitude is obtained. It is given by the correlation function of the vertices $\exp(ikX(z))$ multiplied by combinations of various derivations of the correlation function of the scalar field $X^\mu(z)$ on the Riemann surface.

Among physical states, one has zero-norm states, such as $L_{-1} |0; k\rangle$ (with $k^2 = 0$), $L_{-1} \epsilon_\mu \alpha_{-1}^\mu |0; k\rangle$ (with $k^2 = -2$, $\epsilon^2 = 1$, $\epsilon k = 0$) or $(L_{-2} + 3/2 L_{-1}^2) |0; k\rangle$ (with $k^2 = -2$). For consistency of the theory, these states must decouple from amplitudes, i.e. amplitudes with external zero-norm states must vanish, and such states (as well as all other unphysical states) cannot contribute to pole and cut singularities of physical amplitudes. As is well known, the properties (3) ensure that this is indeed the case for tree amplitudes with arbitrary external physical states (see Ref.[24] for a discussion in the case of tachyon amplitudes) [30].

Open string tree amplitudes have cyclicity properties (see Ref.[24] for the case of tachyon amplitudes) [30], which allow for the introduction of gauge degrees of freedom through Chan-Paton factors [31]. Factorisation of tree amplitudes restricts the choice of these factors [32]; for oriented strings, one can only have $n \times n$ complex matrices, and for unoriented strings one can only have $n \times n$ real matrices or $n \times n$ quaternionic matrices. In each case, these Chan-Paton factors $\lambda_{\alpha\beta}$ must be anti-Hermitian for even mass levels and Hermitian for odd mass levels [32]. These three possibilities lead respectively to gauge symmetries associated with $U(n)$, $SO(n)$ and $USp(n)$. The contribution of these Chan-Paton factors to amplitudes is given by the trace of their product with the same cyclic ordering as that of the corresponding vertices on the boundaries of the world-sheet.

In the case of loop amplitudes, one has to make sure that none of the unphysical (negative-norm) nor zero-norm states contributes to singularities due to intermediate states. In the operator approach used here, this can be implemented at the 1-loop level through Feynman's tree theorem [33], and has to be done so in the case of planar diagrams for bosonic [34] and supersymmetric [35] open string theories. Those results can easily be extended to non-planar and Möbius strip diagrams [30]; they determine the integration measure over the modular parameter of the world-sheet.

Through (3b), we choose to normalize the vertices $\phi(z)$ for (positive-norm) physical states by requiring that the corresponding states $|\phi; k\rangle$ have norm (+1) (with $\langle k|k'\rangle = \delta^{(D)}(k-k')$); we also impose that $\text{tr}\lambda_{\alpha}^2 = \pm 1$. For a process with M external states of momenta k_i^μ ($i = 1, \dots, M$), the S-matrix element is given by

$$S(k_1, \dots, k_M) = \prod_{i=1}^M (2\pi)^{-(D-1)/2} (2k_i^0)^{-1/2} A(k_1, \dots, k_M), \quad (6)$$

where $A(k_1, \dots, k_M)$ is the corresponding amplitude. At 1-loop, we then have [30], for orientable (or planar and non-planar) diagrams

$$A_o^{(1)}(k_1, \dots, k_M) = -ig^M \frac{1}{M} C_o \frac{1}{2} \int_0^1 \frac{dw}{w^2} f^2(w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_i| \delta(z_i - z_i^o) \cdot \text{Tr} R \left[\prod_{i=1}^M \phi_i(z_i) w^{L_o} \right], \quad (7)$$

and for non-orientable (or Möbius strip) diagrams

$$A_n^{(1)}(k_1, \dots, k_M) = -ig^M \frac{1}{M} C_n \frac{1}{2} \int_0^1 \frac{dw}{w^2} f^2(-w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_i| \delta(z_i - z_i^o) \cdot \text{Tr} R \left[\prod_{i=1}^M \phi_i(z_i) w^{L_o} (-1)^N \right], \quad (8)$$

where

$$f(w) = \prod_{n=1}^{\infty} (1 - w^n) , \quad \Omega(w) = [-1, -w] \cup [w, 1] , \quad (9a)$$

and

$$C_0 = 1 , \quad C_N = 0 \quad \text{for } U(n) , \quad (9b)$$

$$C_0 = 1/2 , \quad C_N = 1/2 \quad \text{for } SO(n) , \quad (9c)$$

$$C_0 = 1/2 , \quad C_N = -1/2 \quad \text{for } USp(n) . \quad (9d)$$

In these expressions, g is the open string coupling constant, R stands for the R -ordered product, $\phi_i(z_i)$ are the vertices corresponding to the external physical open string states, the trace is taken over the oscillators α_n^μ , the zero-mode P^μ and the Chan-Paton factors (associated to each $\phi(z)$), and the factor $|z_a| \delta(z_a - z_a^0)$ (with z_a one of the z_i 's and z_a^0 an arbitrary point in $\Omega(w)$) is the gauge-fixing factor for the (real) conformal Killing vector of the cylinder and the Möbius strip.

Eq. (7) represents an integration over the two boundaries $[-1, -w]$ and $[w, 1]$ of a cylinder defined by $(w \leq |z| \leq 1, \text{Im}z \geq 0)$ and the equivalence $(w \sim wz)$. Eq. (8) represents an integration over the single boundary $\Omega(w)$ of a Möbius strip defined by $(w \leq |z| \leq 1, \text{Im}z \geq 0)$ and the equivalence $(w \sim -wz)$. The factors $f^2(\pm w)$ follow from Feynman's tree theorem [34,30]; they lead to the correct partition functions of these theories. The values for C_0 and C_N agree with those given in Ref.[15] in the case of the M -tachyon amplitude; in particular, the relative sign between planar and Möbius diagrams for $SO(n)$ and $USp(n)$ bosonic strings is opposite to that of the corresponding Type I superstrings [24]. This is important for our discussion.

The evaluation of (7) and (8) leads to three different types of contributions. The planar one, with all external vertices attached to the same boundary of the cylinder, is given as a sum over all cyclically inequivalent permutations of the external states, with each term of the form

$$-ig^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) C_0 \frac{n \text{tr}(\pi \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f^{-24}(\lambda) \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \cdot$$

$$\prod_{1 \leq i < j \leq M} \left[\frac{\theta_1(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\theta_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot \left[N_i \text{ derivatives in } \frac{\partial}{\partial v_i} \right] . \quad (10)$$

The non-planar contribution, with the external vertices split into two sets A and B corresponding to the two boundaries of the cylinder, is given as a sum

over all such splits, with permutations of A and B or within A or within B not counted as different splits, with each term of the form

$$\begin{aligned}
 & -ig^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) C_0 \frac{\text{tr}(\prod_{i \in A} \lambda_d^{(i)}) \text{tr}(\prod_{i \in B} \lambda_d^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f^{-24}(\lambda) \int_{R_2} \prod_{i=1}^M dv_i \delta(v_M) \cdot \\
 & \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\theta_1 \text{or } \theta_4(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\theta'_1(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}] \quad (11)
 \end{aligned}$$

The non-orientable contribution is given as a sum over all cyclically inequivalent permutations of the external lines, with each term of the form

$$\begin{aligned}
 & -ig^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) C_N \frac{2^{13} \text{tr}(\prod_{i=1}^M \lambda_d^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \frac{1}{\lambda} f^{-24}(-\lambda) \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \cdot \\
 & \cdot \prod_{1 \leq i < j \leq M} \left[\frac{\theta_1(v_i - v_j | \frac{\ln(-\lambda)}{2i\pi})}{\theta'_1(0 | \frac{\ln(-\lambda)}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}] \quad (12)
 \end{aligned}$$

In those expressions, $\theta_1(v|\tau)$ and $\theta_4(v|\tau)$ are the Jacobi θ -functions, N_i is the excitation level of the i th external state with $\frac{1}{2}k_i^2 + N_i = 1$, the integration region R_1 is defined by $0 \leq v_M \leq v_{M-1} \leq \dots \leq v_1 \leq 1$ (corresponding to the cyclic ordering of the vertices on the boundary), and the integration region R_2 is defined by $0 \leq v_{MA}^A \leq v_{MA-1}^A \leq \dots \leq v_1^A \leq 1$ and $0 \leq v_{MB}^B \leq v_{MB-1}^B \leq \dots \leq v_1^B \leq 1$ with v_1^A and v_1^B corresponding to the vertices in the sets A and B defined above. In (11), $\theta_1(v_i - v_j|\tau)$ or $\theta_4(v_i - v_j|\tau)$ is to be used depending on whether the vertices ϕ_i and ϕ_j are attached to the same boundary or to different boundaries. The gauge-fixing $\delta(v_M)$ in terms of v_M could also be done in terms of any other v_i with the appropriate orderings. Finally, all terms which involve derivatives are combinations of products of derivatives of the corresponding propagators $\ln \theta_1$ and $\ln \theta_4$. We also set $D = 26$.

In obtaining (10) and (11), we used the change of variables

$$\lambda = \exp\left(\frac{4\pi^2}{\ln w}\right), \quad v_i = \frac{\ln z_i}{\ln w}, \quad (13a)$$

and for the unoriented diagrams (12)

$$\lambda = \exp\left(\frac{\pi^2}{\ln w}\right), \quad v_i = \frac{\ln z_i}{2 \ln w}. \quad (13b)$$

Let us emphasize that the expressions (10), (11) and (12) are valid for any 1-loop amplitude with any external physical open string states. The notorious

thing about (10) and (12) is that they only differ by a factor $(C_N C_0^{-1} 2^{13})$ and the replacement $(\lambda \rightarrow -\lambda)$ in the integrands. The origin of the factor 2^{13} can be traced back to the integration over the bosonic zero modes (or to the transformation relating $f(\pm w)$ to $f(\pm \lambda)$). In the case of open superstrings, the expressions are essentially the same (the factors $\lambda^{-1} f^{-24}(\pm \lambda)$ simply do not appear) and the analogous factor 2^5 (which has been established for amplitudes with M massless gauge bosons only [3,8,10,11]) can be traced back to the fermionic zero modes. The discussion which follows thus also applies to those theories.

The amplitudes above have divergences of different physical meaning and origin. We are not concerned here with those which arise when some vertices come close to each other; they correspond to singularities due to intermediate states in different channels [14,24].

A divergence common to all three types of diagrams comes from the region $\lambda \rightarrow 1$ ($w \rightarrow 0$). This divergence is due to the tachyon of the open string, but it does not possess any physical meaning; it is simply a consequence of an inappropriate integral representation of the vacuum loop for such a state [6,20,27]. It can be shown [27] that the correct representation is obtained by inserting a factor w^γ in the integration over w , where γ is an arbitrary parameter which may be different for all three types of diagrams. All integrals must be evaluated with $\gamma > 1$, and only in the final result can one take the analytic continuation to $\gamma = 0$.

All three types of contributions also have divergent behaviours as $\lambda \rightarrow 0$ ($w \rightarrow 1$), with a well-known physical meaning; they correspond to closed string intermediate states [24,36]. For the non-planar diagram (11), the integrand has the form

$$\int_0^1 \frac{d\lambda}{\lambda^2} \exp\left(\frac{4\pi^2 \gamma}{\ln \lambda}\right) (2\pi \lambda^{1/4})^{s_{AB}} G(\lambda) , \quad (14a)$$

where $G(\lambda)$ is an analytic function of $|\lambda| < 1$, with

$$G(\lambda) = \sum_{n=0}^{\infty} G_n \lambda^n , \quad (14b)$$

and $s_{AB} = (\sum_{i \in A} k_i)^2 = (\sum_{i \in B} k_i)^2$ is the invariant energy exchanged between the two boundaries. Eq. (14) is to be evaluated by analytic continuation in s_{AB} , with $s_{AB} > 8$ [24,36]. This leads to:

$$\lim_{\gamma \rightarrow 0} (2\pi)^{s_{AB}} \sum_{n=0}^{\infty} \frac{G_n}{s_{AB}/4 + n - 1} \left[16\pi^2 \gamma \left(\frac{s_{AB}}{4} + n - 1 \right) \right]^{1/2} K_1 \left[\left[16\pi^2 \gamma \left(\frac{s_{AB}}{4} + n - 1 \right) \right]^{1/2} \right] , \quad (15)$$

where $K_1(z)$ is a Bessel function with

$$z K_1(z) = 1 + \sum_{k=0}^{\infty} \frac{(z/2)^{2(k+1)}}{k!(k+1)!} \left[\ln \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(k+2) \right]. \quad (16)$$

Eq.(15) has poles at $s_{AB} = 8(n-1)$ ($n=0,1,\dots$), corresponding to closed string states [36]. The limit $\gamma \rightarrow 0$ can only be taken after the series has been resummed; it cannot be taken in each term separately as the corresponding quantity would diverge. Hence, (15) has logarithmic branch cuts for $(s_{AB}/8 + n - 1) < 0$; this is clearly due to the open string tachyon, and is simply the statement that tachyonic quantum theories are meaningless.

For planar and non-orientable diagrams, the divergences at $\lambda \rightarrow 0$ follow from tadpoles of closed string states at zero momentum. They can also be regulated by analytic continuation in a fictitious momentum inserted in the tadpole [37, 15]. For the planar (resp. Möbius strip) diagrams, we introduce a function $\alpha(t_1)$ (resp. $\beta(t_2)$) of some parameter t_1 (resp. t_2) such that

$$\lim_{t_1 \rightarrow 0} \alpha(t_1) = 0, \quad \lim_{t_2 \rightarrow 0} \beta(t_2) = 0. \quad (17)$$

The integrands of (10) and (12) then take the form, respectively:

$$\int_0^1 \frac{d\lambda}{\lambda^2} \exp\left(\frac{4\pi^2 \gamma_1}{\ln \lambda}\right) \lambda^{\alpha(t_1)} F(\lambda), \quad (18a)$$

$$\int_0^1 \frac{d\lambda}{\lambda^2} \exp\left(\frac{\pi^2 \gamma_2}{\ln \lambda}\right) \lambda^{\beta(t_2)} F(-\lambda), \quad (18b)$$

where the same analytic function $F(\lambda)$ of $|\lambda| < 1$ appears in both terms (for a fixed cyclic ordering of the vertices on the boundary), with

$$F(\lambda) = \sum_{n=0}^{\infty} F_n \lambda^n. \quad (19)$$

In (18), γ_1 and γ_2 are the regulators for the open string tachyon, and it is assumed that the parameters t_1 and t_2 are such that $\alpha(t_1) > 0$ and $\beta(t_2) > 0$. The evaluation of (18a) and (18b) gives respectively:

$$\lim_{\substack{\gamma_1 \rightarrow 0 \\ t_1 \rightarrow 0}} \sum_{n=0}^{\infty} \frac{F_n}{\alpha(t_1) + n - 1} \left[16\pi^2 \gamma_1 (\alpha(t_1) + n - 1) \right]^{\frac{1}{2}} K_1 \left[\left[16\pi^2 \gamma_1 (\alpha(t_1) + n - 1) \right]^{\frac{1}{2}} \right], \quad (20a)$$

$$\lim_{\substack{\gamma_2 \rightarrow 0 \\ t_2 \rightarrow 0}} \sum_{n=0}^{\infty} \frac{(-1)^n F_n}{\beta(t_2) + n - 1} \left[4\pi^2 \gamma_2 (\beta(t_2) + n - 1) \right]^{\frac{1}{2}} K_1 \left[\left[4\pi^2 \gamma_2 (\beta(t_2) + n - 1) \right]^{\frac{1}{2}} \right]. \quad (20b)$$

Here again, the limits $\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0$ cannot be taken inside the summations and logarithmic branch cuts appear in the contribution of the closed string tachyon ($n=0$) due to the open string tachyon.

The quantities (20) are finite as $t_1 \rightarrow 0, t_2 \rightarrow 0$, except for the dilaton tadpole ($n=1$). However, these divergences can be made to cancel between planar and non-orientable diagrams for unoriented open strings of gauge group $SO(n)$ or $USp(n)$, if we choose $\alpha(t_1)$ and $\beta(t_2)$ such that

$$\lim_{t_1, t_2 \rightarrow 0} \frac{\alpha(t_1)}{\beta(t_2)} = \frac{C_0}{C_N} 2^{-13} n. \quad (21)$$

The same clearly holds for Type I superstrings; the change in the relative sign of C_0 and C_N is compensated by the absence of the divergence due to the closed string tachyon (i.e. we then have $\int_0^1 d\lambda \lambda^{-1} F(\pm\lambda)$). By fully using the freedom in regularisation by analytic continuation, any open string theory with gauge group $SO(n)$ or $USp(n)$ can be made (1-loop) finite (up to the problem of mass renormalisation). In particular, choosing $\alpha(t_1) = \beta(t_2)$ (which seems natural [21] and is supported by the approach of Ref.[1]), we have finiteness for $SO(2^{D/2})$ only [7,15], and choosing $4\alpha(t_1) = \beta(t_2)$ we have finiteness for $SO(2^{D/2-2})$ only [8,9].

The regularisation which is the most often used in the evaluation of open superstring anomalies is a Pauli-Villars regularisation [2,3]. Because of the closed string tadpole in (18), this cannot be used for open bosonic strings [27]. However, Pauli-Villars regularisation raises a conceptual problem: in the evaluation [34,35,30] of the integration measure over w through Feynman's tree theorem, it is essential that no additional factors of w^{m^2} are introduced (and that vertex insertions can coincide). Otherwise, one gets additional surface terms necessary to compensate the unphysical cuts in 1-loop diagrams. When factors such as w^γ are introduced (which is necessary only for theories with tachyons), this does not lead to problems since the additional terms are multiplied by γ and vanish as $\gamma \rightarrow 0$. However, when factors such as w^{m^2} are introduced, it is not clear what happens to the unphysical cuts as $m^2 \rightarrow \infty$. On the other hand, the regularisations used in the calculations above are applicable to any string theory, and they avoid this conceptual problem.

In conclusion, by considering the problem of finiteness only, we are still left with many possible consistent quantum open string theories, with any of the gauge groups $SO(n)$ or $USp(n)$. However, decoupling of zero-norm states is still to be implemented; it may lead to a restriction on the acceptable gauge groups, as did the requirement of factorisation at tree-level [32].

3. ANOMALY CANCELLATION

As said before, no restriction is to be expected from the decoupling of massless zero-norm states, namely $L_{-1}|0; k\rangle$ (with $k^2 = 0$). Indeed, the corresponding vertex is $\partial_z e^{ikX(z)}$, and any physical 1-loop amplitude with such a state then vanishes, by an appropriate analytic continuation in the variables $k_i k_j$ (see (7) and (8)). Similarly, the massive zero-norm vector state $L_{-1} \epsilon_\mu \alpha^\mu_1 |0; k\rangle$ (with $k^2 = -2$, $\epsilon^2 = 1$, $\epsilon k = 0$), with the vertex $\partial_z (i \epsilon_\mu \partial_z X^\mu(z) e^{ikX(z)})$, decouples from any physical 1-loop amplitude. We expect the same result to hold for any zero-norm state of the form $L_{-1}|\chi_1; k\rangle$.

The first non-trivial constraint is found for the zero-norm state $(L_{-2} + 3/2 L_{-1}^2)|0; k\rangle$ (with $k^2 = -2$) [20]. Its vertex is

$$\phi(z) = \frac{5}{2} : i k_\mu \partial_z^2 X^\mu(z) e^{ikX(z)} : + \frac{1}{2} (\eta_{\mu\nu} + 3 k_\mu k_\nu) : i \partial_z X^\mu(z) i \partial_z X^\nu(z) e^{ikX(z)} : \quad (22)$$

Here, we only consider 1-loop amplitudes with one such zero-norm state and $(M-1)$ arbitrary physical open string states (some of which may be of zero-norm as well). However, we expect our conclusions to be valid for any zero-norm states since such states can always be expressed as $(L_{-1}|\chi_1; k\rangle + L_{-2}|\chi_2; k\rangle)$ by using the Virasoro algebra.

From the structure of (22), it can be shown [27] that the amplitudes (7) and (8) reduce respectively to

$$A_o^{(n)}(k_1, \dots, k_M) = -ig^M \frac{1}{M} C_o \frac{1}{2} \int_0^1 \frac{dw}{w} w \frac{d}{dw} \left\{ \frac{1}{w} f^2(w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_M| \delta(z_M - z_M^0) \cdot \text{Tr } R \left[: e^{ik_M X(z_M)} : \prod_{i=1}^{M-1} \phi_i(z_i) w^{L_0} \right] \right\}, \quad (23)$$

$$A_N^{(n)}(k_1, \dots, k_M) = -ig^M \frac{1}{M} C_N \frac{1}{2} \int_0^1 \frac{dw}{w} w \frac{d}{dw} \left\{ \frac{1}{w} f^2(-w) \int_{\Omega(w)} \prod_{i=1}^M dz_i |z_M| \delta(z_M - z_M^0) \cdot \text{Tr } R \left[: e^{ik_M X(z_M)} : \prod_{i=1}^{M-1} \phi_i(z_i) w^{L_0} (-1)^N \right] \right\}, \quad (24)$$

where $\phi_i(z_i)$ ($i=1, \dots, M-1$) are the vertices corresponding to the $(M-1)$ arbitrary external states, and $k_M^2 = -2$, z_M correspond to the zero-norm state in (22). The amplitude is thus given by a total surface term in modular space, a result also obtained in Ref.[20] following a different approach and to be expected on general grounds [18,19]. This is also the case for the gauge anomaly in Type I superstrings [5]. In (23) and (24), additional surface terms in z_i ($i=1, \dots, M-1$) vanish by appropriate analytic continuations in $k_i k_j$.

Performing the change of variables given in (13), the amplitude reduces to sums over different cyclic orderings of the vertices, as in (10), (11) and (12), of terms of the form

$$-ig^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_0 \frac{n \operatorname{tr}(\Pi \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\alpha(t_1)} \exp\left(\frac{4\pi^2 \delta_1}{\ln \lambda}\right) \left(-4\pi^2 \lambda \frac{d}{d\lambda}\right) \left\{ \frac{1}{\lambda} f^{-24}(\lambda) \cdot \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\theta_1(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\theta_1'(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}, i=1, \dots, M-1] \right\}, \quad (25)$$

$$-ig^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_0 \frac{\operatorname{tr}(\Pi \lambda_a^{(i)}) \operatorname{tr}(\Pi \lambda_b^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{4\pi^2 \delta}{\ln \lambda}\right) \left(-4\pi^2 \lambda \frac{d}{d\lambda}\right) \left\{ \frac{1}{\lambda} f^{-24}(\lambda) \cdot \int_{R_2} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\theta_{1024}(v_i - v_j | \frac{\ln \lambda}{2i\pi})}{\theta_{1024}'(0 | \frac{\ln \lambda}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}, i=1, \dots, M-1] \right\}, \quad (26)$$

$$-ig^M \delta^{(26)} \left(\sum_{i=1}^M k_i \right) C_N \frac{2^{13} \operatorname{tr}(\Pi \lambda_a^{(i)})}{2\pi} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\beta(t_2)} \exp\left(\frac{\pi^2 \delta_2}{\ln \lambda}\right) \left(-4\pi^2 \lambda \frac{d}{d\lambda}\right) \left\{ \frac{1}{\lambda} f^{-24}(-\lambda) \cdot \int_{R_1} \prod_{i=1}^M dv_i \delta(v_M) \prod_{1 \leq i < j \leq M} \left[\frac{\theta_1(v_i - v_j | \frac{\ln(-\lambda)}{2i\pi})}{\theta_1'(0 | \frac{\ln(-\lambda)}{2i\pi})} \right]^{k_i k_j} \cdot [N_i \text{ derivatives in } \frac{\partial}{\partial v_i}, i=1, \dots, M-1] \right\}, \quad (27)$$

respectively for planar, non-planar and non-orientable contributions. In these expressions, the notations are those defined above; the necessary regularisation factors have already been included, with the appropriate analytic continuations to be performed as explained above.

For non-planar amplitudes, the integrand of (26) has the form

$$\int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{4\pi^2 \delta}{\ln \lambda}\right) \left(-4\pi^2 \lambda \frac{d}{d\lambda}\right) \left[\frac{1}{\lambda} (2\pi \lambda^{1/8})^{s_{AB}} G(\lambda) \right], \quad (28a)$$

with $G(\lambda)$ an analytic function of $|\lambda| < 1$

$$G(\lambda) = \sum_{n=0}^{\infty} G_n \lambda^n, \quad (28b)$$

and s_{AB} defined before. If there were no tachyon in the open string channel, this expression would simply be a total divergence and vanish by analytic continuation in s_{AB} (the contribution at $\lambda = 1$ or $w = 0$ can be shown to vanish in that case). This is the situation in Type I superstrings [2.24]. Here however, we obtain for (28)

$$\lim_{\delta \rightarrow 0} (-4\pi^2) (2\pi)^{s_{AB}} \sum_{n=0}^{\infty} G_n \left[16\pi^2 \delta \left(\frac{s_{AB}}{8} + n - 1 \right) \right]^{1/2} K_1 \left[\left[16\pi^2 \delta \left(\frac{s_{AB}}{8} + n - 1 \right) \right]^{1/2} \right]. \quad (29)$$

This contribution is entirely due to the open string tachyon. It has logarithmic branch cuts and does not vanish. Tachyonic string theories are not only

meaningless because of the unphysical character of their amplitudes, but, in addition, zero-norm states do not decouple in such theories!

Considering the planar and non-orientable contributions, we have respectively for the integrands of (25) and (27):

$$\int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{4\pi^2\delta_1}{\ln\lambda}\right) \lambda^{\alpha(t_1)} \left(-4\pi^2\lambda \frac{d}{d\lambda}\right) \left[\frac{1}{\lambda} F(\lambda)\right], \quad (30a)$$

$$\int_0^1 \frac{d\lambda}{\lambda} \exp\left(\frac{\pi^2\delta_2}{\ln\lambda}\right) \lambda^{\beta(t_2)} \left(-4\pi^2\lambda \frac{d}{d\lambda}\right) \left[\frac{1}{\lambda} F(-\lambda)\right], \quad (30b)$$

with $F(\lambda)$ the same analytic function of $|\lambda| < 1$ in both expressions

$$F(\lambda) = \sum_{n=0}^{\infty} F_n \lambda^n. \quad (31)$$

The evaluation of (30a) and (30b) gives respectively:

$$4\pi^2 F_1 + \lim_{\delta_1 \rightarrow 0} (-4\pi^2) \sum_{n=0}^{\infty} F_n [16\pi^2\delta_1(n-1)]^{1/2} K_1\left[[16\pi^2\delta_1(n-1)]^{1/2}\right], \quad (32a)$$

$$-4\pi^2 F_1 + \lim_{\delta_2 \rightarrow 0} (-4\pi^2) \sum_{n=0}^{\infty} (-1)^n F_n [4\pi^2\delta_2(n-1)]^{1/2} K_1\left[[4\pi^2\delta_2(n-1)]^{1/2}\right]. \quad (32b)$$

The second terms in those results are entirely due to the open string tachyon, as in (29). Note their logarithmic branch cut for $n = 0$. On the other hand, the first terms, proportional to F_1 , correspond to a genuine anomaly, independent of the existence of tachyons and any of the regulators. Summing planar and non-orientable diagrams (and ignoring for a moment the contributions from the open string tachyon), the total amplitude with the zero-norm state (22) is given by

$$-ig^M \delta^{(26)}\left(\sum_{i=1}^M k_i\right) \sum_{\text{cyclic}} 2\pi \text{tr}(\pi_i \lambda_a^{(i)}) C_0\left(n - 2^{13} \frac{C_N}{C_0}\right) F_1, \quad (33)$$

where $\sum_{\text{cyclic}}$ stands for the summation over all cyclically inequivalent permutations of the M external states. The cancellation of this quantity uniquely selects the gauge group $SO(2^{13})$.

From (23) and (24), or (25) and (27), one sees that the anomaly (33) can be interpreted as a tadpole of an unphysical closed string state in the unphysical 1-loop amplitude of an off-mass-shell open string "tachyon"

(with $k_M^2 = -2$) and the $(M-1)$ physical open string states corresponding to the vertices $\phi_i(z_i)$ ($i = 1, \dots, M-1$).

The same analysis is applicable to Type I superstrings. The change in the relative sign of C_0 and C_N is again compensated by the fact that the contribution to the anomaly is now given by F_0 , since there is no tachyon in the closed string sector (which would be responsible for a factor λ^{-1} in the total derivative, as in (30)). Clearly, the group $SO(2^5)$ is then uniquely selected, as is well known by considering the 1-loop amplitude with six massless vectors [2-6]. In this case, the same interpretation in terms of a tadpole of an unphysical closed string state has been given for the gauge anomaly [6]. However, since the anomaly is related to F_0 , it can also be expressed in this case as an unphysical tree-level amplitude on the disk, as is usually done [2,24]. The interpretation in terms of a tadpole seems more general and useful.

4. CONCLUSIONS

We discussed how (1-loop) decoupling of zero-norm states in open string theories selects the gauge group $SO(2^{D/2})$ uniquely, whereas finiteness (of 1-loop amplitudes with external open string states only) can be achieved for any $SO(n)$ or $USp(n)$ gauge group by appropriate regularisation. Although finiteness may also result from a dilaton background field [38,6,12], such a mechanism cannot lead to the cancellation of anomalies since they correspond to tadpoles of unphysical closed string states [6]. We also saw how tachyonic string theories not only violate unitarity, but also break the gauge symmetries associated to the Virasoro algebra, i.e. to world-sheet diffeomorphisms.

These results show that the recent suggestion [1] to regard $SO(2^{D/2})$ open string theories as closed string theories with a $\mathbb{Z}_2$ -orbifold construction in the world-sheet actually leads to all the (possibly) consistent quantum theories of open strings which are known (ignoring problems related to tachyonic states). This proposal thus provides the natural setting for studying the full quantum consistency of open string theories, i.e. finiteness (up to the problem of mass renormalisation) and decoupling of all zero-norm states (not only those of the open string sector); those properties should follow from modular invariance as they do in theories of closed strings only [24,25].

Clearly, the ideas of Ref.[1] can be generalised and used in the construction of many new consistent open string theories (with or without supersymmetry) in dimensions ranging from 10 to 4, and characterized by

different orbifold structures in their world-sheet. Work in these directions is in progress.

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