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CHIRAL SCHWINGER MODEL WITH MASSIVE FERMIONS

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Abstract

We study the chiral Schwinger model with massive fermions. By bosonizing it within the path integral framework, we find an equivalent bosonic Lagrangian with an interaction term of the sine-Gordon type. The consistency of the theory is discussed.

Gauge theories coupled to chiral fermions can lose their gauge invariance at the quantum level. This anomalous breaking of the gauge symmetry sheds doubt on the consistency of such theories. Recently, however, it has been shown by Jackiw and Rajaraman [1] that a U(1) gauge theory chirally coupled to massless fermions in two space-time dimensions (chiral Schwinger model) yields a consistent unitary theory despite the presence of the anomaly. Their argument stems from the fact that one can exploit the ambiguity which appears as a consequence of the lack of gauge invariance of the fermionic determinant. This ambiguity introduces in the effective Lagrangian a quadratic term $(ae^2/8\pi)A_\mu A^\mu$ for the gauge field, which contains an arbitrary parameter a . For $a > 1$ we have a consistent and unitary theory which contains [1,2] in the spectrum two non-interacting degrees of freedom, one which is massless while the other has a mass $m^2 = e^2 a^2 / 4\pi(a-1)$.

In this letter we are going to study the effect of the introduction of a fermion mass. In this case the chiral gauge symmetry (which must be given up anyway at the quantum level even when the fermion is massless) is already broken at the classical level. We shall show that when the fermions are massive the two degrees of freedom present in the massless case couple through an interaction term of the sine-Gordon type. Using this result we shall study how the introduction of a fermion mass affects the range of values of the parameter a which give rise to a sensible quantum field theory.

Let us start by writing the Lagrangian density of the chiral Schwinger model in Euclidean space:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \bar{\psi} (i \not{\partial} + e \not{A} \frac{1-\gamma_5}{2}) \psi + i m_0 \bar{\psi} \psi \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\gamma_0 = \sigma_1$, $\gamma_1 = -\sigma_2$ and $\gamma_5 = i\gamma_0\gamma_1$ so that $\gamma_\mu\gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$ with $\epsilon_{01} = -\epsilon_{10} = 1$.

The generating functional is given by:

$$Z = N \int DA_\mu D\bar{\psi} D\psi \exp \left[- \int d^2x \mathcal{L} \right] \quad (2)$$

Let us parametrize the gauge field as:

$$A_\mu = -\frac{1}{e} (\epsilon_{\mu\nu} \partial_\nu \phi - \partial_\mu \eta) \quad (3)$$

Using the properties of the γ -matrices in two dimensions and the parametrization (3), it is very easy to check that:

$$\gamma_\mu \frac{1}{2} (1 - \gamma_5) A_\mu = \frac{1}{2e} \gamma^\mu (1 - \gamma_5) \partial_\mu (\eta + i\phi) \quad (4)$$

which means that if $m_0 = 0$ the fermions and the gauge field can be decoupled by means of a chiral rotation of the fermionic fields :

$$\begin{aligned} \psi &= \exp\left[i \frac{1 - \gamma_5}{2} (\eta + i\phi)\right] \chi \\ \bar{\psi} &= \bar{\chi} \exp\left[-i \frac{1 + \gamma_5}{2} (\eta + i\phi)\right] \end{aligned} \quad (5)$$

In terms of the new fermionic variables the Lagrangian density is:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \bar{\chi} i \not{\partial} \chi + i m_0 \bar{\chi} \exp[\gamma_5 (\phi - i\eta)] \chi \quad (6)$$

However [3], due to the the chiral anomaly, the fermionic measure is not invariant under the change of variables given by equation (5):

$$D\psi D\bar{\psi} = D\chi D\bar{\chi} J_F \quad (7)$$

The Jacobian J_F for this change of variables has been computed in refs.[4-6] by using the methods developed in [7]. It is given by:

$$J_F = \exp\left[-\frac{1}{8\pi} \int d^2x \left[(1-a)\eta \square \eta - (1+a)\phi \square \phi + 2i\eta \square \phi\right]\right] \quad (8)$$

where a is the regularization parameter mentioned above. Equations (7) and (8) imply that we can write Z as follows:

$$Z = N \int D\phi D\eta D\chi D\bar{\chi} \exp\left[-\int d^2x \mathcal{L}_{eff}(\phi, \eta, \chi)\right] \quad (9)$$

with:

$$\begin{aligned} \mathcal{L}_{eff}(\phi, \eta, \chi) &= \frac{1}{2e^2} \phi \square \square \phi + \frac{(1-a)}{8\pi} \eta \square \eta - \frac{(1+a)}{8\pi} \phi \square \phi + \\ &+ \frac{2i}{8\pi} \eta \square \phi - i \bar{\chi} \not{\partial} \chi + i m_0 \bar{\chi} \exp[\gamma_5 (\phi - i\eta)] \chi \end{aligned} \quad (10)$$

In this paper we shall only consider the case $a \neq 1$. We want to define new bosonic fields in such a way that the effective Lagrangian (10) is diagonalized when $m_0 = 0$. For this purpose, let us introduce a new field σ , which is related with η and ϕ in the following way:

$$\sigma = i(1-a)\eta - \phi \quad (11)$$

In terms of σ the effective Lagrangian takes the form :

$$\begin{aligned} \mathcal{L}_{eff}(\phi, \sigma, \chi) = & \frac{1}{2e^2} \phi \square \phi - \frac{a^2}{8\pi(a-1)} \phi \square \phi + \frac{1}{8\pi(a-1)} \sigma \square \sigma - \\ & - i \bar{\chi} \not{\partial} \chi + i m_0 \bar{\chi} \exp \left[\gamma_5 \frac{a\phi + \sigma}{a-1} \right] \chi \end{aligned} \quad (12)$$

We are going to establish the equivalence of the theory given by the Lagrangian (12) and a certain bosonic Lagrangian [8]. We shall closely follow the method developed in refs. [9-11] to study the bosonization in the path integral framework. The first step in this method is to expand Z in powers of m_0 :

$$Z = \sum_{n=0}^{\infty} \frac{(-i m_0)^n}{n!} \int \prod_{i=1}^n dx_i \langle \bar{\chi}(x_i) \exp \left[\left(\frac{a\phi(x_i) + \sigma(x_i)}{a-1} \right) \gamma_5 \right] \chi(x_i) \rangle \quad (13)$$

where $\langle \dots \rangle$ means the vacuum expectation value in a theory given by the Lagrangian (12) with $m_0 = 0$. In order to calculate this vacuum expectation value we need the propagators for the ϕ and σ fields, which satisfy:

$$\begin{aligned} \left(\frac{1}{e^2} \square - \frac{a^2}{4\pi(a-1)} \right) \Delta \phi(x) &= \delta^2(x) \\ \frac{1}{4\pi(a-1)} \square \Delta \sigma(x) &= \delta^2(x) \end{aligned} \quad (14)$$

The solutions to these equations are :

$$\begin{aligned} \Delta \phi(x) &= \frac{-2(a-1)}{a^2} \ln c \mu |x| - \frac{4\pi(a-1)}{a^2} \Delta(m; x) \\ \Delta \sigma(x) &= 2(a-1) \ln c \mu |x| \end{aligned} \quad (15)$$

where c is a constant related to the Euler constant ($c = e^{\gamma/2}$, $\gamma = 0.5772\dots$), μ is a small mass needed in

order to avoid infra-red divergences, $m^2 = e^2 a^2 / 4\pi(a-1)$ and $\Delta(m; x) = K_0(m|x|)/2\pi$ is the propagator for a free scalar field of mass m in two dimensions. Define:

$$\sigma_{\pm}(x) = \bar{\chi}(x) \frac{1 \pm \gamma_5}{2} \chi(x) \quad (16)$$

Then :

$$Z = \sum_{n=0}^{\infty} \frac{(-im_0)^n}{n!} \int \prod_{i=1}^n dx_i \langle \sigma_+(x_i) \exp\left(\frac{a\phi(x_i) + \sigma(x_i)}{a-1}\right) + \sigma_-(x_i) \exp\left(-\frac{a\phi(x_i) + \sigma(x_i)}{a-1}\right) \rangle \quad (17)$$

The only non-vanishing vacuum expectation values of a product of σ_+ and σ_- are those in which we have an equal number of σ_+ and σ_- . This means that we can write :

$$Z = \sum_{n=0}^{\infty} \frac{(-im_0)^{2n}}{(n!)^2} \int \prod_{i=1}^n dx_i d\gamma_i \langle \exp\left[\frac{a}{1-a} \sum_{i=1}^n (\phi(x_i) - \phi(\gamma_i))\right] \rangle \times \\ \times \langle \exp\left[\frac{1}{a-1} \sum_{i=1}^n (\sigma(x_i) - \sigma(\gamma_i))\right] \rangle \times \langle \prod_{i=1}^n \sigma_+(x_i) \sigma_-(\gamma_i) \rangle \quad (18)$$

Using the well-known expression for the propagator of a massless fermion in two dimensions it is easy to check that :

$$\langle \prod_{i=1}^n \sigma_+(x_i) \sigma_-(\gamma_i) \rangle = \frac{(-1)^n}{(2n)^{2n}} \frac{\prod_{i < j} (|x_i - x_j|^2 |\gamma_i - \gamma_j|^2)}{\prod_{i, j} |x_i - \gamma_j|^2} \quad (19)$$

On the other hand, for any field $\rho(x)$ with propagator $\Delta^\rho(x)$, the following equation is satisfied :

$$\langle \prod_{i=1}^n e^{\beta_i \rho(x_i)} \rangle = \exp\left[\frac{1}{2} \sum_{i=1}^n \beta_i^2 \Delta^\rho(o)\right] \prod_{i < j} e^{\beta_i \beta_j \Delta^\rho(x_i - x_j)} \quad (20)$$

Applying this equation to the σ field, we obtain:

$$\langle \exp\left[\frac{1}{a-1} \sum_{i=1}^n (\sigma(x_i) - \sigma(\gamma_i))\right] \rangle = \exp\left[\frac{2n \Delta^\sigma(o)}{2(a-1)^2}\right] \times \\ \times \exp\left[\frac{1}{(a-1)^2} \left[\sum_{i < j} \Delta^\sigma(x_i - x_j) + \sum_{i < j} \Delta^\sigma(\gamma_i - \gamma_j) - \sum_{i, j} \Delta^\sigma(x_i - \gamma_j) \right]\right] \quad (21)$$

From equation (15) we see that $\Delta^\sigma(0)$ is divergent. In order to regularize this divergence one has to introduce an ultra-violet cut-off Λ , which is a large regulator mass. Proceeding in this way we have :

$$\Delta^\sigma(0) = 2(a-1) \ln \mu/\Lambda \quad (22)$$

So we obtain:

$$\langle \exp \left[\frac{1}{a-1} \sum_{i=1}^n (\sigma(x_i) - \sigma(y_i)) \right] \rangle = \frac{\prod_{i < j} (\mu^2 c^2 |x_i - x_j| |y_i - y_j|)^{\frac{2}{a-1}}}{\prod_{i,j} (\mu c |x_i - y_j|)^{\frac{2}{a-1}}} \times \left(\frac{\mu}{\Lambda} \right)^{\frac{2n}{a-1}} \quad (23)$$

The ϕ contribution can be worked out in the same way with the result:

$$\begin{aligned} \langle \exp \left[\frac{a}{a-1} \sum_{i=1}^n (\phi(x_i) - \phi(y_i)) \right] \rangle &= \left(\frac{m}{\mu} \right)^{\frac{2n}{a-1}} \times \frac{\prod_{i < j} (\mu^2 c^2 |x_i - x_j| |y_i - y_j|)^{\frac{2}{a-1}}}{\prod_{i,j} (\mu c |x_i - y_j|)^{\frac{2}{a-1}}} \times \\ &\times \exp \left[\frac{-4n}{a-1} \left[\sum_{i < j} \Delta(m; x_i - x_j) + \sum_{i < j} \Delta(m; y_i - y_j) - \sum_{i,j} \Delta(m; x_i - y_j) \right] \right] \end{aligned} \quad (24)$$

It is interesting to notice that in this case no ultra-violet cut-off is needed. Taking into account equations (19), (23) and (24) we can write :

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{m_0}{2n} \left(\frac{m}{\Lambda} \right)^{\frac{1}{a-1}} \right)^{2n} \cdot \int \prod_{i=1}^n dx_i dy_i \frac{\prod_{i < j} (|x_i - x_j|^2 |y_i - y_j|^2)}{\prod_{i,j} |x_i - y_j|^2} \times \\ &\times \exp \left[\frac{-4n}{a-1} \left[\sum_{i < j} \Delta(m; x_i - x_j) + \sum_{i < j} \Delta(m; y_i - y_j) - \sum_{i,j} \Delta(m; x_i - y_j) \right] \right] \end{aligned} \quad (25)$$

The two factors that appear inside the integrals in equation (25) can be expressed as vacuum expectation values of scalar fields. In fact if θ is a massless scalar field, we have :

$$\frac{\prod_{i < j} (|x_i - x_j|^2 |y_i - y_j|^2)}{\prod_{i,j} (|x_i - y_j|^2)} = (c\Lambda)^n \left\langle \prod_{i=1}^n e^{2i\sqrt{n}\theta(x_i)} e^{-2i\sqrt{n}\theta(y_i)} \right\rangle \quad (26)$$

On the other hand, let be ρ a scalar field of mass m . Then :

$$\begin{aligned}
 & \exp \left[\frac{-4n}{a-1} \left(\sum_{i < j} \Delta(m; x_i - x_j) + \sum_{i < j} \Delta(m; y_i - y_j) - \sum_{i, j} \Delta(m; x_i - y_j) \right) \right] = \\
 & = \left(\frac{\Lambda}{m} \right)^{\frac{2n}{a-1}} \times \left\langle \prod_{i=1}^n \exp \left(\frac{2i\sqrt{n}}{\sqrt{a-1}} \rho(x_i) \right) \exp \left(\frac{-2i\sqrt{n}}{\sqrt{a-1}} \rho(y_i) \right) \right\rangle
 \end{aligned} \tag{27}$$

Substituting these results in Z we obtain :

$$Z = \sum_{n=0}^{\infty} \frac{\left(\frac{m_0 c \Lambda}{2n} \right)^{2n}}{(n!)^2} \int \prod_{i=1}^n dx_i dy_i \left\langle e^{2i\sqrt{n} \left(\theta(x_i) + \frac{\rho(x_i)}{\sqrt{a-1}} \right)} e^{-2i\sqrt{n} \left(\theta(y_i) + \frac{\rho(y_i)}{\sqrt{a-1}} \right)} \right\rangle$$

(28)

This generating functional corresponds to a theory with Lagrangian density:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} (\partial_\mu \theta)^2 + \frac{1}{2} m^2 \rho^2 + \alpha \cos 2\sqrt{n} \left(\theta + \frac{\rho}{\sqrt{a-1}} \right)$$

(29)

with $\alpha = c\Lambda m_0/\pi$. Equation (29) gives us a bosonized Lagrangian equivalent to our starting Lagrangian (equation(1)). This is the result we were searching for. We observe that in the massless case ($m_0 = 0$) the fields θ and ρ are free and we recover the results of refs.[1,2]. However the mere presence of the mass term in our initial Lagrangian (1) implies that these two fields are not free any more but they interact through the last term in equation (29).

Let us finally consider the impact of the presence of this interaction term on the consistency of the theory. Let us argue, following Coleman[8], that for an interaction term of the form $\cos(\beta_\theta \theta + \beta_\rho \rho)$ it is at least necessary that $\beta_\theta^2 < 8\pi$ and $\beta_\rho^2 < 8\pi$ in order to have a Hamiltonian bounded from below. These values are obtained by trivially extending the variational argument of ref.[8]. In the Lagrangian (29) $\beta_\theta^2 = 4\pi$ and $\beta_\rho^2 = 4\pi/(a-1)$. In order to fulfil the above conditions it is necessary to have $a > 3/2$, to be compared with the $a > 1$ condition that must be verified in the massless case in order to have a non-tachyonic theory.

In conclusion, we have investigated the effect of having massive fermions in the chiral Schwinger model. An equivalent bosonic Lagrangian (equation (29)) has been found in which the fields are no longer free. The interaction term in the bosonic Lagrangian modifies (with respect to the massless case)

the range of values of the regularizing parameter which correspond to a well-behaved quantum theory.

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