Non-Compact Gaugings from Higher Dimensions

C.M. Hull

Blackett Laboratory, Imperial College
London SW7 2BZ, England

N.P. Warner*

C.E.R.N., Geneva, Switzerland

Abstract

The $N = 8$ supergravity theories in 4 (and 5) dimensions with non-compact gauge groups are obtained from 11 (and 10) dimensional supergravities in backgrounds with non-compact internal spaces. The $SO(p, q)$ gaugings are obtained by a consistent truncation of the higher dimensional supergravity in a background that is a warped product of a 4 (or 5) dimensional Einstein space with the hyperboloid $H^{p,q}$ (i.e., the surface $(z^1)^2 + \dots + (z^p)^2 - (z^{p+1})^2 - \dots - (z^{p+q})^2 = r^2$). For the non-semi-simple $CSO(p, q, r)$ gaugings $H^{p,q}$ is replaced by the "cylinder" $H^{p,q} \times \mathbb{R}^r$. These backgrounds are solutions of the higher dimensional field equations precisely when the lower dimensional supergravity has an $SO(p) \times SO(q)$ invariant vacuum solution. The mode analysis and massive spectra in these backgrounds are briefly considered.

*On leave of absence from Dept. of Mathematics, M.I.T., Cambridge, MA 02139, U.S.A.

1. Introduction

It is now virtually beyond doubt that there is a consistent truncation to the massless sector of the S^7 compactification of eleven dimensional supergravity [1-4], and moreover, that this consistent truncation is precisely the gauged $SO(8)$, $N = 8$ supergravity of de Wit and Nicolai [5]. It also seems highly likely that the S^4 compactification of eleven dimensional supergravity [6], and the S^6 compactification of chiral, ten dimensional supergravity [7] can be consistently truncated to gauged $SO(5)$, $N = 4$, seven dimensional supergravity [8], and gauged $SO(6)$, $N = 8$, five dimensional supergravity [9, 10] respectively.

One of the unanswered questions is whether the maximal supergravity theories with non-compact gauge groups can be obtained from higher dimensions. To date, $N = 8$ gauged supergravity theories have been constructed in four dimensions with gauge groups $SO(p, 8-p)$, ($p = 0, \dots, 4$), or with non-semi-simple contractions of these gauge groups [11-14]. In seven dimensions $N = 4$ supergravity theories have been constructed with gauge group $SO(p, 5-p)$ ($p = 0, 1, 2$) [15], and in five dimensions there are gauged $N = 8$ supergravity theories with gauge groups $SO(p, 6-p)$, ($p = 0, 1, 2, 3$) or $SU(3, 1)$ [9]. Surprisingly, these odd-dimensional gauged supergravity theories do not appear to allow gaugings of non-semi-simple contractions of $SO(p, q)$ or $SU(3, 1)$.

In this paper we will show how the $SO(p, q)$ gaugings (and their non-semi-simple contractions) can be obtained from the appropriate higher dimensional supergravity theories. We find that the spheres used to compactify to the $SO(p)$ gaugings are replaced by hyperboloids for the $SO(p, q)$ gaugings, and generalized cylinders for the non-semi-simple contractions. Such a relation between non-compact internal spaces and non-compact gaugings was conjectured in [23]. For reasons that will become apparent later, we will only be able to make conjectures about the higher dimensional solutions which give rise to the $SU(3, 1)$ gauging. In Section 2 of this paper we will review some of the details of the consistent truncation of the S^7 compactification of eleven dimensional supergravity. In Section 3 we use the analytic continuation technique of [11-14] for obtaining the non-compact gaugings in four dimensions in order to analytically continue the S^7 background of eleven dimensional supergravity to some new backgrounds. These are the appropriate backgrounds that lead to the non-compact gaugings. We also discuss symmetry breaking. In Section 4 we consider a more general class of related backgrounds, and determine when they provide solutions to the field equations of ten and eleven dimensional supergravity. It

is found that they satisfy the higher dimensional field equations if and only if the corresponding non-compact gauging has a solution that is invariant under the maximal compact subgroup of the gauge group.

Finally in Section 5, we comment on the mode analysis and massive spectra on these backgrounds, and make some remarks about the $SU(3, 1)$ gauging of $N = 8$ supergravity in five dimensions.

2. Review of the Metric Ansatz

In reference [3] it was shown how one could obtain the eleven dimensional background metric corresponding to some scalar fluctuation in the gauged $SO(8)$, $N = 8$ supergravity in four dimensions. Recall that the 35 scalars and 35 pseudoscalars of $N = 8$ supergravity in four dimensions lie in a coset space $E_{7(7)}/SU(8)$, and can be parameterized by an element $V(x)$ of E_7 , acting on its fundamental representation [5]:

$$V(x) = \begin{pmatrix} u_{ij}{}^{IJ}(x) & v_{ij}{}^{IJ}(x) \\ v^{ij}{}_{IJ}(x) & u^{ij}{}_{IJ}(x) \end{pmatrix} \quad (2.1)$$

In this expression, the index pairs $[ij]$ and $[IJ]$ are both antisymmetric so that u and v can be viewed as 28×28 matrices. Small Roman indices transform under $SU(8)$ and capital Roman indices transform under $SO(8)$. Raising and lowering of indices corresponds to complex conjugation. (In fact, the conventions throughout this paper will be the same as those of reference [3] except that here the Ricci tensors will have the opposite sign, so that de Sitter spaces have positive Ricci scalar.) Define an $SU(8)$ invariant, $SO(8)$ tensor:

$$M_{IJ\,KL} \equiv \frac{1}{2} \left[(u_{ij}{}^{IJ} + v_{ij}{}^{IJ}) (u^{ij}{}_{KL} + v^{ij}{}_{KL}) + (u^{ij}{}_{IJ} + v^{ij}{}_{IJ}) (u_{ij}{}^{KL} + v_{ij}{}^{KL}) \right]. \quad (2.2)$$

Let $\hat{e}_m{}^a$ and $\hat{g}_{mn}$ be the sieben-bein and metric on the round S^7 , and let η^I be the eight Killing spinors [1, 2]. The 28 Killing vectors on S^7 may then be written

$$\hat{K}^m{}_{IJ} (y) = i\hat{e}_a{}^m \hat{\eta}^I(y) \Gamma^a \hat{\eta}^J(y). \quad (2.3)$$

It was shown in [3] that the internal seven dimensional metric of eleven dimensional supergravity that corresponds to the scalar or pseudoscalar field value $V(x)$ is given by

$$\Delta^{-1}(x, y) g^{mn}(x, y) = \frac{1}{8} M_{IJ\,KL} \hat{K}^m{}_{IJ} (y) \hat{K}^n{}_{KL} (y) \quad (2.4)$$

where

$$\Delta(x, y) = \left[\frac{\det g_{mn}(x, y)}{\det \hat{g}_{mn}(y)} \right]^{1/2} = \left[\frac{\det g^{mn}(x, y)}{\det \hat{g}^{mn}(y)} \right]^{-1/2} \quad (2.5)$$

Moreover, if $\hat{g}_{\mu\nu}(x)$ is some background space-time metric when the internal space is a round S^7 , then the space-time metric, $g_{\mu\nu}$, corresponding to $V(x)$ is obtained by multiplying $\hat{g}_{\mu\nu}$ by a "warp" factor, Δ^{-1} .

$$g_{\mu\nu}(x, y) = \Delta^{-1}(x, y) \hat{g}_{\mu\nu}(x). \quad (2.6)$$

In this paper we will only need these results for constant expectation values, V , of the scalar/pseudoscalar matrix. Thus g^{mn} and Δ are functions of y alone. Furthermore, as we will only consider purely scalar expectation values, the pseudoscalars can be set to zero. This means that we are restricting ourselves to an $SL(8, \mathbb{R})$ subgroup of $E_{7(7)}$, so that we can greatly simplify the foregoing by passing to an $E_{7(7)}$ basis in which this $SL(8, \mathbb{R})$ subgroup is diagonal. (See [3] for details.) In particular

$$\Gamma_{CD}^{IJ} (u_{IJ}^{ij} + v^{ij}{}_{IJ}) \Gamma_{AB}^{IJ} = 4\delta_{[A}^{[I} \delta_{B]}^{J]} \quad (2.7)$$

where $S \in SL(8, \mathbb{R})$ acting on its eight dimensional representation, and $(\Gamma^{IJ})_{AB} = (\Gamma^{IJ})_{AB}$ are the twenty-eight antisymmetric Γ -matrices of $SO(8)$ (see [16], appendix C, D). Define

$$P^{AB} \equiv S_C{}^A S_C{}^B \quad (2.8)$$

and

$$\mu^2 \equiv (P^{AB} P^{AC} P^B{}_C P^A{}_C) \quad (2.9)$$

where z^A are cartesian coordinates on $\mathbb{R}^8$. Then one can show [3] that, in these circumstances, the internal metric obtained from (2.4) is a warped version of the flat Euclidean $\mathbb{R}^8$ metric projected on to the quadratic surface:

$$(P^{AB} z^A z^B) = r^2 \quad (2.10)$$

where r is a constant. Specifically, the unit normal to the surface (2.10) is just

$$\hat{n}_A = \frac{1}{\mu} P^{AB} z^B \quad (2.11)$$

and the required internal metric, whose inverse is given by (2.4), is given by

$$g_{AB} = \frac{r^2}{z^2} \mu^{-2/3} (\delta_{AB} - \hat{n}_A \hat{n}_B) \quad (2.12)$$

where c is some real constant. The determinant warp factor, Δ^{-1} , is a power of μ :

$$\Delta = \frac{2}{c^2} \mu^{-4/3} \quad (2.13)$$

Obviously, for a zero scalar expectation value, $S \equiv 1$, the identity matrix, the surface (2.10) is the round seven-sphere, $\mu = r$, and so the warp factor is constant. From (2.12) we obtain the standard metric on the round S^7 , as, of course, we must.

Equations (2.8)–(2.12) give a very simple geometric interpretation of the internal metric corresponding to a purely scalar vacuum expectation value. In addition, equations (2.6) and (2.13) show how the four dimensional components of the 11-metric must also be warped in order to be consistent with the four-dimensional, gauged $SO(8)$, $N = 8$ theory. It is these equations derived only from the requirement of consistency between the gauged $SO(8)$, four dimensional theory and eleven dimensional theories [2.3], that we will use to determine the backgrounds of the eleven dimensional theory that correspond to the non-compact gaugings.

3. Non-compact gaugings, analytic continuation and dimensional reduction

The simplest way of obtaining the non-compact gaugings of $N = 8$ supergravity is by making an analytic continuation of the $SO(8)$ gauged theory. This is discussed in detail in [11–14] so we shall be brief here.

In the ungauged $N = 8$ supergravity of Cremmer and Julia [16] there is a global $E_{7(7)}$ symmetry that is an invariance only of the field equations, not of the action, since it acts on the spin-one fields through duality transformations. Moreover, the $SL(8, \mathbb{R})$ subgroup of $E_{7(7)}$ defined in the last section is a rigid symmetry of the ungauged action. The $SO(8)$ subgroup of $SL(8, \mathbb{R})$ preserves the metric δ_{AB} , and can be gauged by introducing minimal couplings involving $g\delta_{AB}$ (where g is the Yang-Mills coupling constant). For example, the Yang-Mills field strength becomes

$$F_{\mu\nu}^{AB} = 2\partial_{[\mu} A_{\nu]}^{AB} + 2g\delta_{CD} A_{[\mu}^{C(A} A_{\nu]}^{B)D} \quad (3.1)$$

Obviously these minimal couplings break both $E_{7(7)}$ and $SL(8, \mathbb{R})$ down to $SO(8)$. Supersymmetry can be restored to the gauged $SO(8)$ action by adding a scalar potential and fermion "mass" terms [5].

More generally we can gauge the subgroup $CSO(p, q, r)$ of $SL(8, \mathbb{R})$. This is defined as the subgroup which preserves the metric

$$\eta_{AB} = \text{diag} \left(\underbrace{+1, +1, \dots, +1}_{p\text{-times}}, \underbrace{-1, -1, \dots, -1}_{q\text{-times}}, \underbrace{0, 0, \dots, 0}_{r\text{-times}} \right) \quad (3.2)$$

When $r = 0$ this group is simply $SO(p, q)$, while if $r \neq 0$, this group is non-semi-simple. For example, $CSO(7, 0, 1) = ISO(7)$, the isometry group of flat $\mathbb{R}^7$, and $CSO(6, 1, 1) = ISO(6, 1)$, the Poincaré group in seven dimensions. To gauge $CSO(p, q, r)$ we simply replace $g\delta_{AB}$ with $g\eta_{AB}$ in all the minimal couplings, and then make corresponding changes in the scalar potential and fermion "mass"-terms. For example, (3.1) becomes

$$F_{\mu\nu}^{AB} = 2\partial_{[\mu} A_{\nu]}^{AB} + 2g\eta_{CD} A_{[\mu}^{C(A} A_{\nu]}^{B)D} \quad (3.3)$$

In principle we could use any 8×8 matrix, η_{AB} , to make the minimal couplings. However, if we perform an $SL(8, \mathbb{R})$ transformation, S , of the resulting theory, the minimal coupling matrix transforms according to

$$\eta' = S\eta S^T \quad (3.4)$$

Using such a transformation, and the freedom to rescale the coupling constant, g , we can bring any matrix η to the standard form (3.2).

Although it is straightforward to obtain the minimal couplings in this way, the corresponding changes in the scalar potential are, at first sight, somewhat more difficult to find. However, the complete non-compact, supersymmetric gaugings of the Cremmer-Julia theory [16] can be obtained immediately from the supersymmetric gauged $SO(8)$ theory [5] by the analytic continuation technique of [11–14]. Consider making an $SL(8, \mathbb{R})$ transformation of the full $SO(8)$ theory using

$$S_A^B = \text{diag} \left(\underbrace{e^{+\frac{1}{2}t}, \dots, e^{+\frac{1}{2}t}}_{p\text{-times}}, \underbrace{e^{-\frac{1}{2}t}, \dots, e^{-\frac{1}{2}t}}_{q\text{-times}}, \underbrace{e^{-\frac{1}{2}t}, \dots, e^{-\frac{1}{2}t}}_{r\text{-times}} \right) \quad (3.5)$$

where $\beta = p/q$ and t is a parameter. This changes the g -dependent terms but leaves the g independent terms unchanged. We combine this with a rescaling of the coupling constant

$$g \rightarrow g' = (f(t))^2 g \quad (3.6)$$

for some function f to be determined later.

We define $\hat{S} \in GL(8, \mathbb{R})$ by $\hat{S} = |f(t)|S$. Then, according to (3.4) and (3.6), the minimal couplings are replaced by

$$g\delta_{AB} \rightarrow g\eta(t)_{AB} \equiv g\delta_{CD}\hat{S}_A^C\hat{S}_B^D \quad (3.7)$$

So far the resulting theory is still the $SO(8)$ gauging, but in a non-standard basis for the $SO(8)$ Lie algebra. However, analytic continuation to complex or infinite t can lead to a new theory. We first choose

$$f(t) = e^{-\frac{1}{2}t} \quad (3.8)$$

so that $\hat{S}(t)$ and $\eta(t)$ remain finite as $t \rightarrow \infty$. Then

$$\eta_{AB}(t) = \text{diag}(1, \dots, 1, \xi, \dots, \xi) \quad (3.9)$$

where

$$\xi = \exp[-(1 + p/q)t] \quad (3.10)$$

In the limit $t \rightarrow \infty$, $\xi \rightarrow 0$ and (3.9) becomes the $CISO(p, 0, q)$ invariant metric, and hence the minimal couplings are those of a gauged $CISO(p, 0, q)$ theory. Moreover, as this theory is the well-defined limit of a one parameter family of supersymmetric theories, it is itself supersymmetric and is the $CISO(p, 0, q)$ gauged $N = 8$ supergravity theory [11–13].

Analytically continuing to $t = -\frac{i\pi}{(p+q)}$, we have $\xi = -1$, and the theory is gauged $SO(p, q)$ $N = 8$ supergravity.

Finally, the $CISO(p, q, r)$ gaugings can be obtained by acting on the $SO(p, q + r)$ gauging

with

$$S_A^{'B} = \text{diag} \left(\underbrace{e^{\frac{1}{2}t}, \dots, e^{\frac{1}{2}t}}_{p+q \text{ times}}, \underbrace{e^{-\frac{1}{2}t}, \dots, e^{-\frac{1}{2}t}}_{r \text{ times}} \right) \quad (3.11)$$

where

$$\alpha' = \frac{4}{(p+q)} \quad (3.12)$$

and following this with a rescaling

$$g \rightarrow g'(s) = ge^{-\frac{1}{2}t}. \quad (3.13)$$

The $CISO(p, q, r)$ gaugings are obtained in the limit $s \rightarrow \infty$ [12]. Clearly, the $CISO(p, q, r)$ gaugings can also be obtained in one go from the $SO(8)$ gauging by using a suitable choice of a single $SL(8, \mathbb{R})$ transformation and rescaling of g .

This analytic continuation technique gives a direct method of converting one gauged theory into another. By the same token it can be used to directly convert the ansatz for the consistent truncation of the S^7 compactification of 11-dimensional supergravity down to the de Wit-Nicolai theory into an ansatz for the consistent truncation of the 11-dimensional theory in some new background to the gauged $CISO(p, q, r)$ theories. In either situation one takes a one-parameter family of $SL(8, \mathbb{R})$ transformations, $S(t)$, along with the appropriate rescalings of the coupling constant and analytically continues to $t = \infty$ or imaginary t as described above. We now use this method to find eleven-dimensional backgrounds corresponding to the $CISO(p, q, r)$ gaugings.

We first consider backgrounds in which all the scalar fields vanish. The 56-bein, (2.1) is the identity matrix, and acting with the $SL(8, \mathbb{R})$ transformation (3.5), yields a 56-bein given by substituting the $S_A^{'B}$ defined in (3.5) into the right-hand side of (2.7). Then, for all finite real values of t , (2.8)–(2.13) define a family of ellipsoidally distorted seven spheres corresponding to "distorted" versions of the de Wit-Nicolai theory. In order to obtain a well-defined limit as $t \rightarrow \infty$, we must, in analogy to the rescaling of the coupling constant (3.6), rescale the radius r defined in (2.10) and also rescale μ defined in (2.9). This is done simply by replacing S in (2.8) by the corresponding $GL(8, \mathbb{R})$ transformation, $\hat{S} = e^{-\frac{1}{2}t}S(t)$, defined above. As a result P_{AB} is replaced by

$$\hat{P}_{AB} = \delta_{CD}\hat{S}_A^{'C}\hat{S}_B^{'D} \quad (3.14)$$

and μ^2 in (2.9) is replaced by

$$\hat{\mu}^2 = \hat{P}^{AC}\hat{P}^{AD}\hat{C}_z^{'D} = \eta^{AC}(t)\eta^{AD}(t)\hat{C}_z^{'D}, \quad (3.15)$$

where $\eta_{AB}(t)$ is given by (3.9) and (3.10).

For $t = -\frac{i\pi}{(p+q)}$, $\xi = -1$, $\hat{\mu}^2$ is simply $\sum_{A=1}^8 \hat{C}_z^{'A} \hat{C}_z^{'A}$, while for $t = \infty$, $\xi = 0$, $\hat{\mu}^2$ is $\sum_{A=1}^p \hat{C}_z^{'A} \hat{C}_z^{'A}$. Thus $\hat{\mu}^2$ is always positive. The quadratic form (2.10) for the surface becomes

$$r^2 = \eta_{AB}\hat{C}_z^{'A}\hat{C}_z^{'B} = \sum_{A=1}^p (\hat{C}_z^{'A})^2 + \xi \sum_{A=p+1}^8 (\hat{C}_z^{'A})^2. \quad (3.16)$$

Thus for $\xi = 1$ this gives the seven sphere. For $\xi = 0$ we have a generalized cylinder with cross section S^{p-1} and q flat directions, i.e., $S^{p-1} \times \mathbb{R}^q$. (When $p = 1$, $q = 7$, this gives $\mathbb{Z}_2 \times \mathbb{R}^7 = S^0 \times \mathbb{R}^7$.) For $\xi = -1$, equation (3.16) defines a hyperboloid with "cross-section" $S^{p-1} \times S^{7-p}$ which we denote $HP^{p,8-p}$.

For $\xi > 0$, but $\xi \neq 1$, (3.16) describes a family of ellipsoids with one cross section being S^{p-1} , and whose limit as $\ell \rightarrow \infty$, $\xi \rightarrow 0$ is the cylinder $S^{p-1} \times \mathbb{R}^q$. Similarly if $\xi < 0$, but $\xi \neq -1$, one obtains a family of hyperboloids whose limit is the same cylinder.

The unit normal to the surface defined using (3.14) is

$$\hat{n}_A = \frac{1}{\mu} \hat{p}_{AB} z^B \quad (3.17)$$

and the metric corresponding to (2.12) on this surface for real and finite ℓ is

$$g_{AB} = (c^2/r^2) \mu^{-2/3} [\delta_{AB} - \hat{n}_A \hat{n}_B] \quad (3.18)$$

The four-dimensional warp factor, Δ^{-1} , is given by

$$\Delta = \left(\frac{2}{c^2}\right) \mu^{-4/3} \quad (3.19)$$

Once gain, by analytic continuation in ξ , the metric (3.18) gives a background for either the $CSO(p, 0, q)$ gauging ($\xi = 0$) or the $SO(p, q)$ gauging ($\xi < 0$). Moreover, because there is a consistent $N = 8$ truncation about this background for $\xi > 0$, there must be a consistent truncation for $\xi \leq 0$.

The metric (3.18) is positive definite and is that induced on the hyperboloid by its embedding in Euclidean 8-space with metric δ_{AB} . Whereas the indefinite metric η_{AB} in $\mathbb{R}^d$ would induce a metric on the hyperboloid with isometry group $SO(p, 8-p)$, the metric (3.18) only has isometry group $SO(p) \times SO(8-p)$, which should be the unbroken gauge group of the resulting four-dimensional theory. This is indeed the case — In the $SO(p, 8-p)$ gauged $N = 8$ supergravity, the non-compact generators of the gauge group are non-linearly realized and so are spontaneously broken [13], breaking the gauge group down to a compact sub-group.

To obtain the internal space for the $CSO(p, q, r)$ gaugings we start with the hyperboloid corresponding to the $SO(p, q+r)$ gauging and perform the transformation defined by (3.11)–(3.13) in the limit $s \rightarrow \infty$. The result is exactly as described above, but with η_{AB} given by (3.2). Hence the surface (3.16) becomes

$$\eta_{AB} z^A z^B = \sum_{A=1}^p (z^A)^2 - \sum_{A=p+1}^{p+q} (z^A)^2 = r^2 \quad (3.20)$$

and corresponds to a generalized “cylinder” whose cross section is the $p+q-1$ dimensional hyperboloid, $HP^{p,q}$. That is, the surface is $HP^{p,q} \times \mathbb{R}^r$.

As before, the Euclidean metric is projected onto this surface, and thus the metric on the hyperbolic part only has $SO(p) \times SO(q)$ symmetry. The complete metric is invariant under $SO(p) \times SO(q) \times ISO(r)$, where $ISO(r)$ is the Euclidean group in r dimensions. On the other hand, $CSO(p, q, r)$ has a maximal compact subgroup of $SO(p) \times SO(q) \times U(1)^{r(r-1)/2}$. This suggests that some, or possibly all, of the massless $U(1)^{\frac{r(r-1)}{2}}$ gauge fields come from the antisymmetric tensor field A_{mnp} , as in the Cremmer-Julia reduction [16].

So far we have only considered backgrounds that correspond to the non-compact gaugings with all the scalar fields vanishing. The techniques used here can also be used to obtain the background for arbitrary scalar fluctuations. Indeed, it is trivial to obtain the backgrounds for an arbitrary expectation value of the unique $SO(p) \times SO(q)$ invariant scalar field, ϕ . In the $SO(8)$ or $SO(p, q)$ gaugings. This corresponds to taking ξ to be arbitrary, and not merely restricting $\xi = \pm 1$. In the conventions of [5, 12], we have $\xi = \pm \exp\left(-\frac{\sqrt{2}}{4} \phi\right)$ where $\phi > 0$ is the expectation value of this scalar. From equation (3.16) we see that giving this scalar an expectation value rescales a spherical cross-section of the ellipsoid or hyperboloid.

Reference [3] also gave the consistent ansatz for the space-time metric $g_{\mu\nu}(x, y)$. Indeed, if $\hat{g}_{\mu\nu}(x)$ is a four dimensional space-time background for the round S^7 compactification of $d = 4$ supergravity then for a scalar fluctuation, the background space-time metric (see (2.6)) is

$$g_{\mu\nu}(x, y) = \frac{c}{2} \mu^{4/3} (y) \hat{g}_{\mu\nu}(x) \quad (3.21)$$

The analytic continuation of this is a little subtle. Accommodating the $SL(8, \mathbb{R})$ transformation is straightforward, but incorporating the rescaling of the gauge coupling constant requires care since the four dimensional cosmological constant is related to g^2 and also to the deformations of the internal geometry [12]. We believe that the correct ansatz for the space-time metric is (3.21) with μ replaced by $\hat{\mu}$, but $\hat{g}_{\mu\nu}(x)$ will in general become another Einstein metric with a different (possibly zero or positive) cosmological constant. The cosmological constants have been determined in [12] in terms of the four dimensional quantities g and $\phi > 0$, but the problem here is to relate these to the eleven-dimensional geometry. It should be stressed that the only part of the analytic continuation which is not yet known is this change in $\hat{g}_{\mu\nu}(x)$. Equation (3.21) gives the correct y -dependence of $g_{\mu\nu}$.

Finally, it is known that the potential of the $SO(4, 4)$ gauged theory has a critical point

where all the scalar fields vanish. Moreover, the potential of the $SO(5, 3)$ gauging has a critical point when the unique $SO(5) \times SO(3)$ invariant scalar is given an expectation value, but with all the other scalars set to zero. This last critical point corresponds to taking $\xi = -3$ [12]. Both these critical points give a positive cosmological constant to space-time. The other $SO(p, 8-p)$ gaugings have no $SO(p) \times SO(8-p)$ invariant critical points ($4 \leq p \leq 7$). Thus we expect the simple hyperbolic backgrounds described above to provide solutions to the eleven dimensional field equations only for $p = 3, 4$ and 5 ($p = 3$ is equivalent to $p = 5$).

It is also known that the $C\text{-}SO(2, 0, 6)$ gauging has a critical point when all the scalar fields vanish, and the cosmological constant is zero [12]. In addition, the potential is also flat in the $SO(2) \times SO(6)$ invariant direction, and thus there is a solution for any value of $\phi >$. For this solution (3.16) gives $z_1^2 + z_2^2 = r^2$, and the warp factor $\mu = r$ is constant. Thus the corresponding background is a Ricci-flat four-dimensional space-time with a cylindrical internal space $S^1 \times \mathbb{R}^7$. Obviously the truncation of this theory must be closely related to that of Cremmer and Julia [16].

4. Solutions of the higher dimensional field equations

In this section we shall take a slightly more general tack than we did in the previous sections. We shall consider a metric of the form

$$ds^2 = \rho_1^2(y) ds_1^2(x) + \rho_2^2(y) dy^2 \quad (4.1)$$

where ds_1^2 is a metric on some d_1 -dimensional Einstein space-time with cosmological constant Λ , and ds_2^2 is a (Euclidean) metric on a d_2 -dimensional ellipsoid or hyperboloid in $\mathbb{R}^{d_2+1}$. The coordinates in the d_1 -dimensional and d_2 -dimensional spaces will be denoted x^μ and y^m respectively, and z^A will denote Cartesian coordinates in $\mathbb{R}^{d_2+1}$. Motivated by our earlier discussion, we will describe ds_2^2 in terms of a metric in $\mathbb{R}^{d_2+1}$:

$$g_{AB} = (\delta_{AB} - n_A n_B) \quad (4.2)$$

where n_A is the unit normal to the hyperboloid

$$p^{AB} z^A z^B = r^2 \quad (4.3)$$

Defining

$$\mu^2 \equiv (p^{AB} p^{AC} p^B z^C) \quad (4.4)$$

we shall take

$$\rho_1(y) = (\mu/r)^a; \quad \rho_2(y) = (\mu/r)^b \quad (4.5)$$

where a and b are constants. As a notational convenience, we will denote the metrics, connections and curvatures corresponding to ds_1^2 by unprimed symbols, and all quantities corresponding to the metric ds^2 by primed symbols.

We shall seek metrics, ds^2 , for which the components of the Ricci tensor satisfy:

$$R'_{\mu\nu} = f_1(y) g'_{\mu\nu} = f_1 \rho_1^2 g_{\mu\nu} \quad (4.6)$$

$$R'_{\mu m} = 0 \quad (4.7)$$

$$R'_{mn} = f_2(y) g'_{mn} = f_2 \rho_2^2 g_{mn} \quad (4.8)$$

Such metrics will include those which arise from the generalized Freund-Rubin ansätze which one can use for the compactifications of eleven dimensional supergravity to either seven or four dimensions, or for the compactification of ten dimensional supergravity down to five dimensions [7]. Taking into account the Bianchi identities and field equations of the antisymmetric tensor fields in these theories, and the warp factors in the metric, we see that for these compactifications the appropriate values of f_1 and f_2 are:

$$d_1 = 4, d_2 = 7: f_1(y) = -2f_2(y) = -(4/3)\rho_1^{-8}m^2 \quad (4.9)$$

$$d_1 = 7, d_2 = 4: f_1(y) = -\frac{1}{2}f_2(y) = -(2/3)\rho_1^{-14}m^2 \quad (4.10)$$

$$d_1 = 4, d_2 = 5: f_1(y) = -f_2(y) = -4m^2\rho_1^{-10} \quad (4.11)$$

where m is a constant. Note that we are using a different convention from [3], in that the Ricci scalar is positive on a deSitter space.

A straightforward calculation shows that the Ricci tensor of the metric (4.1) automatically has $R'_{\mu m} = 0$, and that

$$\begin{aligned} R'_{\mu\nu} &= R_{\mu\nu} - \frac{(d_1-1)}{\rho_2^2} (\nabla_m \rho_1) (\nabla^m \rho_1) g_{\mu\nu} - \frac{\rho_1}{\rho_2^2} \{ (d_2-2) (\nabla_\mu \rho_2) (\nabla^\mu \rho_2) + \rho_2 (\nabla^2 \rho_2) \} g_{\mu\nu} \\ R'_{mn} &= R_{mn} + \frac{1}{\rho_2^2} \{ (d_2-2) [2 (\nabla_m \rho_2) (\nabla_n \rho_2) - \rho_2 (\nabla_m \nabla_n \rho_2)] \\ &\quad - [(d_2-3) (\nabla^\mu \rho_2) (\nabla_\mu \rho_2) + \rho_2 (\nabla^2 \rho_2)] g_{mn} \} \\ &\quad - \frac{1}{\rho_1 \rho_2} d_1 \{ (\nabla_\mu \rho_1) (\nabla^\mu \rho_2) g_{mn} + [\rho_2 (\nabla_m \nabla_n \rho_1) \\ &\quad - (\nabla_m \rho_1) (\nabla_n \rho_2) - (\nabla_m \rho_2) (\nabla_n \rho_1)] \} \end{aligned} \quad (4.13)$$

where ∇_m is the covariant derivative on ds_2^2 , and all index raising is done with $g^{\mu\nu}$.

Take

$$P^{AB} = \text{diag} \left(\underbrace{1, 1, \dots, 1}_{p\text{-times}}, \underbrace{-c^2, -c^2, \dots, -c^2}_{q\text{-times}} \right) \quad (4.14)$$

where c is a constant, so that

$$\mu^2 = (z^1)^2 + \dots + (z^p)^2 + c^4 [(z^{p+1})^2 + \dots + (z^{p+q})^2] \quad (4.15)$$

and

$$n_A = \frac{1}{\mu} (z^1, \dots, z^p, -c^2 z^{p+1}, \dots, -c^2 z^{p+q}) \quad (4.16)$$

Let

$$m_A = \frac{1}{\mu} (z^1, \dots, z^p, c^4 z^{p+1}, \dots, c^4 z^{p+q}) \quad (4.17)$$

where

$$\nu^2 = (z^1)^2 + \dots + (z^p)^2 + c^8 [(z^{p+1})^2 + \dots + (z^{p+q})^2] \quad (4.18)$$

$$= \mu^2 + c^2(c^2 - 1)(\mu^2 - r^2). \quad (4.19)$$

Using the Gauss-Codacci equations for the curvature R_{ABCD} of an embedded hypersurface,

one can show that the Ricci tensor is

$$\begin{aligned} R_{AB} = & \frac{1}{\mu^2} \left\{ \left[p - c^2 q + 2(c^2 - r^2) - \frac{c^2 r^2}{\mu^2} \right] P^{AB} - c^2 \delta_{AB} \right. \\ & + \left[(p - c^2 q) \left(1 - c^2 + \frac{c^2 r^2}{\mu^2} \right) + 2c^2 - \left(\frac{\nu}{\mu} \right)^2 \right] n_A n_B \\ & \left. + \frac{\nu^2}{\mu^2} m_A m_B - \left(\frac{\nu}{\mu} \right) [(p - c^2 q) - (1 - c^2)] m_A n_B + m_B n_A \right\} \end{aligned} \quad (4.20)$$

Substituting (4.5) and (4.20) into (4.13), one obtains

$$R'_{AB} = c_1 P_{AB} + c_2 m_A m_B + c_3 (m_A n_B + n_A m_B) + c_4 \delta_{AB} + c_5 n_A n_B \quad (4.21)$$

where R'_{mn} is obtained as the restriction of R'_{AB} to the space orthogonal to n_A . It should be noted that R'_{AB} already involves the appropriate projection operators, so that

$$R'_{AB} n^B \equiv 0 \quad (4.22)$$

The coefficients c_1 – c_5 in (4.21) are:

$$c_1 = \frac{1}{\mu^2} \left[k_1 + k_2 \frac{c^2 r^2}{\mu^2} \right] \quad (4.23)$$

$$c_2 = \frac{\nu^2}{\mu^4} [(b+2)k_2 + (b+3) - a(a-b)d_1] \quad (4.24)$$

$$\begin{aligned} c_3 = & \frac{\nu}{\mu^3} \left\{ (1 - c^2)k_2 - k_1 - \left[(1 - c^2) + \frac{c^2 r^2}{\mu^2} \right] \right. \\ & \left. \times [(k_2 + 1)(b+3) - a(a-b)d_1] \right\} \end{aligned} \quad (4.25)$$

$$\begin{aligned} c_4 = & -\frac{c^2}{\mu^2} [(b+2)k_2 + b(d_2 - 1) + 2] - \frac{2bc^4 r^4}{\mu^6} \\ & + \frac{bc^2 r^4}{\mu^4} \left\{ k_1 + k_2 \left[(1 - c^2) + \frac{c^2 r^2}{\mu^2} \right] \right\} \end{aligned} \quad (4.26)$$

$$\begin{aligned} c_5 = & +\frac{c^2}{\mu^2} [(b+2)k_2 + (d_2 - 1)b + 2] + \frac{2bc^4 r^4}{\mu^6} \\ & - \frac{bc^2 r^4}{\mu^4} \left\{ k_1 + k_2 \left[(1 - c^2) + \frac{c^2 r^2}{\mu^2} \right] \right\} \\ & + \frac{1}{\mu^2} \left[(1 - c^2) + \frac{c^2 r^2}{\mu^2} \right] \left\{ [(b+3)(k_2 + 1) - a(a-b)d_1] \right. \\ & \left. \left[(1 - c^2) + \frac{c^2 r^2}{\mu^2} \right] + k_1 - k_2(1 - c^2) \right\} \end{aligned} \quad (4.27)$$

where

$$k_1 = (p-2) - c^2(q-2) \quad (4.28)$$

$$k_2 = [b(d_2 - 2) + ad_1 - 1] \quad (4.29)$$

For the Ricci tensor to have the form (4.8) we must take (for $q \neq 0$)

$$c_1 = c_2 = c_3 \equiv 0 \quad (4.30)$$

$$c_4 = -c_5 \quad (4.31)$$

This implies $k_1 = k_2 = 0$ and

$$(b+3) - a(a-b)d_1 = 0 \quad (4.32)$$

Hence

$$c^2 = \frac{(p-2)}{q-2} \text{ or } p = q = 2 \quad (4.33)$$

$$a^2 = \frac{(3d_2 - 5)}{d_1(d_1 + d_2 - 2)} \quad (4.34)$$

$$b = \frac{1 - ad_1}{(d_2 - 2)} \quad (4.35)$$

and the internal components of the Ricci tensor are

$$R'_{mn} = -\frac{c^2}{\mu^2} \left[2 + (d_2 - 1)b + 2b\frac{c^2 r^4}{\mu^4} \right] g_{mn} \quad (4.36)$$

Given (4.28) and (4.29) one can also show that

$$R'_{\mu\nu} = \left\{ \Lambda - \frac{\alpha c^2}{r^2} \left(\frac{\mu^2}{r^2} \right)^{a-b-1} \cdot \left[(d_2 - 1) + 2c^2 \frac{r^4}{\mu^4} \right] \right\} g_{\mu\nu} \quad (4.37)$$

Any solutions of the form defined by (4.6)–(4.11) require the scalar factors in front of the metric in (4.36) and (4.37) to be simple powers of μ . If we set $c^2 = 0$ this results in a flat space compactification, and this requires $p = 2$, in which case (3.12) describes a cylinder. There are also other trivial alternatives, but the only non-trivial possibility is to take

$$b = -\frac{2}{(d_2 - 1)}; \quad a - b = 1 \quad (4.38)$$

Using this in (4.34) and (4.35) one obtains

$$a = \frac{2}{(d_1 - 1)}; \quad b = -\frac{2}{(d_2 - 1)} \quad (4.39)$$

and a constraint on d_1 and d_2 :

$$(d_2 - 3) = \frac{4}{(d_1 - 3)} \quad (4.40)$$

(We note that this constraint excludes the alternative $p = q = 2$ in (4.33) since this requires $d_2 = 3$.)

There are only three solutions to this constraint, and we treat them in turn:

$$i) \quad d_1 = 4, \quad d_2 = 7$$

$$a = 2/3, \quad b = -1/3$$

$$R'_{\mu\nu} = \left[\left(\Lambda - \frac{4c^2}{r^2} \right) - \frac{4}{3} \frac{c^4 r^2}{\mu^4} \right] g_{\mu\nu}$$

$$R'_{mn} = \frac{2}{3} \frac{c^4 r^2}{\mu^6} g_{mn}$$

Comparing this with (4.9) we see we have a solution, for all choices of the signature of the hyperboloid, if we take $\Lambda = \frac{4c^2}{r^2}$ and $m^2 = \frac{c^2}{r^2}$. We note however that since

$$c^2 = \frac{(p-2)}{(q-2)},$$

there are only two possible choices:

$$p = q = 4 \rightarrow c^2 = 1$$

$$p = 5, \quad q = 3 \rightarrow c^2 = 3$$

(The choice $p = 3, q = 5$ is equivalent to $p = 5, q = 3$.)

These two solutions coincide with the backgrounds discussed in the previous section as vacua of the $SO(4, 4)$ and $SO(5, 3)$ gauged theories. In this section, however, we have confirmed that the four dimensional cosmological constant, Λ , must now be positive, as was found in [12]. We also note in passing that for $p = 7, q = 1$ we also have a solution, but with $c^2 = -5$. This corresponds to the inhomogeneously squashed seven sphere [20].

$$ii) \quad d_1 = 7, \quad d_2 = 4$$

$$a = 1/3, \quad b = -2/3$$

$$R'_{\mu\nu} = \left[\left(\Lambda - \frac{c^2}{r^2} \right) - \frac{2}{3} \frac{c^4 r^2}{\mu^4} \right] g_{\mu\nu}$$

$$R'_{mn} = \frac{4}{3} \frac{c^4 r^4}{\mu^6} g_{mn}$$

Comparison with (4.10) suggests that provided we choose $\Lambda = c^2/r^2$ and $m^2 = \frac{c^4}{r^4}$ then there should be a solution for any choice of signature. However, $p + q = 5$ and

$$c^2 = \frac{(p-2)}{(q-2)}$$

so that there is no non-trivial, non-singular choice. The only option is $p = 4, q = 1$ (or equivalently $p = 1, q = 4$) which leads to $c^2 = -2$, and corresponds to the inhomogeneously squashed 4-sphere [20].

$$iii) \quad d_1 = d_2 = 5$$

$$a = -b = \frac{1}{2}$$

$$R'_{\mu\nu} = \left[\left(\Lambda - \frac{2c^2}{r^2} \right) - \frac{c^4 r^2}{\mu^4} \right] g_{\mu\nu}$$

$$R'_{mn} = \frac{c^4 r^4}{\mu^6}$$

Comparison with (4.11) leads to a solution for all choices of p and q , provided we take $\Lambda = \frac{2c^2}{r^2}$ and $m^2 = \frac{c^2}{4r^2}$. Once again we have

$$c^2 = \frac{(p-2)}{(q-2)}$$

and this time there is only one non-trivial choice: $p = q = 3$, $c^2 = 1$. This must be the background corresponding to the $SO(3) \times SO(3)$ invariant critical point of gauged $SO(3,3)$. $N = 8$ supergravity in five dimensions [9].

For the choice $p = 5$, $q = 1$ (or $p = 1$, $q = 5$) we get $c^2 = -3$, and recover the inhomogeneously squashed five-sphere [20].

5. Discussion

We have shown how to obtain non-compact gauged supergravity theories from higher dimensional field theories. In particular, suppose we have a background solution of a higher dimensional supergravity theory in which there is a complete, consistent truncation to a maximal supergravity theory with compact gauge group in the lower dimensional space-time. Then for each non-compact gauging of the lower dimensional supergravity theory, we can construct a new higher dimensional background, and a truncation that yields this new gauging. The new internal manifold is non-compact.

Though we have resolved the question as to whether the non-compact gaugings have higher dimensional analogues, our results lead to many new questions. In particular, can we dimensionally reduce on these new non-compact manifolds, retaining all the massive modes, and still obtain a well behaved-field theory? Since we are using a non-compact internal manifold, we might expect to obtain a continuous spectrum of massive modes. This sort of problem has been addressed in [21,22]. The continuous spectrum can often be avoided by imposing suitable asymptotic boundary conditions on fluctuations about the internal space. For the non-compact internal manifolds discussed here, it may be possible to obtain "well behaved" modes by analytically continuing the modes from the corresponding sphere. At the very least, such a calculation would suggest appropriate boundary conditions. We also observe that the scale factor $\mu t^{-2/3}$ (or more generally μ^{2h}) in front of the metric (3.18) serves to pinch off the metric at infinity. This pinching off is not strong enough to make the manifold have finite volume, but the presence of these terms may give guidance as to the correct boundary conditions for fluctuations about the background.

Another problem is that of stability. In four dimensional Minkowski space instability is signalled by negative (mass)² states (tachyons). In curved space-time the relation between

instabilities and the mass spectrum is more complicated [17]. However, if the theory has an unbroken supersymmetry in a given background then that background is completely stable.

The $SO(4,4)$ and $SO(5,3)$ critical points described above lead to backgrounds which are unstable [12]. Thus the corresponding eleven dimensional solutions must be unstable. On the other hand if we choose the four-dimensional backgrounds to be electrovac spaces [18], then the eleven dimensional theories can be stable since at least some of them have an unbroken supersymmetry [19].

Another non-compact gauging with a supersymmetric vacuum state is the $SU(3,1)$ gauged $N = 8$ theory in five dimensions, or the four-dimensional theory obtained by dimensionally reducing this on a circle [9]. The vacuum is Minkowski space with the gauge symmetry spontaneously broken to $SU(3) \times U(1) \times U(1)$ (in four dimensions) and an $SU(2)$ global symmetry. It also has an unbroken $N = 2$ supersymmetry and is thus completely stable. We conjecture that this theory can be obtained from the chiral IIB supergravity theory in ten dimensions [9] by dimensional reduction on a non-compact manifold. We would expect to be able to obtain this background by analytically continuing the S^6 dimensional reduction of the ten dimensional IIB theory. However, in five dimensions the analytic continuation of the $SO(6)$ gauged theory to the $SU(3,1)$ theory involved continuing a scalar that corresponds to part of the three index antisymmetric field strength, G_{mnp} , in the ten dimensional theory. While our results here suggest how to obtain the gravitational part of the background, we need the full ansatz for G_{mnp} in order to determine the background completely. This ansatz is as yet unknown. If we can obtain the solution of the IIB field theory which gives rise to the $SU(3,1)$ gauging, then it seems likely that it will provide a solution of the IIB superstring. Presumably such compactifications are related to the string theories obtained in [24]. More generally, it may be of considerable interest to use backgrounds with non-compact internal manifolds to dimensionally reduce other supergravity or superstring field theories.

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