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GROUP THEORETIC APPROACH TO THE SUPERSTRING
AND ITS SUPERMODULI

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ABSTRACT

We show that the group theoretic method can be used to compute the superstring S matrix with Ramond external and internal lines in a manifestly Lorentz invariant way compatible with two-dimensional supersymmetry. Essential use is made of the fact that string vertices carry an induced representation of the super-Virasoro algebra. Supermoduli have a clear interpretation, as arising from the boosting procedure of induced representations.

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1. - INTRODUCTION

Despite the evident promise that string theories would provide a framework in which all interactions would ultimately become unified in a consistent way [1], it was also manifest from the earliest days that our basic understanding of these theories was far from sufficient to enable us to decide on such an issue. To remedy this situation, covariant second quantized string string field theories [2-5] were developed, in order to apply the usual powerful methods of ordinary Lagrangian theories of gauge fields. A Lagrangian theory can potentially address non-perturbative questions like the choice of the vacuum, and perturbative non-renormalization and finiteness theorems. Covariant string field theory has enabled us to clarify the gauge structure responsible for the absence of string states of negative norm and has in particular revealed the importance of the first quantized BRST operator Q in the second quantized BRST theory [6] and the second quantized gauge invariant theory $\langle \chi | Q | \chi \rangle$ [4,7]. However, it has not yet enabled us to solve the non-perturbative problems mentioned above. Although perturbation theory results, which were in essence known since 1971 [8], can be recovered from string field theory it is a cumbersome process. This may be only due to our present technical inability to handle the rather complicated form of the interaction terms. It is also conceivable that this approach has been too conservative in copying too closely the successful Yang-Mills theories, and that some more basically string-like features, and deeper mathematical concepts remain to be discovered and combined to efficiently address non-perturbative questions.

To this end, two new approaches to string theory have been developed in the last year. One goes under the name "group theoretic approach to string theory" [9-15]. The other is based on the mathematics of Grassmannians [16-20]. Another development of interest has been the incorporation of ghosts into the old dual model [8] which builds all graphs out of the propagator [8,21] and three-string vertex [22]. The decoupling of negative norm states is achieved through the simple BRST transformation properties of the ghost extended three vertex [5] and propagator [23]. Sewing these objects together ensures the correct decoupling of negative norm states within multiloop contributions [24].

The group theoretic approach [9-15] generalizes specifically string-like features from the known perturbative results. One of these features is the existence of overlap equations for vertices. They relate via conformal transformations the action of any conformal operator acting on one external string state to its action on the other external string states, without having to know the details of the interior of the graph. The overlaps provide a powerful tool to compute excited string vertices in perturbation theory. The string scattering amplitude is found

by integrating the vertex over the variables upon which it depends, i.e., the Koba-Nielsen variables [25] and moduli [26] with an appropriate measure. While the vertex is obtained through the overlap equations, the measure is obtained by demanding that zero-norm physical states decouple in the amplitude. This reduces the computation of string scattering amplitudes to an exercise in conformal group theory. This general framework allows one to deduce general properties of these amplitudes, like their behaviour under Virasoro gauge transformations, without having to know their explicit form nor to perform any tedious oscillator algebra. One application has been the discovery of the explicit existence of a gauge anomaly at one loop in the open bosonic string, even in 26 dimensions, and its cancellation for the special Chan-Paton group $SO(2^{13})$.

The other specifically string-like feature of the group theoretic approach uses the singular conformal transformations which change the moduli of the Riemann surface; for the bosonic and Neveu-Schwarz strings, this enabled us to related the decoupling of zero-norm states to differential equations on the integration measure on these (super) moduli. The cases of bosonic and Neveu-Schwarz strings are made easier by the existence of the (super) $SU(1,1)$ subalgebra of the (super) Virasoro algebra. This makes the problem of supermoduli for the scattering of Neveu-Schwarz a rather easy extension of the purely bosonic case.

The general superstring amplitudes contain an arbitrary number of external Ramond lines, and in this case the concept and number of supermoduli is less clear. To handle this general problem requires pushing one step further the logic of the group theoretic method. This was done in Ref. [27] where it was shown that excited string vertices fit nicely in the theory of induced representations of (multiple copies of) the Virasoro algebra. In these induced representations, the isotropy group is trivially realized, and corresponds to (super) conformal transformations which leave the (super) moduli inert. These transformations are simply given by the overlaps of the (super) Virasoro generators. The other transformations, which change the (super) moduli correspond to the boosts of the theory of induced representations. Hence, string vertices appear as a coset space. From this observation, one may obtain a general superstring vertex by applying a general superboost to its value determined at an "easy" point, e.g., when all anticommuting moduli vanish. The vertex at this easy point is entirely determined by overlap equations which do not involve anticommuting co-ordinates, and are therefore straightforward. Once this is done, following the general philosophy of the group theoretic method, the integration measure over both commuting and anticommuting moduli is obtained by requiring decoupling of zero-norm states. This is better done after extending the string vertices to contain the ghost oscillators, and making a corresponding extension of the Virasoro algebra, to include antighosts and the BRST charge [27].

In Section 2, we review the basic features of the group theoretic approach, how it determines the excited string vertex and the integration measure over Kobayashi-Nielsen variables and internal loops moduli. The method is presented for the bosonic open string, but in a simple enough form to allow its easy extension to superstrings.

In Section 3, we show that the case of external Neveu-Schwarz lines can be considered as a straightforward generalization of the bosonic case, using a simple two-dimensional superspace approach and the corresponding superconformal algebra. Examination of the general multiloop amplitudes reveals that this straightforward approach is actually able to handle the possibility of internal loops of Ramond strings in a completely Lorentz covariant way. This strongly suggests that a Lorentz covariant approach of the case of external Ramond strings should not be essentially different. At first sight, one apparent problem is that there exists in the Ramond sector no super-extension of $SU(1,1)$ which does not end up containing the full (super) Virasoro algebra. However, considering vertices as induced representations, there is no need for the coset generators, i.e., those outside the isotropy group, to form a closed algebra.

Indeed, in Section 4, we show how viewing string vertices as induced representations of the (super) Virasoro algebra provides a straightforward method for computing the Neveu-Schwarz and Ramond string scattering. In particular, it provides an explicit method for the introduction of supermoduli in the Ramond case. We recall that the first calculations of string scattering in the Ramond sector were done in the light-cone functional approach [28] or needed the insertion of the Brink-Olive projection operator [29] in the intermediate states between fermion emission vertices [30], and had been limited to the case of four external Ramond ground states [28,31-33]. More recent calculations [34,35] have relied on the elegant bosonization [34] of the Neveu-Schwarz-Ramond field, but in this process lose manifest Lorentz covariance and two-dimensional supersymmetry. The calculations presented in the present paper are manifestly Lorentz covariant and automatically incorporate two-dimensional supersymmetry and BRST invariance.

Section 5 shows how vertices and overlap equations transform when one goes from one representation of the Riemann surface to another, in particular from the Schottky to the Fuchsian representations. This shows how various spin structures are naturally found in the group theoretic approach, and is necessary for explicit examples.

In Sections 6 and 7, for trees and loops respectively, we demonstrate that the strategy set out in Section 4 does indeed determine the scattering amplitudes for arbitrary Ramond and Neveu-Schwarz external states.

2. - REVIEW OF GROUP THEORETIC APPROACH

We now review the group theoretic [9-15] approach to the open string S matrix. The scattering of N physical states $|x_1\rangle, |x_2\rangle, \dots, |x_N\rangle$ is given by

$$W = \int \prod_i d\bar{z}_i \prod_r d\sigma_r \bar{f}(z_i, \sigma_r) V(z_i, \sigma_r) |x_1\rangle, \dots, |x_N\rangle. \quad (2.1)$$

In this formula, z_i , $i = 1, 2, \dots, N$ are the external line Koba-Nielsen co-ordinates, v_r , $r = 1, 2, \dots, 3(g-1) \equiv g_1$ are the moduli of the Riemann surface of genus g , the function $f(z_i, v_r)$ is the integration measure. The vertex $V(z_i, v_r)$ is a bra in the Hilbert space which is the direct product of the Hilbert spaces of the N external states. In the group theoretic approach, V is determined by overlap equations; to be precise, if $R^{(i)}(z)$ is a conformal operator of dimension d acting on the i^{th} external line, then V obeys the equation

$$V \frac{R^i(\bar{z}_i)}{(\bar{z}_i)^d} = V \frac{R^j(\bar{z}_j)}{(\bar{z}_j)^d} \left(\frac{d\bar{z}_j}{d\bar{z}_i} \right)^d. \quad (2.2)$$

for any pair (i, j) , $i \neq j$. Such overlap equations for the position operator X^μ were encountered [36] in the light-cone string field theory, where the interaction vertex is a δ function and for the operators H^μ and Γ^μ in the fermion emission vertex [30]. More recently [2,5], overlap equations were used to construct the vertices and their conformal properties in gauge covariant string field theory. It was then realized [9] that any string vertex including those in the dual model satisfies such overlaps for arbitrary conformal operators. An example [37] was the three-string vertex of the bosonic dual model (i.e., the Caneschi-Schwimmer-Veneziano vertex V_{CSV}) and its ghost extension [5]. Overlap equations were used to derive the three Neveu-Schwarz vertex [38] and the ghost extension of the fermion emission vertex [39]. Overlap equations for multistring trees [9,12,13,14,40] and multiloops [10,11,13,15] have also been considered.

The functions $\xi_i(z)$ and $\xi_j(z)$ are analytic functions of the co-ordinate z which vanish at $z = z_i$ and $z = z_j$ respectively [13]

$$\xi_i(z) = (V_i)^{-1} z = \sum_{n=1}^{\infty} a_n^i (z - z_i)^n; (a_n \neq 0) \quad (2.3)$$

Equation (2.2) simply states that the action on V of the conformal operator R is independent of the external line through which it acts, once the conformal transformations (2.3) are taken into account. We note that we may write Eq. (2.3) in terms of the conformal operators L_n acting on line i :

$$V_i = \exp(z_i L_1^{(i)}) \exp\left(-\sum_{n=0}^{\infty} a_n^{(i)} L_n^{(i)}\right) \quad (2.4)$$

for some $a_n^{(i)}$'s.

For each internal loop n , we have the overlap conditions [10-15]

$$V \frac{R^i(\bar{z}^i)}{(\bar{z}^i)^d} = V \frac{R^i(P_n^i \bar{z}^i)}{(P_n^i \bar{z}^i)^d} \left[\frac{d(P_n^i \bar{z}^i)}{d \bar{z}^i} \right]^d \quad (2.5)$$

Using Eq. (2.2) to change the index i in Eq. (2.5), we find that for consistency P_n^i is of the form

$$P_n^i = (V^i)^{-1} P_n V^i$$

For reasons that will be clear below, we insist that P_n is in $SU(1,1)$.

The moduli dependence of V is defined through Eqs. (2.2) and (2.5); there being three parameters in each P_n , and one z_j in each V_j^{-1} . The vertex also depends on $a_n^{(j)}$, $n = 1, \dots, \infty$, $j = 1, \dots, N$. However, as we shall see, the S -matrix is independent of these latter parameters.

Equations (2.2) and (2.5) are defined on the Riemann sphere. For loops, Eq. (2.5) tells us that one may work on the fundamental domain only, corresponding to the group generated by P_n , that is, the region outside all the isometric circles of P_n (Fig. 1). Having chosen P_n in $SU(1,1)$, we recognize here the Schottky representations [26,41] of the Riemann surface. One can further cut the Schottky picture and consider the Fuchsian representation. Corresponding to this further dissection of the Riemann surface, one inherits one extra overlap condition for each loop. We shall see explicit examples of these in Section 7. Of course, all these overlaps just express the fact that the vertex V lives on the Riemann surface independently of the representation of the surface.

Equations (2.2) and (2.5) hold for any conformal operator. For example, in the case of the bosonic string, we may use $Q^\mu(z)$, $P^\mu(z) \equiv -z(d/dz)Q^\mu(z)$, the energy-momentum tensor $L(z)$ (being careful in this case with extra terms due to normal ordering), and the ghosts $\beta(z)$, $\bar{\beta}(z)$ which have conformal weights -1 and $+2$ respectively, and are more usually denoted $c(z)$ and $b(z)$ in the literature. In the case of loops, some care must be taken for $Q^\mu(z)$, and the reader may consult Ref. [13].

Equations (2.2) and (2.5) determine V up to a multiplicative function of the moduli. Given a choice of this function, the decoupling of zero-norm physical states determines the measure $\bar{f}(z_i, v_r)$ [9-15], the final S-matrix being of course independent of this choice, as one may shift it at will between V and $\bar{f}$ in Eq. (2.1) without affecting the left-hand side.

One can integrate Eq. (2.2) to obtain what we called an integrated overlap equation [2,5,37-39,9-15]

$$V \left\{ \sum_{j=1}^N \oint_{\vec{z}=0} d\vec{z}_j \, \phi \frac{R^0(\vec{z}_j)}{(\vec{z}_j)^d} \left[\frac{d\vec{z}_j}{d\vec{z}_i} \right]^{d-1} \right\} = 0 \quad (2.6)$$

An alternative derivation of certain integrated overlap equations from the Polyakov path integral has been given in Ref. [18], and were also discussed in Refs. [42,40,20].

We may identify ϕ as a tensor of rank $1-d$. Equation (2.6) is valid provided that ϕ is defined on the Riemann sphere for trees, and satisfied the relevant automorphic conditions for loops, i.e.

$$\phi(\vec{z}_i) = \phi(P_n^i \vec{z}_i) \left[\frac{d(P_n^i \vec{z}_i)}{d\vec{z}_i} \right]^{1-d} \quad (2.7)$$

In addition, when ξ_i vanishes, i.e., at the Koba-Nielsen points corresponding to the external lines, ϕ must be such that the integrals appearing in Eq. (2.6) are single valued. This is most important in the case of the Neveu-Schwarz-Ramond model, as we shall see in this paper. Otherwise, ϕ is analytic elsewhere. These requirements on ϕ are obvious consequences of the integration procedure leading to Eq. (2.6), which is discussed in Refs. [5,37,9-15].

In order to determine the integration measure $\bar{f}$, the group theoretic approach requires a knowledge of the change of V under changes of the moduli and/or Koba-Nielsen variables. To determine how V 's are related for different values of these quantities, we consider the action of a conformal operator $\mathcal{X}^{(i)}$ where i refers to the external line of V it acts on:

$$V \rightarrow \hat{V} = V \prod_{i=1}^N \mathcal{X}^{(i)} \quad (2.8)$$

Using Eqs. (2.2) and (2.5), one finds that one can change the transformations V_i to $\hat{V}_i$ (hence the Koba-Nielsen variables z_j to $\hat{z}_j$) and the loop overlap transformations P_n to $\hat{P}_n$ (hence the moduli v_r to $\hat{v}_r$) by taking $\mathcal{X}^{(i)}$ of the form

$$\mathcal{U}^i = (V^i)^{-1} \mathcal{U} V^i \quad (2.9)$$

with

$$P_n^{-1} \mathcal{U} \hat{P}_n = \mathcal{U}. \quad (2.10)$$

Some particular cases of these general equations will be more useful in the rest of this paper. For example, making the particularly simple choice [13]

$$(V_i)^{-1} z = z - z_i \quad (2.11)$$

and moving only the Koba-Nielsen points z_i to $\hat{z}_i$, keeping the loop transformations P_n fixed, we have simply:

$$\mathcal{U}^i = \exp(-z_i L_{-1}^{(i)}) \exp(\hat{z}_i L_{-1}^{(i)}) \quad (2.12)$$

Further specializing z_i to differ from $\hat{z}_i$ by an infinitesimal amount, we learn that

$$\frac{\partial V}{\partial z_i} = V L_{-1}^{(i)} \quad (2.13)$$

which is a particularly simple result.

Another simple case corresponds to keeping the Koba-Nielsen points z_j fixed, but changing the moduli v_r by an infinitesimal amount to $\hat{v}_r = v_r + \epsilon h_r(v)$. In such a case, the conformal transformation $\mathcal{U}$ is also infinitesimal, and can be represented by:

$$\mathcal{U} : z \rightarrow z + \epsilon f(z).$$

Equation (2.10) then reads [10,11]

$$P_n(z) + \epsilon f(z) \frac{dP_n}{dz} = \hat{P}_n(z) + \epsilon f(P_n z) \quad (2.14)$$

Equation (2.8) then becomes a differential equation for the vertex:

$$\sum_r h_r(v) \frac{\partial V}{\partial v_r} = V \left\{ \sum_{j=1}^N \oint_{\tilde{\gamma}_j} \frac{L^{(j)}(\tilde{z}_j) f^{(j)}(\tilde{z}_j)}{\tilde{z}_j} + c_f \right\} \equiv V L_f \quad (2.15)$$

where $f^{(j)}$ is the function representing the infinitesimal conformal transformation $\mathcal{U}^{(j)}$ on line j . c_f is a constant, related to normal ordering, which must be determined by taking a suitable state [10,11] (like for example the vacuum) on either side of this equation. It is clear that the overlap equations, being homo-

geneous in V , leave a multiplicative arbitrariness of a function of the moduli (and Koba-Nielsen variables), and the value of c_f is fixed only relative to the choice of this arbitrary multiplicative function. We recall, however, that to find the actual scattering amplitude, we multiply by the measure $\bar{f}$ and integrate over all Koba-Nielsen variables and moduli. The measure $\bar{f}$ is closely related to the constants c_f , and the final result of the integration is independent of the multiplicative arbitrariness in V , as it should be, of course. We shall discuss further this point below.

An important point about Eqs. (2.9) and (2.10) is that there are non-trivial solutions for $\mathcal{U}$ even if $\hat{V}^i = V^i$ and $\hat{P}_n = P_n$. These transformations do not change the moduli nor the Koba-Nielsen variables at all, and leave the vertex inert. They can be found by solving Eqs. (2.9) and (2.10), but also by using the integrated overlaps of Eq. (2.6), taking $R(z)$ to be $L(z)$. The conformal transformations which leave the vertex inert obviously form a subalgebra of the Virasoro algebra. This subalgebra plays a crucial role, conceptually and technically, in the construction of string vertices as induced representations given in Ref. [27] and applied to Ramond strings in Section 4 of this paper.

3. - TWO-DIMENSIONAL SUPERSPACE APPROACH FOR SUPERSTRINGS AND SUPERMODULI

The previous section is written for bosonic strings, but its generalization to superstrings is straightforward, at least when the external lines are bosons (Neveu-Schwarz sector). External Ramond strings generate special features, and will be dealt with more specifically in Sections 4 to 7. However, even with only external Neveu-Schwarz lines, loop diagrams may contain internal Ramond lines, and this possibility does show up in the formalism of this section, as we shall see.

We simply defined superconformal transformations

$$U = \exp\left(\sum_{n \in \mathbb{Z}} a_n L_n + \sum_{r \in \mathbb{Z} + \frac{1}{2}} a_r G_r\right) \quad (3.1)$$

with commuting a_n and anticommuting a_r numbers. These transformations are naturally realized on a superspace, which was first introduced into the spinning string in Ref. [43] and into the dual model in Ref. [44]. Superconformal operators $R(z, \theta)$ depend on commuting z and anticommuting θ variables. The superconformal transformation $(z, \theta) \rightarrow (z', \theta')$ is induced by taking L_n and G_r to have the usual representation:

$$L_n = z^n \left(z \frac{d}{dz} + \frac{1}{2}(n+1) \theta \frac{\partial}{\partial \theta} \right) \quad (3.2)$$

$$G_r = z^{r+\frac{1}{2}} \left(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right).$$

A superconformal operator $R(z, \theta)$ with superconformal weight d transforms as

$$U R(z, \theta) U^{-1} = \left[\frac{z}{z'}, \text{Ber}(z', \theta'; z, \theta) \right]^d R(z', \theta'). \quad (3.3)$$

where

$$\text{Ber}(z', \theta'; z, \theta) = \begin{vmatrix} \frac{dz'}{dz} & \frac{d\theta'}{d\theta} \\ \frac{dz'}{d\theta} & \frac{d\theta'}{dz} \end{vmatrix} \quad (3.4)$$

The overlap equation (2.2) now reads

$$V \left\{ R^i(\bar{z}_i, \eta_i) - \left[\frac{\bar{z}_i}{\bar{z}_j} \text{Ber}(\bar{z}_j, \eta_j; \bar{z}_i, \eta_i) \right]^d R^j(\bar{z}_j, \eta_j) \right\} = 0 \quad (3.5)$$

$$(\bar{z}_i, \eta_i) = (V_i)^{-1}(z, \theta) \quad ; \quad (\bar{z}_j, \eta_j) = (V_j)^{-1}(z, \theta)$$

where V_i and V_j are superconformal transformations.

For the case of external Neveu-Schwarz states, V_j^{-1} is any analytic superconformal transformation which vanishes at the Koba-Nielsen points (z_j, θ_j) . The simplest choice is thus [13]

$$\begin{aligned} \bar{z}_j &= z - z_j - \theta \theta_j \\ \eta_j &= \theta - \theta_j \end{aligned} \quad (3.6)$$

Just as for bosonic strings, we may take the more general choice

$$V_0 = \exp \left(z_0 L_{-1}^{(j)} + \theta_j G_{-\frac{1}{2}}^{(j)} \right) \exp \left(\sum_{n \geq 0} a_n^{(j)} L_n^{(j)} + \sum_{r \geq 0} b_r^{(j)} G_r^{(j)} \right) \quad (3.7)$$

For internal loops, just as in the bosonic case of Eq. (2.5), we have the overlap conditions

$$V R^i(\xi_i, \eta_i) = V \left[\frac{\xi_i}{\xi_i'} \text{Ber}(\xi_i', \eta_i'; \xi_i, \eta_i) \right]^d R^i(\xi_i', \eta_i') \quad (3.8)$$

where

$$(\xi_i', \eta_i') = P_n^i(\xi_i, \eta_i) \quad (3.9)$$

and P_n^i is a superconformal transformation. Again for consistency of Eqs. (3.5) and (3.8), we find that P_n^i is for the form

$$P_n^i = (v^i)^{-1} P_n v^i \quad (3.10)$$

and we insist that P_n is in super $SU(1,1)$, i.e., has three commuting and two anti-commuting parameters.

Integrated overlaps for super conformal operators can be obtained in the same way as in the bosonic case. Indeed, Neveu-Schwarz tree vertices have been determined, using such a superspace formalism, in Refs. [38,12,13], and the superconformal operators

$$Q^\mu(z, \theta) = \varphi^\mu(z) + \frac{\theta}{\sqrt{z}} \mathcal{H}^\mu(z). \quad (3.11)$$

$$\mathcal{H}^\mu(z, \theta) = \sqrt{z} \left(H^\mu(z) + \frac{\theta}{\sqrt{z}} P^\mu(z) \right) \quad (3.12)$$

where

$$\varphi^\mu(z) = + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu z^{-n} - \alpha_0^\mu \ln z - \frac{\partial}{\partial z} \alpha_0^\mu \quad (3.13)$$

$$H^\mu(z) = \sum_r b_r^\mu z^{-r}. \quad (3.14)$$

Q^μ and $\mathcal{H}^\mu$ have superconformal weight zero and one respectively.

The problem of supermoduli changing for vertices in the Neveu-Schwarz sector follows closely the bosonic case, and does not have to be repeated here. The

superconformal transformations that preserve the vertex are given by taking integrated overlaps of the operator $z^{3/2} [G(z) + \theta \sqrt{z} L(z)]$, which has superconformal weight 3.

When one attempts to extend this formalism to the case of Ramond external states, replacing the half-integer moded field $H^\mu(z)$ by the integer moded field

$$H^\mu(z) = \sum_{n=-\infty}^{\infty} d_n z^{-n}. \quad (3.15)$$

one finds the immediate problem that the simple choice (3.6) for the cycling transformations is incompatible with this mode structure as soon as θ_j is non-zero. Of course, this simply reflects the fact that super $SU(1,1)$ made out of $L_0, L_{\pm 1}, G_{\pm \frac{1}{2}}$ in the Neveu-Schwarz sector does not exist in the Ramond sector, where the half-integer moded G 's are replaced by integer moded F 's. Thus, the introduction of anticommuting Koba-Nielsen variables and supermoduli in the Ramond sector is much less trivial than in the Neveu-Schwarz sector, and will be done in next section.

Coming back to the Neveu-Schwarz sector, we now wish to comment on several features of this superspace approach. A general superconformal transformation is of the form

$$z' = g(z) + \theta A(z) \quad (3.16)$$

$$\theta' = g + \theta B \quad (3.17)$$

with

$$A = gB \quad (3.18)$$

$$B^2 = \frac{dg}{dz} + g \frac{dg}{dz}. \quad (3.19)$$

We notice that Eq. (3.19) only determines B^2 ; hence there is a natural sign ambiguity in the transformation of anticommuting variables. For overlap equations between different external lines, Eq. (3.5), this sign ambiguity turns out to be irrelevant if the external states are truncated according to the Gliozzi-Scherk-Olive (GSO) projection [45]. For loop overlaps, Eq. (3.8), this sign depends on the presence or absence of the operator $(-1)^{\sum b_r}$ in the internal loop. To obtain the appropriate GSO projection on internal loops, one must thus sum over vertices corresponding to different choices of this sign.

A further sign ambiguity occurs naturally in loops, and is seen most easily in the Fuchsian representation, where there exist two P_n transformations for each loop. This sign ambiguity corresponding to the other P_n is related to whether the

internal loop involves Ramond or Neveu-Schwarz lines. Here again, one must ultimately sum over vertices corresponding to different choices of this sign. This will be discussed in detail for loops in Section 7.

From this discussion, we see that the two-dimensional superspace and supermoduli approach of this section, when applied to external Neveu-Schwarz lines, is perfectly able to handle intricacies such as the GSO projection, and, more importantly, sees internal Ramond lines. We can understand this last point only by concluding that it should also be able to handle external Ramond lines, and that our past inability to deal with the Ramond sector is only due to the inadequacy of the simple guess of Eq. (3.6), or any similar simple guess. This is indeed what we shall see in next section.

4. - VERTICES AS INDUCED REPRESENTATIONS AND GENERAL STRATEGY FOR RAMOND STRINGS

In Refs. [12] and [13], an alternative but equivalent strategy to the superspace method of Section 3 was proposed to compute Neveu-Schwarz vertices. In this equivalent method, one first calculates the vertex with all the anticommuting Kobayashi-Nielsen variables and supermoduli taking the value zero, that is, one computes $V(z_i, \theta_i = 0)$. This is easily achieved by using Eqs. (2.2) and (2.5) for the conformal operators $P^\mu(z) = \sum \alpha_n^\mu z^{-n}$ and $H^\mu(z) = \sum b_r^\mu z^{-r}$. One then introduces the anticommuting parameters as supermodulus changes, i.e., generalizing Eq. (2.8)

$$V(z_i, \theta_i) = V(z_i, 0) \prod_j \mathcal{U}^j \quad (4.1)$$

In particular, for the case of trees,

$$\mathcal{U}^j = [V_j(z_k, 0)]^{-1} V_j(z_k, \theta_k) \quad (4.2)$$

and for the case of the transformations of Eq. (3.6), we find

$$V(z_j, \theta_j) = V(z_j) \left\{ 1 + \sum_i \theta_i G_{-\frac{1}{2}}^{(i)} + \sum_{i < j} \theta_i \theta_j G_{-\frac{1}{2}}^{(i)} G_{-\frac{1}{2}}^{(j)} + \sum_{i < j < k} \theta_i \theta_j \theta_k G_{-\frac{1}{2}}^{(i)} G_{-\frac{1}{2}}^{(j)} G_{-\frac{1}{2}}^{(k)} + \dots \right\} \quad (4.3)$$

This strategy was taken to its logical conclusion in Ref. [27], where it was shown that the group theoretic method implied that string vertices were induced representations of the Virasoro algebra. In particular, the overlap equations

imply that the vertex at a given point in Koba-Nielsen and moduli space is left inert by a certain set of (super)conformal transformations corresponding to the isotropy group, and that changing moduli and Koba-Nielsen variables as in Eqs. (2.8), (2.13) and (2.15) was equivalent to performing a boost in the theory of induced representations. It was pointed out that [27] if one knows the vertex at any one point in moduli and Koba-Nielsen space, one can compute its isotropy group at that point using the usual overlap equations (2.2) and (2.5) for $L(z) = \sum_{-n} L_{-n} z^n$. Assuming that the vertex is an induced representation of the (super) Virasoro group denoted V with this isotropy group then determines the vertex uniquely.

In fact, when one incorporates ghost oscillators into the theory, it is more natural [27] to consider a graded extension of the Virasoro group to include the ghosts of conformal weight two, $\bar{\beta}_n$ and the BRST charge, Q . We denote this graded extension of V by U . The isotropy group is determined by the overlap Eqs. (2.2) and (2.5) with R being $L(z)$ and $\bar{\beta}(z) = \sum_n \bar{\beta}_{-n} z^n$ and the vertex is annihilated by Q . The full vertex is then an induced representation of U and one finds that it depends on commuting as well as anticommuting parameters. Indeed, one finds that the ghost insertions arise naturally in this way [27].

We now recall the definition of an induced representation. Given a group G with subgroup H , whose generator we denote by H_1 , we denote by K_a the remaining generators of G . At least in the neighbourhood of the origin, an element of the coset G/H is of the form $\exp \sum_a \xi_a K_a$, and so the coset is parametrized by the numbers ξ_a . An induced representation is a set of vectors $\langle \xi |$ on G/H , each carrying a representation of H , such that

$$\langle n | U(h) = \langle n | D_n^m(h^{-1}). \quad (4.4)$$

$$\langle n | U(e^{\vec{\xi} \cdot K}) = \langle n | \xi_a \quad (4.5)$$

This definition illustrates the fact that given the representation of the vectors under H at one point of the coset (in this case, the origin $\xi = 0$), we can define $\langle \xi |$ by the action of $\exp \xi K$ and then deduce the action of the whole group on $\langle \xi |$. Indeed, we deduce that under a general transformation g

$$\langle \xi_a | U(g) = \langle \xi'_a | D_n^m(h^{-1}(\xi, g)) \quad (4.6)$$

where

$$(\exp \vec{\xi} \cdot K) g = h(\vec{\xi}, g) \exp \vec{\xi}' \cdot K \quad (4.7)$$

We observe that implicit in the above definition, in particular Eq. (4.6) is the knowledge of how the K_a 's act on the vectors at a given point. For the case of interest to us, the vectors $\langle \xi \rangle_n$ are the vertex operators, the quantities ξ_a which label them are the (super) moduli and (super) Koba-Nielsen variables, and the group G is multiple copies, one for each external line, of the (super) conformal group and its (super) graded extension to accommodate the BRST charge and the antighosts. The vertex at a given point is determined by the overlap equations, and the group is implemented on the vertex by the L_n 's, whose action is known; simple examples of this action of the group are found in Eq. (2.13) and (2.15). We refer the reader to Ref. [27] for more details.

The strategy we will adopt is to first calculate, using the overlap equations, the vertex $V(z_j, v_r)$ for which the anticommuting parameters are taken to be zero. The isotropy group of the vertex, i.e., those superconformal transformations that leave it inert, is then found also from the overlap identities of Eq. (2.5). We will then assume following Ref. [27] that the vertex is an induced representation of N copies of the superconformal group or its graded extension. Hence, using Eq. (4.5) we may find the vertex including its dependence on the anticommuting variables by a boost.

The purpose of this paper is to extend the group theoretic method to scattering which involves Ramond lines both as external and internal lines in the corresponding graph. One approach would be to use a superspace formalism, however, for this one must know the cycling transformations $V^j(z, \theta)$ and the general theory behind them which was given in Ref. [13] for the bosonic and Neveu-Schwarz theories. However the branch-cut structure of Ramond contributions complicates this approach which we will comment on later.

Let us see how this strategy works for Neveu-Schwarz vertices at tree level for M external lines. Consider first the vertex without its ghost extension. As explained in the previous section, we can evaluate the vertex $V_0(z_j) \equiv V(z_j, \theta_j=0)$ by using the overlaps

$$V_0(z_j) \left\{ \sum_{j=1}^M \oint d\bar{z}_j \epsilon \frac{H^{\dot{j}\mu}(\bar{z}_j)}{\bar{z}_j^{\frac{1}{2}}} \right\} = 0 \quad (4.8)$$

where $\xi_j = (z - z_j)$, and the analogous equation for $P^\mu(\xi_j)$. The vertex for this set of functions ξ_j is given in Refs [12,13]. The isotropy (super) group for V_0 is generated by the (super) transformations, linear superpositions of $G_r^{(i)}$ and $L_n^{(j)}$ such that

$$V_0 \left\{ \sum_{j=1}^m \oint d\bar{z}_j \varphi \frac{G^{(j)}(\bar{z}_j)}{(\bar{z}_j)^{3/2}} \right\} = 0 \quad (4.9)$$

for all allowed ϕ 's, and the analogous equations for $L(z)$. These transformations include

$$\sum_{j=1}^m G_{-\frac{1}{2}}^{(j)}, \quad \sum_{j=1}^m (z_j G_{-\frac{1}{2}}^{(j)} + G_{\frac{1}{2}}^{(j)})$$

and

$$G_{-n-\frac{1}{2}}^{(i)} + \sum_{j=1}^m \sum_{p=0}^{\infty} \binom{-n}{p} \frac{1}{(z_j - z_i)^{n+p}} \bar{g}_{p-\frac{1}{2}}^{(j)} \quad n \geq 1$$

Hence, the anticommuting generators in the coset can be chosen to be $G_{-\frac{1}{2}}^{(j)}$, $j = 1, \dots, M$, except for two values j_1 and j_2 , say, of j , and $G_r^{(j)}$, $r > \frac{1}{2}$, $j = 1, \dots, M$. The latter reflects the arbitrariness in the choice of the transformations V_j which will give identical results on physical states. Consequently, we find in effect the vertex to depend on $M-2$ anticommuting variables, which we denote by Q_j . The general vertex is then given by

$$V(z_j, Q_j) = V_0(z_j) \exp \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^m Q_j G_{-\frac{1}{2}}^{(j)}$$

[see Eq. (4.5)]. We note that this vertex is the same as in Eq. (4.3). Throughout this paper we will use the signs $\sum_{j=1}^{2N(p)}$ and $\prod_{j=1}^{2N(p)}$ to mean the sum over the p objects referred to in the text and if there is no reference then the sum over any p objects from the range indicated. With this convention the above equation reads

$$V(z_j, Q_j) = V_0(z_j) \exp \left(\sum_{j=1}^m (m-2) Q_j G_{-\frac{1}{2}}^{(j)} \right)$$

As explained in Ref. [15], it is convenient to extend the Virasoro algebra to include the BRST operator Q and the superreparametrization antighost fields. In this case, they are the fields $\bar{\beta}(z)$ of conformal dimension 2 and $\bar{g}(z)$ of conformal dimension $3/2$. As $\bar{\beta}(z)$ and $\bar{g}(z)$ have the same dimensions as $L(z)$ and $G(z)$ respectively, they obey identical overlap conditions (except for possible constants induced by normal ordering terms), and so, appear in the isotropy group and the coset in the same way. The vertex dependence on the oscillators of the fields $\beta(z)$, $\bar{\beta}(z)$, $g(z)$, $\bar{g}(z)$ is given by the overlaps. We henceforth use the same symbol $V_0(z_i)$ for the vertex which includes these fields.

As explained above and in Ref. [15], the general vertex includes extra Koba-Nielsen-like variables and is given by:

$$V(z_j, \theta_j, \psi_j, \kappa_j) = V_0(z_j) \exp \left(\sum_{\substack{j=1 \\ j \neq j_1, j_2}}^m (\theta_j G_{-\frac{1}{2}}^j + \kappa_j \bar{g}_{-\frac{1}{2}}^j) \right) \cdot \exp \left(\sum_{\substack{j=1 \\ j \neq j_1, j_2, j_3}}^m \psi_j \bar{\beta}_{-1}^j \right). \quad (4.10)$$

and the scattering amplitude is obtained by integrating over all variables $z_j, \theta_j, \psi_j, \kappa_j$. The integrals over K_j are taken to lead to δ functions of $\bar{g}_{-\frac{1}{2}}$. They are discussed in more detail in Ref. [27]. The result of the ghost algebra computation, i.e., the α_n^μ, b_r^ν part of the vertex agrees with Refs. [12] and [13], while the $\beta, \bar{\beta}$ ghost extension for the three-point case has been previously calculated [5], and the $g, \bar{g}$ ghost extension for the three-point vertex in Ref. [42].

Let us now discuss this strategy in the Ramond case. Consider a multiloop graph with $2N$ fermionic (i.e., Ramond) and M bosonic (i.e., Neveu-Schwarz) external strings, and let us consider the vertex, again denoted $V_0(z_i)$, evaluated when all the anticommuting parameters vanish; then for the simplest choice of cycling $(V^j)^{-1}(z) = z - z_j$ in the bosonic sector, V_0 will obey

$$V_0 \left\{ \frac{\Gamma^{i\mu}(z-z_i)}{\sqrt{z-z_i}} - \frac{\Gamma^{j\mu}(z-z_j)}{\sqrt{z-z_j}} \right\} = 0 \quad (4.11)$$

$$\begin{aligned} & ; i, j = 1, 2, \dots, 2N \\ V_0 \left\{ \frac{\Gamma^{i\mu}(z-z_i)}{\sqrt{z-z_i}} - \frac{\partial \epsilon^{\mu j}(z-z_j)}{\sqrt{z-z_j}} \right\} &= 0; \quad i=1, \dots, 2N \\ & \quad j=2N+1, \dots, 2N+M \\ V_0 \left\{ \frac{\partial \epsilon^{i\mu}(z-z_i)}{\sqrt{z-z_i}} - \frac{\partial \epsilon^{\mu j}(z-z_j)}{\sqrt{z-z_j}} \right\} &= 0; \quad j, i=2N+1, \dots, 2N+M. \end{aligned} \quad (4.12)$$

where $\xi_k = z - z_k$. V_0 also obeys overlap conditions for the loops, which are

$$V_0 \left\{ \frac{\Gamma^{\mu i}(\xi_i)}{\sqrt{\xi_i}} \pm \frac{\Gamma^{\mu i}(P_n^i \xi_i)}{\sqrt{P_n^i \xi_i}} \left[\frac{d(P_n^i \xi_i)}{d \xi_i} \right]^{\frac{1}{2}} \right\} = 0 \quad (4.13)$$

$i=1, \dots, 2N$

$$V_0 \left\{ \frac{H^{\mu j}(\xi_j)}{\sqrt{\xi_j}} \pm \frac{H^{\mu j}(P_n^j \xi_j)}{\sqrt{P_n^j \xi_j}} \left[\frac{d(P_n^j \xi_j)}{d \xi_j} \right]^{\frac{1}{2}} \right\} = 0 \quad (4.14)$$

$j=2N+1, \dots, 2N+M.$

where $P_n^i = (V^1)^{-1} P_n V^1$ and P_n are elements of $SU(1,1)$ as in the purely bosonic case. We will return to the sign ambiguities in Eqs. (4.13) and (4.14) shortly. We shall work with the simplest cycling, given in Eq. (2.11), but it is straightforward although technically more complicated to carry this out in general.

The above equations are defined on the Riemann sphere and Eqs. (4.13) and (4.14) say in essence that the action of Γ or H on V_0 obeys an automorphic condition which enables us to consider the restriction to the fundamental region generated by the group P_n . This is, of course, the Schottky representation of the relevant Riemann surface (see Fig. 4.1). On the Riemann sphere $V_0 (H^{\mu(i)}(\xi_i)/\sqrt{\xi_i})$ is a single-valued function of ξ_i . In particular, it is single-valued, when taken around any of the Koba-Nielsen points and around the isometric circles of P_n , that is, around the A cycles. In contrast, $V_0 (\Gamma^{\mu(i)}(\xi_i)/\sqrt{\xi_i})$ is not single-valued; indeed when taken around any Koba-Nielsen point, it is double-valued as is also the case for the A cycles. We must therefore regard the Riemann sphere as sheeted and we can arrange the N branch cuts to be between any pairing of the $2N$ Koba-Nielsen points where the fermions are emitted.

Following the usual strategy [9-15], one may integrate the overlap equations (4.11) and (4.12) to obtain integrated overlaps of the form

$$V_0 \left\{ \sum_{j=1}^{2N} \oint d\zeta_j \varphi \frac{\Gamma^{\mu(j)}(\zeta_j)}{\sqrt{\zeta_j}} + \sum_{j=2N+1}^{2N+M} \oint d\zeta_j \varphi \frac{H^{\mu(j)}(\zeta_j)}{\sqrt{\zeta_j}} \right\} = 0 \quad (4.15)$$

The derivation of this equation proceeds as for the bosonic case, except that we must take into account the branch cuts discussed above on the Riemann sphere. A Riemann sphere with N branch cuts is actually a Riemann surface of genus $N-1$. It would be quite possible to work on this surface. However, we can avoid the problems that arise from these branch cuts in a particularly simple way, namely to choose ϕ to have the same branch cuts as $V_0 \Gamma^{\mu(j)}(\xi^j)(\xi^j)^{-\frac{1}{2}}$, so that

$$\varphi V_0 \left\{ \sum_i \frac{H^{\mu(i)}(\zeta^i)}{\sqrt{\zeta^i}} + \sum_j \frac{\Gamma^{\mu(j)}(\zeta_j)}{\sqrt{\zeta_j}} \right\}$$

is single-valued. This is the strategy we adopt.

In the derivation of Eq. (4.15), one also has to take account of the valuedness around A cycles, so that, as discussed above, ϕ must be single-valued and double-valued around A cycles that correspond to loops made from Neveu-Schwarz and Ramond strings respectively.

Similarly, we must make sure that the contour integrals around the isometric circles C_n and C'_n in Fig. 1 cancel, and this is ensured by taking the function ϕ to have the same behaviour as $V_0(\Gamma^\mu(z)/\sqrt{z})$ or $V_0(H^\mu(z)/\sqrt{z})$ under the corresponding loop transformation P_n , i.e., Eqs. (4.13) and (4.14). As already, mentioned in Section 3, this sign ambiguity in these equations is related to the GSO projection [45] which is composed of

$$\Gamma^d = \gamma_n \exp i\pi \sum_{n=1}^{\infty} d_{-n}^\mu d_n^\mu \quad \text{and} \quad \Gamma^b = \exp i\pi \sum_{r=\frac{1}{2}}^{\infty} b_{-r}^\mu b_r^\mu$$

respectively, for the Ramond and Neveu-Schwarz sectors. A loop overlap of the form of Eqs. (4.13) and (4.14) arises from taking the conformal field in question along all the propagators and vertices which compose the loop. H^μ and Γ^μ anticommute with their respective Γ 's. If one originally has a given sign in Eq. (4.13), then inserting a factor Γ gives a scattering amplitude which has the opposite sign in this equation. In this straightforward way, we find the well-known result [47] that a sum over spin structures corresponds to a sum over GSO projection and the two types of string, i.e., fermions and bosons, propagating in the loops. Consequently, we must for both types of loops, take $\phi(\xi_i) = \phi(P_{n,i}^i \xi_i)$ and evaluate the vertex V_0^1 and also $\phi(\xi_i) = -\phi(P_{n,i}^i \xi_i)$ and find the vertex V_0^2 . To get the final result, we sum together all these vertices.

We have carried out the discussion of branch cuts and boundary conditions for the conformal operators $H^\mu(z)$ and $\Gamma^\mu(z)$, but it is clear that the discussion applies equally well to other such conformal operators, for example $G(z)$ and $F(z)$ or $\bar{g}(z)$ and $\bar{f}(z)$ or $g(z)$ and $f(z)$.

To summarize, in order to find the vertex, we must take in Eq. (4.15) the function ϕ which is a tensor of rank $\frac{1}{2}$, has the branch cut structure corresponding to the emission of the Ramond strings, has the valuedness around A cycles appropriate to bosons or fermions, and finally consider both possible signs of the automorphic condition under P_n , in order to be able to perform the GSO projection. The final result is obtained by summing over all these possibilities with appropriate weights.

The number of identities of the type of Eq. (2.2) and in particular Eq. (4.15) is the same as the number of ϕ 's one can find that satisfy the required conditions given above. The question now arises as to whether the integrated overlaps uniquely determine V_0 . At first sight, it would seem necessary that to determine the non-ghost part of V_0 depending on conformal a operator $R(z)$ uniquely up to a multiplicative constant one would require a basis of ϕ 's containing functions which have

which have a pole of any arbitrary order at one Koba-Nielsen point and are analytic elsewhere. In this case, one would have an equation of the form

$$V_0 \left\{ r_{-p}^{(i)} + \text{terms involving } r_p^{(i)} \quad p > 0 \right\} = 0$$

where $R(z) = \sum_p r_{-p} z^p$ is the conformal operator in question. This would enable us to simply read off V_0 . For an operator $R(z)$ of conformal dimension d , the corresponding ϕ is a tensor of weight $1-d$. The Riemann-Roch theorem states that such a tensor which is analytic everywhere except at one point must have a pole of at least order $(2d-1)(g-1) + 1$ at that point [46]. For $R(z) = P^\mu(z)$, this means that we have a pole of at least order g . In fact, in this case, one is saved by another mechanism, namely the fact that $P^\mu(z) = -z \partial_z Q^\mu(z)$ allows one to use non-single-valued functions (see Ref. [11] for details on this question). For $d = \frac{1}{2}$, i.e., the case of the Neveu-Schwarz-Ramond field, the Riemann-Roch theorem tells us that there are no missing poles, and so the overlap identities do determine the vertex uniquely. For example, a detailed treatment for excited Neveu-Schwarz string scattering at tree level is given in Refs. [12,13]. To arrive rigorously at this conclusion in the most general case, one must use a version of the Riemann-Roch theorem that applies to double-valued functions. We will see in Section 7 that indeed life is a little more subtle. However, the above reasoning is in broad outline correct and the vertex is uniquely determined by the overlaps.

In the next sections, we will produce the required ϕ 's for the tree and one-loop cases, and discuss the corresponding vertex. To find these functions for more general cases is a question of searching the relevant mathematical literature. To find the vertex given the complete set of functions ϕ involves expanding ϕ in a Laurent expansion around each of the Koba-Nielsen points and solving the resulting equation. This is straightforward apart from the constraints imposed by certain overlap identities involving the mode α_0^μ , and its effects on controlling the way one identifies the spinor indices of the $2N$ external strings. The latter point is discussed in principle, and it is then a technical exercise to find the vertices. This will be given elsewhere. Finally, we note that one can use the overlap identities to compute the scattering of excited states without even knowing the general vertex, by pushing all the d_{-n} 's or b_{-r} 's, $n > 0$, $r > 0$ and carrying out the resulting oscillator algebra.

According to the general strategy, we must not only compute V_0 , but we must also find its isotropy group. This is a straightforward application of the overlap equations. We go through the same steps as above, replacing $z^{-\frac{1}{2}} H^\mu(z)$ by $z^{-3/2} G(z)$ and $z^{-\frac{1}{2}} T^\mu(z)$ by $z^{-3/2} F(z)$, which are conformal fields of dimension $3/2$. We find the unintegrated and integrated overlaps, the latter being

$$V_0 \left\{ \sum_{j=1}^{2N} \oint d\bar{z}_j \phi \frac{F^j(\bar{z}_j)}{(\bar{z}_j)^{3/2}} + \sum_{j=2N+1}^{2N+M} \oint d\bar{z}_j \phi \frac{G^j(\bar{z}_j)}{(\bar{z}_j)^{3/2}} \right\} = 0 \quad (4.16)$$

ϕ is now a tensor of rank $-\frac{1}{2}$, but the same discussion as for $H^\mu(z)$ and $\Gamma^\mu(z)$ on its necessary branch cut structure and automorphic conditions applies.

There are transformations composed of L_n 's which leave the vertex inert; however, they are the same even in form as for the bosonic string and we refer the reader to Ref. [27] for a discussion. We just recall that it leads to $2N+M+g_1$ bosonic moduli, divided into $2N+M$ external Koba-Nielsen variables z_j and $g_1 = 3(g-1)$ internal loop moduli v_r .

Let us consider the consequences for a tadpole graph, with just one external Neveu-Schwarz string, and g loops. The function ϕ , analytic everywhere except at the external Koba-Nielsen point has, by the Riemann-Roch theorem mentioned above, a pole of order at least $2(g-1)+1 = g_2+1$ at that point. This means that transformations of the form

$$G_{-g_2-\frac{1}{2}-t} = G_{-(g_2+\frac{1}{2}+t)} + \sum_{s=-g_2-\frac{1}{2}}^{\infty} a_{ts} G_s \quad t=0,1,2,\dots$$

leave this vertex invariant. Hence, the fermionic part of the isotropy group is generated by G_r , $-g_2-\frac{1}{2} > r$. The fermionic coset generators are then those that remain, i.e., G_r , $r > -g_2-\frac{1}{2}$. As G_s , $s > \frac{1}{2}$ annihilate physical states, we find that although the vertex is boosted by such generators, they have no effect on physical state scattering. As such, the coset space is parametrized in effect by only g_2 anticommuting co-ordinates (i.e., g_2 supermoduli).

In order to find for an arbitrary number of external states of bosonic and fermionic type, the number of anticommuting moduli one examines the overlap equation (4.16) for the isotropy group and then deduces the coset generators. There are in fact $N+M+g_2$ supermoduli as will be shown in Section 7. Again we will see that the reasoning is in general a little more subtle than a naive application of the Riemann-Roch theorem would suggest although the result is correct.

Before the reparametrization of antighosts and the BRST charge come into play, it is appropriate to give our notation and summarize this first part of the section. The coset is parametrized by loop moduli v_r , an infinitesimal change of which is generated by applying L_{-2} , L_{-3} , ... on external lines, and Koba-Nielsen co-ordinates z_j , moved about by the action of L_{-1} . It needs also super Koba-Nielsen co-ordinates, θ_j (see Section 3), and the loop supermoduli θ_r discussed above. Although the coset also depends on co-ordinates corresponding to the action of L_n , $n \geq 0$ and G_r , $r \geq \frac{1}{2}$, S-matrix elements are independent of these, and we

shall set them equal to zero. This choice corresponds to the simplest choice of cycling [see Eq. (3.6)]. As such, the coset vertex is denoted

$$V(v_r, z_j, \theta_r, \theta_j) \quad (4.17)$$

The number of such co-ordinates depends on the precise scattering contribution being considered and will be discussed for trees in Section 6 and loops in Section 7. We only note here the result, that for trees, there are no v_r 's nor θ_r , and $2N+M-3$ z_j 's and $N+M-2$ θ_j 's for $2N$ external Ramond lines and M external Neveu-Schwarz lines. For a graph with one loop, we have $2N+M-1$ z_j 's and $N+M-1$ θ_j 's, one v_r and one θ_r , the distinction between anticommuting loop and tree moduli being somewhat artificial. For each extra loop, one adds three commuting and two anticommuting moduli, that is, for $g \geq 2$ there are $g_1 \equiv 3(g-1)$ v_r 's, $2N+M$ z_j 's, $g_2 \equiv 2(g-1)$ θ_r 's and $N+M$ θ_j 's. The actual scattering amplitude is given by

$$W = \int dv_r dz_j d\theta_r d\theta_j \bar{f} V(v_r, z_j, \theta_r, \theta_j) \quad (4.18)$$

when applied to physical states only. As usual, these are states annihilated by F_n ($n \geq 0$) in the Ramond sector and by $L_0 - \frac{1}{2}$ and G_r ($r \geq 0$) in the Neveu-Schwarz sector. Hence, it is clear that the amplitude is independent of the coset co-ordinates corresponding to L_n , F_n ($n \geq 1$) and G_r ($r \geq 0$), and this justifies taking these co-ordinates equal to zero. One can, of course, take them to be non-zero and for some purposes this is useful. Although L_0 can be set equal to a number on physical states and is not in general zero, the dependence of the scattering amplitude on this co-ordinate is the same for all scattering and so may be neglected as being a multiplicative constant. An exception to this is when this depends on z_j , however, in this case the decoupling procedure for the measure $\bar{f}$ ensures that the scattering is independent of this co-ordinate. The function $\bar{f}$ is the measure, which is a function depending on the integration variables and will be determined below through decoupling of zero-norm physical states.

We can identify

$$V_0(v_r, z_j) = V(v_r, z_j, 0, 0) \quad (4.19)$$

which has isotropy group H_{2N+M}^g . The general vertex $V(v_r, z_j, \theta_r, \theta_j)$ is an induced representation of $V_{2N+M} = V \times \dots \times V$ ($2N+M$ factors) having its isotropy group H_{2N+M}^g at the point $(v_r, z_j, 0, 0)$ trivially realized. The vertex is therefore defined on the coset space V_{2N+M}/H_{2N+M} and is obtained at any point of this coset space by boosting V_0 according to the usual theory of induced representations:

$$V(\sigma_r, z_j, \Theta_r, \Theta_j) = V_0(\tilde{\sigma}_r, \tilde{z}_j) \cdot g(\sigma_r, z_j, \Theta_r, \Theta_j) \quad (4.20)$$

where $g(v_r, z_j, \theta_r, \theta_j)$ is any element in V_{2N+M} in the coset with co-ordinates $(v_r, z_j, \theta_r, \theta_j)$. We may write

$$g(\sigma_r, z_j, \Theta_r, \Theta_j) = g_1(\sigma_r, \tilde{\sigma}_r) g_2(z_j, \tilde{z}_j) \cdot g_3 \quad (4.21)$$

where g_1 and g_2 make the changes their labels indicate. As a result we may write

$$V(\sigma_r, z_j, \Theta_r, \Theta_j) = V_0(\sigma_r, z_j) g_3. \quad (4.22)$$

Since we can compute $V_0(v_r, z_j)$ for any v_r and z_j using the overlap equations, it is redundant to boost it. The exact form of the boosts g_3 depends on the parametrization and choice of coset generators one takes. A parametrization is given by

$$g_3 = \exp \left(\sum_r \Theta_r F_{\phi_r} + \sum_s \Theta_s G_{\phi_s} + \sum_j \Theta_j F_{-1}^j + \sum_j \Theta_j G_{-\frac{1}{2}}^j \right) \quad (4.23)$$

where F_{ϕ_r} and G_{ϕ_r} are appropriate combinations of F_n 's and G_r 's. Although the exact detailed form of all the boosts depends on the exact choice of coset and its parametrization one takes, the final physical answer is always the same; changes in choice of coset and parametrization are balanced by changes in the measure $\bar{f}$ and have no effect when physical states are applied on Eq. (4.18). This statement assumes that certain boundary terms vanish.

Following our general philosophy, the integration measure is determined by the decoupling of zero-norm physical states. In the Neveu-Schwarz sector, these states are of the form

$$G_{-\frac{1}{2}} |\Omega\rangle \quad \text{and} \quad (G_{-\frac{3}{2}} + 2L_{-1} G_{-\frac{1}{2}}) |\Omega'\rangle \quad (4.24)$$

where $L_n |\Omega\rangle = G_r |\Omega\rangle = 0$, $n \geq 1$, $r \geq \frac{1}{2}$ and similarly for $|\Omega'\rangle$; however $L_0 |\Omega\rangle = 0$

$= (L_0 + \frac{1}{2}) |\Omega'\rangle$. While in the Ramond sector, they are of the form

$$F_0 \underline{L}_1 |\Omega\rangle \quad (4.25)$$

where

$$F_n |\Omega\rangle = L_n |\Omega\rangle = (L_0 + 1) |\Omega\rangle = 0 \quad n=1, 2, \dots \quad (4.26)$$

At tree level, decoupling of these states on all external lines is sufficient to determine the measure, through first order differential equations as usual [9]. However, the calculation although manageable, is more involved than for the bosonic or Neveu-Schwarz strings, for two reasons: the zero-norm states (4.18) involve two operators, F_0 and L_{-1} , instead of just one and normal-ordering constants show up because the Ramond vacuum is not $SU(1,1)$ invariant. This is discussed in more detail in Section 6.

In principle, the same procedure applies for loop graphs. However, as for the bosonic case, there are more and more (super)moduli and one thus needs more and more differential equation, i.e., zero-norm decoupling states to determine the measure. The situation is improved by the incorporation of ghosts [14,15] into the vertex, because all zero-norm physical states become of the form $Q|\Lambda\rangle$. The first step is to introduce ghosts into the vertex. The antighost fields are $\bar{f}(z)$ (R sector) and $\bar{g}(z)$ (NS sector) and have conformal dimension $-\frac{1}{2}$.

The ghost extended vertex, still denoted V_0 , obeys integrated overlap equations

$$V_0 \left\{ \sum_{j=1}^{2N} \oint d\bar{z}_j \varphi \frac{\bar{f}^j(\bar{z}_j)}{(\bar{z}_j)^{3/2}} + \sum_{j=2N+1}^{2N+M} \oint d\bar{z}_j \varphi \frac{\bar{g}^j(\bar{z}_j)}{(\bar{z}_j)^{3/2}} \right\} = 0 \quad (4.27)$$

and

$$V_0 \left\{ \sum_{j=1}^{2N} \oint d\bar{z}_j \varphi \bar{z}_j^{\frac{1}{2}} \bar{f}^j(\bar{z}_j) + \sum_{j=2N+1}^{2N+M} \oint d\bar{z}_j \varphi \bar{z}_j^{\frac{1}{2}} \bar{g}^j(\bar{z}_j) \right\} = 0 \quad (4.28)$$

The ϕ function in Eq. (4.20) is a tensor of rank $-\frac{1}{2}$ and in Eq. (4.21) a tensor of rank $3/2$, but of course the discussion on their cut structure and (multi)-valuedness is the same as for the physical fields $H^\mu(z)$ and $\Gamma^\mu(z)$, independently of the dimension. What the dimension does affect, however, is the number of such overlap equations. $\bar{f}(z)$ and $\bar{g}(z)$ have the same dimension as $F(z)$ and $G(z)$, and so

their identities are in one-to-one correspondence and identical in form. In contrast, from the Riemann-Roch theorem, we know that there are $g_2 \equiv 2(g-1)$ analytic tensors of rank $3/2$ on a surface of genus g (as well as tensors with poles, of course). This is similar to the case of the bosonic reparametrization ghost $\bar{\beta}$ considered in Refs. [14,15] which leads to $g_1 = 3(g-1)$ analytic tensors of rank 2. The part of the vertex V_0 which depends on $\beta(z)$ and $\bar{\beta}(z)$ is given in Refs. [14,15]. Determining the part of V_0 which depends on $g(z)$, $\bar{g}(z)$, $f(z)$, $\bar{f}(z)$ is just a matter of following the same method with appropriate adjustments, and this is left as an exercise for the reader. In fact, the vertex is determined uniquely by the overlap equations for the ghost and antighosts, one type only not being sufficiently strong. One finds that the complete set of ϕ 's for the anti-ghost cannot be expressed in terms of functions which have a pole at only one Koba-Nielsen point and these additional identities play an essential role.

We now come to the problem of finding the vertex V at any value of the coset, including the ghost fields. We consider the extended super Virasoro group denoted U , which is generated by the operators L_n , G_r (or F_n), $\bar{\beta}_n$, $\bar{g}_r$ (or $\bar{f}_n$) and the BRST charge Q . Let us call K_{2N+M}^g the isotropy group of the vertex V_0 , where g is the genus of the Riemann surface (number of loops), $2N$ the number of external Ramond lines, M the number of external Neveu-Schwarz lines. K_{2N+M}^g is generated by the transformations which leave the vertex inert [see Eq. (4.16) for example, or Eq. (23) of Ref. [10]]. Let U_{2N+M} be the direct product of $2N+M$ copies of U (one for each external line). The general vertex V is an induced representation of U_{2N+M} with isotropy group K_{2N+M}^g ; in other words, it is defined on the coset U_{2N+M}/K_{2N+M}^g . This coset is parametrized by the co-ordinates v_r , z_j , θ_r , θ_j [Eq. (4.17)] which we have already discussed before introducing the ghosts, together with new co-ordinates, related to the introduction of ghosts, Ψ_r corresponding to loop moduli v_r but associated with the $\bar{\beta}$ oscillators, ϕ_j corresponding to Koba-Nielsen co-ordinates z_j but associated with $\bar{\beta}_{-1}$, together with their super-counterparts for the $\bar{g}$ (or $\bar{f}$) fields, which we denote K_r and κ_j . The Ψ_r and ϕ_j are anticommuting variables, and have been introduced and discussed in Ref. [27]. As the antighost and the corresponding conformal operator have the same conformal dimension [i.e., $\bar{\beta}(z)$ and $L(z)$, $\bar{g}(z)$ and $G(z)$ etc.], they have the same overlaps. As such, there are many v_r 's as Ψ_r 's, z_j 's as ϕ_j 's, and similarly, as many θ_r 's as K_r 's, and as many θ_j 's as κ_j 's. The actual number of co-ordinates depends on the specific vertex being considered. It was mentioned earlier in this section, and will be discussed in more detail for trees in Section 6 and for loops in Section 7.

There is not in general a unique choice of coset generators. As will be discussed elsewhere, this ambiguity can be formulated in the frame work of a gauge fixing problem and handled with the corresponding machinery, i.e., Faddeev-Popov determinants, etc. The problem of how different coset choices are related is then the same as how to go from one gauge fixing term to another. Indeed, one finds that they give the same physical amplitudes up to boundary terms. We note also that it is desirable to choose a gauge which is preserved under the action of Q . In particular, one sees that the gauge choice for the original coset co-ordinates determines that for the ghost related co-ordinates. This point will be discussed elsewhere.

We may draw the following table which summarizes this notation.

	loop moduli	Koba-Nielsen co-ordinates	loop ghost moduli	K-N ghost co-ordinates
bosonic	v_r	z_j	Ψ_r	ϕ_j
fermionic	θ_r	θ_j	K_r	κ_j

where $\Psi_r, \phi_j, \theta_r, \theta_j$ are anticommuting and v_r, z_j, K_r, κ_j are commuting numbers. The vertex V_0 corresponds to taking equal to zero all variables except v_r and z_j , that is, we recognize that

$$V_0(v_r, z_j) = V(v_r, z_j, 0, 0, 0, 0, 0, 0). \quad (4.29)$$

and these are related by a boost

$$\begin{aligned} & V(v_r, z_j, \Phi_r, \Psi_j; \theta_r, \theta_j, K_r, \kappa_j) \\ &= V_0(\tilde{v}_r, \tilde{z}_j) g(v_r, z_j, \Phi_r, \Psi_j, \theta_r, \theta_j, K_r, \kappa_j) \end{aligned} \quad (4.30)$$

where g is any element of U_{2N+M} which is in the coset corresponding to the co-ordinates shown. We may write

$$\begin{aligned}
 & V(v_r, z_j, \Phi_r, \Psi_j ; \Theta_r, \Theta_j, \kappa_r, \kappa_j) \\
 & = V_0(v_r, z_j) g_2 \\
 & = V(v_r, z_j, 0, 0, \Theta_r, \Theta_j, 0, 0) g_4(\Phi_r, \Psi_j, \kappa_r, \kappa_j)
 \end{aligned}
 \tag{4.31}$$

where $g = g_1 g_2$ and g_1 is a boost from $\overset{\circ}{v}_r, \overset{\circ}{z}_j$ to v_r, z_j and the vertex in the last equation is that of Eq. (4.20). The actual scattering amplitude is given by

$$\begin{aligned}
 W = & \int \pi d\sigma_r dz_j d\Phi_r d\Psi_j d\Theta_r d\Theta_j d\kappa_r d\kappa_j \\
 & \cdot \bar{f} V(v_r, z_j, \Phi_r, \Psi_j ; \Theta_r, \Theta_j, \kappa_r, \kappa_j)
 \end{aligned}
 \tag{4.32}$$

where $\bar{f}$ is the measure. Of course, the coset also contains the generators $L_n, \bar{\beta}_n, F_n, \bar{f}_n, n \geq 0$ and $G_r, \bar{g}_r, r \geq \frac{1}{2}$, and so has the corresponding co-ordinates. However, as discussed for the bosonic case, when physical states are applied to W , the result is independent of these co-ordinates, and we may set their values to zero. This corresponds to generalizing the simplest cycling choice.

Physical states $|\chi\rangle$ are those which satisfy $Q|\chi\rangle = 0, \bar{\beta}_0|\chi\rangle = 0$ (and $\bar{f}_0|\chi\rangle = 0$ in the Ramond sector) and have ghost number 1. These are the states of the gauge covariant theory once we impose the gauge fixing condition $\bar{\beta}_0|\chi\rangle = 0$ ($\bar{f}_0|\chi\rangle = 0$) and the equation of motion.

The measure $\bar{f}$ is determined by decoupling zero-norm physical states which are of the form $Q|\Lambda\rangle$. This is discussed for trees in Section 6 and loops in Section 7.

5. - CONFORMAL MAPPING OF VERTICES

In the previous section, we have used the Schottky representation of a Riemann surface, but one can equally well use the Fuchsian representation. We shall indeed find it useful in Section 7. At one loop this means representing the torus by a rectangle rather than the surface on the Riemann sphere external to the two isometric circles of the one generator P_1 and its inverse P_1^{-1} . To move from one

configuration to the other one performs a conformal mapping, however, as we now discuss, this must be carried out with some care.

Let us first consider how to change an integrated overlap equation for an arbitrary conformal operator $R(z)$ of dimension d , under an arbitrary conformal transformation $g: z \rightarrow z = g(z)$ which maps from one representation of the string to another. To specify the map on the vertex V which takes it from one configuration to the other, we must also consider which cycling choice or local co-ordinate systems we will use in each configuration. In particular, if the map on leg i is $\mathcal{X}^i$ then

$$\hat{V} = V \prod_i \mathcal{X}^i \quad (5.1)$$

where $\hat{V}$ is the vertex for the configuration with $\hat{z}$ co-ordinates and V for the configuration with z co-ordinates. Then, from the overlaps, we find (see Section 2), the usual relation

$$\mathcal{X}^j = (V^j)^{-1} V^i \mathcal{X}^i (\hat{V}^i)^{-1} (\hat{V}^j) \quad (5.2)$$

where V^j and $\hat{V}^j$ are the cycling choices in the two configurations. We may denote the corresponding local co-ordinate systems by

$$\xi_i = (\hat{V}^i)^{-1} \hat{z} \quad ; \quad \xi_i = (V^i)^{-1} z \quad (5.3)$$

Solving Eq. (5.2), [cf Eq. (2.9)], we find that

$$\mathcal{X}^j = (V^j)^{-1} g^{-1} (\hat{V}^j) \quad (5.4)$$

where one finds g^{-1} rather than g since examining any overlap with conformal operator R^j , one finds that the transformation acts as

$$(\mathcal{X}^j)^{-1} R^j(\xi_j) \mathcal{X}^j \quad (5.5)$$

We observe that

$$(\mathcal{X}^j)^{-1} \xi_j = ((\hat{V}^j)^{-1} g(V^j)) (V^j)^{-1} z = (\hat{V}^j)^{-1} \hat{z} = \xi_j$$

and so as is to be expected the relevant map takes the local co-ordinate system ξ_j around the Koba-Nielsen point z_j into the local co-ordinate system around the corresponding Koba-Nielsen point $\hat{z}_j$. Consequently we find that

$$\frac{(\mathcal{U}^j)^{-1} R^j(\xi_j)}{(\xi_j)^d} = \left(\frac{d\xi_j}{d\zeta_j} \right)^d \frac{R^j(\xi_j)}{(\xi_j)^d} \quad (5.7)$$

Taking for simplicity the simplest choice of cycling in each configuration, i.e.,

$$\xi_j = (V^j)^{-1} z = (z - z^j) \quad ; \quad \zeta_j = (\hat{V}^j)^{-1} z = z - z_j \quad (5.8)$$

we can map the overlap equation (2.6) to find

$$\hat{V} \left\{ \sum_{j=1}^N \oint d\zeta_j \frac{R^j(\xi_j)}{(\xi_j)^d} \hat{\mathcal{C}} \right\} = 0 \quad (5.9)$$

where

$$\hat{\mathcal{C}}(\zeta) = \mathcal{C}(z(\zeta)) \left(\frac{dz}{d\zeta} \right)^{1-d} \quad (5.10)$$

which transforms as is expected for a tensor of dimension $(1-d)$. In deriving this result, we have used the relation, valid for the above cycling

$$\frac{d\xi_j}{d\zeta_j} = \frac{dz}{d\zeta} \quad \forall j \quad (5.11)$$

When mapping from Schottky to Fuchsian representation, one will find that $\tilde{\phi}$ which is generally single-valued on the former obeys on the latter equations which implement this single-valuedness.

The mapping of the unintegrated overlap equations is even more straightforward as they just involve the local charts ξ_j and ζ_j and the results are obvious. A subtlety occurs, however, when one maps from Schottky to Fuchsian representations. In the Schottky case, the periodicity conditions around A cycles such as single or double-valuedness are in essence built in a straightforward way, while in the Fuchsian case, one has in effect dissected the multiloop Riemann surface again and the corresponding condition must be written down explicitly. Indeed, for each loop we gain in the Fuchsian representation one extra unintegrated overlap in addition to Eq. (2.5) corresponding to the additional identifications which are required to reconstruct the Riemann surface. How to recover these additional requirements is the subject of the remainder of this section, and it is here that the subtleties for fields of half integer conformal dimension occur.

Before discussing this, however, it will be instructive to first consider at tree level the correspondence between representing the string as a strip in the $\mathbf{z}$ plane and the upper half plane which has co-ordinate z , the precise map being

$z = \tau + i\sigma$ (see Fig. 2). The strip represents the world-sheet of the string and if we define $z = \tau + i\sigma$, then $0 \leq \sigma \leq \pi$, while $-\infty < \tau < +\infty$. It is usual, however, to extend the range of σ from $-\pi$ to π by reflection.

The extension of the range of σ corresponds on the Riemann sphere to reflection about the real axis to include the lower as well as upper half plane. In this doubling of the range, we combined the two string variables $x^\mu(\tau, \sigma)$ and its momentum $p^\mu(\tau, \sigma)$ into the one variable

$$p^\mu(\tau, \sigma) = \begin{cases} p^\mu(\tau, \sigma) + \frac{1}{2\pi\alpha'} x^{\mu'}(\tau, \sigma) & \text{for } 0 < \sigma < \pi \\ p^\mu(\tau, \sigma) - \frac{1}{2\pi\alpha'} x^{\mu'}(\tau, \sigma) & \text{for } -\pi < \sigma < 0 \end{cases} \quad (5.12)$$

where $x^{\mu'} = dx^\mu/d\sigma$ or equivalently demanding

$$x^\mu(\tau, -\sigma) = x^\mu(\tau, \sigma) \quad \text{for } 0 < \sigma < \pi. \quad (5.13)$$

The boundary condition on the string is $x^{\mu'}(\sigma=0) = 0 = x^{\mu'}(\sigma=\pi)$ and it is now encoded by $p^\mu(-i\pi) = p^\mu(i\pi)$, the boundary condition at $\sigma = 0$ being automatically encoded.

We may regard the variables as defined on the whole z plane by taking

$$p^\mu(z + 2\pi i) = p^\mu(z). \quad (5.14)$$

On the Riemann sphere $p^\mu(\xi^i)$ is a single-valued function of ξ^i , i.e.,

$$V_0 p^{\mu i}(\xi^i) = V_0 p^{\mu i}(\xi^i e^{2\pi i}) \quad (5.15)$$

and similarly for other bosonic conformal operators. Following the discussion given above on conformal mapping, we find that this equation becomes on the strip

$$\widehat{V}_0 \frac{p^{\mu i}(\xi^i)}{\xi^i} = \widehat{V}_0 \frac{p^{\mu i}(\xi^i + 2\pi i)}{(\xi^i + 2\pi i)} \quad (5.16)$$

The additional factors are a result of our definition of conformal fields and would disappear if we regarded $R(z)/(z)^d$ as the conformal field. In fact, this is common

practice in much of the literature and coincides with the more usual definition of a tensor. Equation (5.16) in fact corresponds to Eq. (5.14) after this redefinition.

The situation for the superstring is analogous. The doubling procedure allows one to incorporate the two components of the world-sheet spinor into one object ϕ^μ by extending the range of $\sigma = \text{Im} \tau$ from 0 to π to $-\pi$ to π (see Ref. [47]). We recall that on the strip $\phi^\mu(\tau+2\pi i) = \pm \phi^\mu(\tau)$ where the upper sign corresponds to a Ramond string while the lower sign is for a Neveu-Schwarz string.

However, to go from one configuration to another requires more care as we are dealing with conformal operators of half integer dimension which involve under a transformation a factor $(dz/dz)^{n+\frac{1}{2}}$, n integer. In particular, if $\tau = \lambda n z$, we have a factor $z^{-n-\frac{1}{2}}$ which is in fact double-valued under $z \rightarrow e^{2\pi i} z$. This point is discussed in Ref. [48] for the tree case we are now considering. On the Riemann sphere $V_0(H(z)/\sqrt{z})$ and $V_0(\Gamma(z)/\sqrt{z})$ have the branch cut structure discussed in the previous section. In particular, around a point z_k , $k \in (2N+1, \dots, 2N+M)$ where a Neveu-Schwarz string is emitted, $V_0(H^{\mu(k)}(\xi_k)/\sqrt{\xi_k})$ is single-valued, i.e.

$$V_0 \frac{H^{\mu k}(\xi_k e^{2\pi i})}{\sqrt{\xi_k e^{2\pi i}}} = V_0 \frac{H^{\mu k}(\xi_k)}{\sqrt{\xi_k}} \quad (5.17)$$

while around a point z_j , $j \in (1, \dots, 2N)$ where a Ramond string is emitted, we have

$$V_0 \frac{\Gamma^{\mu j}(\xi_j)}{\sqrt{\xi_j}} = - V_0 \frac{\Gamma^{\mu j}(\xi_j e^{2\pi i})}{\sqrt{\xi_j e^{2\pi i}}} \quad (5.18)$$

This agrees with the fact that on the Riemann sphere, they have the expansions

$$H^{\mu k}(\xi_k) = \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (\xi_k)^\nu b_{-\nu}^{\mu(k)}$$

and

$$\Gamma^{\mu j}(\xi_j) = \sum_{n \in \mathbb{Z}} (\xi_j)^n d_{-n}^{\mu(j)}$$

Let g be the map from the Riemann sphere to the strip (i.e., $g: z \rightarrow \mathfrak{z} = g(z) = \ln z$). Carrying out this mapping, in accord with the discussions earlier in this section, we find that Eqs. (5.17) and (5.18) become:

$$\hat{V}_0 \frac{H^{\mu\kappa}(\mathfrak{z}_\kappa)}{\sqrt{\mathfrak{z}_\kappa}} = - \hat{V}_0 \frac{H^{\mu\kappa}(\mathfrak{z}_\kappa + 2\pi i)}{\sqrt{\mathfrak{z}_\kappa + 2\pi i}} \quad (5.19)$$

$$\hat{V}_0 \frac{\Gamma^{\mu j}(\mathfrak{z}_j)}{\sqrt{\mathfrak{z}_j}} = \hat{V}_0 \frac{\Gamma^{\mu j}(\mathfrak{z}_j + 2\pi i)}{(\mathfrak{z}_j + 2\pi i)} \quad (5.20)$$

where $\bar{V}_0 = V_0 g^{-1}$.

This result is to be expected as it is $H^\mu(\mathfrak{z})\mathfrak{z}^{-\frac{1}{2}}$ or $\Gamma^\mu(\mathfrak{z})\mathfrak{z}^{-\frac{1}{2}}$ which are conformal fields in the more conventional sense and so it is they which are to be identified with $\psi^\mu(\mathfrak{z})$ which is also a conformal tensor in the usual sense. Equations (5.19) and (5.20) then correspond to the usual boundary condition imposed on ψ^μ on the world sheet.

To be more explicit, let us consider only two external legs which are both either Neveu-Schwarz or Ramond lines. We may on the plane take these to be at $z = 0$ or $z = \infty$ corresponding on the world sheet to the points $\mathfrak{z} = -\infty$ and $+\infty$ respectively. In the Ramond case, we have, on the upper half plane, a branch cut between $z = 0$ and ∞ and this becomes on the strip the condition $V_0 \psi^\mu(\mathfrak{z} + 2\pi i) = V_0 \psi^\mu(\mathfrak{z})$, while for the Neveu-Schwarz case we have no branch cut on the upper half plane, and find on the world strip that $V_0 \psi^\mu(\mathfrak{z} + 2\pi i) = -V_0 \psi^\mu(\mathfrak{z})$.

Let us now return to the problem at hand, namely to map from the Schottky to the Fuchsian representation of a multiloop Riemann surface and in particular to deduce the behaviour around the A cycles which was discussed in the previous section for the Schottky case. To be specific, let us consider the one loop case which is required in Section 7. In this case the Schottky group is generated by one element P_1 which we may take to be $P_1(z) = \omega z$. The isometric circles are $|z| = 1$ and $|z| = \omega$ and the fundamental region $\omega < |z| < 1$. The map to the rectangle is given by $\mathfrak{z} = \ln z$ (see Fig. 2). In fact, as for the tree case, one

doubles the Riemann surface to obtain a surface without boundary. This corresponds, in fact, to a reflection in the z plane about $\text{Im } z = 0$.

Suppose now that we have only bosonic, i.e., Neveu-Schwarz external states, then we have around an A cycle, i.e., around the circle of Fig. 3

$$V_0 \frac{H^{\mu i}(z)}{\sqrt{z}} = \pm V_0 \frac{H^{\mu i}(ze^{2\pi i})}{\sqrt{ze^{2\pi i}}} \quad (5.21)$$

where the $+$ and $-$ signs correspond to the propagation of a Neveu-Schwarz and Ramond string respectively around the loop. Mapping to the rectangle, i.e., the z plane we find that the equation becomes

$$\hat{V}_0 \frac{H^{\mu i}(z+2\pi i)}{\sqrt{z+2\pi i}} = \mp \hat{V}_0 \frac{H^{\mu i}(z)}{\sqrt{z}} \quad (5.22)$$

We note that the same change of sign as in the tree case, the origin being the fact that H^μ has dimension one half.

This explains the signs used in Section 7. In particular, we will note there that the (even,even) boundary condition leads to a gauge, i.e., BRST anomaly and we know that this is a result of a fermionic string propagating in the loop. Examining Eqs. (5.21) and (5.22), we indeed find that the lower sign (i.e., $+$) corresponds to fermion propagation. The situation with external fermion lines is more complicated, as one has both fermions and bosons propagating in the loop. However, the different boundary conditions are taken into account by the N branch points due to the $2N$ external fermions and the signs change between the configurations in the way described above. The multiloop case proceeds in the same way.

6. - TREES

In this section, we wish to explain in detail how to compute string scattering at tree level by implementing the discussion in Section 4. As noted in that section, one can perform this computation with or without including ghost oscillators.

The first step is to determine the vertex V_0 by means of the overlap equations (4.15), (4.27) and (4.28). We recall that these equations are valid if ϕ has poles at only Koba-Nielsen points, is defined on the Riemann surface and mirrors exactly the branch cut structure of the vertex times the relevant conformal operator. The latter requirement implies that we must take ϕ to be of the form

$$\phi = Q f(\bar{z}_k) \quad (6.1)$$

where $Q \equiv \prod_{j=1}^{2N} \sqrt{\xi_j}$ and f is a single-valued function of ξ_j , $j = 1, 2, \dots, 2N+M$. We recall that if the overlap equation involves conformal operators of weight d , then ϕ is a tensor of weight $1-d$. As such, in order for ϕ to be defined on the Riemann surface, it must behave at most like $\xi_j^{2(d-1)}$ for large $\xi_j \approx \xi$ as $\xi \rightarrow \infty$. This point is discussed at length in Ref. [14] and it is most easily seen by taking our two co-ordinate patches on the sphere in such a way that all the Koba-Nielsen points are in one patch and then ϕ must be analytic in the other patch. To go from one patch to the other, we make the transformation $\zeta = 1/\xi$. Consequently, $f(\xi)$ of Eq. (6.1) must behave at most like $f(\xi) \sim \xi^{-N+2(d-1)}$ for large $\xi_j \sim \xi$ as $\xi \rightarrow \infty$.

We now consider the overlaps which determine the dependence of the vertex $V_0(z_k)$ on the oscillators α_n^μ , d_n^μ and b_r^μ . Very roughly speaking, the vertex $V_0(z_j)$ without the ghost oscillator contribution is of the form

$$\left\{ \prod_{i=1}^{2N+M} \leq 0 \right\} \exp \sum_{i,j} \sum_{n,m} \left\{ \alpha_n^{\mu i} b_{nm}^{ij} \alpha_m^{\mu j} + d_n^{\mu i} e_{nm}^{ij} b_m^{\mu j} + d_n^{\mu i} \hat{e}_{nm}^{ij} d_m^{\mu j} + b_n^{\mu i} \hat{e}_{nm}^{ij} b_m^{\mu j} \right\} \quad (6.2)$$

where b_{nm}^{ij} and e_{nr}^{ij} 's are functions of z_k . The α_n^μ piece is the same as for the bosonic string (see Refs. [9] and [13]) as it is determined by the P^μ overlap which is specified only by its conformal weight being one. The d_n^μ and b_r^μ dependence of V_0 is determined by the overlap of Eq. (4.15), as we now discuss this evaluation is conceptually straightforward except for the d_0^μ piece whose anticommutator $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$ requires to be dealt with carefully.

In this overlap as $\Gamma(z)$ and $H(z)$ have conformal dimension $\frac{1}{2}$ and so ϕ is a tensor of rank $\frac{1}{2}$. Let us first consider the case where ϕ has no poles at the points where Neveu-Schwarz strings are emitted, i.e., at z_j , $j = 2N+1, \dots, 2N+M$. We may, according to the above discussion take

$$\phi^{(p)} = Q \prod_{k=1}^{2N} (\bar{z}_k)^{-1} \quad (6.3)$$

where as usual $\prod^{(N+1)}$ means the product over only $N+1$ $(\xi_k)^{-1}$ from $(\xi_1)^{-1}, \dots, (\xi_{2N})^{-1}$. The overlap identity (4.15) for this ϕ^P is of the form

$$V_0 \left\{ \sum_{k=1}^{2N} a_0^{(P)k} d_0^{k\mu} + \sum_{j=1}^{2N} \sum_{n=1}^{\infty} a_n^{(P)j} d_n^{j\mu} + \sum_{j=2N}^{2N+M} \sum_{r=\frac{1}{2}}^{\infty} a_r^{(P)j} b_r^j \right\} = 0. \quad (6.4)$$

the coefficients $a_n^{(P)j}$ being read off from the Laurent expansion of ϕ^P about z_j in the obvious way. At first sight, there would seem to be $2N!/(N-1)!(N+1)!$ such identities, but in fact by subtracting one choice of $\prod_k^{(N+1)}$ from another, we obtain an identity of the same type. One can verify this explicitly, however, its converse would imply the existence of an identity with less than $N+1$ d_0 's. However, such an identity would arise from a ϕ which is not defined on the Riemann sphere.

A little thought shows that there are in fact only N such independent identities. Consequently, we may express any N d_0 's in terms of the remaining N d_0 's.

To be concrete, we may choose in Eq. (6.3) the $N+1$ ξ_k 's in the product to be consecutive ξ_k 's starting with ξ_1 and then ξ_2 , etc., until we have N identities. In other words, we take

$$\varphi^{(P)} = \phi \prod_{i=1}^{N+1} \left(\frac{1}{\xi_{p+i-1}} \right) \quad p = 1, \dots, N \quad (6.5)$$

$$= \begin{cases} a_0^{(P)j} \xi_j^{-\frac{1}{2}} + \sum_{n=0}^{\infty} a_n^{(P)j} (\xi_j)^{n+\frac{1}{2}} & \text{for } j = p, \dots, p+N \\ \sum_{n=0}^{\infty} a_n^{(P)j} (\xi_j)^{n+\frac{1}{2}} & \text{for } j = 1, \dots, p-1, p+N+1, 2N \\ \sum_{r=\frac{1}{2}}^{\infty} c_r^{(P)j} (\xi_j)^{r-\frac{1}{2}} & \text{for } j = 2N+1, \dots, 2N+M. \end{cases} \quad (6.6)$$

We note that due to ϕ having the correct branch cut structure and so the corresponding square roots, the Laurent expansions are much more complicated in practice than the bosonic operators.

For a vertex with only two fermionic legs, we have only one identity which is of the form $V_0(d_0^{\mu(1)} + id_0^{\mu(2)} + \text{terms which involve } d_n, n \geq 1 \text{ and } b_r, r \geq \frac{1}{2} = 0)$. Since V_0 has two spinor indices. This identity states in essence that we may in fact identify the two spinor indices from the two legs. The presence of the d_n and b_r in the identity being due to the $d_0 d_n$ and $d_0 b_r$ terms in the exponential factor in the vertex.

More generally, the N identities enforce the identification of spinor indices on any chosen pairing of the $2N$ points. We will discuss this point further elsewhere.

We now turn our attention to the other identities that follow from different choices of ϕ . We now consider ϕ 's that lead to d_{-m} on one leg with $d_n, n \geq 0, b_r, r \geq \frac{1}{2}$ on the other legs, namely:

$$\mathcal{Q} = \mathcal{Q} \frac{1}{(\bar{z}_i)^m} \prod_{\substack{k=1 \\ k \neq i}}^{2N} (N+1-m) \frac{1}{(\bar{z}_k)} ; i \in [1, 2, \dots, 2N] \quad (6.7)$$

where $\prod_k^{(N+1-m)}$ is now the product over $N-m+1$'s not including i . From this ϕ , we find an identity of the form

$$V_0 \left\{ \sum_{p=m}^1 e_{-p}^{ii} d_{-p}^{mi} + \sum_{j=1}^{2N} \sum_{n=0}^{\infty} e_n^{ij} d_n^j + \sum_{j=2N+1}^{2N+m} \sum_{r=\frac{1}{2}}^{\infty} e_r^{ij} b_r^j \right\} = 0 \quad (6.8)$$

where the coefficients are those in the Laurent expansion of $\mathcal{Q}$ about the point z_j . Once $m \geq N+1$, then the product in (6.7) required for convergence is unnecessary. This and the previous identity determine the part of the vertex which depends on $d_n^{\mu}, n \geq 0$.

We now consider ϕ 's which have poles at Neveu-Schwarz Koba-Nielsen points. By taking a ϕ of the form

$$\mathcal{Q} = \mathcal{Q} \frac{1}{(\bar{z}_i)^m} \prod_k^{(N+1-m)} (\bar{z}_k)^{-1} ; i \in [2N+1, \dots, 2N+m] \quad (6.9)$$

we find the identity

$$V_0 \left\{ \sum_{p=m}^1 f_{-p}^{ii} b_{-p+\frac{1}{2}}^i + \sum_{j=1}^{2N} \sum_{n=0}^{\infty} f_n^{ij} d_n^j + \sum_{j=2N+1}^{2N+M} \sum_{r=\frac{1}{2}}^{\infty} \hat{f}_r^{ij} b_r^j \right\} = 0 \quad (6.10)$$

Equation (6.10) determines the dependence of the vertex on b_r^μ , $r > \frac{1}{2}$. By a more clever choice of ϕ , we can arrange for each of Eqs. (6.8) and (6.10) to involve only d_{-m}^μ or b_{-r}^μ respectively.

The above arguments show that the overlap equation (4.15) determines uniquely, up to a multiplicative factor, the dependence of the vertex V_0 on b_r^μ and d_n^μ . Since the α_n^μ part is the same as for the bosonic theory and is given for the simple cycling under consideration here in Ref. [13], we have determined V_0 when we choose not to include the ghost oscillator contribution. The explicit computation V_0 will be given elsewhere.

We now discuss the isotropy group H_{2N+M}^0 of the tree vertex $V_0(z_j)$ without ghost oscillators discussed above. The isotropy group is determined by the overlap equations which involve the super Virasoro operators. The transformations involving L_n which leave the vertex inert are the same as in the bosonic vertices with $2N+M$ external strings and are discussed in detail in Refs. [14,15,27]. The part of the isotropy group generated by F_n 's and G_r 's is determined by the overlap equation (4.16). In this equation ϕ is a tensor of rank $-\frac{1}{2}$ and so it can as $\xi_j \sim \xi \rightarrow \infty$ behave at most like $(\xi)^1$. That is f in Eq. (6.1) must behave like ξ^{1-N} . The ϕ 's which have poles only at the fermionic Koba-Nielsen points and which saturate the above bound are

$$\phi^{(p)} = \varphi \prod_{k=1}^{2N} (N-1) \left(\frac{1}{3k} \right) \quad (6.11)$$

The corresponding overlap identity is

$$V_0 \left\{ \sum_{k=1}^{2N} \sum_{n=0}^{N-1} \hat{a}_{-1}^{(p)k} F_{-1}^k + \sum_{j=1}^{2N} \sum_{n=0}^{\infty} \hat{a}_n^{(p)j} F_n^j + \sum_{r=-\frac{1}{2}}^{\infty} \sum_{j=2N+1}^{2N+M} \hat{c}_r^{(p)j} G_r^j \right\} = 0 \quad (6.12)$$

where the $\hat{a}$'s and $\hat{c}$'s are the coefficients in the Laurent expansion of $\phi^{(p)}$. We

observe that such identities involve $N-1$ F_{-1} 's and one realizes that there are only $(N+2)$ independent identities of this type. A basis for these $(N+2)$ ϕ 's is given by

$$\begin{aligned} \phi^{(p)} &= \phi \prod_{\ell=1}^{N-1} \left(\frac{1}{\xi_{p+\ell-1}} \right) \quad p = 1, \dots, N+2 \quad (6.13) \\ &= \begin{cases} \hat{a}_{-1}^{(p)} (\xi_j)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \hat{a}_n^{(p)j} (\xi_j)^{n+\frac{1}{2}} ; j = p, \dots, p+N-2 \\ \sum_{n=0}^{\infty} \hat{a}_n^{(p)j} (\xi_j)^{n+\frac{1}{2}} ; j = 1, \dots, p-1, p+N-1, \dots, 2N \\ \sum_{r=\frac{1}{2}}^{\infty} \hat{c}_r^{(p)j} (\xi_j)^{r+\frac{1}{2}} ; j = 2N+1, \dots, 2N+M. \end{cases} \quad (6.14) \end{aligned}$$

In effect we may solve any F_{-1} in terms of $N-2$ chosen F_{-1} 's, M $G_{-\frac{1}{2}}$'s and F_n^j , $n > 0$, $j = 1, \dots, 2N$ and G_r^k , $r > \frac{1}{2}$, $k = 2N+1, \dots, 2N+M$.

By taking a ϕ of the form

$$\phi = \phi \left[\prod_{k=1}^{2N} (\xi_k)^{-1} \right] \xi^e \quad e \in [2N+1, \dots, 2N+M] \quad (6.15)$$

we find an identity of the form of Eq. (6.12) except for the $G_{-\frac{1}{2}}$ on leg λ has been removed at the expense of one more F_{-1} . Continuing in the same vein, we may swap $G_{-\frac{1}{2}}$'s for F_{-1} 's in such a way that there are still in total $M+N-2$ F_{-1} 's and $G_{-\frac{1}{2}}$'s. Indeed, we see that for all the identities with no F_{-n} with $n < 2$ and G_{-r} , $r < 3/2$ involve a total $M+N-1$ $G_{-\frac{1}{2}}$'s and F_{-1} 's.

It is straightforward to choose a ϕ which relates F_{-m}^j , $m > 2$ on one leg in terms of F_n , $n > -1$ and G_r , $r > -\frac{1}{2}$ and similarly for G_{-r} , $r > 3/2$. While the isotropy group H_{2N+M}^0 is fixed, there is a number of ways of choosing the coset generators which generate the rest of V^{2N+M} and this point will be discussed elsewhere. However, one convenient choice of fermionic coset generation is given by the set

$$\{ \text{any } M+N-2 \text{ generators chosen from the } F_{-1} \text{'s and } G_{-\frac{1}{2}} \text{'s; } G_r^j, j = 2N+1, \dots, 2N+M, r > \frac{1}{2} \text{ and } F_n^j, j = 1, \dots, 2N, n > 0 \} \quad (6.16)$$

Following Section 4 we are now in a position to define the vertex at any point of the coset

$$(6.17)$$

Since F_n , $n > 0$ and G_r , $r > \frac{1}{2}$ annihilate the physical states on which the vertex acts, the dependence of the vertex of these coset directions is irrelevant and we therefore have $N+M-2$ remaining anticommuting parameters denoted θ_j , $j = 1, \dots, N+M-2$. The vertex, which is an induced representation of V_{2N+M} is given by

$$V(z_j, \Theta_j) = V_0(z_j) g(\Theta_j). \quad (6.18)$$

where we may take

$$g(\Theta_j) = \exp \left(\sum_{j=1}^{2N(N-2)} \Theta_j F_{-1}^j + \sum_{j=2N}^{2N+M} \Theta_j G_{-\frac{1}{2}}^j \right) \quad (6.19)$$

In accord with our convention $\sum_{j=1}^{(N-2)}$ is a sum over our chosen $N-2$ F_{-1} 's. One could also boost the bosonic Koba-Nielsen co-ordinates, but we can compute $V_0(z_j)$ for any z_j with equal ease so it is not necessary. The actual scattering amplitude is given by

$$W = \int \prod_{j=1}^{2N+M} (2N+M-3) dz_j \prod_{j=1}^{2N+M} d\Theta_j V(z_j, \Theta_j) \bar{f} \quad (6.20)$$

The measure is found by decoupling zero-norm physical states, as usual and it will be discussed further below. The count of moduli is in agreement with what one would obtain by sewing, as will be discussed in the next section.

Let us now turn to the incorporation of ghosts into the vertex. The overlap identities for $\bar{g}$ and $\bar{f}$ are given in Eq. (4.27). However, as $\bar{g}(z)$ and $\bar{f}(z)$ have conformal weight $3/2$, these identities are the same as those discussed for $G(z)$ and $F(z)$ above. The overlap identities for $g(z)$ and $f(z)$ are given in Eq. (4.28) and the corresponding ϕ is a tensor of rank $3/2$. As such it is allowed to diverge only as $(\xi_j)^{-3}$ as $\xi_j \rightarrow \infty$. As for the case of the β ghosts of the bosonic string, one can find additional identities by taking ϕ to have poles at more than one Koba-Nielsen point. In the appendix of Ref. [27] we constructed the ghost vertex in the absence of fermion lines to illustrate the general procedure.

Implementing the discussion in Section 4, we find the isotropy group K_{2N+M}^0 contains the L_n , G_r , F_m discussed above, $\bar{\beta}_n$, $\bar{g}_r$, $\bar{f}_n$ in a one-to-one correspondence with these as well as Q . The vertex which is an induced representation of U_{2N+M} is defined on U_{2N+M}/K_{2N+M}^0 and is given by

$$V(z_j, \Theta_j, \kappa_j) = V_0(z_j) g(\Theta_j, \kappa_j). \quad (6.21)$$

where $g(\Theta_j, \kappa_j)$ is an element of U_{2N+M} of the coset with co-ordinates Θ_j , κ_j . One

is free to choose this parametrization how one likes, however, one such choice is

$$g(\omega_j, \kappa_j) = \exp\left(\sum_{j=1}^{2N} (N-2) \omega_j \bar{F}_{-1}^j + \sum_{j=2N+1}^{2N+M} \omega_j \bar{G}_{-\frac{1}{2}}^j\right) \\ \exp\left(\sum_{j=1}^{2N+M} \frac{2N+M}{(2N+M-3)} \psi_j \bar{\beta}_{-1}^j\right) \exp\left(\sum_{j=1}^{2N} (N-2) \kappa_j \bar{f}_{-1}^j + \sum_{j=2N+1}^{2N+M} \kappa_j \bar{g}_{-\frac{1}{2}}^j\right) \quad (6.22)$$

The scattering amplitude is given by

$$\int \prod_{j=1}^{2N+M} (dz_j d\psi_j) \prod_{j=1}^{2N+M} (N+M-2) d\omega_j d\kappa_j \bar{f} V(z_j, \omega_j, \kappa_j) \quad (6.23)$$

One should interpret the integrations over the κ_j in Eq. (6.23) as leading to δ functions of $\bar{f}_{-1}^j$ and $\bar{g}_{-\frac{1}{2}}^j$. In this way, as $\delta(\bar{f}_{-1})$ and $\delta(\bar{g}_{-\frac{1}{2}})$ have the opposite ghost number of $\bar{f}_{-1}$ and $\bar{g}_{-\frac{1}{2}}$, the resulting vertex has the correct ghost number to give a non-zero result on physical states.

It only remains to explain how to compute the measure by insisting that zero-norm physical states decouple. We may perform this computation with ghosts and use the BRST formalism or without ghosts and use our knowledge of the explicit form of zero-norm physical states. We first pursue the latter course, the states being in the Neveu-Schwarz sector

$$G_{-\frac{1}{2}} |\Omega\rangle \quad \text{and} \quad (G_{-\frac{3}{2}} + 2L_1 G_{-\frac{1}{2}}) |\Omega\rangle' \quad (6.24)$$

where

$$L_n |\Omega\rangle = G_n |\Omega\rangle = 0 \quad n \geq \frac{1}{2}; n \geq 0 \quad (6.25)$$

and

$$(L_0 + 1) |\Omega\rangle = 0 = L_n |\Omega'\rangle = G_n |\Omega'\rangle; n \geq 1, n \geq \frac{1}{2}. \quad (6.26)$$

in the Ramond sector

$$(L_1 - \frac{1}{2} F_1 F_0) |\Omega\rangle. \quad (6.27)$$

where

$$F_n |\Omega\rangle = L_n |\Omega\rangle = 0, n > 0 \quad (L_0 + 1) |\Omega\rangle = 0 \quad (6.28)$$

We note that there are really two such spurious states as if $|\Omega\rangle$ obeys the above constraints so does $F_0 |\Omega\rangle$. The zero-norm physical states may be written as the order ϵ term of an infinitesimal operator $g_s = \exp \epsilon S$ where S is a sum of F 's, L 's and G 's on a state. For example, for the first state of Eq. (6.24), $g_s = \exp \epsilon G_{-\frac{1}{2}}$ while for the Ramond states we use the action of one such operators, one on $|\Omega\rangle$ and another one on $F_0 |\Omega\rangle$. Consequently we must consider the action of g_s on the vertex. However, we know that the vertex is an induced representation of the group V_{2N+M} and so we may compute the action of any superconformal operators on it and in particular g_s . It follows from the fact that V_{2N+M} is a group that the action of g_s^i on leg i of our coset representation $g(\Theta)$ may be written as:

$$\begin{aligned} g(\Theta) g_s^i &\equiv \exp \left(\sum_{j=1}^{2N} (N-2) F_{-1}^j + \sum_{j=2N+1}^{2N+M} \Theta_j G_{-\frac{1}{2}}^j \right) g_s^i \\ &= h \exp \left\{ \epsilon \sum_j a_j^i L_{-1}^j \right\} \exp \left\{ \sum_{j=1}^{2N} (N-2) \Theta_j F_{-1}^j + \sum_{j=2N+1}^{2N+M} \Theta_j G_{-\frac{1}{2}}^j \right\} \end{aligned} \quad (6.29)$$

• v

where h is an element of the isotropy group of V_0 and $\hat{\Theta}_j = \Theta_j + \epsilon \chi_j^i$ and

$$\begin{aligned} v &= \exp \left[\epsilon \left\{ \sum_{j=1}^{2N} \sum_{n=0}^{\infty} a_n^j F_n^j + \sum_{j=2N+1}^{2N+M} \sum_{r=\frac{1}{2}}^{\infty} a_r^j G_r^j \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^{2N+M} \sum_{n=0}^{\infty} b_n^j L_n^j \right\} \right] \cdot \{ \text{constant} \}. \end{aligned} \quad (6.30)$$

Applying this result to the vertex of Eq. (6.18) we have:

$$\begin{aligned} V(z, \Theta_j) g_s &= V_0(z,) \exp \left\{ \epsilon \sum_j a_j^i L_{-1}^j \right\} \\ &g(\Theta_j + \epsilon \chi_j^i) \cdot v \\ &= V(z_j + \epsilon a_j^i, \Theta_j + \epsilon \chi_j^i) \cdot v \\ &\quad \cdot \exp \left(\epsilon \sum_j a_j^i C_j \right) \end{aligned} \quad (6.31)$$

In deriving this result, we have used the equation

$$\frac{\partial V_0(z_j)}{\partial z_j} = V_0(\zeta_1^j + c^j) \quad (6.32)$$

where c^j is a constant. For L_{-1} on a Neveu-Schwarz string $c^j = 0$.

Decoupling requires that W of Eq. (6.20) satisfy

$$W |x\rangle_1 \dots |x\rangle_{2N+M} = 0 \quad (6.33)$$

if $|x\rangle_i$ is a zero-norm physical state. Hence, in particular it should contain no term of order ϵ if $|x\rangle_i = g_s^i |\Omega\rangle$. Hence we find that

$$0 = \int \pi' dz_j \pi' d\omega_j V \left\{ - \sum_j \left\{ \frac{\partial}{\partial z_j} (a^j \bar{f}) + a^j c^j \bar{f} \right\} + \eta^i \bar{f} - \sum_j' \frac{\partial}{\partial \omega_j} (x^i_j \bar{f}) \right\} = 0 \quad (6.34)$$

where η^i are the terms which result from the factor r acting on physical states and $|\Omega\rangle$. Neglecting boundary terms we find differential equations which determine the measure.

Let us apply this strategy to the first state of equation (6.24). We find that $\chi^{ij} = \delta^{ij}_\epsilon$, $a^{ij} = \theta^i \delta^{ij}$, $\eta^i = 0$, we recall that $c^j = 0$ and as a result

$$\left(\frac{\partial}{\partial \omega_i} + \omega_i \frac{\partial}{\partial z_i} \right) \ln \bar{f} = 0 \quad i = 2N+1, \dots, 2N+M \quad (6.35)$$

This result implies that $\bar{f}$ is independent of ω_i and z_i , $i = 2N+1, \dots, 2N+M$.

The decoupling for the Ramond states is more complicated as we have a state with two terms, however, the principle is the same and the result will be reported elsewhere.

We now discuss the determination of the measure when we include the ghost oscillators. In this case we may write our spurious state as the order ϵ term in the transformation $g = \exp \epsilon Q \equiv \exp \epsilon \left(\sum_{j=1}^{2N+M} Q^j \right)$ acting on a state. The first step in this procedure is to compute the action Q induces on the coset.

A parametrization which leads to particularly simple results is given by taking

$$g(\psi_j, \omega_j, k_j) = \exp \left(\sum_j^{(2N+M-3)} \psi_j \bar{\beta}_{-1}^j \right) \exp \sum_j^{(N-2)} (\omega_j \bar{F}_{-1}^j + k_j \bar{f}_{-1}^j) \exp \sum_{j=2N+1}^{2N+M} (\omega_j \bar{G}_{-\frac{1}{2}}^j + k_j \bar{g}_{\frac{1}{2}}^j) \quad (6.36)$$

As a result we find that

$$WQ = 0 = \int \prod_{j=1}^{2N+M} [dz_j d\psi_j] \prod_{j=1}^{2N+M} (N+M-2) d\omega_j dk_j V(z_j, \psi_j, \omega_j, k_j) \sum_j' \psi_j \left(\frac{\partial \bar{f}}{\partial z_j} + c_j \bar{f} \right) - \sum_j k_j \frac{\partial \bar{f}}{\partial \omega_j} \quad (6.37)$$

and hence the measure must satisfy

$$\frac{\partial}{\partial z_j} (\ln \bar{f}) + c_j = 0, \quad \frac{\partial \bar{f}}{\partial \omega_j} = 0 \quad (6.38)$$

When the ghosts are present, for the same reasons in the bosonic case, it is possible to remove all the constants such as c_j (see end of Section 7) which occur in the moduli changing equations by multiplying V by a suitable function. Having carried out this manoeuvre, the derivation of the measure is simplified and we see that for the parametrization above it is a constant.

7. - LOOPS

We now discuss in more detail the general approach given in Section 4 for computing string scattering. Let us recall what must be shown. First, we must demonstrate that the overlap equations (4.15) determine the vertex uniquely up to a multiplicative function. We must use the overlap equation (4.16) to compute the isotropy group of the vertex and then characterize the group transformations that lie in the coset. By boosting with these latter coset generators, we find the vertex at all points of the coset. We finally find the measure by demanding that zero-norm physical states decouple. We recall that we can carry out this programme with or without incorporating ghost oscillators and in this section we will pursue both possibilities.

For the sake of concreteness, most of this section will largely concentrate on one-loop scattering. There are four possible types of graphs corresponding to whether we have internal Neveu-Schwarz or Ramond strings propagating inside the loop and whether or not we insert a projector. As is well known [47], this corresponds to the possible spin structures. At the level of formalism we must decide whether to compute in Schottky or Fuchsian representation. We choose the latter as it is technically slightly simpler. The torus is therefore represented by the rectangle with height $2\pi i$ and length $2\pi w$ (see Fig. 3).

To begin with, let us consider only Neveu-Schwarz external lines which allows us to discuss the loop effects before we must incorporate the branch-cut structure due to the external fermionic strings. In this case Eq. (4.15) becomes

$$V_0 \left\{ \sum_{k=1}^M \oint d\bar{z}_k \frac{H^{\mu(k)}(\bar{z}_k)}{\sqrt{\bar{z}_k}} \right\} = 0 \quad (7.1)$$

We recall from Section 4 that

$$V \frac{H^{\mu R}}{\sqrt{z}} = \begin{cases} \frac{V H^{\mu R}(z+2\pi i)}{\sqrt{z+2\pi i}} & \text{for Ramond internal line} \\ - \frac{V H^{\mu R}(z+2\pi i)}{\sqrt{z+2\pi i}} & \text{for Neveu-Schwarz internal line} \end{cases} \quad (7.2)$$

while inserting the relevant projector corresponds to the boundary conditions

$$\frac{V H^{\mu\nu}(z)}{\sqrt{z}} = \begin{cases} -V \frac{H^{\mu\nu}(z+\ell n w)}{\sqrt{z+\ell n w}} & \text{without projector} \\ +V \frac{H^{\mu\nu}(z+\ell n w)}{\sqrt{z+\ell n w}} & \text{with projector} \end{cases} \quad (7.3)$$

In order for (7.1) to hold ϕ must have exactly the same boundary conditions as H has in Eqs. (7.2) and (7.3), that is

$$\phi(z) = \begin{cases} \phi(z+2\pi i) & \text{for Ramond internal line} \\ -\phi(z+2\pi i) & \text{for Neveu-Schwarz internal line} \end{cases} \quad (7.4)$$

while

$$\phi(z) = \begin{cases} \phi(z+\ell n w) & \text{with projector} \\ -\phi(z+\ell n w) & \text{without projector} \end{cases} \quad (7.5)$$

For every ϕ which has the appropriate boundary conditions and poles only at Koba-Nielsen points, we find from (7.1) an identity on V_0 . Let us adopt the notation $(+,+)$ $(-,+)$ $(-,+)$ $(-,-)$ for the four boundary conditions, the first entry corresponding to the signs in Eq. (7.2) while the second entry corresponds to those in Eq. (7.3). In order to determine V_0 we require, at first sight, an identity which contains b_{-s}^{μ} , $s > \frac{1}{2}$ on one leg only and otherwise b_r^{μ} , $r > \frac{1}{2}$. In particular, this means we should have a ϕ which has a single pole of order one in the rectangle. We will treat the case of $(+,+)$ separately later but functions ϕ which have a single first order pole at a Koba-Nielsen point and have the correct boundary conditions are

$$\phi_{-+}(z) = \left\{ \zeta_1(z-z_j) - \zeta_1(z-z_j+2\pi i) + \zeta_1(2\pi i) \right\} \quad (7.6)$$

where ζ_1 is the Weierstrass ζ function with periods $4\pi i$ and $\ell n w$

$$\varphi_{-}(z) = \left\{ \zeta_2(z - z_j) - \zeta_2(z - z_j + 2\pi i) - \zeta_2(z - z_j + \ln w) + \zeta_2(z - z_j + 2\pi i + \ln w) \right\} \quad (7.7)$$

where ζ_2 is the Weierstrass ζ function with periods $4\pi i$ and $2\ln w$

$$\varphi_{+}(z) = \left\{ \zeta_3(z - z_j) - \zeta_3(z - z_j + \ln w) + \zeta_3(\ln w) \right\} \quad (7.8)$$

where ζ_3 has periods $2\pi i$ and $2\ln w$. We observe that to gain a change of sign when we translate by a required period we have used a Weierstrass ζ function with double the period in the required direction and subtracted from it the same function with its pole displaced by the original period. In order to verify the boundary conditions we have used the relation

$$\begin{aligned} \zeta(z + 2\omega_1) &= \zeta(z) + 2\zeta(\omega_1) \\ \zeta(z + 2\omega_2) &= \zeta(z) + 2\zeta(\omega_2) \end{aligned}$$

where ζ has periods $2\omega_1$ and $2\omega_2$. Substituting into Eq. (7.1) we find an identity with $b_{-\frac{1}{2}}^{\mu j}$ and terms involving $b_r^{\mu k}$, $r > \frac{1}{2}$. These identities determine the $b_{\frac{1}{2}}^{\mu j}$ dependence of the vertex.

By differentiating the functions of Eqs. (7.6), (7.7) and (7.8) with respect to z successively we obtain functions ϕ with the required boundary conditions and have a pole of any desired order at one Koba-Nielsen point:

$$\frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (\varphi_{-+}) = \begin{cases} \left(\frac{1}{z_j}\right)^n + \sum_{p=0}^{\infty} a_{np}^{jk} (\bar{z}_j)^p & \text{when expanded about } z_j \\ \sum_{p=0}^{\infty} a_{np}^{jk} (\bar{z}_k)^p & \text{when expanded about } z_k \end{cases} \quad (7.9)$$

$n \geq 1$

As such we find that our identity becomes

$$V_0^{(-+)} \left\{ b_{-n+\frac{1}{2}}^{\mu j} + \sum_{k=1}^m \sum_{p=0}^{\infty} a_{np}^{jk} b_{p+\frac{1}{2}}^{\mu k} \right\} = 0 \quad (7.10)$$

$n \geq 1$

which implies that

$$V_6^{(-+)} = \left\{ \prod_{i=1}^m \langle 0 | \right\} \exp \frac{1}{2} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \sum_{j,k=1}^m b_{n-\frac{1}{2}}^{\mu j} a_{np}^{jk} b_{p+\frac{1}{2}}^{\mu k}. \quad (7.11)$$

and similarly for ϕ_{--} and ϕ_{+-} . Thus for these boundary conditions the vertex is uniquely determined.

Let us now return to the case ϕ_{++} which corresponds to an internal fermionic string which has a projector $\Gamma = \exp i \pi \sum_n d_n^\dagger d_n$ inserted. Clearly, we cannot have in the rectangle a single pole of first order as integrating round the boundary implies the residue of such an object vanishes.

For ϕ 's with poles of order two and above we may take the Weierstrass P function and its derivatives

$$\phi_{++}(z) = \frac{(-1)^{n-1}}{n!} \frac{d^{n-1}}{dz^{n-1}} \wp(z - z_j). \quad (7.12)$$

These lead to identities which contain $b_{-r}^{\mu j}$, $r > 3/2$ and otherwise b_s^{μ} , $s > \frac{1}{2}$ and as such determine the dependence of the vertex on $b_r^{\mu j}$, $r > 3/2$, $j = 1, \dots, m$. Unlike for the other boundary conditions, considering only ϕ 's which are analytic except at only one Koba-Nielsen point does not provide a complete basis for all allowed ϕ 's. In particular, a ϕ which satisfies the $(+,+)$ boundary conditions and has only poles at Koba-Nielsen points and is not included in such a basis is

$$\phi_{++} = \left\{ \wp(z - z_i) - \wp(z - z_j) \right\}. \quad (7.13)$$

The resulting identity is of the form

$$0 = V_0^{(+,+)} \left\{ b_{-\frac{1}{2}}^{\mu i} - b_{-\frac{1}{2}}^{\mu j} + \text{terms involving } b_r^{\mu k} \quad r \geq \frac{1}{2} \right\} \quad (7.14)$$

As a result, the vertex is determined by using the ϕ_{++} of Eqs. (7.12) and (7.13) except for a dependence on a factor of the form

$$\left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu j} \right) \quad (7.15)$$

In other words, using the Laurent expansions of the ϕ 's, we obtain an expression for the vertex of the form

$$V_0 = \left\{ \prod_{i=1}^m \langle 0 | \right\} \exp \frac{1}{2} \sum_{i,j=1}^m \sum_{p=1}^{\infty} \sum_{m=0}^{\infty} b_{p-\frac{1}{2}}^{\mu i} E_{pm}^{ij} b_{m+\frac{1}{2}}^{\mu j} \cdot \exp k \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu j} \right) \dots \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu_{10} j} \right) \epsilon_{\mu_1 \dots \mu_{10}} \quad (7.16)$$

where E_{pm}^{ij} are known, but k is undetermined. At first sight, it may appear that the overlaps do not determine the vertex uniquely due to the fact that we have a missing pole. This is reminiscent of the situation for the multiloop bosonic string. However, in this case one may express the conformal operator of interest $P^{\mu}(z)$ as the derivative of another conformal operator, i.e., $Q^{\mu}(z)$ which obeys also overlap equations. The net effect of this is to allow functions ϕ which are not single-valued and can have poles of any order at only one Koba-Nielsen point [10,11].

In the case of interest here, the cure results from the presence of a further identity, namely we may for a $(+,+)$ boundary condition only take $\phi = \text{constant}$. This results in the identity

$$V_0 \left\{ \sum_{j=1}^m b_{\frac{1}{2}}^{\mu j} \right\} = 0 \quad (7.17)$$

This identity is unusual insomuch as it does not involve any destruction operators as viewed from the usual vacuum contained in the left in V_0 . It can only be implemented by including a prefactor which consists of the ten factors $(\sum_{j=1}^M b_{\frac{1}{2}}^{\mu j})$, $\mu = 0, 1, \dots, 9$ multiplying the vertex itself. Namely

$$V_0^{++} = \prod_{i=1}^m \langle 0 | \exp \sum_{i,j=1}^m \sum_{p=1}^{\infty} \sum_{m=0}^{\infty} b_{p-\frac{1}{2}}^{\mu i} E_{pm}^{ij} b_{m+\frac{1}{2}}^{\mu j} \cdot \exp \left\{ k \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu j} \right) \dots \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu_{10} j} \right) \epsilon_{\mu_1 \dots \mu_{10}} \right\} \cdot \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu_1 j} \right) \dots \left(\sum_{j=1}^m b_{\frac{1}{2}}^{\mu_{10} j} \right) \epsilon_{\mu_1 \dots \mu_{10}} \quad (7.18)$$

We observe that we have resolved the ambiguity in the determination of the vertex. Since $b_{\frac{1}{2}}^{\mu}$ is anticommuting the expression (7.18) is independent of k and so we may set $k = 0$.

The existence of this identity answers another problem, namely how one gets a gauge anomaly which we know for a general open string to be present [49], and require an $\varepsilon^{\mu_1, \dots, \mu_{10}}$ symbol. In fact, the possibility of an anomaly becomes a rather obvious feature of this way of calculating string amplitudes.

We may therefore conclude that the vertex at one loop is indeed uniquely determined by the overlap equations. The method of procedure at higher loops is straightforward in principle. As mentioned in Section 4, the naive application of the Riemann-Roch theorem implies there are no missing poles. Although the actual situation, as at one loop, may be likely to be a little more complicated, the overlap identities will determine the vertex uniquely. Let us now turn to the isotropy group of the vertex V_0 with only bosonic external lines. The part of the group, H_M^1 generated by the L_n 's is the same as for the bosonic graph with M legs and is discussed in Ref. [27]. The part generated by the G 's is determined by the overlap

$$V_0 \left\{ \sum_{k=1}^M \oint d\bar{z}_k \phi \frac{G^k(\bar{z}_k)}{(\bar{z}_k)^{3/2}} \right\} = 0 \quad (7.19)$$

The discussion of boundary conditions on $VH(z)/\sqrt{z}$ of Eqs. (7.3) and (7.4) applies unmodified to $V G(z)/z^{3/2}$ with $H/\sqrt{z}$ replaced by $G/z^{3/2}$. Hence corresponding to the four types of graphs we have the four types of boundary conditions of Eqs. (7.2) and (7.3) which we must impose on ϕ . We note that on the rectangle, dz is invariant under the two automorphic transformations (i.e., $z \rightarrow z+2\pi i$, $z \rightarrow z+\ln w$), hence an automorphic tensor ϕ of any rank must obey the same equations. This makes the discussion at one loop particularly simple; indeed we may use the same ϕ 's as for the H overlap of equations (7.4) and (7.5).

Let us consider first the contributions with boundary conditions $(+,-), (-, \pm)$; using the ϕ 's of Eqs. (7.6), (7.7) and (7.8) and their derivatives we find substituting in Eq. (7.19) a relation of the form

$$V_0 G_{n-\frac{1}{2}}^j \equiv V_0 \left\{ G_{n-\frac{1}{2}}^j + \sum_{i=1}^M \sum_{m=0}^{\infty} a_{p^m}^{ji} G_{m-\frac{1}{2}}^i \right\} = 0 \quad (7.20)$$

Hence the isotropy group contains

$$G_{-3/2}^j, G_{-5/2}^j, \dots \quad (7.21)$$

We note that squaring these combinations we find the isotropy group for the L_n 's.

It is straightforward to list the generators of the coset, namely

$$\left\{ G_r^j, \quad j=1, \dots, M, \quad r \geq -\frac{1}{2} : L_{-2}^i \text{ for given } i \in [1, \dots, M] ; L_{-1}^j, j=1, \dots, M \text{ except } j=i, \text{ say}; \right. \quad (7.22)$$

$$\left. L_m^j, \quad j=1, \dots, M, \quad m \geq 0 \right\}$$

As usual, although the vertex depends on transformations generated by G_r^j , $r \geq \frac{1}{2}$ and L_m^j , $m \geq 0$ the scattering amplitude does not and so we may regard the coset as generated by $G_{-1/2}^j$, $j=1, \dots, M$ and L_{-2}^j , L_{-1}^j , $j=1, \dots, M$. As a result, we have M supermoduli Θ_j for these three graphs and we may write the general vertex without ghosts as

$$V(u, z_j; \Theta_j) = V_0(u, z_j) \exp \sum_{j=1}^M \Theta_j G_{-1/2}^j. \quad (7.23)$$

Let us now consider the contribution with $(+,+)$ boundary conditions for which the isotropy group is somewhat different. Substituting the ϕ_{++} 's of Eq. (7.12), we find isotropy relations of the form

$$V_0 G_{-p-1/2}^j \equiv V_0 \left\{ G_{-p-1/2}^j + \text{terms with } G_r^k, r \geq -\frac{1}{2} \right\} \quad (7.24)$$

$p \geq 2.$

and from Eq. (7.13) relations of the form

$$V_0 \left\{ G_{-3/2}^i - G_{-3/2}^j + \text{terms with } G_r^k, r \geq -\frac{1}{2} \right\} \quad (7.25)$$

while the $\phi_{++} = \text{constant}$ implies the relation

$$V_0 \left\{ \sum_{j=1}^M G_{-1/2}^j \right\} = 0 \quad (7.26)$$

Hence the coset generators may be chosen as

$$\left\{ \sum_{j=1}^m G_{-3/2}^j, G_{-1/2}^j \quad j = 1, \dots, i-1, i+1, \dots, m, G_{-1/2}^j \right. \\ \left. j = 1, \dots, m \quad v \geq \frac{1}{2} ; L_2^i \text{ for some } i \in (1, \dots, m) \right. \\ \left. L_1^j, j = 1, \dots, i-1, i+1, \dots, m, L_m^j \quad j = 1, \dots, m \quad m \geq 0 \right\}. \quad (7.27)$$

Thus we see we still have the same number of fermionic coset generators as with the other possible boundary conditions but in effect one $G_{-1/2}$ has been swapped for a $G_{-3/2}$. The vertex is given by

$$V(u, z_j, \Theta_j) = V_0(u, z_j) \exp \left\{ \sum_{\substack{j=1 \\ j \neq i_1}}^m \Theta_j G_{-1/2}^j \right. \\ \left. + \Theta_{i_1} \sum_{j=1}^m G_{-3/2}^j \right\} \quad (7.28)$$

We now give a generic discussion of how the gauge anomaly arises from this vertex. After integration over Θ_j we find a vertex which must contain a term of the form

$$V(u, z_j, \Theta_j) = V_0(u, z_j) \left\{ \prod_{\substack{j=1 \\ j \neq i_1}}^m G_{-1/2}^j \right\} \left\{ \sum_{j=1}^m G_{-3/2}^j \right\} \quad (7.29)$$

The gauge anomaly arises from a graph with six external legs. On one leg we act with $G_{-1/2}$ while on the remaining five we put a state of the form $|\chi\rangle = b_{-1/2}^\mu A |0\rangle$. We recall from Eq. (7.18) that V_0 contains a factor of ten $b_{-1/2}^\mu$'s with an $\varepsilon^{\mu_1, \dots, \mu_{10}}$ symbol. We annihilate five of these against the $b_{-1/2}^\mu$'s in the external states and the other five against the five $G_{-1/2}$'s ($\sim k_\mu b_{-1/2}^\mu$) which are prefactors of the vertex. This creates the product of five field strengths times the $\varepsilon^{\mu_1, \dots, \mu_{10}}$ symbol (i.e., $\varepsilon^{\mu_1 \dots \mu_{10}} F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \dots F_{\mu_9 \mu_{10}}$).

The $G_{-1/2}$ on the one external leg creates with the $G_{-3/2}$ prefactor an L_{-2} acting on the vertex. Using the moduli changing properties of the vertex discussed in Section 2, we find that we may express this as a total divergence which yields a term at the boundary which is in general non-zero and finite. The latter part of

the calculation once one finds L_{-2} is very similar to the open bosonic string where the anomaly was derived in Ref. [10]. This anomaly was also found independently in Ref. [50]. It was found that the open theory was anomalous for all groups except $SO(2^{13})$. This gauge group results from having to balance the two one-loop graphs against each other. We encourage the reader to fill in the details in the above discussion and arrive at $SO(2^5)$ as the only anomaly free group.

Let us generalize the above considerations to include contributions with external fermionic strings. We now consider a contribution with $2N$ fermionic and M bosonic strings. As discussed in Section 4, we must encode into ϕ N branch cuts, corresponding to the $2N$ emitted fermionic strings. However, this must be consistent with ϕ being automorphic and having poles only at Koba-Nielsen points. Let us first consider the case where the boundary condition is such that

$$\phi(z) = \phi(z + \ln w) \quad (7.30)$$

and we choose to take our N branch cuts so that they do not cross the edges of the rectangle.

Since ϕ^2 satisfies $(+,+)$ boundary conditions and is single-valued, it must have as many zeros as poles if we include their order. However, we must not introduce zeros of order one in ϕ^2 which do not begin and end at the fermionic Koba-Nielsen points as these will lead in ϕ to branch cuts which are not allowed. To construct the required functions we require the Weierstrass $\sigma(z)$ function which has no poles in the rectangle, vanishes only at $z = 0$ and transforms under the periods $2\omega_1 = 2i\pi$ and $2\omega_2 = \ln w$ as

$$\sigma(z + 2\omega_i) = -\sigma(z) \exp 2\eta_i(z + \omega_i) \quad (7.31)$$

where $\zeta(\omega_i) \equiv \eta_i$. Consider the function

$$\phi(z) = \frac{\sigma\left(z + \frac{1}{2} \sum_{i=1}^{2N} z_i - \sum_{i=1}^{2N(N+1)} z_i\right) \prod_{i=1}^{2N} \sigma(z - z_i)}{\left\{ \prod_{i=1}^{2N(N+1)} \sigma(z - z_i) \right\}} \equiv S(z) \quad (7.32)$$

where $\prod^{N+1}$ and $\sum^{N+1}$ are a product and a sum respectively over any $N+1$ σ 's that vanish at any $N+1$ Koba-Nielsen points. Using Eq. (7.31) one may easily verify that

$$\phi(z + 2\omega_i) = \phi(z). \quad (7.33)$$

The correct branch cut structure is encoded by the factors under the square roots. Demanding that the $\exp 2\eta_1 z$ term in Eq. (7.31) cancel, tells us that there are equal numbers of σ 's upstairs and downstairs, while the zero at $-\frac{1}{2} \sum_{i=1}^{2N} z_i + \sum_i^{(N+1)} z_i$ is a result of requiring that remaining factors cancel.

Substituting the ϕ of Eq. (7.32) into Eq. (4.15), we obtain an equation of the form

$$V_0 \left\{ \sum_{j=1}^{2N(N+1)} a_j^{\mu_j} d_0^{\mu_j} + \sum_{n=0}^{\infty} \sum_{j=1}^{2N} a_n^j d_n^{\mu_j} + \sum_{r=\frac{1}{2}}^{\infty} \sum_{j=2N+1}^{2N+M} c_r^j b_r^{\mu_j} \right\} = 0 \quad (7.34)$$

There are in fact N such independent equations. We thus recover at the loop level the analogue of the tree level Eq. (6.4) which allows us to identify the spinor indices on any N pairs of the $2N$ lines.

We may construct suitable ϕ 's by multiplying $S(z)$ of Eq. (7.32) by any automorphic function, for example, we may consider

$$\varphi(z) = S(z) \frac{d^n}{dz^n} \mathcal{P}(z - z_j) \quad \begin{matrix} j \in [1, \dots, 2N+M] \\ n = 0, 1, 2, \dots \end{matrix} \quad (7.35)$$

or

$$\varphi(z) = S(z) \left\{ \zeta(z - z_i) - \zeta\left(z + \frac{1}{2} \sum_{i=1}^{2N} z_i - \sum_{i=1}^{2N(N+1)} z_i\right) \right\} \quad (7.36)$$

Despite what one may at first sight think, the function $\zeta(z + \frac{1}{2} \sum_{i=1}^{2N} z_i - \sum_i^{(N+1)} z_i)$ does not lead to a pole since $S(z)$ has a zero of order one at the same point.

Substituting these functions ϕ into Eq. (4.15), we find that we arrive at equations with a term b_{-r}^j , $r > \frac{1}{2}$ for any j or d_{-n}^j , $n > 1$ for any j and terms which involve oscillators b_s , $s > \frac{1}{2}$, d_n , $n > 0$. As such the overlaps completely determine the vertex, the discussion paralleling that for the tree case of Section 6. We note that the above ϕ has the boundary condition $\phi(z) = \phi(z + 2\pi i)$ at the beginning and end of the rectangle (i.e., at $z = 0$ and $z = \lambda n w$). However, when z is between fermionic Koba-Nielsen points, $\phi(z) = \pm \phi(z + 2\pi i)$ the sign which occurs being related to the choice of the N branch cuts. This is to be expected since one has both bosonic and fermionic strings propagating in the loop. One must, however, sum over all spin structures, i.e., the boundary conditions $(+, -)$, $(+, +)$, $(-, +)$ and

(-, -). This is achieved in much the same way as in the case without fermion lines by taking the same ϕ 's as for (+, +) but doubling the periods in the appropriate directions. For example, if we have the boundary condition

$$\varphi(z) = -\varphi(z + \ln w) \quad (7.37)$$

then we can find a suitable basis of functions by considering

$$\varphi_1(z) - \varphi_1(z + \ln w) \quad (7.38)$$

where ϕ_1 is the ϕ used above except that we now take it to have periods $2\pi i$ and $2\ln w$.

It is straightforward to find the isotropy group H_{2N+M}^1 associated with the above vertex. One simply substitutes the functions ϕ found above into Eq. (4.16) and one arrives at the fermionic generators of the isotropy group, the bosonic generators being the same as for a one loop bosonic graph with $2N+M$ legs (see Ref. [27]). The fermionic coset generators are therefore given by

$$\left\{ \begin{array}{l} \text{any choice of } N+M \text{ from } F_{-1}^j, \quad j=1, \dots, 2N \\ \text{and } G_{-\frac{1}{2}}^j, \quad j=2N+1, \dots, 2N+M; \quad F_n^j \quad n \geq 0, j=1, \dots, 2N \\ ; G_r^j \quad j=2N+1, \dots, 2N+M, \quad r \geq \frac{1}{2} \end{array} \right\} \quad (7.39)$$

As a result, we have $N+M$ supermoduli at one loop in contrast to $N+M-2$ at tree level. We recall that at the tree level there is a difference between the number of H-T overlaps and G-F overlaps due to the different behaviour that tensors of rank $\frac{1}{2}$ and $-\frac{1}{2}$ are required to have as $\xi_j \rightarrow \infty$.

We may write the vertex which is an induced representation of V_{2N+M} and defined on the coset V_{2N+M}/H_{2N+M}^1 as

$$\begin{aligned} V(\nu, z_j, \otimes_j) &= V_0(\nu, z_j) \exp \left\{ \sum_{j=1}^{2N} \nu_j \otimes_j F_{-1}^j \right. \\ &+ \left. \sum_{j=2N+1}^{2N+M} \nu_j \otimes_j G_{-\frac{1}{2}}^j \right\} \end{aligned} \quad (7.40)$$

It is instructive to verify that one finds, by sewing, the same number of supermoduli. We may build any graph by sewing, with the appropriate propagators, the three Neveu-Schwarz vertex [38], V_B and the two fermion one boson vertex V_F (see appendix). While the latter vertex has no supermoduli, the former vertex has one supermodulus which is discussed in detail in Refs. [38] and [12]. The boson propagator carries no supermodulus although the fermionic propagator, which is a fancy version of F_0^{-1} , must carry one. Consequently, a contribution with I_F fermionic propagators, I_B bosonic propagators, V_F two fermion one boson vertices and V_B three boson vertices has

$$I_F + V_B \quad (7.41)$$

supermoduli. Using the equations $2V_F = 2N+2I_F$, $V_F+3V_B = M+2I_B$, and that the number of loops $g = I_F + I_B - (V_B + V_F) + 1$ we find the number of supermoduli to be given by

$$2(g-1) + M + N \quad (7.42)$$

This agrees with the number found at tree level in Section 6 and the number at one loop above. It also agrees with the number in a multiloop contribution as predicted by applying in Section 4 the Riemann-Roch theorem to the overlap equations.

It is straightforward to introduce ghost oscillators into the vertex using the overlaps of Eqs. (4.27) and (4.28). The $\bar{g}$, $\bar{f}$ have the same conformal weight as G and F and so we may substitute the ϕ 's discussed above in their overlap equation.

The isotropy group K_{2N+M}^1 is given by the generators of H_{2N+M}^1 plus any $N+M$ of the $2N+M$ $\bar{f}_{-1}^j$'s, $\bar{g}_{-\frac{1}{2}}^j$ as well as g_r^j , $r > \frac{1}{2}$, $\bar{f}_n^j$, $n > 0$. The vertex is an induced representation of I_{2N+M} and is given by

$$\begin{aligned} V(u, z_j, \psi, \psi_j, \omega_j, k_j) &= V_0(u, z_j) \\ \exp(\psi \bar{\beta}_{f_1}) &\exp\left(\sum_{\substack{j=1 \\ j \neq i_1}}^{2N+M} \psi_j \bar{\beta}_{-1}^j\right) \exp\left\{\sum_{j=1}^N (\omega_j E_1^j + k_j \bar{f}_{-1}^j)\right\} \\ &\exp\left\{\sum_{j=2N+1}^{2N+M} (\omega_j G_{-\frac{1}{2}}^j + k_j \bar{g}_{-\frac{1}{2}}^j)\right\} \end{aligned} \quad (7.43)$$

$\bar{\beta}_{f_1}$ is the replacement of L_n by $\bar{\beta}_n$ in L_{f_1} where L_{f_1} is the conformal transformation that induces a change in the loop moduli according to the equation

$$\frac{\partial V_0}{\partial \sigma} = V_0 (\angle_{f_1} + c_1) \quad (7.44)$$

The scattering amplitude being given by

$$W = \int d\sigma d\psi \prod_{j=1}^{2N+M} [dz_j d\psi_j] \prod_{j=1}^N d\omega_j dk_j \prod_{j=2N+1}^{2N+M} d\omega_j dk_j$$

$$\int V(\omega, z_j, \psi, \psi_j; \omega_j, k_j) \quad (7.45)$$

The integrals over κ_j are taken to mean δ functions and its derivatives.

For the case of only Neveu-Schwarz external lines, the discussion is as above if we set $\omega_j = \psi_j = z_j = k_j = 0$ for $j = 1, \dots, 2N$. The exception is for the boundary condition $(+, +)$ where one of the $G_{-\frac{1}{2}}$ and $\bar{G}_{-\frac{1}{2}}$ are replaced by $G_{-3/2}$ and $\bar{G}_{-3/2}$. We leave the reader to fill in the details of this case.

The multiloop case proceeds in a straightforward way as discussed in Section 4. Having computed the vertex V_0 from the P^μ , $H^\mu - \Gamma^\mu$, β , $\bar{\beta}$, $g-f$, $\bar{g}-\bar{f}$ overlaps, one then computes its isotropy group and then boosts with the remaining generators. One will find that in the coset one has $2N+M+g_1$ ($g_1 \equiv 3(g-1)$) bosonic generators and $N+2M+g_2$ ($g_2 \equiv 2(g-1)$) supergenerators with a corresponding number of bosonic and supermoduli.

One could also find such multiloop contributions by sewing together repeatedly the two-fermion one-boson vertices and the three-boson vertex when ghost extended. The correct propagator to use is found by demanding that the final multiloop vertex satisfy the requisite overlap properties. The discussion is an extension of that given in Refs. [13] and [27] for the bosonic theory. As discussed above, one recovers the correct number of moduli and supermoduli.

It remains to discuss how the measure is determined by the decoupling of zero-norm physical states which we carry through in the presence of ghosts. Let us also, just for this discussion, introduce a streamlined notation; all the g_1+M+2N bosonic moduli are denoted by v_r , $r = 1, \dots, g_1+2N+M$ and all the g_2+M+N supermoduli by θ_s , $s = 1, \dots, g_2+N+M$. The superconformal generators F_m^j and G_r^j are denoted collectively by $\mathcal{F}^j$ where each type of leg it acts on dictates whether it is a F_n or a G_r . Similarly the antighosts $\bar{g}$ and $\bar{f}$ we denote collectively by $\bar{\mathcal{F}}$. We have constructed at least in principle the vertex $V_0(v_r)$ which contains the ghost

oscillators and boosted it by the $N+M+g_2$ superboosts parametrized by Θ_s to yield the vertex $V(v_r, \Theta_s)$ which is of the generic form

$$V(v_r, \Theta_s) = V_0(v_r) \exp \sum_s \Theta_s \mathcal{F}_s \quad (7.46)$$

where $\mathcal{F}_s$ are the g_2+N+M supercoset generators. The behaviour of $V_0(v_r)$ under the Virasoro algebra is discussed in Ref. [27]. In particular, we know how to change the moduli v_r , i.e., Eq. (2.15) by a conformal transformation. Dragging this transformation through $\exp \Theta_s \cdot \mathcal{F}_s$, and using any necessary overlap equations we find an equation of the type

$$\frac{\partial V}{\partial v_r}(v_r, \Theta_s) = V(L_{\psi_r} + c_r) \quad (7.47)$$

where

$$L_{\psi_r} = L_{\psi_0} + \sum_s \Theta_s \mathcal{F}_{\psi_1, s} + \sum_{s, t} \Theta_s \Theta_t L_{\psi_2, s, t} + \dots \quad (7.48)$$

and c_r is a function of v_r and Θ_s . The L 's denote conformal transformations composed of L_n 's while the $\mathcal{F}_\psi$'s are composed of G_r 's and F_n 's. Similarly, we may, by knowing the explicit form of $V(v_r, \Theta_s)$ discover that

$$\frac{\partial V}{\partial \Theta_s}(v_r, \Theta_s) = V \{ \mathcal{F}_{\phi_s} + d_s \} \quad (7.49)$$

where d_s is an anticommuting function of Θ_s and v_r and

$$\mathcal{F}_{\phi_s} = \mathcal{F}_{\phi_0} + \sum_s \Theta_s L_{\phi_1, s} + \sum_{s, t} \Theta_s \Theta_t \mathcal{F}_{\phi_2, s, t} + \dots \quad (7.50)$$

We now follow the analogous arguments for the bosonic case given in Ref. [15]. To find the measure we will be required to drive Q through the vertex including ghost prefactors. In carrying out this step we will need certain identities which we now derive. Equations (7.47) and (7.49) imply that V must satisfy certain integrability conditions:

$$\begin{aligned} 0 = \frac{\partial^2 V}{\partial v_r \partial \Theta_s} - (v \leftrightarrow s) &= V \left\{ \frac{\partial L_{\psi_r}}{\partial \Theta_s} - \frac{\partial L_{\psi_s}}{\partial v_r} \right. \\ &+ [L_{\psi_r}, L_{\psi_s}] \left. \right\} + V \left\{ \frac{\partial c_r}{\partial \Theta_s} - \frac{\partial c_s}{\partial v_r} \right\} \quad (7.51) \end{aligned}$$

v_r, s
 $v \leftrightarrow s$

$$0 = \frac{\partial^2 V}{\partial \Theta_s \partial \Theta_v} + (v \leftrightarrow s) = V \left\{ \frac{\partial \mathcal{F}_{\phi_v}}{\partial \Theta_s} + \frac{\partial \mathcal{F}_{\phi_s}}{\partial \Theta_v} \right. \\ \left. + [\mathcal{F}_{\phi_s}, \mathcal{F}_{\phi_v}] \right\} + V \left\{ \frac{\partial d_v}{\partial \Theta_s} + \frac{\partial d_s}{\partial \Theta_v} \right\} \quad \forall v, s \quad (7.52)$$

$$\frac{\partial^2 V}{\partial \Theta_s \partial \sigma_v} - \frac{\partial^2 V}{\partial \sigma_v \partial \Theta_s} = 0 = V \left\{ \frac{\partial \mathcal{L}_{\psi_v}}{\partial \Theta_s} - \frac{\partial \mathcal{F}_{\phi_s}}{\partial \sigma_v} \right. \\ \left. + [\mathcal{F}_{\phi_s}, \mathcal{L}_{\psi_v}] \right\} + V \left\{ \frac{\partial c_v}{\partial \Theta_s} - \frac{\partial d_s}{\partial \sigma_v} \right\} \quad \forall v, s \quad (7.53)$$

These equations we recognize as preserving identities i.e., overlap equations. They result from the overlap equations (4.16) and the analogue for $L(z)$ for V_0 when transformed to the point with anticommuting co-ordinate Θ_s , i.e., dragged through $\exp \Theta \cdot \mathcal{F}$. However, $\bar{\beta}$, L and $\bar{\mathcal{F}}$ and $\mathcal{F}$ have respectively the same conformal weight and hence their preserving identities on V_0 are in one-to-one correspondence and identical in form. Further their anticommutation relations with the super-Virasoro group are the same in form, for example

$$(G_s, \bar{g}_v) = -2 \bar{\beta}_{v+s}, \quad \{G_s, G_v\} = 2 \mathcal{L}_{v+s} \\ \{G_s, \bar{\beta}_n\} = (\frac{n}{2} - s) \bar{g}_{n+s}, \quad (G_s, \mathcal{L}_n) = (s - \frac{n}{2}) G_{n+s} \quad (7.54)$$

As such their preserving identities on $V(v_r, \Theta_s)$ are in one-to-one correspondence and identical in form. Consequently, we deduce the existence of the identities

$$V \left\{ \frac{\partial \bar{\beta}_{\psi_v}}{\partial \Theta_s} - \frac{\partial \bar{\beta}_{\psi_s}}{\partial \sigma_v} + [\mathcal{L}_{\psi_v}, \bar{\beta}_{\psi_s}] \right\} = 0 \quad (7.55)$$

$$V \left\{ \frac{\partial \bar{\beta}_{\phi_s}}{\partial \omega_r} + \frac{\partial \bar{\beta}_{\psi_r}}{\partial \omega_s} + [\bar{\beta}_{\phi_s}, \bar{\mathcal{F}}_{\psi_r}] \right\} = 0 \quad (7.56)$$

$$V \left\{ \frac{\partial \bar{\beta}_{\psi_r}}{\partial \omega_s} + \frac{\partial \bar{\beta}_{\phi_s}}{\partial \omega_r} - [\bar{\beta}_{\phi_s}, \bar{\mathcal{L}}_{\psi_r}] \right\} = 0 \quad (7.57)$$

where β_{ψ_r} and $\bar{\beta}_{\phi_s}$ are obtained from L_{ψ_r} and $\bar{\mathcal{F}}_{\phi_s}$ respectively by the replacements $L_n \rightarrow \bar{\beta}_n$, $G_r \rightarrow \bar{g}_r$, $F_n \rightarrow \bar{f}_n$. We observe that Eqs. (7.55) to (7.57) have no constants since this is forbidden by ghost number counting. However, as

$$V \bar{\phi} = 0 \quad (7.58)$$

we can apply Q to Eqs. (7.55) to (7.57) and deduce Eq. (7.51) to (7.53). We notice that there are then no constants in these latter equations and so

$$\frac{\partial c_s}{\partial \omega_r} - \frac{\partial c_r}{\partial \omega_s} = 0 = \frac{\partial d_r}{\partial \omega_s} + \frac{\partial d_s}{\partial \omega_r} = \frac{\partial c_r}{\partial \omega_s} - \frac{\partial d_r}{\partial \omega_r} \quad (7.59)$$

As a result we deduce that there exists a $\ln \bar{f}$ such that

$$c_s = \frac{\partial}{\partial \omega_s} \ln \bar{f}, \quad d_r = \frac{\partial}{\partial \omega_r} \ln \bar{f} \quad (7.60)$$

Scaling the vertex V by $\bar{f}$ then removes the constants from the moduli changing Eqs. (7.47) and (7.49). As a change is as good as a rest let us insert, following Ref. [15], the requisite antighost factors by hand and define the vertex

$$U = V(\omega_r, \omega_s) \prod_{s=1}^{g_2 + M + 2N} \bar{\beta}_{\phi_s}^{\frac{g_2 + M + 2N}{2}} \prod_{r=1}^{g_1 + 2N + M} \bar{\beta}_{\psi_r} \quad (7.61)$$

These factors guarantee that U has the correct ghost number and decoupling

properties as we shall see. We observe that $\delta(\bar{\beta}_{\phi_r}) = \bar{\beta}_{\phi_r}$ and hence it is really a string of δ functions. Now, we find that

$$\begin{aligned}
 u \phi = 0 &= V(\sigma_r, \Theta_s) \prod_{s=1}^{g_2+2N+M} \delta(\bar{\beta}_{\phi_s}) \\
 &\cdot \sum_{n=1}^{g_1+2N+M} \bar{\beta}_{\psi_1} \dots L_{\psi_n} \dots \bar{\beta}_{\psi_{g_1+2N+M}} (-1)^{g_1+2N+M-n} \\
 &+ V(\sigma_r, \Theta_s) \sum_{s=1}^{g_2+2N+M} \delta(\bar{\beta}_{\phi_s}) \dots (-\mathcal{F}_{\phi_s} \delta'(\bar{\beta}_{\phi_s}) \dots \delta(\bar{\beta}_{\phi_{g_2+2N+M}}) \dots \\
 &\cdot (-1)^{g_1+2N+M} \dots \delta(\bar{\beta}_{\phi_s}) \dots \delta'(\bar{\beta}_{\phi_s}) \dots \delta(\bar{\beta}_{\phi_{g_2+2N+M}}) \dots
 \end{aligned} \quad (7.62)$$

We have used the formula

$$[\delta(\bar{\beta}_{\phi}), \phi] = -\mathcal{F}_{\phi} \delta'(\bar{\beta}_{\phi}) + \frac{1}{2} [\mathcal{F}_{\phi}, \bar{\beta}_{\phi}] \delta''(\bar{\beta}_{\phi}) \quad (7.63)$$

Moving L_{ψ_n} and $\mathcal{F}_{\phi_s}$ towards V and using Eqs. (7.55), (7.56) and (7.57) and Eqs. (7.47) to (7.49) we find that

$$\begin{aligned}
 u \phi &= \sum_s \left(\frac{\partial}{\partial \Theta_s} - d_s \right) \left\{ V \delta(\bar{\beta}_{\phi_1}) \dots \delta'(\bar{\beta}_{\phi_s}) \dots \right. \\
 &\delta(\bar{\beta}_{\phi_{g_2+2N+M}}) \prod_{\nu} \bar{\beta}_{\psi_{\nu}} (-1)^{g_1+2N+M} \left. \right\} \\
 &+ \sum_n \left(\frac{\partial}{\partial \sigma_n} - c_n \right) \left\{ V \left\{ \prod_s \delta(\bar{\beta}_{\phi_s}) \right\} \bar{\beta}_{\psi_1} \dots \bar{\beta}_{\psi_n} \right. \\
 &\dots \bar{\beta}_{\psi_{g_1+2N+M}} (-1)^{g_1+2N+M-n} \left. \right\}
 \end{aligned} \quad (7.64)$$

Using this relatively simple result, it is straightforward to find the first order differential equations for the measure implied by decoupling. The scattering amplitude

$$W = \int \prod_{\hat{\alpha}} d\sigma_{\hat{\alpha}} \prod_s d\Theta_s u \bar{F} \quad (7.65)$$

and $WQ = 0$ implies that

$$\frac{\partial \ln \bar{f}}{\partial \psi_n} + c_n = 0, \quad \frac{\partial \ln \bar{f}}{\partial \Theta_s} + d_s = 0 \quad (7.66)$$

As discussed above, the scale of the vertex V is not fixed by the overlap equations and can be chosen arbitrarily. This, however, affects the constants c_r and d_s and the final result is independent of the original choice after multiplication by the measure. We can scale the vertex by $\bar{f}$ and then $c_n = d_s = 0$ and the "measure" is one.

The ghost number of $\delta(\bar{p}_{\phi})$ is the opposite of the ghost number of $\bar{p}_{\phi}$. Taking the ghost number operator to be

$$N = \left(\sum_{n=-1}^{\infty} \bar{\beta}_n \beta_n - \sum_{n=2}^{\infty} \bar{\beta}_{-n} \beta_n \right) - \sum_{p \geq 0} (\bar{\beta}_{-p} \bar{b}_p + \bar{b}_{-p} \beta_p) \quad (7.67)$$

one can show along the lines of Ref. [14] that $V(v_r, \Theta_s)$ has ghost number $N+M-(g-1)$ and consequently U has ghost number $2N+M$ as it should in order to give a non-zero result when applied to physical states which have ghost number one.

The vertex of Eq. (7.61) can be written as an integral over the vertex

$$V(\psi_r, \Theta_s, \bar{\Psi}_r, k_s) = V(\psi_r, \Theta_s) \exp \sum_s \bar{\Psi}_s \bar{\beta}_{\psi_s} + \sum_r k_r \bar{b}_{\phi_r}. \quad (7.68)$$

One can repeat the above steps along the lines of Eq. (69) of Ref. [27] for the vertex in the form in which the antighosts are part of the group.

It is appropriate at this point to make a comment on the general procedure for determining the measure. The decoupling of zero-norm physical states implies a first order differential equation for the measure as derived above. Having found the measure, one can then substitute it into the boundary term which was discarded in this process and examine if the result vanishes or not. Indeed, in general it does not; for example in the open bosonic string it vanishes only for the gauge group $SO(2^{13})$ [10,50]. This failure represents a BRST anomaly which one may then attempt to cancel. To do this we add counter terms which are necessarily terms at

the boundary and cancel the anomaly when Q is applied to them. This procedure is similar to that in field theory where one attempts to cancel any anomaly by adding finite local counterterms.

CONCLUSION

We have shown how the group theoretic approach to string theory can be applied to compute the superstring scattering S matrix. In particular, it provides a systematic understanding of how the supermoduli arise in the theory as a result of the string vertex carrying an induced representation of the super Virasoro group. One could easily proceed backwards from the results of this paper to work out the shifts on the supercoset induced by the Ramond and Neveu-Schwarz generators F_n and G_r respectively. While those for the Neveu-Schwarz case are those expected from a straightforward extension of the Koba-Nielsen variables to include anticommuting co-ordinates, those for the Ramond generators are rather involved. This results from the fact that the generators $L_0, L_{\pm 1}$ have a straightforward extension to include $G_{\pm \frac{1}{2}}$ while the only extension in the Ramond case which closes to form an algebra involves the entire super Virasoro algebra. Indeed in the computation of the coset transformation, one requires the relation $F_{-1}^2 \approx L_{-2}$ and one eliminates L_{-2} in terms of $L_n, n > 1$ by using an overlap relation. Having found these transformations, one can introduce an appropriate superspace formalism and cast all the results of this paper in it.

Although we have worked with the open superstring the extension to the closed superstring is straightforward, the general theory being the same, only details such as the form of conformal operators being different. One can also compute the scattering of twisted strings using the group theoretic method. In particular, these authors have computed the four vertex for arbitrary excited twisted string scattering.

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APPENDIX

In this appendix we consider in some more detail the vertex V_0 for one Neveu-Schwarz and two Ramond external lines. For the simple cycling transformation of Eq. (2.11), V_0 satisfies the following overlap equations

$$V_0 \left[\frac{1}{\sqrt{z-z_1}} r^{(1)}(z-z_1) - \frac{1}{\sqrt{z-z_2}} H^{(2)}(z-z_2) \right] = 0, \quad (A.1)$$

$$V_0 \left[\frac{1}{\sqrt{z-z_3}} r^{(3)}(z-z_3) - \frac{1}{\sqrt{z-z_1}} r^{(1)}(z-z_1) \right] = 0, \quad (A.2)$$

where z_1, z_2, z_3 are the Koba-Nielsen variables of the three external lines. We shall only consider the piece of the vertex involving the Neveu-Schwarz b_r^μ and Ramond d_n^μ oscillators and still call it V_0 , the α_n^μ piece being as for the bosonic string. We shall suppress the space-time index μ which is irrelevant for the present discussion. The fermionic vertex V_0 can be written as quadratic exponential:

$$V_0 = \langle 0 | \exp \sum_{n,m,r,s} \left[A_{nm}^{31} d_n^{(3)} d_m^{(1)} + A_{nm}^{33} d_n^{(3)} d_m^{(3)} + A_{nm}^{11} d_n^{(1)} d_m^{(1)} \right. \\ \left. + A_{nr}^{32} d_n^{(3)} b_r^{(2)} + A_{nr}^{12} d_n^{(1)} b_r^{(2)} + A_{rs}^{22} b_r^{(2)} b_s^{(2)} \right]. \quad (A.3)$$

The determination of the coefficients $A_{nm}^{31}, A_{nm}^{33}, A_{mn}^{11}$ is based on performing on Eq. (A.2) the operation

$$\frac{1}{2i\pi} \oint_{z=z_3} \frac{dz}{(z-z_3)^n \sqrt{(z-z_3)(z-z_1)}} \left[1 - \frac{1}{2} \frac{z-z_3}{z_1-z_3} + \frac{1}{2} \left(-\frac{1}{2}\right) \frac{1}{2!} \left(\frac{z-z_3}{z_1-z_3} \right)^2 + \dots \right. \\ \left. \dots + (-1)^n \frac{\frac{1}{2}(-\frac{1}{2}) \dots (\frac{1}{2}-n+1)}{n!} \left(\frac{z-z_3}{z_1-z_3} \right)^n \right]. \quad (A.4)$$

Let us first examine the case $n = 0$. Distorting the integration contour, this shows that V_0 must satisfy the following equation:

$$\begin{aligned}
 & V_0 \frac{1}{\sqrt{z_3 - z_1}} \left[d_0^{(3)} + \sum_{m=1}^{\infty} (-1)^m \binom{-\frac{1}{2}}{m} d_m^{(3)} (z_1 - z_3)^{-m} \right] \\
 & + V_0 \frac{1}{\sqrt{z_1 - z_3}} \left[d_0^{(1)} + \sum_{m=1}^{\infty} (-1)^m \binom{-\frac{1}{2}}{m} d_m^{(1)} (z_3 - z_1)^{-m} \right] \quad (A.5) \\
 & + V_0 \times \text{terms involving } b_{r>0}^{(2)} = 0.
 \end{aligned}$$

For general n , the function in brackets in Eq. (A.4) is designed to select on line 3 the creation operator $d_{-n}^{(3)}$ and only annihilation operators. Expanding around $z = z_3$ and around $z = z_1$ gives the following equations for V_0 , which are generalizations of Eq. (A.5) to arbitrary n :

$$\begin{aligned}
 & V_0 \frac{1}{\sqrt{z_3 - z_1}} \left[d_{-n}^{(3)} + \sum_{m=1}^{\infty} d_m^{(3)} \frac{m}{m+n} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{m} (-1)^{n+m} (z_1 - z_3)^{-n-m} \right. \\
 & + V_0 \frac{1}{\sqrt{z_1 - z_3}} \left[d_0^{(1)} (z_1 - z_3)^{-n} (-1)^n \binom{-\frac{1}{2}}{n} + \sum_{m=1}^{\infty} d_m^{(1)} (z_1 - z_3)^{-n-m} d_{n+m} \right] \quad (A.6) \\
 & + V_0 \times \text{terms involving } b_{r>0}^{(2)} = 0.
 \end{aligned}$$

In this equation, α_{nm} is defined by

$$\frac{\sqrt{(1-x)(1-y)}}{1-x-y} = \sum_{n,m \geq 0} \alpha_{nm} x^n y^m \quad (A.7)$$

We can solve the set of Eqs. (A.5) and (A.6) by first choosing a vacuum $\langle 0 |$ which satisfies

$$\langle 0 | \left(\frac{d_0^{(3)}}{\sqrt{z_3 - z_1}} + \frac{d_0^{(1)}}{\sqrt{z_1 - z_3}} \right) = 0 \quad (A.8)$$

As the operator in this equation squares to zero, the solution is unique, and essentially identifies the two spinors of the Ramond lines 1 and 3. We furthermore have to use the following rather simple identity

$$\langle 0 | \left[\exp A d_0 d_m \right] d_{-n} = \langle 0 | \left[\exp A d_0 d_m \right] A \left(d_0 + \frac{1}{2} A d_m \right) \quad (A.8)$$

In this equation, the term $\frac{1}{2} A d_m$ of course comes from the fact that d_0 does not square to zero. Armed with these tools, we then find the solution for V_0 :

$$\begin{aligned}
 V_0 = & \langle 0 | \exp \left[d_0^{(3)} \sqrt{\frac{z_3 - z_1}{z_1 - z_3}} \sum_{m=1}^{\infty} (-1)^m \begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} d_m^{(1)} (z_3 - z_1)^{-m} \right. \\
 & + d_0^{(1)} \frac{\sqrt{z_1 - z_3}}{\sqrt{z_3 - z_1}} \sum_{m=1}^{\infty} (-1)^m \begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} d_m^{(3)} (z_1 - z_3)^{-m} \\
 & - \sum_{n,m=1}^{\infty} d_m^{(3)} d_n^{(3)} \frac{m-n}{4(m+n)} (z_1 - z_3)^{-m-n} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} (-1)^{n+m} \\
 & - \sum_{n,m=1}^{\infty} d_m^{(1)} d_n^{(3)} \frac{\sqrt{z_3 - z_1}}{\sqrt{z_1 - z_3}} (z_1 - z_3)^{-n-m} d_{n+m} \\
 & - \sum_{n,m=1}^{\infty} d_m^{(1)} d_n^{(1)} \frac{m-n}{4(m+n)} (z_3 - z_1)^{-m-n} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} (-1)^{n+m} \\
 & \left. + \text{terms involving } b_{2>0}^{(2)} \right]. \tag{A.9}
 \end{aligned}$$

We notice that the coefficients of these various terms have the correct antisymmetry properties. For this, Eq. (A.8) is crucial, a fact also noted by A. Jevicki (private communication). The terms involving $b_r^{(2)} d_n^{(3)}$ and $b_r^{(2)} d_n^{(1)}$ are obtained by expanding the integrand of Eq. (A.4) around $z = z_2$, and are not particularly illuminating. The coefficients α_{nm} of Eq. (A.7) themselves are quite complicated; for example

$$\begin{aligned}
 \alpha_{n1} &= (-1)^n \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \cdots \left(-\frac{1}{2}-n+1\right)}{n!} \left(-\frac{1}{4}-n\right), \\
 \alpha_{n2} &= (-1)^n \frac{\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) \cdots \left(-\frac{1}{2}-n+1\right)}{n!} \times \frac{32n^2 + 40n + 9}{32}. \tag{A.10}
 \end{aligned}$$

On this example, we see once more the complications which the presence of external Ramond lines produces on the explicit form of the vertex, even for the simple cycling of Eq. (2.11). This complication is of course due to the presence of the square roots, in Eq. (A.7) for example, made necessary by the mode structure of the Ramond field. Fortunately, as noted repeatedly in this paper, all properties of the vertex can be obtained from the overlaps it obeys.

For completeness, we give the NS-R-R vertex V_0 when the cycling transformation is the same as for the original Caneschi-Schwimmer-Veneziano vertex, i.e., $(T_j)^{-1}(z) = (V_{j-1})^{-1}(V_j)(z) = (1-z)^{-1}$ for $j = 1, 2, 3$ [37]. It satisfies the overlaps

$$V_0 \left[\frac{\Gamma^{(3)}(z)}{\sqrt{z}} + \frac{1}{\sqrt{1-z}} \Gamma^{(1)}\left(\frac{1}{1-z}\right) \right] = 0, \quad (\text{A.11})$$

$$V_0 \left[\frac{\Gamma^{(3)}(z)}{\sqrt{z}} + \frac{1}{\sqrt{z(z-1)}} H^{(2)}\left(\frac{z-1}{z}\right) \right] = 0. \quad (\text{A.12})$$

To obtain the oscillator form, one performs on (A.11) the operation

$$\frac{1}{2i\pi} \oint_{|z|=1} \frac{dz}{z^{n+\frac{1}{2}}}, \quad (\text{A.13})$$

and on (A.12) the operation

$$\frac{1}{2i\pi} \oint_{|z|=1} \frac{dz}{z^2} \times \left(\frac{z-1}{z}\right)^{-n-\frac{1}{2}} z^{\frac{1}{2}} \left[1 - \frac{1}{z} \frac{z-1}{z} + \dots + (-1)^{n-\frac{1}{2}} \frac{\left(\frac{1}{2}\right) \dots \left(\frac{1}{2}-n+\frac{3}{2}\right)}{(n-\frac{1}{2})!} \left(\frac{z-1}{z}\right)^{n-\frac{1}{2}} \right] \quad (\text{A.14})$$

The answer is built on the vacuum $\langle 0|$ defined by (A.8) and writes

$$\begin{aligned} V_0 = & \langle 0| \exp \sum_{\substack{n=\frac{1}{2} \\ n=0}}^{\infty} (-1)^{n-\frac{1}{2}} \binom{n-\frac{1}{2}}{n-\frac{1}{2}} d_n^{(3)} b_n^{(2)} \\ & - i \sum_{n,m=0}^{\infty} (-1)^m \binom{-n-\frac{1}{2}}{m-n} d_m^{(3)} d_n^{(1)} \\ & - \frac{1}{4} \sum_{n,m=1}^{\infty} d_m^{(1)} d_n^{(1)} \frac{m-n}{m+n} (-1)^{n+m} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{m} \\ & - \sum_{n=1}^{\infty} d_n^{(1)} d_0^{(1)} (-1)^n \binom{-\frac{1}{2}}{n} \\ & + \frac{i}{2} \sum_{\substack{n=\frac{1}{2} \\ n=1}}^{\infty} d_n^{(1)} b_n^{(2)} (-1)^{n+\frac{1}{2}} \frac{n+n}{n-n} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{n-\frac{1}{2}} \\ & - i \sum_{n=\frac{1}{2}}^{\infty} d_0^{(1)} b_n^{(2)} (-1)^{n-\frac{1}{2}} \binom{-\frac{1}{2}}{n-\frac{1}{2}} \\ & - \frac{1}{4} \sum_{n,s=\frac{1}{2}}^{\infty} b_s^{(2)} b_n^{(2)} \frac{s-n}{s+n} (-1)^{n+s-1} \binom{-\frac{1}{2}}{s-\frac{1}{2}} \binom{-\frac{1}{2}}{n-\frac{1}{2}}. \end{aligned} \quad (\text{A.15})$$

We note that this vertex can already be found in Ref. [30], although not in exactly this form.

In Ref. [32] some interesting identities which were derived played an important role in the development of superstring theory. These identities concerned the transition and reflection of super Virasoro operators by the fermion emission vertex [39]. It is appropriate to show here how these identities may be obtained in a rather simple way using the relevant overlap equation. Using the cycling $(T_j)^{-1}(z) = (1-z)^{-1}$ for $j = 1, 2, 3$ given above, we find that the tree vertex [30] $V^{(3)}$, derived above and for which lines 3 and 2 are fermionic and line 1 bosonic the overlap equation.

$$V^{(3)} \left[\oint_{z=0} \frac{dz}{z} \varphi(z) F^{(3)}(z) - \oint_{z=1} \frac{dz}{z} \varphi \frac{1}{(z-1)^{3/2}} F^{(2)}\left(\frac{z-1}{z}\right) + \oint_{z=\infty} \frac{dz}{z} \varphi \frac{z^{3/2}}{(1-z)^{3/2}} G^{(1)}\left(\frac{1}{1-z}\right) \right] = 0 \quad (\text{A.16})$$

For an explicit derivation, see Eq. (26) of Ref. [37]. To make contact with the vertex used in Ref. [32], we twist on line 2, namely:

$$\bar{V}^{(3)} = V^{(3)} e^{-L_2^{(2)}} \exp i\pi \left[L_0^{(2)} - \frac{1}{2} p_r^{(2)2} \right] \quad (\text{A.17})$$

Carrying out this transformation, Eq. (A.16) becomes

$$\bar{V}^{(3)} \left[\oint_{z=0} \frac{dz}{z} \varphi(z) F^{(3)}(z) + \oint_{z=1} \frac{dz}{z} \frac{z^{3/2}}{(z-1)^{3/2}} \varphi F^{(2)}(1-z) + \oint_{z=\infty} \frac{dz}{z} \varphi \frac{z^{3/2}}{(1-z)^{3/2}} G^{(1)}\left(\frac{1}{1-z}\right) \right] = 0 \quad (\text{A.18})$$

We find the fermion emission vertex by taking on line 3 the ground state which is a massless spin $\frac{1}{2}$ denoted by $|k\rangle$ where $F_n |k\rangle = L_n |k\rangle = 0$, $n > 0$. One defines

$$\bar{V}^{(3)} |k\rangle_3 \equiv W \quad (A.19)$$

Provided ϕ is an analytic function at $z = 0$ then putting $|k\rangle_3$ on Eq. (4.18) gives no contribution. Carrying this step out, making the change of variable $\xi = (1-z)^{-1}$, rescaling ϕ and taking the contour around $\xi = \infty$ to be around $\xi = 0$ for $|\xi| \gg 1$ we can obtain the result

$$W \left[\oint_{\substack{\xi=0 \\ |\xi|>1}} \frac{d\xi}{\xi} \phi \left(1 - \frac{1}{\xi}\right)^{1/2} F^{(2)}\left(\frac{1}{\xi}\right) + \oint_{\substack{\xi=0 \\ |\xi|<1}} \frac{d\xi}{\xi} \phi \left(1 - \frac{1}{\xi}\right)^{1/2} G^{(1)}(\xi) \right] = 0 \quad (A.20)$$

Identifying the lines 2 and 1 in the appropriate way, we recover the identity of Ref. [32].

We observe that ϕ had to be analytic at $z = 0$, i.e., at $\xi = 1$, but it can have a pole at $\xi = 0$ or $\xi = \infty$ which are the Koba-Nielsen points corresponding to strings 2 and 1. This agrees with the restriction on ϕ found in Ref. [32]. We note, however, that it is a consequence of demanding that the identity gives no contribution when one line is taken to be a ground state.

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Fig. 1: The Schottky representation of a Riemann surface

Fig. 2: Map from strip to the upper half plane

Fig. 3: Mapping the torus to the rectangle