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STABILITY OF PARTICLE ORBITS

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ABSTRACT. The correspondence between the stability of particle orbits in synchrotrons and storage rings and the standard maps of modern nonlinear dynamics is discussed. The problem is exposed in the language of particle accelerator physics. Examples are given how the equations for synchrotron and betatron oscillations can be brought into the form of standard maps.

1 Introduction

This paper discusses the relation between the stability of particle orbits in synchrotrons and storage rings and the standard maps of modern nonlinear dynamics. Chapter 2 exposes the problem of particle orbit stability in terms of nonlinear oscillations in the vicinity of a closed reference trajectory, using the language of particle accelerator physics (Courant and Snyder 1958, Steffen 1965, Bruck 1966, Kolomensky and Lebedev 1966, Blewett 1977). Chapter 3 gives analytical and simulation results for synchrotron oscillations, shows that for special cases the equations of motion can be brought into the form of Chirikov's standard map, and compares the early results for the stability limit to recent ones for the standard map. Chapter 4 shows how the equations for betatron oscillations can be brought into the form of the Hénon or Hénon and Heiles maps in special cases which, however, are too unrealistic to be quantitatively useful for modern synchrotrons and storage rings.

2 Synchrotrons and Storage Rings

The particle motion in a synchrotron or storage ring, a "machine", is mathematically described in a curvilinear coordinate system. The independent variable is the pathlength s along the reference trajectory which passes through the centres of all elements in an ideal machine without alignment and excitation errors. The transverse coordinates x and y have their origin on the reference trajectory. In a planar machine x is in the plane of the machine, and y is perpendicular to it.

Particles perform oscillations in three degrees of freedom with six dependent canonical variables $x, p_x, y, p_y, \Delta E, \phi$. Oscillations in the variables x and p_x are called horizontal

betatron oscillations. Oscillations in the variables y and p_y are called vertical betatron oscillations. Oscillations in the variables ΔE and ϕ are called synchrotron oscillations. The oscillatory behaviour is achieved by restoring forces acting on the particles in all three degrees of freedom. In a real machine with errors, all three degrees of freedom are coupled. To simplify the presentation, the three degrees of freedom are treated separately in this paper.

2.1 BETATRON (TRANSVERSE) OSCILLATIONS

Betatron oscillations are caused and controlled by magnets. In modern machines, the functions of bending and focusing are separated. In earlier machines, combined-function magnets, i.e. dipoles with quadrupole and sometimes sextupoles components were bending and focusing at the same time. The approximately circular orbit is obtained by bending the beam with homogeneous-field dipoles. Bending radius ρ , particle momentum p and dipole field B are related by $\rho[m] = 3.34p[GeV/c]/B[T]$.

Focusing is achieved by using alternating gradients in the quadrupoles, in contrast to weak, constant-gradient focusing in betatrons and classical cyclotrons. Quadrupoles focus in one transverse direction, and defocus in the other. The change in p_x is proportional to the offset x , and the change in p_y is proportional to the offset y , with equal and opposite proportionality constants in the two directions. Therefore, the signs of the gradient in successive quadrupoles must alternate in order to achieve focusing in both directions. The quadrupole strength is described by the normalized gradient $K = (dB_y/dx)/(B\rho)$. The quantity $B\rho$ is called the magnetic rigidity of the particle. The gaps between the dipole, quadrupole, etc. magnets are called straight sections. In a synchrotron consisting only of straight sections, ideal dipoles and ideal quadrupoles, and with small betatron amplitudes, the betatron oscillations can be approximated by the following linear equations of motion:

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2(s)} + K(s) \right] x = \frac{1}{\rho(s)} \frac{\Delta p}{p} \quad \frac{d^2y}{ds^2} - K(s)y = 0 \quad (1)$$

Here, $\Delta p/p$ is the relative momentum error, trivially related to ΔE , and $K(s)$ is periodic with the circumference C , i.e. $K(s+C) = K(s)$. The linear map for one turn, which transforms the vector (x_n, x'_n) at the beginning of the n th turn into the vector (x_{n+1}, x'_{n+1}) at the beginning of the $(n+1)$ th turn, can be written in matrix form:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\gamma \sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} \quad (2)$$

A similar matrix operates on the vector (y_n, y'_n) . The slopes x' and y' are trivially related to the transverse momenta p_x and p_y . The reduced instantaneous betatron wavelength is called $\beta(s)$, $\alpha(s)$ is related to the derivative of β with respect to s , $\alpha = -(d\beta/ds)/2$, and $\gamma(s)$ is shorthand for $(1 + \alpha^2)/\beta$. The tune Q can be much larger than one in alternating-gradient machines. This is in contrast to constant-gradient machines for which $0 < Q < 1$. The map for m turns can be obtained from the above matrix by changing the argument of the trigonometric functions to $2\pi mQ$. Hence, if Q is real and not an integer, the linear motion is bounded for $m \rightarrow \infty$.

The linear equations of transverse motion are not good enough for modern machines, because nonlinearities are present in them both intentionally and unintentionally (Peggs and Talman 1986). The main sources of nonlinearities are (i) sextupoles, (ii) field errors, and (iii) the beam-beam effect.

Intentionally installed sextupoles are the predominant source of nonlinearities in most existing machines. They combat chromatic effects of quadrupoles, arising from the simple fact that a quadrupole focuses a particle with a higher momentum less than one with a lower momentum. These chromatic effects are unacceptable for two reasons: (i) The momentum spread in beam causes a spread of tunes in proportion to the chromaticity. This tune spread must be smaller than the spacing between resonances which will be discussed in Section 4.3. Hence, the chromaticity must be smaller than some limit, usually less than natural value. This effect becomes more pronounced with increasing machine tune Q . (ii) The chromaticity must be positive to suppress the head-tail effect – a particular current dependent effect which would limit the beam current far below the design value (Sacherer 1973, Sacherer 1977).

Practical dipoles, quadrupoles, etc., have unintentional field errors. Their effect was of much concern for early alternating-gradient synchrotrons in the late 1950's. It later was of less concern because the sextupoles which had to be installed to combat the head-tail effect were much stronger sources of nonlinearities. More recently, the effects of field errors cause much concern for large machines with superconducting magnets of small aperture. The most appropriate way of describing the field of dipoles, quadrupoles, sextupoles, and higher-order multipoles is by a multipole expansion:

$$B_x + iB_y = B_0 \sum_1^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r} \right)^{n-1} \quad (3)$$

Here B_0 is a scaling magnetic field, e.g. the field on the reference orbit in a dipole, the a_n are coefficients describing the magnitude of the skew multipoles with poles in the median plane, the b_n are coefficients describing the magnitude of the regular multipoles with gaps in the median plane, and r defines the transverse scale. A typical value is $r = 1$ cm. In this notation, $b_1 = 1$ is the regular dipole component, b_2 is the regular quadrupole component, b_3 is the regular sextupole component, and b_6 would be the regular duodecapole component. In the US, the multipole coefficients are labelled differently (Jackson 1986). Multipole fields with $n \geq 1$ increase with x and y .

In storage rings with colliding beams of e^+ and e^- , or $\bar{p}$ and p , the two beams exert the beam-beam force on each other, a nonlinear electromagnetic force arising from the smooth charge and current distributions in the beams. This beam-beam effect is distinct from the violent collisions between the particles in the beams. Their observation is the whole purpose of colliding-beam storage rings. For two round beams with Gaussian density distributions and unity rms radius, the radial restoring force drops like r^{-1} for $r \gg 1$, i.e. decreases with r unlike the multipole forces, and is given by:

$$F_y + iF_x \propto r^{-1} \left[1 - \exp \left(-\frac{1}{2}r^2 \right) \right] \quad (4)$$

The consequences of these nonlinearities are discussed in Chapter 4.

2.2 ENERGY (SYNCHROTRON) OSCILLATIONS

Energy or synchrotron oscillations are caused and controlled by radio-frequency (RF) cavities which are needed in synchrotrons and storage rings in order to accelerate the particle by electric fields for one of the following reasons: (i) in synchrotrons to gain energy, (ii) in e^+e^- storage rings to compensate energy losses due to synchrotron radiation, (iii) in $\bar{p}p$ storage rings to prevent the particles spreading around the circumference due to their momentum spread, but rather to remain “bunched”. The electric fields needed for the acceleration are produced in RF cavities at typical frequencies of a few hundred MHz, and with peak fields of the order of MV/m.

If one specializes to one RF cavity along the circumference with circular frequency ω and a voltage

$$V(t) = V_{RF} \sin \omega t \quad (5)$$

the difference equations of synchrotron motion for energy error ΔE and phase ϕ are:

$$\Delta E_{n+1} - \Delta E_n = eV_{RF}(\sin \phi_n - \sin \phi_s) \quad \phi_{n+1} - \phi_n = -\frac{2\pi h\eta}{\beta^2} \frac{\Delta E_{n+1}}{E_s} \quad (6)$$

Here $h = \omega/\omega_s$ is the harmonic number, i.e. the ratio of RF frequency ω and revolution frequency ω_s , η is the momentum compaction, a measure of the relative change of the particle revolution frequency with the relative momentum error $\delta p/p$, β is the particle velocity measured in units of the speed of light, E_s is the total energy of the synchronous particle which has $\Delta E_s = 0$, and ϕ_s is the phase of the elliptic fixed point, and hence the constant phase of the synchronous particle which remains at the elliptic fixed point. In the standard treatment of synchrotron oscillations, the difference equations above are transformed into differential equations, by distributing the voltage V_{RF} around the circumference and dividing by the revolution period $T = C/v$, where v is the relativistic particle velocity. The differential equations are:

$$\frac{d}{dt} \left(\frac{\Delta E}{\omega_s} \right) = \frac{eV_{RF}}{2\pi} (\sin \phi - \sin \phi_s) \quad \frac{d\phi}{dt} = -\frac{h\eta\omega_s}{\beta^2 E_s} \Delta E \quad (7)$$

These equations can be derived from a time-independent Hamiltonian which therefore is an exact integral of the motion. They have two fixed points. For $\eta > 0$, the elliptic fixed point is in the range $0 \leq \phi_s < \pi/2$, and the hyperbolic one at $\pi - \phi_s$. For $\eta < 0$, the two fixed points are interchanged. The border of stability is given by the separatrix passing through the hyperbolic fixed point.

In the language of accelerator physics, the stable area inside the separatrix is called a bucket. Synchrotron oscillations are called librations in modern nonlinear dynamics. The rotations of nonlinear dynamics were used to accumulate the beams in the $\bar{p}p$ storage rings ISR at CERN (Keil 1975) by phase-displacement acceleration. The beam occupies the area inside the bucket and oscillates around the elliptic fixed point. It is called a bunch. The presence of the buckets excited by the RF system prevents the beam from spreading around the circumference due to the finite energy spread. Without an RF system, the energy error ΔE would be constant, and the phases would vary linearly as implied by the difference equation for the phases. The shape of the bucket for $\phi_s = \pi$ is shown in Figure 1, its half

height in the ΔE direction is given by:

$$\left(\frac{\Delta E}{E_s}\right)_b = \sqrt{\frac{\beta^2 e V_{RF}}{\pi h \eta E_s} [(\pi - 2\phi_s) \sin \phi_s - 2 \cos \phi_s]} \quad (8)$$

The synchrotron tune Q_s , i.e. the number of synchrotron oscillations in a turn, is obtained by linearizing the equations in the vicinity of ϕ_s :

$$Q_s = \sqrt{\frac{h \eta \cos \phi_s e V_{RF}}{2\pi \beta^2 E_s}} \quad (9)$$

3 Stability of Synchrotron Oscillations

3.1 SIMULATION OF SYNCHROTRON OSCILLATIONS

The first simulation of the difference equations for synchrotron oscillations (Symon, Steben and Laslett 1966) was done for much higher values of Q_s than used in practice then and now. The results look quite different from the results for synchrotron oscillations described by differential equations. In Figure 2, islands appear inside the bucket, and stochastic layers occur near the separatrix, in contrast to the smooth trajectories shown in Figure 1.

3.2 OVERLAPPING OF RESONANCES

An RF system running at harmonic number h has h stable buckets along the circumference C of the machine, at the energy E_s . At energy errors $\pm(\Delta E/E_s)_h$ there are other strings with $h \pm 1$ buckets, etc. Imposing the condition that the buckets from different harmonic numbers do not overlap (Chirikov 1979), i.e. that the momentum difference be larger than twice the half height of the bucket $(\Delta E/E_s)_b$, leads to a condition which by simple algebra can be transformed into an upper limit $Q_s \leq 1/4$:

$$\left(\frac{\Delta E}{E_s}\right)_h = \frac{\beta^2}{h|\eta|} \geq 2 \left(\frac{\Delta E}{E_s}\right)_b \quad (10)$$

3.3 RELATION TO STANDARD MAP

It was noticed by Dôme 1984 that in the case $\phi_s = \pi$, the difference equations for the synchrotron oscillations can be brought into the form of the standard map (Chirikov 1979). The operations needed are scaling from ΔE to I and introducing the abbreviation K , which should not be confused with the strength of the quadrupoles.

$$I = \frac{2\pi h \eta}{\beta^2} \frac{\Delta E}{E_s} \quad K = \frac{2\pi h \eta e V_{RF}}{\beta^2 E_s} = (2\pi Q_s)^2 \quad (11)$$

The equations for the synchrotron oscillations now have the form of the standard map:

$$I_{n+1} - I_n = K \sin \phi_n \quad \phi_{n+1} - \phi_n = I_{n+1} \quad (12)$$

The standard map has been studied in great detail as a function of the stochasticity parameter K . The transition between local stochasticity, with the stochastic regions closely

Figure 1: Stable orbits and separatrix for $\phi_s = \pi$. Phase ϕ in units of 2π , energy error ε in units of bucket height ΔE_b .

bounded by KAM surfaces, and global stochasticity was determined by Chirikov 1979 who gave $K \approx 0.989$, while Greene 1979 found $K = 0.9716$, which corresponds to $Q_s = 0.1502$. This figure is about 3/5 of the figure obtained by the simple overlap criterion. This limit for the synchrotron tune Q_s has not been tested in practice, since all existing machines have smaller values of Q_s and more than one RF system: The e^+e^- colliding-beam storage ring LEP under construction at CERN (Wyss 1987) has $Q_s \approx 0.1$ and 2 RF systems. Since in comparisons with the stochasticity limit, Q_s must be divided by the number of RF systems, LEP will operate about a factor of three below the stochasticity limit. Furthermore, the height of the bunches is about an order of magnitude smaller than the height of the buckets. Therefore, the particles will reach the neighbourhood of the separatrix only about once per day, due to the stochastic excitation of their synchrotron oscillations by the random process of emitting synchrotron radiation.

4 Stability of Betatron Oscillations

The stability of betatron oscillations is under active study in the machines listed in Table 1. The SppS (Evans 1987), LHC (Brianti and Hübner 1987) and SSC (Jackson 1986) are hadron machines, LEP (Wyss 1987) is an e^+e^- machine. The stochastic excitation of the oscillations in all three degrees of freedom and their damping by synchrotron radiation are very pronounced in LEP, and almost negligible in the other machines. The SppS is

Figure 2: Synchrotron Oscillations for $Q_s = 0.14, \phi_s = \pi$. Phase ϕ in units of 2π , energy error ε in units of bucket height ΔE_b .

in operation, LEP is under construction, LHC and SSC are being studied. The required beam lifetime in all machines is of the order of a day which corresponds to the number of revolutions shown in Table 1. In the SppS it takes about a day to produce the required number of $\bar{p}$'s. In the other machines a much smaller beam lifetime would also be quite inconvenient.

Machine	SppS	LEP	LHC	SSC	
Particles	$\bar{p}p$	e^+e^-	pp	pp	
Beam Energy	0.315	0.055	8	20	TeV
Circumference	6911	26659	26659	82944	m
Revolution period	23.1	88.9	88.9	276.7	μs
Revolutions	3.7×10^9	9.7×10^8	9.7×10^8	3.1×10^8	d^{-1}
Damping rate	7.2×10^{-9}	26.9	8.4×10^{-6}	1.1×10^{-5}	s^{-1}

Table 1: Characteristic Features of Modern Machines

The beams in all these machines have a small but finite size, because their betatron and synchrotron amplitudes are small but finite. There are two reasons for this: (i) Ion sources and electron guns do not deliver beams smaller than some limit, in particular at high intensity, (ii) too small beams cannot be used because the internal forces due to their own space charge and the beam-beam effect would be too strong. Therefore,

all existing machines have been, and all future machines will be designed with adequate physical aperture, which is typically about 10 rms beam radii, and of the order of 5 cm by 5 inches. The dynamic aperture is that set of initial conditions in amplitude I_{m0} and phase ϕ_{m0} for which:

$$|I_m(t)| \leq M_m \quad \text{for} \quad 0 \leq t \leq T \quad (13)$$

where M_m is related to physical aperture, m labels the degrees of freedom, and T is of the order of the required beam lifetime. In the older machines the dynamic aperture was larger than the physical aperture. This is no longer the case in modern machines. Therefore the criterion for the choice of the aperture changes. The dynamic aperture must be adequate for the beam, and the physical aperture and other machine parameters, e.g. tolerances on multipole fields, must be chosen such that the dynamic aperture needed is actually achieved.

4.1 EXPERIMENTS IN EXISTING MACHINES

In the context of the design studies for the LHC and the SSC, dedicated experiments have been performed, and are still continuing, in the SPS (Evans 1988) and in the Tevatron (Chao 1988). The essential ingredients of these experiments are: (i) Deliberate nonlinearities were excited in order to reduce the dynamic aperture such that it is smaller than the physical aperture of the SPS and Tevatron. (ii) Experimental observations were carefully compared with computer simulations, in order to find to what extent they agree, and therefore to check the reliability of computer simulations for machines which do not yet exist. (iii) Several days of dedicated machine time were invested in these experiments. (iv) Special diagnostic equipment was built. In the SPS experiment, the measured dynamic aperture was about a factor of two below that found by tracking over a few thousand turns. In the Tevatron experiment, good agreement was obtained between observation and simulation of the phase space distortion for smaller sextupole excitations, while discrepancies are found at higher sextupole excitation. The measured dynamic aperture was smaller than that found by simulation over a few hundred turns. Experiments are very fast compared to computer simulation: In the SPS, 4×10^{15} particle $\times$ turns occur in a second, while computer simulation takes typically a fraction of a second per turn for a few particles. However, experiments cannot observe individual particles, but only first and second moments of the particle distribution functions.

4.2 COMPUTER SIMULATION

Computer simulations of the beam-beam effect in $e^+ e^-$ storage rings (Keil 1981, Myers 1981, Myers 1983, Peggs and Talman 1981, Piwinski 1981) are in agreement with observations in existing machines. The sextupole effects in LEP (Turner 1984) have also been simulated. The simulation can last for several damping times during which the particles "forget" their initial conditions due to the random excitation of their betatron and synchrotron oscillations by synchrotron radiation. Phenomena causing an amplitude growth with time scales long compared to the damping time are unimportant. Computer simulation has also been used for the study of the multipole effects in LHC (Brianti and Hübner 1987) and SSC (Jackson 1986). In these machines, the damping times are of the order

of the required beam lifetimes, and phenomena causing amplitude growth over this time scale, e.g. Arnold diffusion, are important. However, computer simulation over time scales beyond seconds or minutes in a storage ring is impossible. Keil 1986 quotes a conjecture by Dragt: If one can establish by computer simulation over $n \approx 10^6$ turns that the “spread” in the particle orbits grows by diffusion, i.e. behaves like $\sqrt{n}$, and if the observed spread is ≈ 0.01 in relative units, then one may extrapolate by another two orders of magnitude to 10^8 turns, and expect the spread to be of the order of 0.1 which would be small enough to be useful for LHC and SSC. Dragt 1984 describes Lie-algebraic techniques for observing diffusion.

4.2.1 Horizontal Betatron Oscillations with One Sextupole. A crude model of an actual machine is a periodic machine consisting of FODO cells (Keil 1977) with one infinitesimally short sextupole of finite integrated normalized strength $L_s K_s = L_s (d^2 B_y / dx^2) / B\rho$ in each cell, installed at a symmetry point such that α vanishes. The map for the horizontal betatron oscillations through such a cell is given by:

$$X_{n+1} = X_n \cos \mu + X'_n \beta \sin \mu + \frac{1}{2} L_s K_s \beta X_n^2 \sin \mu \quad (14)$$

$$X'_{n+1} = -\frac{X_n}{\beta} \sin \mu + X'_n \cos \mu + \frac{1}{2} L_s K_s X_n^2 \cos \mu \quad (15)$$

Here, μ is the phase advance through the cell, and β is taken at the sextupole. The transformations (Michelotti 1986, Peggs and Talman 1986)

$$\psi = 2\pi - \mu \quad x = \frac{1}{2} L_s K_s \beta X \quad y = \frac{1}{2} L_s K_s \beta^2 X' \quad (16)$$

bring the cell map into the standard form studied by Hénon 1969:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \cos \psi - (y_n - x_n^2) \sin \psi \\ x_n \sin \psi + (y_n - x_n^2) \cos \psi \end{pmatrix} \quad (17)$$

Figure 3 shows the maximum initial amplitude a as a function of ψ in the interval $0.025(0.025)0.4975$. Particles were launched at $a = 2(-.025)0.025$, and the largest a for which the elongation $|x| < 4$ for 10000 turns is shown. By scaling with the machine-dependent parameters $L_s K_s$ and β , it is possible to reduce the three-parameter cell map to the one-parameter Hénon map, and to use his numerical results to determine the maximum amplitude X in a specific machine. In LHC, the strongest sextupoles have $L_s K_s \approx 0.17 \text{ m}^{-2}$ and $\beta = 169 \text{ m}$. Reading $x \approx 0.8$ from Figure 3, one finds $X \approx 55 \text{ mm}$, which is a factor of four to five above the result of computer simulation (Brianti and Hübner 1987). The Hénon map has dips in the stability limit due to third and fourth order resonances when the phase advance per cell becomes $\psi/2\pi = 1/4$ or $\psi/2\pi = 1/3$. A real machine like the LHC is not simply periodic. Therefore, such resonances may occur there when the fractional part of the tune Q_x is a simple rational fraction like $1/3$ or $1/4$. Since the number of cells N_c in the LHC is a few hundred, and since tune and phase advance are related by $Q \approx N_c \mu / 2\pi$, this happens in the LHC at much smaller intervals in μ than in the Hénon map.

Periodic cells containing more than one sextupole cannot be described exactly by quadratic maps. With two sextupoles in a periodic cell, the exact map is of fourth order. If the terms of third and fourth order are left out, the remaining quadratic map does

Figure 3: Maximum Stable Initial Amplitude for the Hénon Map.

not preserve area. It might be possible to make it area-preserving by slight modifications of its coefficients, without loss of significant information.

4.2.2 Periodic Machine with Averaged Forces. Lee and Tepikian 1985 have considered a periodic machine in which the variations of the focusing due to the quadrupoles and of the nonlinearities due to the sextupoles along the circumference are ignored and their average values are taken. The Hamiltonian then becomes independent of the azimuth and takes the form:

$$H = \frac{1}{2} (X'^2 + Y'^2) + \frac{1}{2} \langle K \rangle (X^2 + Y^2) + \langle K_s \rangle \left(XY^2 - \frac{1}{3} X^3 \right) \quad (18)$$

Here, the coefficient $\langle K \rangle$ is related to the betatron wavelength λ by $\lambda/2\pi = 1/\sqrt{\langle K \rangle}$ and the coefficient $\langle K_s \rangle$ is related to the sextupole strength. The transformations

$$x = \langle K_s \rangle X / \langle K \rangle \quad y = \langle K_s \rangle Y / \langle K \rangle \quad p_x = \langle K_s \rangle X' / \langle K \rangle^{3/2} \quad p_y = \langle K_s \rangle Y' / \langle K \rangle^{3/2} \quad (19)$$

bring the Hamiltonian into the standard form studied by Hénon and Heiles 1964:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + \left(x^2 y - \frac{1}{3} y^3 \right) \quad (20)$$

The machine physics parameters $\langle K \rangle$ and $\langle K_s \rangle$ appear only in the scaling to the Hénon and Heiles map which has no parameters at all. Stochastic layers appear in the map when $\frac{1}{12} < H < \frac{1}{8}$. Lee and Tepikian take the lower figure to arrive at the dynamic aperture in a

Figure 4: Comparison of Methods for Obtaining the Dynamic Aperture. Full line computer simulation, dotted line Lee and Tepikian, dashed line Hagel and Moshhammer.

machine with the sextupole strengths adjusted such that the chromaticity is compensated. Their results are included in Figure 4.

4.3 ANALYTICAL TECHNIQUES

4.3.1 Perturbation Theory. First-order perturbation theory was used to study the effects of nonlinear resonances (Schoch 1957, Guignard 1976) which occur when the tunes Q_x, Q_y, Q_s are related by:

$$n_x Q_x + n_y Q_y + n_s Q_s = p \quad (21)$$

where n_x, n_y, n_s, p are integers. Resonances with $n_s = 0$ are betatron resonances, resonances with $n_s \neq 0$ are synchro-betatron resonances. The stopband width of the betatron resonances is proportional to the driving term in the magnetic field, and varies with the betatron amplitude a like a^{n-2} . Their average distance from the working point Q_x, Q_y drops rapidly with their order $n = |n_x| + |n_y|$: from 10^{-2} for $n = 5$ to 2×10^{-3} for $n = 10$ (Keil and Leroy 1975). As long as these stopband width are small compared to the spacing, the resonances are isolated, and can be avoided by the choice of the working point. At the betatron amplitude where width and spacing become comparable, chaos sets in and an effective upper limit for the amplitude is reached. Conversely, for a given amplitude the overlap yields a tolerance for the nonlinear driving terms in the magnetic field. In order to

avoid isolated resonances, the working point has to be stable to better than, and the tune spreads in the beam have to be smaller than the resonance spacing.

When the synchrotron motion is also involved, e.g. because of a small residual chromaticity, the betatron resonances acquire sidebands at distances $\pm n_s Q_s$. Since the total width of all sidebands is larger than the width of the betatron resonance, a smaller amplitude limit arises. Similar things happen when the tunes Q_x and Q_y are modulated because of ripple in the power supplies. In the SppS, resonances of order 10 reduce the beam lifetime.

Second-order perturbation theory has been used to compute the tune shift with amplitude due to sextupoles (Hagedorn 1957), a major contribution to the tune spread in the beam, and to calculate the excitation of fourth-order resonances by sextupoles. Donald 1976 minimizes these terms as well as several other chromatic terms from first-order perturbation theory by having more than two sextupole circuits, and by adjusting their excitation such that the chromaticity always vanishes. Second-order perturbation theory has also been used to analyse the topology of phase space in the neighbourhood of resonances (Michelotti 1987). The orbits obtained are in good but not perfect agreement with computer simulation. The differences can be explained by the fact that simulation includes effects of all orders while perturbation theory does not.

Hagel and Moshhammer 1988 use second-order perturbation theory to find the dynamic aperture. They derive an analytic approximation for the n -th order perturbation contributions to the equations for horizontal and vertical betatron motion by using perturbation results up to second order. Therefore D'Alembert's convergence criterion can directly be applied to the perturbation series. The agreement between the analytical results and computer simulation for the dynamic aperture for LEP in two dimensions is good.

4.3.2 Hamilton-Jacobi Equation. Warnock and Ruth 1987 consider nearly integrable systems. They seek a canonical transformation such that the new Hamiltonian depends only on the new action. The generating function for this transformation is periodic along the circumference of the machine. It is appropriate to expand it into a double Fourier series in azimuth and phase. The Hamilton-Jacobi equation can thus be solved numerically, taking only a finite number of Fourier components. The invariant surfaces are given explicitly, and the orbits can be obtained by solving a nonlinear equation. In one dimension, the convergence of this method is excellent at small amplitudes. The method is effective even on the border of chaotic regions and on the separatrices of isolated broad resonances, and leads to a criterion for the transition to chaos.

4.4 COMPARISON OF METHODS.

A comparison of the dynamic apertures obtained by the various methods for a periodic machine with cells similar to the LEP cells, and sextupoles close to the horizontally focusing and horizontally defocusing quadrupoles is shown in Figure 4. In all cases, the sextupole strengths are adjusted such that the chromaticity is compensated. The tracking results show dips in the dynamic aperture a in the vicinity of third and fourth order resonances. The method of Lee and Tepikian cannot show them because their Hamiltonian is averaged around the circumference. The method of Hagel and Moshhammer is close to the tracking

results in the neighbourhood of these resonances, but fails at lower phase advances where resonances of higher order than four occur.

The standard maps are only rough models of modern machines, and their properties can only be used as an order of magnitude guide for the transition from regular to stochastic motion. However, the underlying principle of reducing the number of parameters to the minimum before nonlinear maps are studied in great detail is also applicable to the maps describing machines. The analytical techniques discussed are promising and deserve being developed into reliable tools which can be applied to a wide variety of machines.

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