CERN-TH.5031/88

SUPER BELTRAMI DIFFERENTIALS VIA 2D SUPERGRAVITY

S. Uehara^{*)}

CERN - Geneva
and
Department of Mathematics
King's College, London WC2R 2LS, U.K.

and

Yukinori Yasui
Department of Physics
Osaka City University
Osaka 558, Japan

ABSTRACT

Super Beltrami equations which describe deformations of super Riemann surfaces are given. Two kinds of super Beltrami differentials of weight $(-\frac{1}{2}, 0)$ and $(-1, \frac{1}{2})$ respectively are introduced, which exactly agrees with 2D supergravity theory. Solutions are given for the linearized Beltrami equations. Mappings between the super Beltrami differentials and super $3/2$ - differentials are given properly.

^{*)}On leave of absence from Research Institute for Theoretical Physics, Hiroshima University, Takehara, Hiroshima 725, Japan.

String and superstring theories have been vigorously investigated from various angles. In some approaches to string theory, the theory of Riemann surfaces has been shown to play an important role [1-6]. Hence the theory of super Riemann surfaces (SRSs) [7] should be useful for attacking many problems in superstring theory.

In this paper we investigate super Teichmüller deformations of SRSs of genus $g \geq 2$ by the use of super Beltrami differentials. Rabin [8] examined the same subject, however, the deformations had to be restricted due to a special gauge choice for his super Beltrami differentials. He investigated deformations of split SRS, however, he was unable to discuss those of the non-split SRS properly. Here we do not choose any particular gauge for our super Beltrami differentials which are different from his, and a marked SRS can be a non-split one. The plan of this paper is the following: we begin with a brief review of SRSs and proceed to super Beltrami equations, where the ordinary Beltrami equations are reviewed in a certain way and super Beltrami equations are obtained through 2D supergravity theory. We give solutions for the linearized super Beltrami equations, and then we give mappings between the super Beltrami differentials and super 3/2-differentials. The final part is devoted to some comments. Detailed analysis will be given in a separate publication [9].

A super Riemann surface is a one-dimensional complex supermanifold with a superconformal structure parametrized by a local complex supercoordinate $Z = (z, \theta)$. The SRS is represented in many ways as the Riemann surface is. Here the SRS of genus $g \geq 2$ is represented as $SH/S\Gamma$, where SH is the super upper half plane $\{Z | \text{Im } z > 0\}$ and $S\Gamma$ is a discrete subgroup (a super Fuchsian group) of the super Möbius transformations $SPL(2, R)$ [7, 8]

$$\tilde{Z} = \frac{az+b}{cz+d} + \theta \frac{\alpha z + \beta}{(cz+d)^2}, \quad (1)$$

$$\tilde{\theta} = \frac{\alpha z + \beta}{cz+d} + \frac{\theta}{cz+d} (1 + \frac{1}{2} \beta \alpha), \quad ad - bc = 1,$$

where a, b, c and d are real Grassmann even parameters and α and β are Grassmann odd ones with $\bar{\alpha} = i\alpha$, $\bar{\beta} = i\beta$ ^{*}). We introduce a standard metric in SH [10],

$$ds^2 = E^{++} E^{--} - 2 E^+ E^-, \quad (2)$$

where

^{*}) Here the law of complex conjugation is defined by the relations $\overline{X+Y} = \bar{X} + \bar{Y}$, $\overline{XY} = \bar{Y}\bar{X}$.

$$\begin{aligned} E^{++} &= Y^{-1}(dz + \theta d\theta) , \quad E^{--} = Y^{-1}(d\bar{z} - \bar{\theta} d\bar{\theta}) \quad (= \overline{E^{++}}) , \\ E^{+} &= Y^{-3/2} \left\{ \frac{i\theta - \bar{\theta}}{2} dz + \left(Y + \frac{1}{2} \right) d\theta \right\} , \\ E^{-} &= Y^{-3/2} \left\{ -\frac{\theta + i\bar{\theta}}{2} d\bar{z} + \left(Y + \frac{1}{2} \right) d\bar{\theta} \right\} \quad (= \overline{E^{+}}) , \end{aligned} \quad (3)$$

$$Y \equiv \text{Im } z + \frac{1}{2} \theta \bar{\theta} . \quad (4)$$

E^A 's are a set of basis one-forms taking a superconformally flat form [11], i.e., the E^A 's are obtained from the flat basis $\overset{\circ}{E}^A$'s,

$$\overset{\circ}{E}^{++} = dz + \theta d\theta , \quad \overset{\circ}{E}^{--} = d\bar{z} - \bar{\theta} d\bar{\theta} , \quad \overset{\circ}{E}^{+} = d\theta , \quad \overset{\circ}{E}^{-} = d\bar{\theta} , \quad (5)$$

by a super Weyl transformation with the parameter Y^{-1} *). Note that ds^2 in (2) is a superanalog of the Poincaré metric $|dz|^2/(\text{Im } z)^2$.

Next we proceed to super Beltrami equations. Before giving super Beltrami equations, we review the ordinary Beltrami equation [12]: a Riemann surface can be represented by H/Γ where H is the upper half plane and Γ is a Fuchsian group. Given a Beltrami differential μ ($|\mu| < 1$) which is a $(-1,1)$ tensor in H/Γ , the Beltrami equation

$$\partial_{\bar{z}} w = \mu(z) \partial_z w , \quad z \in H , \quad (6)$$

has a unique homeomorphism solution by a suitable normalization. A new conformal structure is defined for the Riemann surface by declaring w to be an isothermal coordinate. We associate with μ a Riemann metric

$$ds^2 = \lambda(z) |dz + \mu d\bar{z}|^2 , \quad \lambda > 0 , \quad (7)$$

which is a general form. With w satisfying (6) this becomes

$$ds^2 = \frac{\lambda}{|\partial_z w|^2} |dw|^2 , \quad (8)$$

*) Following Ref. [11] faithfully, one should adopt $ds^2 = \tilde{E}^{++}\tilde{E}^{--} - \tilde{E}^{+}\tilde{E}^{-}$ [= (1/4)• ds^2 in (2)] with $\tilde{E}^A$'s being obtained from $\overset{\circ}{E}^A$'s by a super Weyl transformation with the parameter $(2Y)^{-1}$.

which can be transformed into the flat metric by a Weyl transformation. On the other hand, the 2D gravity (metric tensor) has three degrees of freedom and they are involved in λ and μ in (7). Furthermore, the above-mentioned fact implies that μ involves all the other degrees of freedom of the 2D gravity than that associated with the Weyl transformation.

We rewrite the Beltrami equation (6) as

$$\partial_z w^- = h_+ \quad , \quad \partial_{\bar{z}} w^- = h_- \quad , \quad (9)$$

where complex-valued $h_{\pm}$ are not arbitrary but they must satisfy the integrability condition in order that $w(z, \bar{z})$ exists,

$$\partial_{\bar{z}} h_+ = \partial_z h_- \quad . \quad (10)$$

Hence h_+ is determined by h_- up to an analytic function and the degree of freedom of the Beltrami differential μ is involved in h_- . Equations (9) and (10) can again be rewritten as a relation between the sets of basis one-forms in z and w coordinates,

$$\dot{e}^a_{(w)} = \dot{e}^b_{(z)} h_b^a(z, \bar{z}) \quad (a, b = \pm) \quad , \quad (11)$$

where

$$\dot{e}^+_{(w)} = dw^- \quad , \quad \dot{e}^+_{(z)} = dz \quad , \quad \dot{e}^-_{(\cdot)} = \overline{\dot{e}^+_{(\cdot)}} \quad , \quad (12)$$

and h_a^b 's are complex-valued and satisfy the integrability conditions

$$\partial_{\bar{z}} h_+^+ = \partial_z h_-^+ \quad , \quad \partial_{\bar{z}} h_+^- = \partial_z h_+^- \quad (h_+^+ = \overline{h_+^-}) \quad . \quad (13)$$

Then only h_-^+ (or $h_+^- = \overline{h_-^+}$) is independent and it involves the degree of freedom of μ .

Now we can present super Beltrami equations considering a super extension of the ordinary Beltrami equation in the form of (11). First we count the degrees of freedom of proper super Beltrami differentials. In the superspace formulation of 2D supergravity theory, the supervielbein E_M^A 's are basic ingredients, which correspond to the zweibein in 2D gravity theory. The supervielbein, however, is not completely independent but is constrained, and hence six components out of the total of 16 can be independent superfields [11]. This implies that we need two kinds of super Beltrami differentials which are complex-valued unconstrained superfields because two degrees of freedom out of the six correspond to the super Weyl

[11] and local Lorentz transformations. Furthermore, the 2D supergravity theory tells us that both of the super Beltrami differentials are Grassmann odd because the independent components of $H_A^B \equiv E_A^M \delta E_M^B$ [11] are H_+^+ , H_+^{++} , H_-^{++} and their c.c. and the degrees of freedom of the super Weyl and local Lorentz transformations are carried by H_+^+ and its c.c. .

We consider a superanalog of (11). A naive one is, of course, given by

$$\overset{\circ}{E}^A(W) = \overset{\circ}{E}^B(Z) \overset{\circ}{H}_B^A \quad (A, B = \pm\pm, \pm), \quad (14)$$

where $\overset{\circ}{E}^A$'s are given in (5) and $W = (w, \eta)$. The integrability conditions yield that $\overset{\circ}{H}_A^B$ should take the following form

$$[\overset{\circ}{H}_A^B] = \begin{matrix} & \begin{matrix} ++ & -- & + & - \end{matrix} \\ \begin{matrix} ++ \\ -- \\ + \\ - \end{matrix} & \begin{bmatrix} D\xi + h^2 & D\bar{\psi} - \bar{k}^2 & Dh & D\bar{k} \\ \bar{D}\psi - k^2 & \bar{D}\bar{\xi} + \bar{h}^2 & \bar{D}k & \bar{D}\bar{h} \\ \xi & \bar{\psi} & h & \bar{k} \\ \psi & \bar{\xi} & k & \bar{h} \end{bmatrix} \end{matrix} \quad (15)$$

with

$$\begin{aligned} \bar{D}h &= Dk, & 2hk + D\psi - \bar{D}\xi &= 0, \\ D\bar{h} &= \bar{D}\bar{k}, & 2\bar{h}\bar{k} + \bar{D}\bar{\psi} - D\bar{\xi} &= 0, \end{aligned} \quad (16)$$

where the spinor covariant derivatives D and $\bar{D}$ are given by

$$D = \partial_\theta + \theta \partial_z, \quad \bar{D} = -\partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}. \quad (17)$$

Equations (16) imply that ψ and ξ (or $\bar{\psi}, \bar{\xi}$) are independent Grassmann odd superfields and hence they should involve the super Beltrami differentials. However, this naive superanalog is not appropriate for super Beltrami equations. One of the reasons is that the super Beltrami differentials which should have definite weights under ST , cannot be defined simply with the components of $\overset{\circ}{H}_A^B$. A proper superanalog of (11) is found to be [9]

$$E^A(W) = E^B(Z) H_B^A, \quad (18)$$

where E^A 's are given in (3). The integrability conditions yield some relations among the components of H_B^A and we find similarly that H_+^{++} and H_-^{++} are

independent Grassmann odd superfields and they involve the super Beltrami differentials. In fact, the super Beltrami differentials (ν, σ) are identified with [9]

$$\nu = H_+^{++}/(Y^{-1/2} H_{++}^{++}) , \quad \sigma = H_-^{++}/(Y^{1/2} H_{++}^{++}) . \quad (19)$$

ν and σ are of weights $(-\frac{1}{2}, 0)$ and $(-1, \frac{1}{2})$, respectively [9]. Note that E^+ is of weight $(-1/4, 1/4)$ under ST (or $SPL(2, R)$), while E^+ does not have a definite weight. Neither of them, however, is a superconformal tensor [7]. The super Beltrami equations are given by

$$\begin{aligned} D w - \eta D \eta &= \nu \left\{ \partial_z w + \eta \partial_z \eta - \frac{i\theta - \bar{\theta}}{2Y} (D w - \eta D \eta) \right\} , \\ \bar{D} w - \eta \bar{D} \eta &= -\sigma \left\{ \partial_{\bar{z}} w + \eta \partial_{\bar{z}} \eta - \frac{i\theta - \bar{\theta}}{2Y} (D w - \eta D \eta) \right\} . \end{aligned} \quad (20)$$

One may consider that σ represents a breakdown of superanalyticity, and ν represents a breakdown of superconformal structure.

Next, we solve the linearized super Beltrami equations and calculate the super Schwarzian derivative [7,8], which leads to a superanalog of the Bers embedding [13]. The super Beltrami differentials are originally defined in SH but we extend them as $\nu = \sigma = 0$ in SL, the super lower half plane^{*}). We consider the super Beltrami equations in SL as well. We normalize the mapping $(z, \theta) \rightarrow (w, \eta)$ at $(0, 0)$, $(1, 0)$ and $(\infty, 0)$ as

$$w = 0 \quad \text{and} \quad \eta = 0 \quad \text{at} \quad (0, 0) , \quad (21a)$$

$$w = 1 \quad \text{at} \quad (1, 0) , \quad (21b)$$

$$w = \infty \quad \text{and} \quad \eta = \text{finite} \quad \text{at} \quad (\infty, 0) . \quad (21c)$$

The condition (21c) means that $w/z^2 \rightarrow 0$ and $\eta/z \rightarrow 0$ as $z \rightarrow \infty$ with $\theta = 0$. We can take such a normalization because of the fact that given a solution $W(Z)$ of the super Beltrami equation (20), $\tilde{W} = A(W)$, $A \in SPL(2, R)$ is also a solution of the same super Beltrami equation.

Rewriting w , η and σ as

$$w = z + \dot{w} , \quad \eta = \theta + \dot{\eta} , \quad \nu = \dot{\nu} , \quad \sigma = \dot{\sigma} , \quad (22)$$

^{*}) Another extension will be discussed in Ref. [9].

we obtain the linearized Beltrami equations,

$$\left\{ \begin{array}{l} D\dot{w} - \dot{\eta} - \theta D\dot{\eta} = \dot{\nu} , \quad \bar{D}\dot{w} - \theta \bar{D}\dot{\eta} = -\dot{\sigma} \quad \text{in } SH , \\ D\dot{w} - \dot{\eta} - \theta D\dot{\eta} = 0 , \quad \bar{D}\dot{w} - \theta \bar{D}\dot{\eta} = 0 \quad \text{in } SL , \\ \dot{w} = 0 \quad \text{at } (0,0) \text{ and } (1,0) , \\ \dot{\eta} = 0 \quad \text{at } (0,0) , \\ \dot{w}/z^2 \rightarrow 0 , \quad \dot{\eta}/z \rightarrow 0 \quad \text{for } z \rightarrow \infty \text{ with } \theta = 0 . \end{array} \right. \quad (23)$$

This linearized super Beltrami equations have a proper solution which is given by the use of potential integrals [9],

$$\begin{aligned} \dot{w}(Z) = & \frac{1}{2} \theta \dot{\nu}(Z) + \bar{\theta} \left\{ \dot{\sigma}(Z) - \frac{1}{2} \theta D\dot{\sigma}(Z) \right\} + \frac{1}{\pi} \int_H d^2t \, R^{(0,1)}(t,z) \partial_{\bar{\theta}} (1 - \theta \partial_{\theta}) \dot{\sigma}(t,\theta) \\ & - \frac{\theta}{2\pi} \int_H d^2t \, R^{(0)}(t,z) \partial_{\bar{\theta}} \partial_{\theta} \dot{\sigma}(t,\theta) , \end{aligned} \quad (24)$$

$$\begin{aligned} \dot{\eta}(Z) = & -\frac{1}{2} \dot{\nu}(Z) - \frac{1}{2} \bar{\theta} D\dot{\sigma}(Z) - \frac{1}{2\pi} \int_H d^2t \, R^{(0,1)}(t,z) \partial_{\bar{\theta}} \partial_{\theta} \dot{\sigma}(t,\theta) \\ & + \frac{\theta}{2\pi} D^2 \int_H d^2t \, R^{(0,1)}(t,z) \partial_{\bar{\theta}} \dot{\sigma}(t,\theta) , \end{aligned}$$

where

$$R^{(0,1)}(t,z) = \frac{1}{t-z} - \frac{1-z}{t} + \frac{z}{1-t} , \quad R^{(0)}(t,z) = \frac{1}{t-z} - \frac{1}{t} . \quad (25)$$

Note that both $\dot{\nu}(z)$ and $\dot{\sigma}(z)$ become zero on the "real" axis, $\bar{z} = z$, $\bar{\theta} = i\theta$ [9].

The solution is superconformal in SL and has a super Schwarzian derivative [7],

$$S(z,\theta) = \frac{D^4 \eta}{D^3 \eta} - 2 \frac{D^3 \eta}{D^2 \eta} \frac{D^2 \eta}{D \eta} . \quad (26)$$

Plugging the solution into (26) we get the super Schwarzian derivative in the linearized approximation,

$$\begin{aligned} \dot{S}(z,\theta) = & D^4 \dot{\eta} \\ = & -\frac{1}{\pi} \int_H d^2t \left[\frac{\partial_{\theta} \partial_{\bar{\theta}} \dot{\sigma}(t,\theta)}{(z-t)^3} - 3\theta \frac{\partial_{\bar{\theta}} \dot{\sigma}(t,\theta)}{(z-t)^4} \right] \\ = & -\frac{1}{\pi} \int_{SH} d^2t d\xi d\bar{\xi} \frac{\dot{\sigma}(t,\xi)}{(z-t-\theta\xi)^3} , \quad (z,\theta) \in SL . \end{aligned} \quad (27)$$

This is valid in SL and hence we have in SH, by use of the symmetry,

$$\dot{S}(Z) = -\frac{1}{\pi} \int_{SH} d^2t d\xi d\bar{\xi} \frac{\overline{\dot{S}(t, \xi)}}{(z - \bar{t} + i\theta \bar{\xi})^3}, \quad Z \in SH. \quad (28)$$

We can easily see that this $\dot{S}(Z)$ is a superanalytic differential of weight $(3/2, 0)$.

Let $SB(S\Gamma)$ and $SQ(S\Gamma)$ be the linear space of super Beltrami differentials^{*)} and 3/2-differentials, respectively. Then Eq. (28) yields an anti-linear mapping

$$\Lambda : (\nu, \sigma) \equiv \vec{\mu} \longrightarrow S[\vec{\mu}] \quad (29)$$

from $SB(S\Gamma)$ to $SQ(S\Gamma)$. On the other hand, let S be an element of $SQ(S\Gamma)$, then $Y^2 \bar{S}$ has weight $(-1, \frac{1}{2})$. We may define a mapping

$$\Lambda^* : S \longrightarrow (\sigma, 4i Y^2 \bar{S}) \quad (30)$$

from $SQ(S\Gamma)$ to $SB(S\Gamma)$. Using a superanalog of the Bergman type identity

$$F(z) = \frac{i2}{\pi} \int_H d^2t \frac{(Im t)^2}{(\bar{t} - z)^4} F(t), \quad F: \text{analytic in } H, \quad (31)$$

we find that $\Lambda \Lambda^*$ is the identity mapping of $SQ(S\Gamma)$. Thus Eq. (28) represents an anti-linear one-to-one mapping of the quotient space $SB(S\Gamma)/N(S\Gamma)$ onto $SQ(S\Gamma)$, where $N(S\Gamma)$ is the subspace of $SB(S\Gamma)$ for which $S[\vec{\mu}] = 0$. Note that the super Beltrami differentials of $(\nu, (1 - \bar{\theta}\partial_{\bar{\theta}})\sigma)$ belong to $N(S\Gamma)$, which means that ν and the $\bar{\theta}$ -independent components of σ do not contribute at least to infinitesimal deformations of SRSs. The Riemann-Roch theorem tells us that $SQ(S\Gamma)$ and $SB(S\Gamma)/N(S\Gamma)$ are $3g-3$ even and $2g-2$ odd dimensional complex linear spaces. The tangent and cotangent spaces at $S\Gamma$ of the super Teichmüller space $ST(S\Gamma)$ will be identified with $SB(S\Gamma)/N(S\Gamma)$ and $SQ(S\Gamma)$, respectively. The integral pairing

$$\langle \vec{\mu}, S \rangle = \int_{SH/S\Gamma} d^2z d\theta d\bar{\theta} \sigma(Z) \dot{S}(Z), \quad (32)$$

$$\vec{\mu} = (\nu, \sigma) \in SB(S\Gamma), \quad S \in SQ(S\Gamma),$$

induces a natural pairing of the underlying tangent and cotangent spaces since $d^2z d\theta d\bar{\theta}(\sigma S)$ is $S\Gamma$ -invariant. Taking account of Eq. (30), we find that the associated inner product

^{*)} $SB(S\Gamma)$ is a linear space in which the addition is defined by $(\nu_1, \sigma_1) + (\nu_2, \sigma_2) = (\nu_1 + \nu_2, \sigma_1 + \sigma_2)$.

$$\langle S_1, S_2 \rangle = \int_{SH/SP} d^2z d\theta d\bar{\theta} \gamma^2 \bar{S}_1 S_2, \quad S_1, S_2 \in SQ(ST), \quad (33)$$

is the superanalog of the Weil-Petersson cometric for $ST(ST)$.

We have given the super Beltrami equations in which two kinds of super Beltrami differentials of weight $(-\frac{1}{2}, 0)$ and $(-1, \frac{1}{2})$ have been introduced. These results exactly agree with the 2D supergravity theory. We have given the solutions of the linearized super Beltrami equations and also mappings between the super Beltrami differentials and the superanalytic $3/2$ -differentials. A natural pairing between them has been defined, which leads to a superanalog of the Weil-Petersson's scalar product. In some parts our explanations might lack mathematical rigorousness but they should be refined.

Finally, we make some comments. As was mentioned below Eq. (31), some part of our super Beltrami differentials has been found not to contribute to infinitesimal deformations of SRSs. Hence we may use the super Beltrami differentials of special form

$$(\nu, \sigma) = (0, -\bar{\theta} \mu(z, \bar{z}, \theta)) \quad (34)$$

in order to describe infinitesimal deformations of SRSs. This restriction on our super Beltrami differentials corresponds exactly to the gauge choice discussed in Ref. [8]. The restriction leads to the following super Beltrami equations [8]

$$\begin{cases} D w - \gamma D \gamma = 0 \\ \bar{D} w - \gamma \bar{D} \gamma = \bar{\theta} \mu (\partial_z w + \gamma \partial_z \gamma) \end{cases} \quad (35)$$

One can easily see from (24) and (34) that the solutions are independent of $\bar{\theta}$ in this case, as is mentioned in Ref. [8]. However, if we begin with Eq. (35) or if we choose the gauge from the beginning, we cannot see straightforwardly that the super Schwarzian derivative calculated with the solutions yields a superanalytic differential of weight $(3/2, 0)$. Furthermore we cannot discuss finite deformations properly in that gauge.

The theory of SRSs should be useful for calculating the Polyakov partition functions for a closed superstring. One may use only the theory of Riemann surfaces to analyze them regarding fermions as matters living on the Riemann surfaces, however, once one takes the superspace formulation of superstring, the theory of SRSs is naturally required. In order to construct the formula of multi-loop amplitude for a closed superstring which will be a superanalog of the formula by D'Hoker and Phong [5], there still remain important problems. Our analysis presented here will be a basis for attacking those problems. In fact,

we have evaluated the differential $d\ell$ of the "super"length ℓ [10,14,15] of a closed geodesic on the tangent vector $\vec{\mu} = (v, \sigma)$,

$$\text{Re}(\vec{\mu}, d\ell) = -\frac{1}{\pi} \text{Re} \int_{\text{SH/Sr}} d^2z d\theta d\bar{\theta} \sigma(\bar{z}) \Phi_{\ell}(z) , \quad (36)$$

where $\Phi_{\ell}(z)$ is of weight $(3/2, 0)$ and it depends on the parameters of the element, which is associated with ℓ , of $\text{S}\Gamma$. This formula is important to examine a super-analog of the Weil-Petersson's measure. Details will be reported in Ref. [9].

ACKNOWLEDGEMENTS

We would like to thank P. Howe, A. Rogers, W.J. Harvey, L.H. Hodgkin and P. Teofilatto for discussions. One of us (Y.Y.) is grateful to the members of the particle theory group at King's College for their hospitality. The other (S.U.) would also like to thank the Theoretical Physics Division of CERN for the warm hospitality extended to him.

REFERENCES

- [1] A.M. Polyakov, Phys. Lett. 103B (1981) 207.
- [2] O. Alvarez, Nucl. Phys. B216 (1983) 125.
- [3] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.
- [4] D. Friedan and S. Shenker, Phys. Lett. B175 (1986) 287.
- [5] E. D'Hoker and D.H. Phong, Nucl. Phys. B264 (1986) 513; Phys. Rev. Lett. 56 (1986) 912.
- [6] A.A. Belavin and V.G. Knizhnik, Phys. Lett., B168 (1986) 201.
- [7] D. Friedan, in Proc. Workshop on Unified String Theories, eds. D. Gross and M. Green, (World Scientific, Singapore, 1986);
E. Martinec, Nucl. Phys. B281 (1987) 157;
L. Crane and J.M. Rabin, Comm. Math. Phys. 113 (1988) 243.
- [8] J.M. Rabin, Phys. Lett. 190B (1987) 40.
- [9] S. Uehara and Y. Yasui, in preparation.
- [10] S. Uehara and Y. Yasui, Phys. Lett. 202B (1988) 530; "'Super' Selberg's Trace Formula from Chaotic Model", King's College preprint (1987).
- [11] P. Howe, J. Phys. A12 (1979) 393.
- [12] L. Ahlfors, Lectures on quasi-conformal mappings (Van Nostrand, New York, 1966).
- [13] L. Bers, Ann. of Math. 91 (1970) 570.
- [14] S. Matsumoto and Y. Yasui, "Chaos on the Super Riemann Surface", Osaka preprint OCU-121 (1987), to appear in Progr. Theor. Phys.
- [15] A.M. Baranov, Yu.I. Manin, I.V. Frolov and A.S. Schwarz, Sov. J. Nucl. Phys. 43 (1986) 670; Comm. Math. Phys. 111 (1987) 373.