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On the Unitary Representations of $N = 2$ and $N = 4$ Superconformal Algebras

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ABSTRACT

We discuss the relationship between the unitary representations of $N=2$ and $N=4$ superconformal algebras. We describe how the $N=4$ irreducible representations decompose into those of $N=2$ algebra. We also describe the modular properties of $N=4$ superconformal characters.

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It has been known for some time that extended superconformal algebras with $N=2$ and $N=4$ world-sheet supersymmetry describe string compactification on complex manifolds of $SU(n)$ holonomy [1,2]. In these algebras, the Ramond (R) and Neveu-Schwarz (NS) sectors of the theory transform into each other under spectral flow, and space-time bosons and fermions necessarily pair up into supergravity multiplets. Recently, Gepner has given an explicit construction of modular invariant partition functions for string propagation on manifolds of $SU(n)$ holonomy making use of discrete representations of the $N=2$ algebra, and he reproduced known quantum (Hodge) numbers of Calabi-Yau and K_3 manifolds [3]. This construction gives exact results on string compactification beyond those based on low energy effective Lagrangian methods [4], and will have some important phenomenological implications.

The use of infinite-dimensional Lie algebras in the study of complex manifolds is also of considerable interest from a mathematical point of view.

Representation theory of $N=2$ algebra has been well understood, in particular in its discrete series via its connection to para-fermion theory [5]. Representation theory of $N=4$ algebra, on the other hand, has been the subject of recent investigations [6,7]. In our previous studies we have classified its unitary representations and derived their character formulas [6]. In this article, we shall discuss the relationship between $N=4$ and $N=2$ algebras in order to facilitate the algebraic analysis of string compactification. We shall derive branching rules for the decomposition of $N=4$ representations into those of the $N=2$ algebra and also discuss the modular properties of $N=4$ character formulas.

This completes our study of the representation theory of the $N=4$ algebra. In a separate article, we shall discuss conformal field theory on K_3 surface [8].

Let us first recapitulate our previous results on the classification of unitary representations of the $N=4$ algebra [6].

Unitary representations exist for discrete values of the central charge $c = 6k$ ($k=1,2,3,\dots$). For these values of c , the $N=4$ algebra contains a level- k $SU(2)$ Kac-Moody algebra as its sub-algebra. The highest weight states of the $N=4$ algebra are labelled by two quantum numbers h and ℓ where h is

the eigenvalue of the Virasoro operator L_0 and ℓ is the isospin. Unitarity puts a lower bound on the allowed values of h : $h \geq k/4$ in Ramond and $h \geq \ell$ in Neveu-Schwarz sectors respectively. There exist two distinct classes of representations in the $N=4$ theory.

(1) massless representations;

$$h = k/4 \quad \ell = 0, 1/2, \dots, k/2 \quad \text{in Ramond sector}$$

$$h = \ell \quad \ell = 0, 1/2, \dots, k/2 \quad \text{in NS sector}$$

These representations saturate the unitarity bound and possess unbroken $N=4$ world-sheet supersymmetry. The ground states of these representations are not paired in bosons and fermions and carry non-zero Witten index. In compactified string theory, these representations describe non-trivial topology of manifolds with $c_1 = 0$ (vanishing first Chern class) and generate massless supergravity multiplets [9].

(2) massive representations;

$$h > k/4 \quad \ell = 1/2, 1, \dots, k/2 \quad \text{in Ramond sector}$$

$$h > \ell \quad \ell = 0, 1/2, 1, \dots, k/2 - 1/2 \quad \text{in NS sector}$$

These representations have ground states with equal numbers of bosons and fermions, and thus vanishing Witten index. In string theory, they describe massive supergravity multiplets.

The topology of compactifying manifolds is coded into the massless representations. However, massless representations mix, under modular transformations, with massive representations, and thus the latter also enter in modular invariant partition functions.

Character formulas of the $N=4$ theory [6] are somewhat involved. They contain two parameters z and y which describe the isospin and fermion quantum numbers of the representation content.

massive representations;

$$ch^R(k, \ell; z, y) = q^{h-\ell^2/(k+1)+1/8} F^R(z, y) \chi_{k-1}^{\ell-\frac{1}{2}}(z) \quad \text{R sector} \quad (1)$$

$$ch^{NS}(k, \ell; z, y) = q^{h-(\ell+1/2)^2/(k+1)+1/8} F^{NS}(z, y) \chi_{k-1}^{\ell}(z) \quad \text{NS sector} \quad (2)$$

where

$$F^R(z, y) = z \prod_{n=1} \frac{(1 + yzq^n)(1 + y^{-1}zq^n)(1 + yz^{-1}q^{n-1})(1 + y^{-1}z^{-1}q^{n-1})}{(1 - q^n)} \quad (3)$$

$$F^{NS}(z, y) = \prod_{n=1} \frac{(1 + yzq^{n-\frac{1}{2}})(1 + y^{-1}zq^{n-\frac{1}{2}})(1 + yz^{-1}q^{n-\frac{1}{2}})(1 + y^{-1}z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)} \quad (4)$$

and $\chi_k^\ell(z)$ is the character function of the level- k and isospin ℓ representation of affine $SU(2)$ algebra :

$$\begin{aligned} \chi_k^\ell(z) &= \frac{q^{(\ell+1/2)^2/(k+2)-1/8}}{\prod_{n=1} (1 - q^n)(1 - z^2q^n)(1 - z^{-2}q^{n-1})} \sum_m q^{(k+2)m^2+(2\ell+1)m} \\ &\times \left\{ z^{2((k+2)m+\ell)} - z^{-2((k+2)m+\ell+1)} \right\} \end{aligned} \quad (5)$$

massless representations;

In the case of massless representations, $\chi_k^\ell(z)$ is replaced by new functions

$$\begin{aligned} \chi_k^{R,\ell}(z, y) &= \frac{q^{(\ell+1/2)^2/(k+2)-1/8}}{\prod_{n=1} (1 - q^n)(1 - z^2q^n)(1 - z^{-2}q^{n-1})} \sum_m q^{(k+2)m^2+(2\ell+1)m} \\ &\times \left\{ \frac{z^{2((k+2)m+\ell)}}{(1 + yz^{-1}q^{-m})(1 + y^{-1}z^{-1}q^{-m})} - \frac{z^{-2((k+2)m+\ell+1)}}{(1 + yzq^{-m})(1 + y^{-1}zq^{-m})} \right\} \end{aligned} \quad (6)$$

$$\begin{aligned} \chi_k^{NS,\ell}(z, y) &= \frac{q^{(\ell+1/2)^2/(k+2)-1/8}}{\prod_{n=1} (1 - q^n)(1 - z^2q^n)(1 - z^{-2}q^{n-1})} \sum_m q^{(k+2)m^2+(2\ell+1)m} \\ &\times \left\{ \frac{z^{2((k+2)m+\ell)}}{(1 + yzq^{m+\frac{1}{2}})(1 + y^{-1}zq^{m+\frac{1}{2}})} - \frac{z^{-2((k+2)m+\ell+1)}}{(1 + yz^{-1}q^{m+\frac{1}{2}})(1 + y^{-1}z^{-1}q^{m+\frac{1}{2}})} \right\} \end{aligned} \quad (7)$$

and the characters read

$$ch_o^R(k, \ell; z, y) = q^{k/4-\ell^2/(k+1)+1/8} F^R(z, y) \chi_{k-1}^{R,\ell-\frac{1}{2}}(z, y) \quad (8)$$

$$ch_o^{NS}(k, \ell; z, y) = q^{\ell-(\ell+1/2)^2/(k+1)+1/8} F^{NS}(z, y) \chi_{k-1}^{NS,\ell}(z, y) \quad (9)$$

The coexistence of massless and massive representations is a characteristic feature of extended superconformal algebra in the continuum range (as

opposed to the discrete range $c < 3$ of the $N=2$ algebra). The general rule is that when h reaches its unitarity bound, a massive representation is decomposed into a sum of massless representations

$$\begin{aligned} ch^R(h = k/4, k, \ell; z, y) &= ch_o^R(k, \ell; z, y) \\ &+ (y + y^{-1})ch_o^R(k, \ell - \frac{1}{2}; z, y) \\ &+ ch_o^R(k, \ell - 1; z, y) \end{aligned} \quad (10)$$

$$\begin{aligned} ch^{NS}(h = \ell, k, \ell; z, y) &= ch_o^{NS}(k, \ell; z, y) \\ &+ (y + y^{-1})ch_o^{NS}(k, \ell + \frac{1}{2}; z, y) \\ &+ ch_o^{NS}(k, \ell + 1; z, y) \end{aligned} \quad (11)$$

In the Ramond sector, the ground state of a massive representation contains bosons (fermions) with isospin ℓ and $\ell-1$ and fermions (bosons) with isospin $\ell-1/2$ and their conjugates, and thus possesses vanishing Witten index. On the other hand, massless characters carry Witten index

$$q^{-k/4}ch_o^R(k, \ell; z = 1, y = -1) = 2\ell + 1$$

$N=4$ characters (equations (1,2,8,9)) have quasi-periodicity in the z and y variables. This is a manifestation of the isomorphism of representations in the $N=4$ theory [10]. Under a shift $z \rightarrow z q^{1/2}$ (i.e. $\theta \rightarrow \theta + 2\pi\tau$ for $z = e^{i\theta/2}$), we find

$$ch_o^R(k, \ell; zq^{1/2}, y) = q^{-k/4}z^{-k}ch_o^{NS}(k, \frac{k}{2} - \ell; z, y), \quad (12)$$

$$ch_o^{NS}(k, \ell; zq^{1/2}, y) = q^{-k/4}z^{-k}ch_o^R(k, \frac{k}{2} - \ell; z, y), \quad (13)$$

Thus, the Ramond and Neveu-Schwarz sectors are transformed into each other with ℓ replaced by $k/2 - \ell$. Under a full shift $z \rightarrow zq$, the characters transform in the following way

$$ch_o^R(k, \ell; zq, y) = q^{-k}z^{-2k}ch_o^R(k, \ell; z, y), \quad (14)$$

$$ch_o^{NS}(k, \ell; zq, y) = q^{-k}z^{-2k}ch_o^{NS}(k, \ell; z, y), \quad (15)$$

This is the same transformation law as for the affine character function $\chi_k^t(zq) = q^{-k} z^{-2k} \chi_k^t(z)$. Equations (14) and (15) are consistent with the fact that N=4 contains SU(2) as sub-algebra. The N=4 characters may then be expanded into a sum of affine characters with z-independent branching functions. The shift of $\theta \rightarrow \theta + \eta 4\pi\tau$ corresponds to a change in the moding of N=4 operators :

$$\begin{aligned} L_n &\rightarrow L_n + 2 \eta T_n^3 + \eta^2 c/6 \delta_{n,0}, \\ T_n^\pm &\rightarrow T_{n \pm 2\eta}^\pm, \\ T_n^3 &\rightarrow T_n^3 + \eta c/6 \delta_{n,0}, \\ G_r^\pm &\rightarrow G_{r \mp \eta}^\pm \text{ and } \bar{G}_s^\pm \rightarrow \bar{G}_{s \pm \eta}^\pm \end{aligned}$$

$T_n^\pm, T_n^3$ are the SU(2) currents and $G_r^\pm$ ($\bar{G}_s^\pm$) are an SU(2) doublet (its conjugate) of supercharge operators. The N=4 algebra remains invariant under this transformation.

The quasi-periodicity in the y variable corresponds to another change of moding of the N=4 operators : $G_r^\pm \rightarrow G_{r+\rho}^\pm, \bar{G}_s^\pm \rightarrow \bar{G}_{s-\rho}^\pm$ with the other operators unchanged. This is also an isomorphism of the N=4 algebra. In the case of k=1 for instance, we find, under a half shift $y \rightarrow y q^{1/2}$,

$$\begin{aligned} ch_o^R(k=1, \ell=0; z, yq^{1/2}) &= q^{1/4} (ch_o^{NS}(k=1, \ell=0; z, y) \\ &\quad + y^{-1} ch_o^{NS}(k=1, \ell=\frac{1}{2}; z, y)), \end{aligned} \quad (16)$$

$$\begin{aligned} ch_o^{NS}(k=1, \ell=0; z, yq^{1/2}) &= q^{-1/4} ((1-q) ch_o^R(k=1, \ell=0; z, y) \\ &\quad - yq ch_o^R(k=1, \ell=\frac{1}{2}; z, y)), \end{aligned} \quad (17)$$

Similar formulas hold for other representations. Under a full shift, the Ramond and Neveu-Schwarz sectors transform into themselves.

The N=4 algebra is reduced to that of N=2 when we identify $G_r = G_r^+ + \bar{G}_r^-$, $G_s = G_s^- + \bar{G}_s^+$ and $T_n = 2T_n^3$ as the supercharge and U(1) current operators of the N=2 algebra. The variable y is put to 1 when we discuss the correspondence between N=4 and N=2 ($y \neq 1$ discriminates $G^\pm$ from $\bar{G}^\pm$ and breaks the N=2 symmetry).

In the following, we concentrate on the case $k=1$ ($c=6$) relevant for compactification on K_3 surface. When $k=1$, the massive characters reduce to (see (1) and (2))

$$ch^R(k=1, \ell = \frac{1}{2}; z, y) = q^h F^R(z, y) \quad (18)$$

$$ch^{NS}(k=1, \ell = 0; z, y) = q^h F^{NS}(z, y) \quad (19)$$

and at $y=1$, the massless characters (8),(9) are rewritten in a more transparent form as

$$ch_o^R(k=1, \ell = 0; z) = \sum_m q^{m^2/2+m/2+1/4} z^{m+1/2} \frac{1}{1+zq^m} f^R(z), \quad (20)$$

$$ch_o^R(k=1, \ell = \frac{1}{2}; z) = \sum_m q^{m^2/2+m/2+1/4} z^{m+1/2} \frac{zq^m - 1}{1+zq^m} f^R(z), \quad (21)$$

$$ch_o^{NS}(k=1, \ell = 0; z) = \sum_m q^{m^2/2} z^m \frac{zq^{m-1/2} - 1}{1+zq^{m-1/2}} f^{NS}(z), \quad (22)$$

$$ch_o^{NS}(k=1, \ell = \frac{1}{2}; z) = \sum_m q^{m^2/2} z^m \frac{1}{1+zq^{m-1/2}} f^{NS}(z), \quad (23)$$

where

$$f^R(z) = (z^{1/2} + z^{-1/2}) \prod_{n=1} \frac{(1+zq^n)(1+z^{-1}q^n)}{(1-q^n)^2}, \quad (24)$$

$$f^{NS}(z) = \prod_{n=1} \frac{(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2})}{(1-q^n)^2}, \quad (25)$$

The equivalence of (20) ~ (23) and (8),(9) with $k=1$ may be checked by inspecting the residues in the complex z -plane.

In order to discuss the branching of the $N=4$ representations into those of $N=2$, let us recall [1] some properties of unitary representations of $N=2$ algebra in the continuum range. Highest weight states are parametrized by two quantum numbers h and Q where Q is the (relative) $U(1)$ charge. (The net $U(1)$ charge is given by $Q \pm 1/2$ in the Ramond sector). Unitary representations exist in the interior of a polygonal domain in (h, Q) -plane

which is bounded by the lines $g_r(c, h, Q) = 0$

$$g_r = 2h - 2rQ + \left(\frac{c}{3} - 1\right)\left(r^2 - \frac{1}{4}\right) - \frac{1}{4}, \quad r \in Z \quad \text{in R sector,} \quad (26)$$

$$g_r = 2h - 2rQ + \left(\frac{c}{3} - 1\right)\left(r^2 - \frac{1}{4}\right), \quad r \in Z + \frac{1}{2} \quad \text{in NS sector.} \quad (27)$$

(See Fig.1). In this domain, the Kac determinant does not vanish ($g_r > 0$ for all r) and the theory is manifestly unitary. These representations have spontaneously broken world-sheet supersymmetry and are the analogue of massive representations of the $N=4$ algebra. We therefore call them $N=2$ massive representations. Unitary representations also exist on a vanishing line $g_r = 0$ (for each r) where the Kac determinant vanishes. The representation is unitary if $g_r = 0$, $g_{r+\text{sign}(r)} \leq 0$ and $f_{1,2} \geq 0$. Here the function $f_{1,2}$ is given by

$$f_{1,2} = 2\left(\frac{c}{3} - 1\right)\left(h - \frac{c}{24}\right) - Q^2 + \frac{1}{4}\left(\frac{c}{3} + 1\right)^2 \quad \text{in R sector,} \quad (28)$$

$$f_{1,2} = 2\left(\frac{c}{3} - 1\right)h - Q^2 - \frac{1}{4}\left(\frac{c}{3} - 1\right)^2 + \frac{1}{4}\left(\frac{c}{3} + 1\right)^2 \quad \text{in NS sector,} \quad (29)$$

The Kac determinant acquires an extra zero when $f_{1,2} = 0$. These representations occur at the unitarity bound and are the analogue of the massless representations of the $N=4$ theory. We call them $N=2$ massless representations.

In view of the correspondence $T_0 = 2T_0^3$, we are interested in $N=2$ representations with integral $U(1)$ charge. We note that when $c=6$, integral (half-integral) values of Q occur in the NS (R) sector at double points (where $g_r = 0$ and $g_{r+1} = 0$ intersect in the (h, Q) -plane) or at triple points (where $g_r = 0$, $g_{r+3} = 0$ and $f_{1,2} = 0$ intersect simultaneously). See Fig.1.

double points,

R sector

$$h = h_m = \frac{m^2}{2} + \frac{m}{2} + \frac{1}{4}, \quad Q = \pm Q_m = \pm\left(m + \frac{1}{2}\right), \quad m = 1, 2, \dots \quad (30)$$

NS sector

$$h = h_m = \frac{m^2}{2}, \quad Q = \pm Q_m = \pm m, \quad m = 1, 2, \dots \quad (31)$$

triple points,

R sector

$$h = h_{m'} = \frac{m^2}{2} + \frac{3m}{2} + \frac{1}{4}, \quad Q = \pm Q_{m'} = \pm(m + \frac{3}{2}), \quad m = 0, 1, 2, \dots \quad (32)$$

NS sector

$$h = h_{m'} = \frac{m^2}{2} + m - \frac{1}{2}, \quad Q = \pm Q_{m'} = \pm(m + 1), \quad m = 1, 2, \dots \quad (33)$$

The character formulas are given by
massive representations;

$$ch^{N=2,R}(h, Q; z) = q^h z^Q f^R(z), \quad (34)$$

$$ch^{N=2,NS}(h, Q; z) = q^h z^Q f^{NS}(z), \quad (35)$$

massless representations [11];

double points,

$$ch_o^{N=2,R}(h_m, \pm Q_m; z) = \frac{q^{h_m} z^{\pm Q_m}}{1 + z^{\pm 1} q^m} f^R(z), \quad (36)$$

$$ch_o^{N=2,NS}(h_m, \pm Q_m; z) = \frac{q^{h_m} z^{\pm Q_m}}{1 + z^{\pm 1} q^{m-1/2}} f^{NS}(z), \quad (37)$$

triple points,

$$ch_o^{N=2,R}(h_{m'}, \pm Q_{m'}; z) = \frac{q^{h_{m'}} z^{\pm Q_{m'}} (1 - q)}{(1 + z^{\pm 1} q^m)(1 + z^{\pm 1} q^{m+1})} f^R(z), \quad (38)$$

$$ch_o^{N=2,NS}(h_{m'}, \pm Q_{m'}; z) = \frac{q^{h_{m'}} z^{\pm Q_{m'}} (1 - q)}{(1 + z^{\pm 1} q^{m-1/2})(1 + z^{\pm 1} q^{m+1/2})} f^{NS}(z), \quad (39)$$

The special case of $h=1/4$ and $Q=\pm 1/2$ ($h=0$ and $Q=0$) in the R (NS) sector may be regarded as a double (triple) point with $m=0$. We note that R and NS characters are interchanged as $z \rightarrow z q^{1/2}$, while the transformation $z \rightarrow zq$ shifts m by 1 in each character.

The decomposition law of massive characters at the unitarity bound is given by

$$\begin{aligned} ch^{N=2}(h = h_m, Q = \pm Q_m) &= ch_o^{N=2}(h = h_m, Q = \pm Q_m) \\ &+ ch_o^{N=2}(h = h_{m+1}, Q = \pm Q_{m+1}) + ch_o^{N=2}(h = h_{m'}, Q = \pm Q_{m'}) \end{aligned} \quad (40)$$

for both the R and NS sectors.

Now, we can immediately read off the branching rules for the N=4 characters from (20) ~ (23)

$$\begin{aligned}
ch_o^{N=4,R}(k=1, \ell=0) &= \sum_{\text{double points}} ch_o^{N=2,R} \\
&= \sum_{m=0}^{\infty} ch_o^{N=2,R}(h_m, Q_m) + \sum_{m=1}^{\infty} ch_o^{N=2,R}(h_m, -Q_m)
\end{aligned} \tag{41}$$

$$\begin{aligned}
ch_o^{N=4,R}(k=1, \ell=\frac{1}{2}) &= \sum_{\text{triple points}} ch_o^{N=2,R} \\
&= \sum_{m=0}^{\infty} ch_o^{N=2,R}(h_m, Q_m) + \sum_{m=0}^{\infty} ch_o^{N=2,R}(h_m, -Q_m)
\end{aligned} \tag{42}$$

$$\begin{aligned}
ch_o^{N=4,NS}(k=1, \ell=0) &= \sum_{\text{triple points}} ch_o^{N=2,NS} \\
&= \sum_{m=0}^{\infty} ch_o^{N=2,NS}(h_m, Q_m) + \sum_{m=1}^{\infty} ch_o^{N=2,NS}(h_m, -Q_m)
\end{aligned} \tag{43}$$

$$\begin{aligned}
ch_o^{N=4,NS}(k=1, \ell=\frac{1}{2}) &= \sum_{\text{double points}} ch_o^{N=2,NS} \\
&= \sum_{m=1}^{\infty} ch_o^{N=2,NS}(h_m, Q_m) + \sum_{m=1}^{\infty} ch_o^{N=2,NS}(h_m, -Q_m)
\end{aligned} \tag{44}$$

The set of all double and triple points is invariant under the SU(2) transformation $m \rightarrow m \pm 1$ and N=2 is enhanced into N=4 symmetry.

On the other hand, the branching rule for the massive representation is

given by ($H > 0$)

$$\begin{aligned}
ch^{N=4,NS}(h = H, \ell = 0) &= q^H \prod_{n=1}^{\infty} \frac{(1 + zq^{n-\frac{1}{2}})^2 (1 + z^{-1}q^{n-\frac{1}{2}})^2}{(1 - q^n)} \\
&= \sum_m q^{H+m^2/2} z^m \times \prod_{n=1}^{\infty} \frac{(1 + zq^{n-\frac{1}{2}})(1 + z^{-1}q^{n-\frac{1}{2}})}{(1 - q^n)^2} \\
&= \sum_m ch^{N=2,NS}(h = H + \frac{m^2}{2}, Q = m)
\end{aligned} \tag{45}$$

Similar relations hold in the R sector.

As H approaches zero, $ch^{N=2,NS}(h = m^2/2, Q = m)$ is decomposed according to (40). Thus

$$ch^{N=4,NS}(h = 0, \ell = 0) = 2 \sum_{\text{double points}} ch_o^{N=2,NS} + \sum_{\text{triple points}} ch_o^{N=2,NS} \tag{46}$$

and we recover the $N=4$ massless-massive relation (11).

Let us now discuss the modular properties of $N=4$ characters. In the following, the character functions are defined with a factor $q^{-c/24} = q^{-1/4}$ and z is put to 1 for simplicity. Under the transformation S ($\tau \rightarrow -1/\tau$), the massless characters transform as

$$\begin{aligned}
ch_o^R(\ell = 0; -1/\tau) &= ch_o^{NS'}(\ell = 1/2; \tau) \\
&+ \int_{-\infty}^{\infty} \frac{d\alpha}{2 \cosh \pi \alpha} ch^{NS'}(h = \frac{\alpha^2}{2} - \frac{1}{8}; \tau)
\end{aligned} \tag{47}$$

$$\begin{aligned}
ch_o^{NS}(\ell = 1/2; -1/\tau) &= -ch_o^{NS}(\ell = 1/2; \tau) \\
&+ \int_{-\infty}^{\infty} \frac{d\alpha}{2 \cosh \pi \alpha} ch^{NS}(h = \frac{\alpha^2}{2} - \frac{1}{8}; \tau)
\end{aligned} \tag{48}$$

Here NS' is the NS sector with the $(-1)^F$ insertion

$$ch^{NS'}(h; \tau) = ch^{NS}(1, 0; z = 1, y = -1) \tag{49a}$$

$$ch_o^{NS'}(\ell = 1/2; \tau) = - \sum_m \frac{q^{m^2/2} (-1)^m}{1 - q^{m-1/2}} \times \prod_{n=1}^{\infty} \frac{(1 - q^{n-1/2})^2}{(1 - q^n)^2} \tag{49b}$$

Under the transformation T,

$$\begin{aligned} ch_o^R(\ell = 0; \tau + 1) &= ch_o^R(\ell = 0; \tau), \\ ch_o^{NS}(\ell = 1/2; \tau + 1) &= ich_o^{NS'}(\ell = 1/2; \tau), \\ ch_o^{NS'}(\ell = 1/2; \tau + 1) &= ich_o^{NS}(\ell = 1/2; \tau). \end{aligned}$$

Similar transformation laws hold for another pair (R ; $\ell = 1/2$) and (NS,NS'; $\ell = 0$). Equations (47) and (48) can be derived using Mordell's formula [12]

$$\begin{aligned} f(\tau) &= \frac{1}{\eta(\tau)} \int_{-\infty}^{\infty} d\alpha \frac{q^{\alpha^2/2}}{2 \cosh \pi \alpha} \\ &= h_1(\tau) + h_2(-1/\tau) \\ &= h_3(\tau) + h_3(-1/\tau) \end{aligned} \tag{50}$$

$$h_1(\tau) = \frac{1}{\eta(\tau)\theta_4(\tau)} \sum_m \frac{q^{m^2/2-1/8}(-1)^m}{1 - q^{m-1/2}} \tag{51}$$

$$h_2(\tau) = \frac{1}{\eta(\tau)\theta_2(\tau)} \sum_m \frac{q^{m^2/2+m/2}}{1 + q^m} \tag{52}$$

$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(\tau)} \sum_m \frac{q^{m^2/2-1/8}}{1 + q^{m-1/2}} \tag{53}$$

which occur in analytic number theory. $f(\tau) = f(-1/\tau)$ is a famous result due to Kronecker. As we see in equations (47) and (48), the massless representations mix with a continuum of massive ones under modular transformations.

It is of great interest to construct the most general modular invariants directly from the transformation laws (47) and (48). An indirect and restricted way to find modular invariants is to use the tensor product of N=2 discrete representations [3]. The SU(2) orbits of tensor products of N=2 discrete characters (i.e. expressions which are invariant under the transformation $z \rightarrow zq$) are expanded into sums of N=4 characters. We can identify the suitable mixtures of N=4 massless and massive characters transforming under finite-dimensional representations of the modular group [8]. Modular invariants are formed by taking the sum of all SU(2) invariant orbits.

Finally we discuss the branching law of N=4 representations into those of affine SU(2) algebra. In the case of k=1 for instance, we may expand

$$ch_o^R(\ell = 1/2; z, y) = A(y) \chi_1^{1/2}(z) + B(y) \chi_1^0(z), \quad (54)$$

$$ch_o^R(\ell = 0; z, y) = C(y) \chi_1^0(z) + D(y) \chi_1^{1/2}(z), \quad (55)$$

A, C are even and B, D are odd functions of y and thus, (54) and (55) are a decomposition of the Hilbert space into bosonic and fermionic states. In the NS sector, the functions A, B and C, D are exchanged.

Explicitly, the branching functions are given by

$$C(y) = \frac{1}{\eta(\tau)^2} \sum_m q^{-m^2} y^{2m} \left(\sum_{n=m}^{\infty} - \sum_{n=-m-1}^{-\infty} \right) q^{2(n+1/4)^2} \quad (56)$$

$$D(y) = \frac{-1}{\eta(\tau)^2} \sum_m q^{-(m+1/2)^2} y^{2m+1} \left(\sum_{n=m+1}^{\infty} - \sum_{n=-m-1}^{-\infty} \right) q^{2(n+1/4)^2} \quad (57)$$

and $A = q^{-1/8} \chi_1^0 / \eta - 2D$, $B = q^{-1/8} \chi_1^{1/2} / \eta - 2C$. (56) and (57) can be derived by making use of the quasi-periodicity in the y variable (16),(17). At y=1, the branching functions can be expressed in terms of the h_3 function,

$$C = \frac{\chi_1^0}{(\chi_1^0)^2 + (\chi_1^{1/2})^2} + h_3 \chi_1^{1/2} \quad (58)$$

$$D = \frac{-\chi_1^{1/2}}{(\chi_1^0)^2 + (\chi_1^{1/2})^2} + h_3 \chi_1^0 \quad (59)$$

and thus, they transform exactly like (47), (48) with Mordell's integral under modular transformations.

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Figure Captions

Fig. 1: Unitary representations of $N=2$ algebra at $c=6$ in the Ramond (Fig.1a) and Neveu-Schwarz (Fig.1b) sectors. Solid lines give $g(r) = 0$ and the parabola is $f_{1,2} = 0$. Dots and crosses represent double and triple points, respectively.

Fig. 1.a.

Fig. 1.b.