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DILATATIONS AND THE COSMOLOGICAL CONSTANT

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A B S T R A C T

The rôle of dilatation symmetry in explaining the smallness of the cosmological constant is investigated. Dilatation-invariant theories necessarily lead to a zero cosmological constant, but only if the vacuum is invariant. This is physically unacceptable, and dilatation invariance must be broken (at least) spontaneously, in which case we do not find any constraint on the cosmological constant. We show, too, that theories in which dilatation invariance is broken by anomalies due to renormalization do not lead to a vanishing cosmological constant without a fine-tuning of parameters. Similarly, theories with an (approximate) dilatation symmetry in the renormalizable sector require fine-tuning to eliminate the cosmological constant.

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1. - INTRODUCTION

The fact that space-time, as we observe it, is to a very high precision Minkowski means that the cosmological constant is extremely small [1]. More precisely, let us write Einstein's equation with an effective cosmological constant term $\lambda g_{\mu\nu}$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -8\pi G_N T_{\mu\nu} - \lambda g_{\mu\nu} \\ &= -8\pi G_N \tilde{T}_{\mu\nu} \end{aligned} \quad (1.1)$$

Then the fact that space-time is approximately flat means that, to a considerable accuracy, the trace of $\tilde{T}_{\mu\nu}$, which is proportional to the scalar curvature, vanishes. Setting the trace of $\tilde{T}_{\mu\nu}$, $\tilde{T}^\mu_\mu \approx \Lambda^4$, we note that the experimental bounds on Λ are that it satisfy $\Lambda \lesssim 10^{-12}$ GeV [2].

This is tiny in comparison to the mass scales associated with elementary particle physics, and has led many physicists to suppose that the actual value is zero. The important point is, however, that the cosmological constant must vanish (or very nearly vanish) after the many phase transitions that have occurred in the evolution of the Universe. Phase transitions are characterized by explicit mass scales which have an important bearing on the physics of those eras, and at the same time, provide potentially enormous contributions to Λ . Our failure to understand why these contributions should cancel is the so-called "cosmological constant problem".

Even if one needs to understand the physics at high scales to completely solve the cosmological constant problem, one might suppose it possible to understand the result in the context of an effective low-energy field theory in which the massive modes are integrated out. In this case, without loss of generality, one may study this effective Lagrangian and ask what mechanisms could conceivably reduce the cosmological constant. An obvious candidate is a symmetry which would require the cosmological constant to vanish. Such a symmetry should have a non-trivial action on the metric (to forbid the cosmological term $\sqrt{-g}\lambda$ in the Lagrangian) as well as on the matter fields [3].

Let us put this another way. The vacuum state of a theory with n scalar fields is determined by their field equations:

$$\phi_{a;\mu}{}^\mu + \frac{\partial V}{\partial \phi_a} = 0 \quad (a=1, \dots, n) \quad (1.2)$$

If in addition, one demands that the cosmological constant vanish, then there is a further condition on the vacuum state:

$$\langle T_{\mu}^{\mu} \rangle = 0 \quad (1.3)$$

Since there are now $n+1$ equations and only n fields, there is, in general, no non-trivial solution. However, suppose we have a particular non-trivial solution. Then, one of the $n+1$ conditions must be redundant, and there must be only n independent conditions satisfied by the fields in the vacuum state.

The question one must ask now is whether this redundancy is accidental. If so, then at each order of perturbation theory the accident needs to be maintained: this is fine-tuning! If, however, the redundancy is not accidental, then there would seem to be an extra "ingredient" missing which should tell us that the first n conditions are sufficient to fix the last. Such a missing "ingredient" is most likely a symmetry; and the natural candidate is dilatation (or Weyl) invariance. (Note that exact global supersymmetry is an example where there would exist non-trivial solutions but it cannot be used for supersymmetry is known to be broken.)

It is well known that exact dilatation (or Weyl) invariance forbids any explicit mass scales appearing in the theory, in particular, the mass scales associated with phase transitions. The prescription to render a given theory dilatation-invariant (see Coleman [4] and Eq. (2.2)) is to replace each mass scale μ by a field σ , the dilaton, such that it always appears in the combination $\mu e^{\sigma/f}$.

However, once we invoke dilatation invariance and remove the potentially-large contributions to Λ from the mass scales associated with phase transitions, the question arises whether the phase transition will take place at all. As has been stressed many times (see for example [3]), in a Weyl-invariant theory the only way of getting a non-zero v.e.v. for a field is in a "flat direction". That is, there must exist a continuously connected set of vacuum expectation values for the scalar fields along which the vacuum energy does not change. This leads to the (unsatisfactory?) situation in which the fundamental mass scale is undetermined. The alternative is to break the dilatation invariance explicitly by introducing a mass scale without the corresponding dilatation factor of $e^{\sigma/f}$.

In this paper we consider whether dilatation invariance, or some form of approximate dilatation invariance, can lead to an acceptably small cosmological constant. In Section 2 we consider theories in which (i) the dilatation symmetry is broken only by renormalization, (ii) the dilatation symmetry is exact and (iii) the dilatation symmetry appears only in the renormalizable sector of the theory. Only in the trivial free case is the cosmological constant found to naturally small.

These conclusions are illustrated in Section 3 by specific examples, and we show how recent work [5,6] on the subject relates to our general analysis.

2. - DILATATIONS AND THE COSMOLOGICAL CONSTANT

In this section we will discuss whether dilatation symmetry may play a rôle in explaining why the cosmological constant is so small. We assume that the dilatation symmetry is spontaneously broken at a scale f and that the field σ is the resulting Goldstone boson. Under this (global) symmetry the transformation properties are [4]:

$$\begin{aligned} x'_\mu &= e^{-\alpha} x_\mu \\ \sigma' &= \sigma + \alpha f \\ \phi_i' &= e^{\alpha D_i} \phi_i \\ g'_{\alpha\beta} &= g_{\alpha\beta} \end{aligned} \quad (2.1)$$

Here ϕ_i represents any field of dimension D_i and all transformed fields are evaluated at the transformed space-time x'_μ . We may rewrite this transformation in the active form by noting that, if we wish a theory to be invariant under general co-ordinate invariance of space-time as required for relativistic invariance, associated with each space-time integration $\int d^4x$, there should be a factor $\sqrt{g}$. Then Eq. (2.1) is equivalent to a global Weyl symmetry

$$\begin{aligned} x'_\mu &= x_\mu \\ \sigma' &= \sigma + \alpha f \\ \phi_i' &= e^{\alpha D_i} \phi_i \\ g'_{\alpha\beta} &= e^{-2\alpha} g_{\alpha\beta} \end{aligned} \quad (2.2)$$

This form is often more convenient when dealing with questions concerning the cosmological constant for it involves a Weyl rescaling of the metric and this is directly related to the cosmological constant. Thus for a general local variation of the metric $\delta g_{\alpha\beta}$, we have for the matter action

$$\delta S_M = -\frac{1}{2} \int \sqrt{g} T^{\alpha\beta} \delta g_{\alpha\beta} d^4x \quad (2.3)$$

where $T^{\alpha\beta}$ is the matter energy momentum tensor, and under a general (local) Weyl rescaling $\delta g_{\alpha\beta} = \delta\rho g_{\alpha\beta}$ we have

$$g_{\alpha\beta} T^{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S_M}{\delta \rho} \quad (2.4)$$

[for global Weyl rescaling, Eq. (2.4) is true up to the addition of a total divergence which has no effect when vacuum averages are only considered] i.e., the trace of the energy-momentum tensor is related to the variation of the matter action under Weyl rescalings of the metric. From Eq. (2.3) we see that any relation involving dilatation invariance will involve this trace.

Now let us discuss the implications of dilatation symmetry for theories possessing varying degrees of invariance under this symmetry. We write the action in the form $(S_E + S_M)$ where S_E is the gravitational action. The usual form of Einstein gravity (without a cosmological constant) corresponds to the choice

$$S_E = -\frac{1}{2}(\delta\pi G_N)^{-1} \int \sqrt{g} R d^4x \quad (2.5)$$

and Einstein's equations, Eq. (1.1), follow immediately from $(\delta(S_E + S_M)/\delta g^{\alpha\beta}) = 0$. [For simplicity we restrict the analysis to a Robertson-Walker metric.]

Due to the dimensionful constant in Eq. (2.5), S_E is not invariant under dilatations, as is obvious using Eq. (2.2). It may be made invariant, however, by a change in the action $S_E \rightarrow S_E^D$

$$S_E^D = -\frac{1}{2}(\delta\pi G_N)^{-1} \int \sqrt{g} R e^{2\sigma/f} \quad (2.6)$$

Which of these forms (if either) is correct depends on the underlying theory and is beyond the scope of the effective field theory analysis. In what follows we will analyze the implications of both possibilities. Taking the trace of Einstein's equations using S_E^D gives

$$-(\delta\pi G_N)^{-1} R e^{2\sigma/f} = g_{\alpha\beta} T^{\alpha\beta}_M \quad (2.7)$$

If we used S_E instead, the factor $e^{2\sigma/f}$ would be absent on the left-hand side of Eq. (2.7).

2.1 Theories with anomalous dilatation symmetry

We turn now to the form of S_M , assuming that at the classical level, at least, it is dilatation invariant. The classical action $S_M(\phi, \sigma, g)$ is obtained from the conventional action simply by accompanying any dimensionful parameter of dimension D by a factor $e^{D\sigma/f}$ [4]. Beyond the classical level we may expect dilation symmetry to be broken by the presence of the renormalization scale in the renormalized theory and by anomalies associated with the non-invariance of the functional

measure under dilatations. The general form of S_M that results for scalar fields is

$$S_M(\phi, \sigma, g) = \int d^4x \left\{ -\sqrt{g} e^{4\sigma/f} \left[V_{\text{eff}}(\phi_i e^{-D_i \sigma/f}) - \frac{\sigma}{f} \Delta(\sigma, \phi_i e^{-D_i \sigma/f}) \right] \right\} \quad (2.8)$$

In this expression, V_{eff} is a function only of $\bar{\Phi}_i \equiv \phi_i e^{-D_i \sigma/f}$ and gives a term which is manifestly invariant under dilatations since $\bar{\Phi}_i$ is invariant. The anomalous term is the term proportional to $\sigma/f \Delta(\sigma, \bar{\Phi}_i)$, and if evaluated perturbatively about a flat metric, we have $\Delta(\sigma, \bar{\Phi}_i) = \Delta(0, \bar{\Phi}_i)^*$. However, as

discussed in Ref. [5], non-perturbative effects may yield a more complicated σ dependence.

We are now in a position to extract some information about the cosmological constant following from the equations of motion for the σ field which give for a static solution

$$\begin{aligned} \langle -(8\pi G_N)^{-1} R e^{2\sigma/f} \rangle &= \langle 4 e^{\sigma/f} (V_{\text{eff}}(\bar{\Phi}_i) - (\frac{\sigma}{f} + \frac{1}{4}) \Delta(\sigma, \bar{\Phi}_i)) \\ &\quad - \frac{1}{4} \frac{\sigma}{f} \frac{\partial \Delta(\sigma, \bar{\Phi}_i)}{\partial (\sigma/f)} \rangle \end{aligned} \quad (2.9)$$

Here, the outer brackets indicate that all fields are to be set to their vacuum expectation values (v.e.v.'s). Equation (2.1) may be directly compared to the equation obtained varying the metric and taking the trace:

$$\begin{aligned} \langle -(8\pi G_N)^{-1} R e^{2\sigma/f} \rangle &= \langle 4 e^{4\sigma/f} (V_{\text{eff}}(\bar{\Phi}_i) - \frac{\sigma}{f} \Delta(\sigma, \bar{\Phi}_i)) \\ &\quad - \frac{1}{2} \frac{\sigma}{f} g_{\alpha\beta} \frac{\partial \Delta(\sigma, \bar{\Phi}_i)}{\partial g_{\alpha\beta}} \rangle \end{aligned} \quad (2.10)$$

Comparison of Eqs. (2.9) and (2.10) gives

*) If the dilatation current has an anomalous trace $\partial_\mu J_D^\mu \equiv \theta \neq 0$, then the anomalous term in $\mathcal{L}_{\text{eff}}$ is $\int \theta d\sigma$. Clearly if $\Delta(\sigma, \bar{\Phi}_i) = \Delta(0, \bar{\Phi}_i)$ then $\theta = \Delta(0, \bar{\Phi}_i)/f$.

$$\langle e^{4\sigma/f} (\Delta(\sigma, \bar{\Phi}_i) - \frac{1}{4} \frac{\alpha}{f} (\frac{\delta}{\delta(\sigma/f)} - 2 \delta^{\alpha\beta} \frac{\delta}{\delta g^{\alpha\beta}}) \Delta(\sigma, \bar{\Phi}_i)) \rangle = 0 \quad (2.11)$$

This is the condition found in Ref. [5] by taking the divergence of the dilatation current $\partial_\mu J_D^\mu \equiv \theta$, and corresponds to the vanishing of the anomalous trace, θ , of the energy momentum tensor. We note that Weyl invariance of Δ implies $\langle (\partial/\partial(\sigma/f) - 2g^{\alpha\beta} \partial/\partial g^{\alpha\beta}) \Delta \rangle = 0$.

As may be seen from Eqs. (2.9) and (2.10) it does not, by itself, guarantee the vanishing of the cosmological constant for there is still the non-anomalous piece to include. As we will discuss more fully in the next section, in general, we expect to have to fine tune parameters to make this piece vanish. Thus we conclude that models with anomalous dilatation symmetry do not solve the cosmological constant problem.

2.2 Theories with exact dilatation symmetry

In Ref. [6], a model was written down in which there remains a non-anomalous dilatation symmetry, even after including one-loop radiative corrections. This is, in principle, a possibility as the renormalization scale, which normally introduces the anomalies, may itself originate in the spontaneous breakdown of some more unified (finite?) theory. In this case the renormalization scale will be a physical quantity and may be associated with the correct power of e^σ to leave the dilatation symmetry unbroken provided, of course, the underlying theory was itself dilatation invariant.

For such a theory, Eq. (2.8) is modified simply by omitting the anomalous term proportional to Δ . As a result, Eqs. (2.9) and (2.10) are identical and read

$$\langle -(8\pi G_N)^{-1} R e^{2\sigma/f} \rangle = \langle 4 e^{4\sigma/f} V_{eff}(\bar{\Phi}_i) \rangle \quad (2.12)$$

Exact dilatation invariance of the vacuum, $dV/d\sigma = 0$, requires $\sigma = -\infty$ which in turn ensures the cosmological constant vanishes but this is the trivial non-interacting case. The non-trivial case corresponds to σ finite and then the cosmological constant is non-zero in general. In fact the v.e.v. of σ is undetermined and we have a flat direction. In Section 3 we discuss the model of Ref. [6] and show that it indeed satisfies Eq. (2.12).

2.3 Theories with approximate dilatation symmetry

There remains one more possibility which, in many ways, we find more reasonable. It is that one sector of the theory should possess an enlarged symmetry, in this case a dilatation symmetry, broken in higher-order by the radiative effects of the other sectors of the theory which are not dilatation invariant. For example, the renormalizable terms of an effective field theory describing massless states are dilatation invariant. Masses in such a theory may arise via scalar fields acquiring v.e.v.'s via dimensional transmutation [7], and in this case the only breaking of the dilatation symmetry will be by the anomalies associated with renormalization.

3. - MODEL CALCULATIONS

In this section we will illustrate the general results of Section 2 by specific examples in the hope of gaining some insight into the possible solution of the cosmological constant problem in realistic theories.

3.1 A model with an anomalous dilatation symmetry

We start with the example discussed in detail by Buchmüller and Dragon [6]. They considered a model describing a complex scalar field ϕ and the dilaton σ . At tree level the dilatation-invariant Lagrangian appearing in Eq. (2.8) has the form

$$V_{eff}^0(\sigma, \bar{\Phi}) = a^4 + \mu^2 \bar{\Phi}^+ \bar{\Phi} + \frac{\lambda}{2} (\bar{\Phi}^+ \bar{\Phi})^2 \quad (3.1)$$

where $\bar{\Phi} = \phi e^{-\sigma/f}$ and a , μ and λ are constants. Assuming the dilatation-invariant form for the gravitational interaction, as given in Eq. (2.6) leads to Eq. (2.12) at tree level. It is clear from this equation that if the cosmological constant is to vanish for non-trivial constant solutions of σ and ϕ , the form of V_{eff} must be restricted with $a^4 = \mu^4/2\lambda$ for $\mu^2 < 0$, giving

$$V_{eff}^0(\sigma, \bar{\Phi}) = \frac{\lambda}{2} \left(\frac{\mu^2}{\lambda} + \bar{\Phi}^+ \bar{\Phi} \right)^2 \quad (3.2)$$

Even at tree level, the irrelevance of the exact dilatation symmetry to the cancellation of the cosmological constant is clear for the choice $a^4 = \mu^4/2\lambda$ is imposed here by hand. However, we may suppose that this result follows from the structure of the underlying theory (e.g., from supersymmetry), and ask whether the vanishing of the cosmological constant persists in higher order.

Buchmüller and Dragon evaluated the one-loop radiative corrections for the model given by Eq. (3.2) treating σ as a classical background field and imposing renormalization conditions such that in the renormalized effective potential all mass parameters have the same coupling to the dilaton field as in the classical Lagrangian^{*}). This leads to the form of Eq. (2.8) with $\Delta(\sigma, \bar{\Phi}) \equiv \Delta(0, \bar{\Phi})$, and hence from Eq. (2.11), $\langle e^{4\sigma/f} \Delta(\sigma, \bar{\Phi}) \rangle = 0$.

From Eq. (2.10) we expect then that

$$\langle -(8\pi G_N)^{-1} R e^{2\sigma/f} \rangle = \langle 4 e^{4\sigma/f} V_{\text{eff}}(\bar{\Phi}) \rangle \quad (3.3)$$

For finite v.e.v.s of σ the cosmological constant will vanish only if the full non-anomalous effective potential $\langle V_{\text{eff}}(\bar{\Phi}) \rangle$ vanishes. To see that this is unlikely, note that the v.e.v. of $\bar{\Phi}$ is separately determined by the condition $\langle \partial V_{\text{eff}}(\bar{\Phi}) / \partial \bar{\Phi} \rangle = 0$, so that the condition $\langle V_{\text{eff}}(\bar{\Phi}) \rangle = 0$ will occur only if the parameters λ and μ are fine-tuned. This may readily be verified using the explicit form of $V_{\text{eff}}(\bar{\Phi})$ given in Ref. [6].

To conclude this section we turn now to the suggestion of Peccei, Sola and Wetterich [5] who discuss the possibility of cancelling the cosmological constant by letting Δ in Eq. (2.11) have a more complicated σ dependence, such as might result from, e.g., gluon-like condensates. We recast the discussion of Section 2.1 into their language. The key observation they made was to note that the anomalous divergence of the dilatation current is linked to T_{μ}^{μ} :

$$\partial_{\mu} J_D^{\mu} = T_{\mu}^{\mu} - f \frac{\delta V}{\delta \sigma} - \mathcal{D}_i \phi_i \frac{\delta V}{\delta \phi_i} \equiv 0 \quad (3.4)$$

Here V stands for all the potential terms including the $\sigma \Delta$ term. The field equations for the dilaton and matter fields are

$$\begin{aligned} \sigma_{;\mu}^{\mu} &\propto T_{\mu}^{\mu} - f \frac{\delta V}{\delta \sigma} \\ \phi_{i;\mu}^{\mu} &= - \frac{\delta V}{\delta \phi_i} \end{aligned} \quad (3.5a)$$

so that the stationary values are just given by

^{*}) In a sense, these renormalization conditions violate scale invariance minimally [6].

$$\begin{aligned} \langle T_{\mu}^{\mu} - f \frac{\delta V}{\delta \sigma} \rangle &= 0 \\ \langle \frac{\delta V}{\delta \phi_i} \rangle &= 0 \end{aligned} \quad (3.5b)$$

Hence, the dilaton field equation drives the anomalous divergence $\langle \theta \rangle$ to zero [see Eq. (2.11)]. However, as the authors themselves point out [5], for this to have a bearing on the cosmological constant, $\langle T_{\mu}^{\mu} \rangle$ must be given by $\langle \theta \rangle$. This means that we need an extra "ingredient" to ensure that $\langle \delta V / \delta \sigma \rangle = 0$, and only in this case can we conclude that the cosmological constant vanishes.

The important point is that in this case it is the extra ingredient and not the axion-like mechanism [5] which is responsible for $\Lambda = 0$. So we are back where we started: looking for a reason (extra ingredient) why the cosmological constant is zero.

3.2 A model with unbroken dilatation symmetry

After computing the radiative corrections in the model given by Eq. (3.2), Buchmüller and Dragon [6] sought to include the effects of gravity by arguing that the metric couples to ϕ only in the dilatation-invariant† form $g_{\mu\nu} \exp(2\sigma/f)$. This led them to propose that the form of Eq. (2.8) should be modified to read

$$\begin{aligned} S_H(\phi, \sigma, g) = \int d^4x \, -\sqrt{g} \, e^{4\sigma/f} \bigg[V_{\text{eff}}(\bar{\Phi}) \\ - \left(\frac{\sigma}{f} + \frac{1}{4} \ln(-\sqrt{g}) \right) \Delta(0, \bar{\Phi}_i) \bigg] \end{aligned} \quad (3.6)$$

We will return in a moment to a discussion of whether this is reasonable, but first let us consider the implications for the cosmological constant. Equation (3.6) is clearly invariant under the dilatation transformations given in Eq. (2.2) and so should be covered by the discussion of Section 2.2. From Eq. (2.12) we see the condition that the cosmological constant should vanish is $\langle e^{4\sigma/f} V_{\text{eff}}(\bar{\Phi}) \rangle$ vanishes. Once again this requires fine tuning of the parameters a , μ and λ if this is to be true (with $\sigma \neq -\infty$), with the v.e.v. of $\bar{\Phi}$ determined by the condition $\langle \partial V_{\text{eff}}(\bar{\Phi}) / \partial \Phi \rangle = 0$.

Let us now return to a discussion of the form of Eq. (3.6). The appearance of the $\log(-\sqrt{g})$ term is worrying since, at first sight, it suggests general covariance is broken. Here we will show how the calculation of radiative corrections do not, in

general, lead to such terms while preserving dilatation invariance. Since the final theory, including radiative corrections, is to be dilatation invariant the initial theory should clearly be so too. The form of Eq. (3.2) is manifestly dilatation invariant after replacing the fields σ and ϕ by the fields σ and $\bar{\phi}$. At the quantum level, however, this will not be true because the functional measure $\mathcal{D}\sigma \mathcal{D}\bar{\phi}$ is not invariant under dilatations. What one can do is to define a new functional measure invariant under dilatations and this will guarantee that the theory, including quantum corrections, is dilatation invariant. To do this first note that the conventional definition for the measure $\mathcal{D}\bar{\phi}$ is given by

$$\mathcal{D}\bar{\phi} = \prod_i da_i \sqrt{|(e_n, e_m)|} \quad (3.7)$$

where e_n is any choice of basis for the functional space, $\phi = \sum_i a^i e_i$ ($i=1,2,\dots$) and $(,)$ is the inner product on the function space. This volume element is clearly (formally) invariant under changes of basis, an essential ingredient in having a basis-free definition of the functional measure. The conventional inner product for a scalar model in the presence of a background gravitational field g_{mn} is given by

$$(\xi, \eta)_g = \int \sqrt{g} \xi \eta d^4x \quad (3.8a)$$

for the two scalar field configurations ξ and η . Using this definition, under a dilatation transformation

$$\mathcal{D}\bar{\phi} \rightarrow \mathcal{D}\bar{\phi} \sqrt{|e^{-2\sigma} g|} \quad (3.8b)$$

Now, we may observe that there is a straightforward way to construct a quantum theory, based on a dilatation-invariant classical theory, which has vanishing Weyl, or dilatation, anomaly and preserves general co-ordinate invariance. This is to assume the functional volume element is formally equivalent to a new inner product

$$(\xi, \eta)_u = \int \sqrt{g} e^{2\sigma/f} \xi \eta d^4x \quad (3.9a)$$

which immediately leads to an invariance of the measure

$$\mathcal{D}\bar{\phi} \rightarrow \mathcal{D}\bar{\phi} \quad (3.9b)$$

Here $V_{\text{eff}}(\sigma, \bar{\phi})$ may now contain derivatives of the σ field. These do not contribute when the vacuum average is taken.

From this it is clear that choosing a regularization procedure which respects both general co-ordinate invariance and Weyl or dilatation invariance leads to a

quantum theory which is both general co-ordinate invariant and dilatation invariant. [Note that we are dealing with global Weyl invariance as defined in Eq. (2.2), not local Weyl symmetry.] Accordingly, we start with the classical Lagrangian of Eq. (3.2) and using the measure defined through Eq. (3.9) we may evaluate the one-loop corrections using the heat kernel method with ζ_A -function regularization. This immediately gives

$$S_H(\phi, \sigma, g) = \int d^4x \sqrt{-g} e^{4\sigma/f} V_{\text{eff}}(\bar{\Phi}) \quad (3.10)$$

Comparison with Eq. (3.6) shows that in both cases we have a dilatation-invariant quantum theory, but with this natural generalization of the measure, the troublesome $\ln(-g)$ term is not needed. Note that this is the modification given by Eq. (3.8) to the measure, i.e., attaching $\sqrt{g}$ which eliminates the $\ln g$ -term in Eq. (3.6) as given in Ref. [6]. The addition of the extra factor of $e^{2\sigma/f}$ as in Eq. (3.9a) now also eliminates the $\sigma\Delta$ -term. Of course, as far as the cosmological constant is concerned, this does not alter the conclusion reached above that $\langle e^{4\sigma/f} V_{\text{eff}}(\bar{\Phi}) \rangle$ should vanish and that this will only be possible through fine tuning of parameters (unless $\sigma = \infty$). This argument is flawed if V_{eff} acquires non-trivial σ -dependence in higher (non-renormalizable) order. In the next section we consider the possibility that this σ dependence arises because the dilatation symmetry is only approximate. Here we discuss the possibility that the σ dependence of V_{eff} occurs in the unbroken theory with $\Delta(\sigma, \bar{\Phi}) \equiv 0$. Equations (2.14) and (2.13) now simply give

$$\langle 4e^{\sigma/f} \left(V_{\text{eff}}(\sigma, \bar{\Phi}) + \frac{1}{4} \frac{\partial V_{\text{eff}}(\sigma, \bar{\Phi})}{\partial(\sigma/f)} \right) \rangle = 0 \quad (3.11)$$

$$\begin{aligned} & \langle 4e^{\sigma/f} \left(V_{\text{eff}}(\sigma, \bar{\Phi}) + \frac{1}{2} g^{\alpha\beta} \frac{\partial}{\partial g^{\alpha\beta}} V(\sigma, \bar{\Phi}) \right) \rangle \\ & = \langle (8\pi G_N)^{-1} R \rangle \end{aligned} \quad (3.12)$$

The left-hand side of Eq. (3.12) now vanishes by Eq. (3.11) and using

$$g^{\alpha\beta} \frac{\partial}{\partial g^{\alpha\beta}} V_{\text{eff}}(\sigma, \bar{\Phi}) = \frac{1}{2} \frac{\partial V}{\partial(\sigma/f)}$$

which follows since the σ dependence of V_{eff} appears only through the combination $g_{\alpha\beta} e^{2\sigma/f}$ as the full theory is Weyl invariant. This implies the vanishing of the

cosmological constant, but is acceptable only if Eq. (3.11) has a solution with $\langle \sigma \rangle$ finite without fine tuning.

In fact, in the presence of a background gravitational field, V_{eff} does have σ -dependence introduced through the effect of (non-renormalizable) terms in the action proportional to $R e^{2\sigma/f}$. These terms do not, however, lead to a non-trivial solution to Eq. (3.11) as, clearly, $R \rightarrow 0$ implies $\langle \sigma \rangle \rightarrow -\infty$. We have not been able to identify any contribution which allows for a non-trivial solution to Eq. (3.11) in the Weyl-invariant case.

3.3 A model with an approximate dilatation symmetry

Here we will consider a model realizing the idea presented in Section 2.3 that the renormalizable sector should have a dilatation symmetry, but the gravitational sector should not. The simplest possibility is just to take the form of Eq. (3.1) for the renormalizable sector and to take the form of Eq. (2.6) for the gravitational sector. This is not quite what we were arguing for in Section 2.3 because of the explicit mass terms in Eq. (3.1) but it will be good enough to illustrate the possibilities.

Let us consider first what happens at tree level. Equations (2.13) and (2.14) give

$$\begin{aligned} \langle -(8\pi G_N)^{-1} R \rangle &= \langle 4 e^{4\sigma/f} V_{\text{eff}}^0(\bar{\Phi}) \rangle \\ 0 &= \langle 4 e^{4\sigma/f} V_{\text{eff}}^0(\bar{\Phi}) \rangle \end{aligned} \quad (3.13)$$

Clearly, Eqs. (3.13) require that the cosmological constant vanish. However, unless we impose the fine-tuning constraint $a^4 = \mu^4/2\lambda$ to give Eq. (3.4), there will be no solution to Eq. (3.13) and to $\langle \partial V_{\text{eff}}/\partial \bar{\Phi} \rangle = 0$, other than the uninteresting one $\sigma = -\infty$.

Things change when we include radiative corrections for now, following the analysis of Ref. [6], the renormalizable Φ interactions with the background σ field give for Eq. (2.13) and (2.14) the form

$$\begin{aligned} \langle -(8\pi G_N)^{-1} R \rangle &= \langle 4e^{4\sigma/f} (V_{eff}(\bar{\Phi}) - \frac{\sigma}{f} \Delta(0, \bar{\Phi})) \rangle \\ 0 &= \langle 4e^{4\sigma/f} (V_{eff}(\bar{\Phi}) - (\frac{\sigma}{f} + \frac{1}{4}) \Delta(0, \bar{\Phi})) \rangle \end{aligned} \quad (3.14)$$

where [6]

$$\begin{aligned} \Delta(0, \bar{\Phi}) &= -(\gamma_1(\lambda)\mu^4 + \gamma_2(\lambda)\mu^2 \bar{\Phi}^\dagger \bar{\Phi} \\ &\quad + \frac{\beta(\lambda)}{\lambda} \frac{1}{2} \lambda (\bar{\Phi}^\dagger \bar{\Phi})^2) \end{aligned} \quad (3.15)$$

and $V_{eff}(\bar{\Phi})$ is given in Ref. [6].

Now, due to the appearance of the σ factor in the second of Eqs. (3.14) there will be a non-trivial solution to this equation and to the $\bar{\Phi}$ field equation:

$$\langle e^{4\sigma/f} \left(\frac{\partial(V_{eff}(\bar{\Phi}) - \Delta(0, \bar{\Phi}) \frac{\sigma}{f})}{\partial \bar{\Phi}} \right) \rangle = 0$$

for some non-vanishing finite values of σ and $\bar{\Phi}$. However, this is a rather minimal improvement as, from the first of Eqs. (3.14) we find in this case

$$\langle -(8\pi G_N)^{-1} R \rangle = -\langle e^{4\sigma/f} \Delta(0, \bar{\Phi}) \rangle \quad (3.16)$$

and, again, the vanishing of the anomalous term Δ will occur only for special values of the parameters [cf. Eq. (3.15)].

Is there any way this unfortunate result may be evaded? Consider what happens if $\langle \Delta(0, \bar{\Phi}) \rangle$ vanishes in Eq. (3.4). Then indeed the cosmological constant vanishes, but it seems that, again, this will happen only for the uninteresting case $\sigma = -\infty$ because we have to solve the simultaneous equations

$$\begin{aligned} \langle e^{4\sigma/f} V_{eff}(\bar{\Phi}) \rangle &= 0 \\ \langle e^{4\sigma/f} \left(\frac{\partial(V_{eff}(\bar{\Phi}) - \Delta(0, \bar{\Phi}) \frac{\sigma}{f})}{\partial \bar{\Phi}} \right) \rangle &= 0 \end{aligned} \quad (3.17)$$

This time, however, there is a possible non-trivial solution provided $\langle \partial \Delta(0, \bar{\Phi}) / \partial \bar{\Phi} \rangle = 0$, for then there is a σ dependence in the second of Eqs. (3.17).

To summarize, the conditions on $\Delta(0, \bar{\Phi})$ necessary (but not sufficient) for Eq. (3.14) to ensure vanishing of the cosmological constant are

$$\begin{aligned} (i) \quad & \langle \Delta(0, \bar{\Phi}) \rangle = 0 \\ (ii) \quad & \langle \frac{\partial \Delta(0, \bar{\Phi})}{\partial \bar{\Phi}} \rangle \neq 0 \end{aligned} \quad (3.18)$$

A glance at Eq. (3.15) shows this is not the case in the model considered here except for exceptional values of the parameters. The only way to ensure Eq. (3.18) without the fine tuning of parameters is to have $\Delta(0, \bar{\Phi}_i)$ of the form $\Phi_i f(\Phi_{j \neq i})$ where Φ_i has vanishing v.e.v.'s. However, this will induce a linear term in Φ_i in the full potential $V_{\text{eff}}(\phi_i) + \sigma/s \Delta(0, \bar{\Phi}_i)$ which will in turn induce a v.e.v. for Φ_i in contradiction with our initial requirement. Thus we conclude again that the cosmological constant will only vanish as a result of fine-tuning of parameters.

4. - CONCLUSIONS

The conclusions one may draw from the above arguments are unfortunately rather negative. Dilatation (or Weyl) invariance does not seem to shed any light on the cosmological constant problem, and we suspect that the solution cannot be understood in terms of an effective four-dimensional field theory alone. Perhaps a true understanding can only be found at the level of the superstring [8] - but this is the subject for another paper!

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