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QUANTUM BREMSSTRAHLUNG IN ALMOST UNIFORM FIELDS

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A B S T R A C T

Inhomogeneity effects in quantum bremsstrahlung, in the case of a weakly non-uniform deflecting field, are calculated in the extreme quantum limit.

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1. Introduction.

The quantum theory of bremsstrahlung energy loss, for highly relativistic particles in quasiclassical orbits, was developed by J. Schwinger and a series of Soviet theorists [1 - 7]. In particular the case of a uniform deflecting field was worked out in detail. The subject has become topical because of the beamstrahlung problem in linear colliders[8-17]. Initially the field has been treated as locally uniform, and the result averaged over the various strengths present in the bunch. The question arises as to on what scale the field must be approximately uniform for this treatment to be good.

When the bremsstrahlung process is largely classical, the important characteristic length is

$$L_C = \rho/\gamma \tag{1}$$

where ρ is the radius of curvature of the track, and γ is the usual relativistic Fitzgerald contraction factor. A condition for the process to be largely classical is that the typical classical photon energy

$$\gamma^3/\rho \tag{2}$$

be small compared with the initial particle energy $\epsilon = m\gamma$. When, on the contrary,

$$\tau = (\gamma^3/\rho)/m\gamma = \gamma^2/(m\rho) \tag{3}$$

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is large compared with 1, we have the 'ultra quantum' limit,[2], with the photon typically taking a substantial fraction of the primary energy. For this extreme quantum case, it was pointed out [17] that the characteristic length for the convergence of the integrals involved is not L_C , but the much greater

$$L_Q = \gamma^{1/3} L_C = \gamma^{1/3} \rho/\gamma = (\gamma/m)^{1/3} (\rho/\gamma)^{2/3} \quad (4)$$

It was suggested therefore that nonuniformity and end effects would be relatively unimportant if the bunch were longitudinally uniform and long on this scale.

The study of small nonuniformities has been initiated by Pisin Chen [10], as regards the first corrections, proportional to

$$(L_Q \cdot B^{-1} dB/dz)^2 \quad \text{and} \quad L_Q^2 B^{-1} d^2 B/dz^2 \quad (4)$$

where B is the deflecting force, and z is distance along the particle trajectory. We present here a calculation of the coefficients of these terms. Denoting by I the average energy loss per unit time, and by I_0 the approximation ignoring inhomogeneity, we find

$$I/I_0 = 1 - .156 (L_Q \cdot B^{-1} dB/dz)^2 + .117 L_Q^2 B^{-1} d^2 B/dz^2 \quad (5)$$

for

$$\gamma \gg 1, \quad \gamma \gg 1 \quad (6)$$

where L_Q is the local value of the convergence length (3).

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In application to the beamstrahlung problem, the approximation of small nonuniformity fails near the bunch ends, and special study of end effects is required [14-17].

2. Departure

We start with the following formula [7] for the mean energy dE radiated into a frequency range $d\omega$ and solid angle do by a relativistic electron:

$$dE = (e^2 \omega^2 / (4\pi^2)) d\omega do \int dt' \int dt'' A e^{i\phi} \quad (7)$$

with

$$A = [(\epsilon^2 + \epsilon'^2) / (2\epsilon'^2)] (v(t') \cdot v(t'') - 1) + (1/2)(\omega/\epsilon')^2 (m/\epsilon)^2 \quad (8)$$

$$\phi = (\epsilon/\epsilon') \omega [(t' - t'') - n \cdot (r(t') - r(t''))] \quad (9)$$

where

$$\epsilon = \text{initial electron energy} \quad (10)$$

$$\epsilon' = \epsilon - \omega \quad (11)$$

$r(t)$ = position on classical orbit unperturbed by radiation

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$$\mathbf{v}(t) = (d/dt)\mathbf{r}(t) \quad (12)$$

$$\mathbf{n} = \text{unit vector in photon direction} \quad (13)$$

$$e^2 = 1/137 \quad (14)$$

$$\hbar/(2\pi) = c = 1 \quad (15)$$

With

$$\delta = E/\epsilon, \quad \xi = \epsilon'/\epsilon, \quad (1-\xi) = \omega/\epsilon \quad (8)$$

$$(16)$$

the formulae can be rewritten

$$d\delta = d\xi \text{ do } (e^2\omega^2/4\pi^2) \int dt' \int dt'' A e^{i\phi} \quad (17)$$

$$A = [(1/2\xi^2)+(1/2)] (\mathbf{v}(t') \cdot \mathbf{v}(t'') - 1) + (1/2)[(1 - \xi)/\xi]^2/\gamma^2 \quad (18)$$

$$\phi = (\omega/\xi)(t' - t'' - \mathbf{n} \cdot \mathbf{r}) \quad (19)$$

$$\mathbf{r} = \mathbf{r}(t') - \mathbf{r}(t'') \quad (20)$$

Performing the integral over solid angle, i.e. over all directions of $\mathbf{n}$, gives

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$$d\delta/d\xi = (e^2/\pi) \omega x \int dt' \int dt'' (A/r) \sin(\omega r/\xi) \cos(\omega(t'-t'')/\xi) \quad (21)$$

with

$$r = |r|(t'-t'')/|t'-t''| \quad (22)$$

In

$$\sin(\omega r/\xi) \cos(\omega(t'-t'')/\xi) = \{(\sin(\omega(r-t'+t'')/\xi) + \sin(\omega(r+t'-t'')/\xi))\}/2 \quad (23)$$

the second term oscillates very quickly for a particle travelling with nearly the velocity of light. Neglecting the variation of v on this scale, and noting that r is well approximated by $(t'-t'')$ and that

$$\int dt/t \sin(Wt) = \pi \quad (24)$$

we have

$$d\delta/d\xi = (1/2)(e^2/\pi) \omega \xi \int dt' \int dt'' (A/r) \{\pi \delta(t'-t'') - \sin(\theta)\} \quad (25)$$

where

$$\theta = (\omega/\xi)(t'-t''-r) \quad (26)$$

It is convenient to introduce normalized variables, denoted by capital letters, such that

$$t = (\rho/\gamma)T, \quad r = (\rho/\gamma)R, \quad z = (\rho/\gamma)Z \quad (27)$$

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$$v_x = V_x/\gamma, \quad v_y = V_y/\gamma \quad (28)$$

$$(x, y) = (\rho/\gamma)((X, Y)/\gamma) \quad (29)$$

where ρ is some typical radius of curvature. Then

with

$$T = (T' + T'')/2, \quad \tau = (T' - T'') \quad (30)$$

and taking advantage of the symmetry of the integrand to restrict the integration to positive τ , we have

$$d\delta/d\xi = (e^2/\pi)\Omega \int dT \{ \int_0^\infty d\tau (F/R) \sin(\theta) - (\pi/2)F_0 \} \quad (31)$$

where

$$F = \gamma^2(1+\xi^2)(1-v(t') \cdot v(t'')) - (1-\xi)^2 \quad (32)$$

$$F_0 = 2\xi \quad (33)$$

$$\Omega = (1 - \xi)/(2\xi\tau) \quad (34)$$

We suppose that the unperturbed classical orbit deviates little from a straight line [18], and drop terms of higher order than second in the deviation. Take Cartesian coordinates with origin at $r((t' + t'')/2)$ and z axis in the direction of $v((t' + t'')/2)$. Then, for large γ , and small deviations

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$$v(t') = (0, v'_x, v'_y, 1 - 1/(2\gamma^2) - v'^2_x/2 - v'^2_y/2) \quad (35)$$

$$v(t'') = (0, v''_x, v''_y, 1 - 1/(2\gamma^2) - v''^2_x/2 - v''^2_y/2) \quad (36)$$

(where $v' = v(t')$, etc.),

$$(1 - v(t') \cdot v(t''))\gamma^2 = 1 + (1/2)(V'_x - V''_x)^2 + (1/2)(V'_y - V''_y)^2 \quad (37)$$

so that

$$F = (1 + \xi^2)(1 + (1/2)(V'_x - V''_x)^2 + (1/2)(V'_y - V''_y)^2) - (1 - \xi)^2 \quad (38)$$

In

$$r = \{(z' - z'')^2 + (x' - x'')^2 + (y' - y'')^2\}^{1/2} \quad (39)$$

the z contribution is dominant, so that to second order in small transverse deviations

$$r = z' - z'' + \{(x' - x'')^2 + (y' - y'')^2\} / \{2(z' - z'')\} \quad (40)$$

In terms of normalized variables then (with an extra factor γ in

X and Y , as compared with Z ; see (27)(29))

$$(\tau - R) = \tau - (Z' - Z'') - \{(X' - X'')^2 + (Y' - Y'')^2\} / (2\tau\gamma^2) \quad (41)$$

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(approximating $(Z'-Z'')$ by τ in the small transverse contribution).

Then

$$\begin{aligned}\theta &= 2\Omega\gamma^2(\tau-R) \\ &= \psi - \Omega\{(X'-X'')^2 + (Y'-Y'')^2\}/\tau\end{aligned}\quad (42)$$

where

$$\psi = 2\Omega\gamma^2\{\tau - (Z'-Z'')\} \quad (43)$$

Differentiating this last with respect to τ ,

$$d\psi/d\tau = 2\Omega\gamma^2\{1 - (1/2)(dZ'/dT' + dZ''/dT'')\} \quad (44)$$

Neglecting any variation of γ , we have to a sufficient approximation

$$dZ'/dT' = \{1 - \gamma^{-2} - (dX'/dT')^2 - (dY'/dT')^2\}^{1/2} \quad (45)$$

$$= 1 - (1 + V'_x{}^2 + V'_y{}^2)/(2\gamma^2) \quad (46)$$

and likewise for dZ''/dT'' . Then

$$d\psi/d\tau = \Omega\{1 + (V'_x{}^2 + V'_y{}^2 + V''_x{}^2 + V''_y{}^2)/2\} \quad (47)$$

Let us write (31) in the form

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$$\delta = (e^2/\pi)\Omega \int d\xi \int dT \quad S(\infty, T, \xi) \quad (48)$$

Then for fixed ξ and T we have to solve the following equations. For $\tau = 0$,

$$X' = X'' = Y' = Y'' = 0 \quad (49)$$

$$V'_x = V''_x = V'_y = V''_y = 0 \quad (50)$$

$$\psi = 0 \quad (51)$$

$$S = -\pi\xi \quad (52)$$

And as τ varies

$$dX'/d\tau = V'_x/2, \quad dX''/d\tau = -V''_x/2 \quad (53)$$

$$dY'/d\tau = V'_y/2, \quad dY''/d\tau = -V''_y/2 \quad (54)$$

$$dV'_x/d\tau = -K_x(T')/2, \quad dV''_x/d\tau = +K_x(T'')/2 \quad (55)$$

$$dV'_y/d\tau = -K_y(T')/2, \quad dV''_y/d\tau = +K_y(T'')/2 \quad (56)$$

(where the quantity K , in general a function of T , is the deflecting force in units of that defining the typical radius of curvature ρ).

$$d\psi/d\tau = \Omega\{1 + (V'^2_x + V'^2_y + V''^2_x + V''^2_y)/2\} \quad (57)$$

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$$\theta = \psi - \Omega\{(X'-X'')^2 + (Y'-Y'')^2\}/\tau \quad (58)$$

$$F = (1+\xi^2)\{1 + ((V'_x-V''_x)^2 + (V'_y-V''_y)^2)/2\} - (1-\xi)^2 \quad (59)$$

$$dS/d\tau = F/\tau \quad (60)$$

Integration of this last to $\tau = \infty$ and substitution in (48) gives δ .

3. Uniform field.

Take

$$K_x = 1, \quad K_y = 0 \quad (61)$$

Then

$$V'_x = -\tau/2, \quad V''_x = +\tau/2, \quad V'_y = V''_y = 0 \quad (62)$$

$$X' = X'' = -\tau^2/8, \quad Y' = Y'' = 0. \quad (63)$$

$$d\psi/d\tau = \Omega(1 + \tau^2/4) \quad (64)$$

$$\theta = \psi = \Omega(\tau + \tau^3/12) \quad (65)$$

$$F = (1 + \xi^2)(1 + \tau^2/2) - (1 - \xi)^2 \quad (66)$$

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whence

$$d^2\delta/d\xi dT = (e^2/\pi)\Omega\{ \int_0^\infty (d\tau/\tau) F \sin\theta - F_0\pi/2 \} \quad (67)$$

in agreement with ref[17], eqs. (35),(36). This gives, in terms of Airy functions,

$$(d^2\delta/d\xi dT) = -e^2(1-\xi)/(\xi\Upsilon)\{(1+\xi^2)Ai'(f)/f + \xi \int_f^\infty dv Ai(v)\} \quad (68)$$

with

$$f = \{(1-\xi)/(\xi\Upsilon)\}^{2/3} \quad (69)$$

This is one form of the result of Sokolov and Ternov ([15],eq.8). As pointed out by them, it simplifies for large Υ :

$$(d^2\delta/d\xi dT) = (e^2/\pi)\{(1-\xi)/(\xi\Upsilon)\}^{1/3}(1+\xi^2)(-\pi Ai'(0)) \quad (70)$$

where

$$-\pi Ai'(0) = .813 \quad (71)$$

One comes directly to the asymptotic form (70) by noting in (67) that the integral converges when the phase oscillates rapidly. For large Υ , and so large Ω (for ξ not very small) this happens only at large τ . The τ^3

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term then dominates in the phase, and the τ^2 in F. Retaining only these gives (70).

More precisely, remembering that our unit of time, or distance, is $L_C = \rho/\gamma$, we see that in the extreme quantum case the characteristic time or length at which rapid oscillation sets in is of order

$$L_{Q\xi} = (2\Omega)^{-1/3} \rho/\gamma = (x\Upsilon/(1-x))^{1/3} (\rho/\gamma) \quad (72)$$

which for ξ not very small or large is of order L_Q .

Integrating (70) over ξ , we have in the central region

$$d\delta/dt = 1.165 (e^2/\pi) L_Q^{-1} = 1.165 (e^2/\pi) (m\gamma/\rho^2)^{1/3} \quad (73)$$

where t is time in ordinary units, in whatever reference frame is used.

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4. Nonuniform field

Take now, in the neighbourhood of $T=0$,

$$K_x(T) = 1 + aT + bT^2/2, \quad K_y(T) = cT + dT^2/2 \quad (74)$$

Then for $T=0$, from (53)-(60),

$$V'_x = -\tau/2 - a\tau^2/8 - b\tau^3/48 \quad (75)$$

$$V''_x = +\tau/2 - a\tau^2/8 + b\tau^3/48 \quad 4(76)$$

$$V'_y = -c\tau^2/8 - d\tau^3/48 \quad (77)$$

$$V''_y = -c\tau^2/8 + d\tau^3/48 \quad (78)$$

$$X' = -\tau^2/8 - a\tau^3/48 - b\tau^4/384 \quad (79)$$

$$X'' = -\tau^2/8 + a\tau^3/48 - b\tau^4/384 \quad (80)$$

$$Y' = -c\tau^3/48 - d\tau^4/384 \quad (81)$$

$$Y'' = +c\tau^3/48 - d\tau^4/384 \quad (82)$$

$$d\psi/d\tau = \Omega(1 + \tau^2/4 + a^2\tau^4/64 + b\tau^4/48 + c^2\tau^4/64) \quad (83)$$

(to second order in a and c , and first order in b and d)

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$$\psi = \Omega \{ \tau + \tau^3/12 + (a^2 + c^2)\tau^5/320 + b\tau^5/240 \} \quad (84)$$

$$\theta = \Omega \{ \tau + \tau^3/12 + (a^2 + c^2 + 3b)\tau^5/720 \} \quad (85)$$

$$F = 2\xi + (1 + \xi^2)(\tau^2/2 + b\tau^4/24) \quad (86)$$

$$d^2\delta/d\xi dT = \Omega e^2/\pi \{ -\xi\pi + \int_0^\infty (d\tau/\tau) H \} \quad (87)$$

where (to second order in a and c, and first order in b and d)

$$H = \{ 2\xi + (1 + \xi^2)(\tau^2/2 + b\tau^4/24) \} \sin\theta_0 \\ + \{ (a^2 + c^2 + 3b)\Omega\tau^5/720 \} \{ 2\xi + (1 + \xi^2)\tau^2/2 \} \cos\theta_0 \quad (88)$$

where

$$\theta_0 = \Omega(\tau + \tau^3/12) \quad (89)$$

We evaluate this only for very small Ω (that is, in the extreme quantum case). Then only the highest powers of τ in θ_0 and in the polynomials in ( ) need be retained. Then

$$d^2\delta/d\xi dT = \Omega e^2/\pi \{ \int_0^\infty d\tau J \} \quad (90)$$

where

$$J = \{ (1 + \xi^2)(\tau/2 + b\tau^3/24) \} \sin(\Omega\tau^3/12) \\ + \{ (a^2 + c^2 + 3b)(\Omega\tau^6/1440)(1 + \xi^2) \} \cos(\Omega\tau^3/12) \quad (91)$$

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The integrals give Airy functions:

$$\int_0^\infty d\tau \tau \sin(\Omega\tau^3/12) = 4^{2/3}\Omega^{-2/3}\pi(-\text{Ai}'(0)) \quad (92)$$

$$\int_0^\infty d\tau \cos(\Omega\tau^3/12) = 4^{1/3}\Omega^{-1/3}\pi\text{Ai}(0) \quad (93)$$

Differentiating this last with respect to Ω

$$\int_0^\infty d\tau \tau^3 \sin(\Omega\tau^3/12) = 4^{4/3}\Omega^{-4/3}\pi\text{Ai}(0) \quad (94)$$

And differentiating again with respect to Ω

$$\int_0^\infty d\tau \tau^6 \cos(\Omega\tau^3/12) = -4^{10/3}\Omega^{-7/3}\pi\text{Ai}(0) \quad (95)$$

So

$$d^2\delta/d\xi dT = e^2(-\text{Ai}'(0))4^{2/3}(1+\xi^2) \{\Omega^{1/3} + \Omega^{-1/3}\kappa(-4a^2 - 4c^2 + 3b)\}/2 \quad (96)$$

where

$$\kappa = \{\text{Ai}(0)/(-\text{Ai}'(0))\}4^{2/3}/180 = .0192029 \quad (97)$$

To obtain (for example) the mean energy loss per unit T , the energy-loss-weighted distribution (96) has to be integrated over ξ . This involves the integrals

$$\int_0^1 d\xi \{(1-\xi)/(2\xi)\}^{1/3}(1+\xi^2) = \pi 2^{-1/3}3^{-1/2}64/81 \quad (98)$$

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$$\int_0^1 d\xi \{(2\xi)/(1-\xi)\}^{1/3} (1+\xi^2) = \pi 2^{1/3} 3^{-1/2} 82/81 \quad (99)$$

Then one finds that the effect of the nonuniformity is to multiply the result (73) for the uniform case by a factor

$$1 + r^{2/3} (3b - 4a^2 - 4c^2) \kappa' \quad (100)$$

where

$$\kappa' = \{Ai(0)/(-Ai'(0))\} 41/1440 = .039056 \quad (101)$$

When the direction of the deflection is constant, ($c=0$), this is equivalent to (5).

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18. It suffices that this be so over distances required for the
convergence of the integral, of order L_Q in the ultra quantum case.