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THE BACKGROUND-QUANTUM SPLIT SYMMETRY IN
TWO-DIMENSIONAL σ -MODELS, A REGULARIZATION
INDEPENDENT PROOF OF ITS RENORMALIZABILITY

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ABSTRACT

A generic, non-linear, background-quantum split is translated into a B.R.S. symmetry. The renormalization of the resulting Slavnov-Taylor identity is analyzed in the class of two-dimensional σ -models with Wess-Zumino term which suggests the adoption of a regularization independent method. We discuss the cohomology of the linearized nilpotent operator derived from the Slavnov-Taylor identity. In particular, the cohomology class with zero Faddeev-Popov charge ensures the stability of the action, while the fact that the cohomology class with one unit of Faddeev-Popov charge is empty ensures the absence of anomalies.

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1. Introduction

The background field method as an effective computational tool has a long history in quantum field theory. The first attempt to derive the method within the Feynman path-integral approach is due to Jackiw [1] who employed a constant background field to obtain the effective potential of a scalar theory.

Later, its usefulness became particularly evident in gauge theories, since it allows to compute the quantum corrections in a way which manifestly respects the symmetry of the problem [2].

More recently, the method has been widely used to perform perturbative calculations in two-dimensional non-linear σ models [3] where, in order to maintain the covariance of the theory, it is necessary to perform a background-quantum split which is non-linear.

This circumstance, coupled with the fact that in two dimensions the quantized field is not multiplicatively renormalized [4], has brought about the necessity of reconsidering the method. In particular, Howe, Papadopoulos and Stelle [5] have proposed a formulation which properly takes into account the non-multiplicative renormalization properties of the quantum fields while respecting the manifest covariance and the computational request of considering diagrams with only background field external legs. The analysis of these authors is presented for a purely bosonic non-linear σ model without Wess-Zumino term and it is centred on the validity of a non-linear Ward identity which describes classically the background-quantum split and which is to be implemented at the quantum level by the B.R.S. technique. In this framework the use of dimensional regularization is sufficient to ensure the validity of the B.R.S. identity.

However the dimensional regularization procedure cannot be straightforwardly applied to the case where a Wess-Zumino term is present or to supersymmetric models. For these reasons it seemed to us necessary to reanalyze the background-quantum split identity in a more general context which assumes no regularization. In our treatment, we shall also translate the classical non-linear Ward operator into a B.R.S. operator, so that the stability and renormalizability properties can be reduced to the identification of the cohomology spaces in the null and charge-one Faddeev-Popov sectors respectively. The results are the expected ones if the background-quantum split is locally defined. Indeed in such a case we shall prove that the model is anomaly free and satisfies a stability condition, where stability

means that the cohomologically non-trivial counterterms correspond to a redefinition of the metric and torsion potential. Moreover, counterterms corresponding to a redefinition of the total field are cohomologically trivial. Let us emphasize that although in this paper we shall be concerned with a generic non-linear background-quantum split, the actual way in which the split is to be performed is dictated by the necessity of maintaining manifest covariance and possible isometries of the model; we shall see that these requirements are compatible with our identity, but they do not explicitly appear in the identity itself.

We shall also not be concerned with the possible infrared pathologies of the model, our aim being that of analyzing the ultraviolet renormalizability of the split symmetry. For this reason we shall follow the approach of ref.[5], and not introduce explicitly a mass term in the action. Of course such a mass term is usually included as an infrared regulator ; however, in a minimal renormalization scheme [6], the breaking to the split symmetry introduced by the mass term remains soft.

2. The Model

Let Ξ be the two-dimensional space-time and M a Riemannian n -dimensional manifold with metric g . The field of the σ -model is a map :

$$\Phi : \Xi \rightarrow M \quad (2.1)$$

The action of the model, including the Wess-Zumino term, reads :

$$\begin{aligned} S(\Phi) = & \frac{1}{2} \int d^2x g_{ij}(\Phi) (\partial^\mu \Phi^i) (\partial_\mu \Phi^j) \\ & + \frac{1}{2} \int d^2x b_{ij}(\Phi) \epsilon^{\mu\nu} (\partial_\mu \Phi^i) (\partial_\nu \Phi^j) \end{aligned} \quad (2.2)$$

where b_{ij} is skewsymmetric, and its curl defines a torsion on M [7]. A specification of the functions $g_{ij}(\Phi)$ and $b_{ij}(\Phi)$ identifies the model up to a Φ^i field diffeomorphism. The background-quantum split corresponds to a (arbitrary) separation of the field Φ^i as :

$$\Phi^i(x) = \varphi^i(x) + \chi^i(\varphi, \xi) \quad (2.3)$$

where $\varphi^i(x)$ is the classical background part and $\xi^i(x)$ is taken as quantum field.

The functions $\chi^i(\varphi, \xi)$ are formal power series in ξ^i :

$$\chi^i(\varphi, \xi) = \xi^i + \sum_{n=2}^{\infty} \frac{1}{n!} a_{j_1 \dots j_n}^i(\varphi) \xi^{j_1} \dots \xi^{j_n} \quad (2.4)$$

where the coefficients $a_{j_1 \dots j_n}^i$ are functions of φ .

One has to note that the usual covariant expansion based on the geodesic normal coordinates [5] is a particular case of (2.4). In this case, the functions $a_{j_1 \dots j_n}^i(\varphi)$ are the appropriate covariant derivatives of the Riemann connection $\Gamma_{jk}^i(\varphi)$ and the quantum field ξ^i becomes a vector on M . As shown by Howe, Papadopoulos and Stelle [5], the non-linear splitting (2.3) induces a symmetry for the action $S(\Phi)$. The symmetry derives from the possibility of varying the fields φ^i and ξ^i in a way such that the total variation of Φ^i vanishes. Let us

consider, indeed, the local transformations :

$$\begin{cases} \delta_\eta \varphi^i(x) = \eta^i(x) \\ \delta_\eta \xi^i(x) = F_j^i(\varphi, \xi) \eta^j(x) \end{cases} \quad (2.5)$$

where F_j^i is determined by the requirement that Φ^i be invariant, *i.e.*:

$$\frac{\partial \chi^i}{\partial \xi^j} F_k^j + \frac{\partial \chi^i}{\partial \varphi^k} + \delta_k^i = 0 \quad (2.6)$$

The equation (2.6) implies that :

$$\left(\frac{\partial F_k^i}{\partial \varphi^j} - \frac{\partial F_j^i}{\partial \varphi^k} \right) + \left(\frac{\partial F_k^i}{\partial \xi^l} F_j^l - \frac{\partial F_j^i}{\partial \xi^l} F_k^l \right) = 0 \quad (2.7)$$

and, by (2.4) :

$$F_k^i(\varphi, 0) = -\delta_k^i \quad (2.8)$$

The condition (2.7) shows that the transformation (2.5) is Abelian. Notice that a quantum field redefinition of the type :

$$\xi^i \rightarrow \xi^i + \epsilon R^i(\varphi, \xi) \quad (2.9)$$

is compatible with the invariance of the field :

$$\Phi^i = \Phi^i + \epsilon \frac{\partial \chi^i}{\partial \xi^k} R^k \quad (2.10)$$

provided we also redefine the transformation law of the quantum fields so that it satisfies to first order in ϵ eq.(2.6), *i.e.* :

$$F_j^i \rightarrow F_j^i + \epsilon \left(\frac{\partial F_j^i}{\partial \xi^k} R^k - \frac{\partial R^i}{\partial \xi^k} F_j^k - \frac{\partial R^i}{\partial \varphi^j} \right) \quad (2.11)$$

The above F_j^i also obey condition (2.7) which denotes that the new split transformations remain Abelian. We shall see that redefinitions of the type given in (2.9) are exactly of the kind induced by radiative corrections ; accordingly we shall say that the perturbations

under which the split symmetry is stable are those given in (2.9) when accompanied by a redefinition as in (2.11). From $\delta_\eta \Phi^i = 0$ it follows that :

$$\delta_\eta S(\Phi) = 0 \quad (2.12)$$

The transformations in (2.5) being non-linear, we need to translate them into a B.R.S. identity in order to discuss their quantum implementability ; this is easily achieved introducing the anticommuting dimensionless ghost fields $c^i(x)$ and the operators defined by :

$$\begin{cases} s\varphi^i(x) = c^i(x) \\ s\xi^i(x) = F_j^i(\varphi, \xi)c^j(x) \\ sc^i(x) = 0 \end{cases} \quad (2.13)$$

which, due to (2.7) is nilpotent. By adding to the action $S(\Phi)$ a source term :

$$\Sigma = S(\Phi) + \int d^2x \gamma_i F_k^i c^k \quad (2.14)$$

one has the Slavnov-Taylor identity :

$$\int d^2x \left(c^i \frac{\delta \Sigma}{\delta \varphi^i} + \frac{\delta \Sigma}{\delta \gamma_i} \frac{\delta \Sigma}{\delta \xi^i} \right) = 0 \quad (2.15)$$

The sources γ_i have canonical dimension two, and c^i have a positive unit of Faddeev-Popov charge, so that γ_i carry a negative unit of Faddeev-Popov charge. The quantum implementability of equation (2.15) is ensured if we can prove that the action is stable and the symmetry is anomaly free. In particular, the stability requirement implies that the classical invariant action is, modulo a diffeomorphism, the general solution of the split symmetry and that the B.R.S. symmetric counterterms induced by the renormalization procedure correspond, according to our previous remark, to a quantum field renormalization and transformation law redefinition as in (2.9), (2.11).

The use of the B.R.S. technique allows us to discuss both the stability and the anomaly problems by analyzing the cohomological properties of the linearized nilpotent Slavnov-

Taylor operator D_Σ :

$$\begin{cases} D_\Sigma = \int d^2x \left(c^i \frac{\delta}{\delta \varphi^i} + \frac{\delta \Sigma}{\delta \gamma_i} \frac{\delta}{\delta \xi^i} + \frac{\delta \Sigma}{\delta \xi^i} \frac{\delta}{\delta \gamma_i} \right) \\ D_\Sigma D_\Sigma = 0 \end{cases} \quad (2.16)$$

in the space of the integrated dimension two local functionals of the variables ξ^i , c^i , φ^i , γ_i and limited to the sector with zero Faddeev-Popov charge (stability) or with a positive unit of Faddeev-Popov charge (anomaly).

3. Cohomology of D_Σ

As a functional differential operator D_Σ acts on a space whose elements are the two-dimensional integrated local functionals which depend on the variables ξ^i , φ^i , c^i , γ_i .

A generic element X of this space can be parametrized as :

$$X = \int d^2x \left(\gamma_i P^i + c^{\mu\nu} A_{\mu\nu} + \eta^{\mu\nu} B_{\mu\nu} \right) \quad (3.1)$$

where P^i is a function of c^i , ξ^i , φ^i and does not contain any space-time derivative, while $A_{\mu\nu}$ and $B_{\mu\nu}$ are functions of c^i , ξ^i , φ^i with two space-time derivatives whose Lorentz indices are contracted with the skewsymmetric tensor $c^{\mu\nu}$ for $A_{\mu\nu}$ and with the space-time flat metric $\eta^{\mu\nu}$ for $B_{\mu\nu}$. To study the cohomology of D_Σ we introduce the counting operator [8] :

$$N = 1 + \int d^2x \left(c^i \frac{\delta}{\delta c^i} + \xi^i \frac{\delta}{\delta \xi^i} - \gamma_i \frac{\delta}{\delta \gamma_i} \right) \quad (3.2)$$

N decomposes the linear space X into subspaces $X^{(\nu)}$ according to its eigenvalues $\nu = 0, 1, 2, \dots$; and induces a separation in D_Σ

$$D_\Sigma = \sum_{n=0}^{\infty} D_\Sigma^{(n)} \quad (3.3)$$

with :

$$\left[N, D_\Sigma^{(n)} \right] = n D_\Sigma^{(n)} \quad (3.4)$$

From (2.16) $D_\Sigma^{(0)}$ and $D_\Sigma^{(1)}$ are given by :

$$\begin{cases} D_\Sigma^{(0)} = - \int d^2x c^i \frac{\delta}{\delta \xi^i} \\ D_\Sigma^{(1)} = \int d^2x \left(c^i \frac{\delta}{\delta \varphi^i} + c^k F_{kl}^i(\varphi) \xi^l \frac{\delta}{\delta \xi^i} + \frac{\delta S(\varphi)}{\delta \varphi^i} \frac{\delta}{\delta \gamma_i} - c^k F_{kl}^i(\varphi) \gamma_l \frac{\delta}{\delta \gamma_i} \right) \end{cases} \quad (3.5)$$

where :

$$F_{kl}^i(\varphi) = \frac{\partial F_{kl}^i(\varphi, \xi)}{\partial \xi^l} \Big|_{\xi=0}, \quad S(\varphi) = S(\Phi) \Big|_{\xi=0} \quad (3.6)$$

The nilpotency of D_Σ implies that :

$$\sum_{p=0}^n D_\Sigma^{(p)} D_\Sigma^{(n-p)} = 0 \quad n = 0, 1, \dots \quad (3.7)$$

from which it follows that $D_\Sigma^{(0)}$ is nilpotent :

$$D_\Sigma^{(0)} D_\Sigma^{(0)} = 0 \quad (3.8)$$

According to the general theory [9], the cohomology of D_Σ is isomorphic to a subspace of the cohomology of $D_\Sigma^{(0)}$, so the first step consists of the identification of the cohomology space of $D_\Sigma^{(0)}$.

3.1 FROM FUNCTIONALS TO FUNCTIONS

To discuss the cohomology of $D_\Sigma^{(0)}$, we shall first translate the action of $D_\Sigma^{(0)}$ on the space of local functionals to an equivalent action in the space of functions [8].

Recalling that in the space X we always have the freedom of integrating by parts, we shall use this freedom to eliminate in X and $D_\Sigma^{(0)}X$ all contributions with a two-dimensional d'Alembertian acting on any field variable. Denoting collectively with $\theta^\alpha(x) \equiv \{\xi^i, \varphi^i, c^i\}$ and taking into account the above constraint we find that $D_\Sigma^{(0)}$ in (3.5) induces in the space $X(x)$ of local functions of θ^α , γ_i :

$$X(x) = \{\gamma_i P^i(\theta) + G_{\alpha\beta}(\theta)(\partial_\mu \theta^\alpha)(\partial^\mu \theta^\beta) + H_{\alpha\beta}(\theta)\epsilon^{\mu\nu}(\partial_\mu \theta^\alpha)(\partial_\nu \theta^\beta)\} \quad (3.9)$$

the ordinary differential operator :

$$d^{(0)} = - \left(c^i \frac{\partial}{\partial \xi^i} + (\partial_\mu c^i) \frac{\partial}{\partial (\partial_\mu \xi^i)} \right) \quad (3.10)$$

Now the integrated cohomology equation $D_\Sigma^{(0)}X = 0$ becomes equivalent to :

$$d^{(0)}X(x) = \partial^\mu Y_\mu(x) \quad (3.11)$$

In order to discuss the solution of (3.11) we proceed as follows ; first we realize that $d^{(0)}$ does not mix terms with opposite parity, i.e. :

$$X_g(x) = \gamma_i P^i + G_{\alpha\beta}(\partial_\mu \theta^\alpha)(\partial^\mu \theta^\beta) \quad (3.12)$$

$$X_h(x) = H_{\alpha\beta}\epsilon^{\mu\nu}(\partial_\mu\theta^\alpha)(\partial_\nu\theta^\beta) \quad (3.13)$$

Analogously we can distinguish on the right-hand side of (3.11) :

$$Y_\mu = Y_{g\mu} + Y_{h\mu} \quad (3.14)$$

with :

$$Y_{g\mu} = Y_{g\alpha}(\theta)(\partial_\mu\theta^\alpha) \quad (3.15)$$

$$Y_{h\mu} = Y_{h\alpha}(\theta)\epsilon_{\mu\nu}(\partial^\nu\theta^\alpha) \quad (3.16)$$

and therefore eq. (3.11) is equivalent to the two equations :

$$d^{(0)}X_g(x) = \partial^\mu(Y_{g\alpha}\partial_\mu\theta^\alpha) \quad (3.17)$$

$$d^{(0)}X_h(x) = (\partial^\mu\theta^\beta)\frac{\partial Y_{h\alpha}}{\partial\theta^\beta}\epsilon_{\mu\nu}\partial^\nu\theta^\alpha \quad (3.18)$$

3.2 EVEN PARITY COHOMOLOGY OF $d^{(0)}$

Let us first examine (3.17) ; we see that the right-hand side always contains a term $\partial_\mu\partial^\mu\theta^\alpha$ which, due to the choice (3.12), does not appear on the left-hand side. Therefore (3.17) implies $Y_{g\alpha} = 0$ and

$$d^{(0)}X_g(x) = 0 \quad (3.19)$$

To solve (3.19) we recall that $D_\Sigma^{(0)}$, and hence $d^{(0)}$, commutes with the filtering operator N , so that the equation can be separately analyzed in each eigenspace $X_g^{(\nu)}(x)$ belonging to the eigenvalue ν of N . $X_g^{(\nu)}$ is a space of polynomials in the variables $(\xi^i, \partial_\mu\xi^i, c^i, \partial_\mu c^i, \gamma_i)$ with coefficients which are functions of the background field $\varphi(x)$; therefore we can embed it in a Fock space with respect to these variables.

This allows us to define an adjoint operator :

$$d^{(0)\dagger} = - \left(\xi^i \frac{\partial}{\partial c^i} + (\partial_\mu \xi^i) \frac{\partial}{\partial (\partial_\mu c^i)} \right) \quad (3.20)$$

so that the cohomology space of $d^{(0)}$ in each subspace $X_g^{(\nu)}(x)$ coincides with the kernel of

the Laplace-Beltrami operator :

$$\begin{aligned}\Delta &= \left\{ d^{(0)}, d^{(0)\dagger} \right\} \\ &= c^i \frac{\partial}{\partial c^i} + \xi^i \frac{\partial}{\partial \xi^i} + (\partial_\mu c^i) \frac{\partial}{\partial (\partial_\mu c^i)} + (\partial_\mu \xi^i) \frac{\partial}{\partial (\partial_\mu \xi^i)}\end{aligned}\tag{3.21}$$

It is therefore apparent that the cohomology space of $d^{(0)}$ does not contain either the variables c^i , ξ^i or their space-time derivatives $(\partial_\mu c^i)$, $(\partial_\mu \xi^i)$ and hence it can be identified with :

$$X_g^{(0)}(x) = \{ \omega_{ij}(\varphi) (\partial_\mu \varphi^i) (\partial^\mu \varphi^j) + \gamma_i \Lambda^i(\varphi) \}\tag{3.22}$$

where ω_{ij} and Λ^i are functions of the sole background field φ^i .

3.3 ODD PARITY COHOMOLOGY OF $d^{(0)}$

To analyse the remaining equation (3.18) it will be useful to employ the language of forms. To the generic element (3.13) of $X_h(x)$ we associate the space-time 2-form :

$$H(x) = dx^\mu \wedge dx^\nu H_{\alpha\beta}(\partial_\mu \theta^\alpha)(\partial_\nu \theta^\beta)\tag{3.23}$$

and to (3.16) the space-time 1-form :

$$Y_h(x) = dx^\mu Y_{h\alpha}(\partial_\mu \theta^\alpha)\tag{3.24}$$

Then $\tilde{d} = dx^\mu \frac{\partial}{\partial x^\mu}$ is the space-time exterior derivative and equation (3.18) becomes :

$$d^{(0)} H(x) = \tilde{d} Y_h(x)\tag{3.25}$$

We also introduce the "ghosts" :

$$\tilde{\theta}^\alpha = (\partial_\mu \theta^\alpha) dx^\mu\tag{3.26}$$

which will enable us to write the space-time exterior derivative $\tilde{d}$ as a nilpotent differential operator which anticommutes with $d^{(0)}$. In order to achieve this, it is necessary to specify

the commutation rule of dx^μ with the B.R.S. ghosts $c^i(x)$; in the following, it will be useful to choose them to anticommute, i.e. :

$$dx^\mu c^i = -c^i dx^\mu \quad (3.27)$$

From (3.27), we deduce the commutation relations :

$$\begin{cases} \theta^\alpha \bar{\theta}^\beta = (-1)^{N_g(\theta^\alpha)(N_g(\theta^\beta)+1)} \bar{\theta}^\beta \theta^\alpha \\ \bar{\theta}^\alpha \bar{\theta}^\beta = (-1)^{(N_g(\theta^\alpha)+1)(N_g(\theta^\beta)+1)} \bar{\theta}^\beta \bar{\theta}^\alpha \end{cases} \quad (3.28)$$

where $N_g(\theta^\alpha)$ is the Faddeev-Popov charge of θ^α . With these notations, we have :

$$\begin{cases} H(x) = \tilde{H}_{\alpha\beta} \bar{\theta}^\alpha \bar{\theta}^\beta \\ Y_h(x) = \hat{Y}_{h\alpha} \bar{\theta}^\alpha \end{cases} \quad (3.29)$$

where $\tilde{H}_{\alpha\beta}$ and $\hat{Y}_{h\alpha}$ differ from $h_{\alpha\beta}$ and $y_{h\alpha}$ at most by a sign, and :

$$\begin{cases} \tilde{d} = (-1)^{N_g(\theta^\alpha)} \bar{\theta}^\alpha \frac{\partial}{\partial \theta^\alpha} \\ d^{(0)} = - \left(c^i \frac{\partial}{\partial \xi^i} + \bar{c}^i \frac{\partial}{\partial \bar{\xi}^i} \right) \end{cases} \quad (3.30)$$

so that $\tilde{d}$ is also a nilpotent operator. In this language, the filtering operator N is rewritten as :

$$N = 1 + \xi^i \frac{\partial}{\partial \xi^i} + c^i \frac{\partial}{\partial c^i} + \bar{c}^i \frac{\partial}{\partial \bar{c}^i} + \xi^i \frac{\partial}{\partial \xi^i} - \gamma_i \frac{\partial}{\partial \gamma_i} \quad (3.31)$$

We can now repeat the Fock space construction and define the adjoint operator :

$$d^{(0)\dagger} = - \left(\xi^i \frac{\partial}{\partial c^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{c}^i} \right) \quad (3.32)$$

and the Laplace-Beltrami operator :

$$\Delta = \left\{ d^{(0)}, d^{(0)\dagger} \right\} = c^i \frac{\partial}{\partial c^i} + \xi^i \frac{\partial}{\partial \xi^i} + \bar{c}^i \frac{\partial}{\partial \bar{c}^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{\xi}^i} \quad (3.33)$$

These operators obey the algebra :

$$\begin{cases} \left\{ d^{(0)}, \tilde{d} \right\} = 0 \\ \left\{ d^{(0)\dagger}, \tilde{d} \right\} = 0 \end{cases} \quad (3.34)$$

In eq. (3.25), $\tilde{d}Y_h(x)$ belongs to the kernel of $d^{(0)}$ and is orthogonal to the kernel of

$d^{(0)\dagger}$, hence a particular solution for Π is :

$$\hat{H} = d^{(0)\dagger} \Delta^{-1} \tilde{d} Y_h = \tilde{d} X_h \quad (3.35)$$

with $X_h = -\Delta^{-1} d^{(0)\dagger} Y_h$. Then eq. (3.25) becomes :

$$d^{(0)}(\Pi - \hat{H}) = 0 \quad (3.36)$$

and the solution is :

$$H(X) = h^{(0)} + d^{(0)}Z + \tilde{d}X_h \quad (3.37)$$

where $h^{(0)}$ belongs to the kernel of the Laplace-Beltrami operator Δ :

$$h^{(0)} = t_{ij}(\varphi) \bar{\varphi}^i \varphi^j \quad (3.38)$$

Care should be taken, however, that two antisymmetric tensors t_{ij} and t'_{ij} differing by a curl :

$$t_{ij} - t'_{ij} = \frac{\partial b_j}{\partial \varphi^i} - \frac{\partial b_i}{\partial \varphi^j} \quad (3.39)$$

identify the same cohomology class of $d^{(0)}$ modulo $\tilde{d}$, i.e. :

$$h^{(0)} - h'^{(0)} = \tilde{d}(b_j \varphi^j) \quad (3.40)$$

3.4 COHOMOLOGY OF $D_{\Sigma}^{(0)}$

If we retranslate the results in (3.22),(3.37) to the integrated functionals we can conclude that the cohomology of $D_{\Sigma}^{(0)}$ is spanned by :

$$\delta = \int d^2x (\omega_{ij}(\varphi) (\partial_\mu \varphi^i) (\partial^\mu \varphi^j) + \epsilon^{\mu\nu} t_{ij}(\varphi) (\partial_\mu \varphi^i) (\partial_\nu \varphi^j) + \gamma_i \Lambda^i(\varphi)) \quad (3.41)$$

One sees that the cohomological space of $D_{\Sigma}^{(0)}$ decomposes in two subspaces according to the eigenvalues $\nu = 0$ and $\nu = 1$ of N , i.e. :

$$\delta = \delta^{(0)} + \delta^{(1)} \quad (3.42)$$

with :

$$\delta^{(0)} = \int d^2x \gamma_i \Lambda^i(\varphi) \quad (3.43)$$

$$\delta^{(1)} = \int d^2x (\omega_{ij}(\partial_\mu \varphi^i)(\partial^\mu \varphi^j) + \epsilon^{\mu\nu} t_{ij}(\partial_\mu \varphi^i)(\partial_\nu \varphi^j)) \quad (3.44)$$

3.5 COHOMOLOGY OF D_Σ

Having determined the cohomology space of $D_\Sigma^{(0)}$, we now turn to the study of D_Σ . We first expose the method and the results obtained [8]. The first task is to determine the general form of a D_Σ -cocycle. We shall show iteratively that any solution of the equation $D_\Sigma X = 0$ can be written as :

$$X = l + D_\Sigma \hat{X} \quad (3.45)$$

where $l = \delta^{(1)} + \sum_{n=2}^{\infty} l^{(n)}$ is a D_Σ -cocycle with zero ghost number uniquely determined, modulo a D_Σ -coboundary, by $\delta^{(1)}$. It is easily shown that, $\delta^{(1)}$ being given by (3.44), a possible form for l is :

$$l = \int d^2x (\omega_{ij}(\Phi)(\partial_\mu \Phi^i)(\partial^\mu \Phi^j) + \epsilon^{\mu\nu} t_{ij}(\Phi)(\partial_\mu \Phi^i)(\partial_\nu \Phi^j)) \quad (3.46)$$

It corresponds to an arbitrary redefinition of the metric and torsion potential. It is already clear from (3.45), (3.46) that the split symmetry has no anomalies, since its cohomology does not contain Faddeev-Popov ghosts c^i . What remains to be checked is the condition for two D_Σ -cocycles $l = \delta^{(1)} + \sum_{n=2}^{\infty} l^{(n)}$ and $l' = \delta'^{(1)} + \sum_{n=2}^{\infty} l'^{(n)}$ to belong to the same cohomology class of D_Σ . It will be shown that the condition is that $\delta^{(1)}$ and $\delta'^{(1)}$ differ by a redefinition of the classical field, i.e.:

$$\delta'^{(1)} - \delta^{(1)} = \int d^2x \frac{\delta S(\varphi)}{\delta \varphi^i(x)} v^i(\varphi) \quad (3.47)$$

for arbitrary vector-valued functions $v^i(\varphi)$.

In the language of the total field Φ^i , the transformation (3.47) corresponds to a redefinition

$$\Phi^i \rightarrow \Phi^i + v^i(\Phi) \quad (3.48)$$

Thus the functions $\omega_{ij}(\Phi)$ and $t_{ij}(\Phi)$ identify a cohomology class of D_Σ only up to a field redefinition.*

* This is the ambiguity, already discussed in the work of H.P.S. [5] of those counterterms which can be described either as field redefinitions, or as metric and torsion renormalizations.

This exactly matches the situation for the classical action $S(\Phi)$ in (2.2) which is fixed, modulo a Φ^i diffeomorphism, by the functions $g_{ij}(\Phi)$ and $b_{ij}(\Phi)$. Notice that no terms proportional to the external sources $\gamma_i(x)$ appear in the cohomology of D_Σ , which implies that the corresponding counterterms are generated by a D_Σ -variation.

$$D_\Sigma \int d^2x \gamma_i R^i(\varphi, \xi) = \int d^2x \frac{\delta S(\Phi)}{\delta \xi^i} R^i + \int d^2x \gamma_k \left(\frac{\partial F_j^k}{\partial \xi^i} R^i - F_j^i \frac{\partial R^k}{\partial \xi^i} - \frac{\partial R^k}{\partial \varphi^j} \right) \epsilon^j \quad (3.49)$$

They obviously correspond to a renormalization of the quantum field :

$$\xi^i \rightarrow \xi^i + R^i(\varphi, \xi) \quad (3.50)$$

together with a redefinition of the split transformation F_j^i according to :

$$F_j^i \rightarrow F_j^i - \frac{\partial R^i}{\partial \xi^l} F_j^l + \frac{\partial F_j^i}{\partial \xi^l} R^l - \frac{\partial R^i}{\partial \varphi^j} \quad (3.51)$$

One has to note that the terms in (3.51), identical to those appearing in (2.11), cannot be viewed as γ_i sources and ξ^i quantum field renormalizations ; in particular, while the first two terms on the right-hand side of (3.51) can be reabsorbed by a (φ, ξ) dependent renormalization of the sources γ_i and of the fields ξ^i , the last one cannot. According to the discussion in Section 2, they are to be interpreted as a change in the split symmetry[†] within the allowed class of choices, i.e. the ones which are Abelian and keep the redefined total field invariant.

Let us now turn to the proof of eqs. (3.45),(3.47). Decomposing X according to the eigenvalues of the operator N , i.e. :

$$X = \sum_{n=0}^{\infty} X^{(n)} \quad (3.52)$$

and using (3.3) one has the equations :

$$\sum_{p=0}^n D_\Sigma^{(p)} X^{(n-p)} = 0 \quad (3.53)$$

[†] This is similar to what happens in the renormalization of non-semi-simple gauge theories, where the radiative corrections induce a change in the representation of the Abelian factors.

We proceed inductively on the index n ; at the level $n = 0$:

$$D_{\Sigma}^{(0)} X^{(0)} = 0 \quad (3.54)$$

whose general solution is :

$$X^{(0)} = \delta^{(0)} + D_{\Sigma}^{(0)} \hat{X}^{(0)} \quad (3.55)$$

Next at $n=1$:

$$D_{\Sigma}^{(0)} X^{(1)} + D_{\Sigma}^{(1)} X^{(0)} = 0 \quad (3.56)$$

which, by (3.55) and (3.7), gives :

$$D_{\Sigma}^{(0)} (X^{(1)} - D_{\Sigma}^{(1)} \hat{X}^{(0)}) + D_{\Sigma}^{(1)} \delta^{(0)} = 0 \quad (3.57)$$

We compute the last term on the left-hand side by (3.5), (3.43) to get :

$$D_{\Sigma}^{(1)} \delta^{(0)} + D_{\Sigma}^{(0)} \int d^2 x \xi^i \left(\frac{\partial \Lambda^i}{\partial \varphi^i} - F_{ik}^i(\varphi) \Lambda^k \right) \gamma_l = \int d^2 x \frac{\delta S(\varphi)}{\delta \varphi^i} \Lambda^i(\varphi) \quad (3.58)$$

and notice that the right-hand side belongs to the cohomology space of $D_{\Sigma}^{(0)}$ and thus (3.57) is inconsistent unless :

$$\int d^2 x \frac{\delta S(\varphi)}{\delta \varphi^i} \Lambda^i(\varphi) = 0 \quad (3.59)$$

Equation (3.59) expresses the fact that the action $S(\varphi)$ possesses the invariance $\varphi^i \rightarrow \varphi^i + \Lambda^i(\varphi)$. In such a case, we have to use a modified B.R.S. operator which properly takes into account the isometries and which yields a quite different renormalization analysis [8]. Therefore we consider a generic model with no additional symmetry, so that (3.59) implies $\Lambda^i(\varphi) = 0$. It then follows that the general solution of (3.57) is :

$$X^{(1)} = \delta^{(1)} + D_{\Sigma}^{(1)} \hat{X}^{(0)} + D_{\Sigma}^{(0)} \hat{X}^{(1)} \quad (3.60)$$

One more step at $n = 2$; we find :

$$D_{\Sigma}^{(0)} X^{(2)} + D_{\Sigma}^{(1)} X^{(1)} + D_{\Sigma}^{(2)} X^{(0)} = 0 \quad (3.61)$$

which by (3.55) with $\delta^{(0)} = 0$ and (3.60) becomes :

$$D_{\Sigma}^{(0)} (X^{(2)} - D_{\Sigma}^{(1)} \hat{X}^{(1)} - D_{\Sigma}^{(2)} \hat{X}^{(0)}) + D_{\Sigma}^{(1)} \delta^{(1)} = 0 \quad (3.62)$$

Now, $\delta^{(1)}$ being a $D_{\Sigma}^{(0)}$ -cocycle, $D_{\Sigma}^{(1)} \delta^{(1)}$ is a $D_{\Sigma}^{(0)}$ -cocycle with filtration number equal to 2.

However, the cohomology space of $D_\Sigma^{(0)}$ with filtration number equal to 2 is empty, and thus there exists $l^{(2)}$ such that :

$$D_\Sigma^{(1)}\delta^{(1)} + D_\Sigma^{(0)}l^{(2)} = 0 \quad (3.63)$$

The procedure clearly iterates for $n > 2$ exactly as in the $n = 2$ case, using the emptiness of the cohomology spaces of $D_\Sigma^{(0)}$ for $n \geq 2$, and leads to the result in (3.45). Concerning the equation (3.47) let us look at the condition that l in (3.45) be a D_Σ -coboundary :

$$l = D_\Sigma h \quad (3.64)$$

Again we decompose this equation with respect to the filtration number and proceed inductively. At $n = 0$ we have :

$$0 = D_\Sigma^{(0)}h^{(0)} \quad (3.65)$$

with general solution :

$$h^{(0)} = \int d^2x \gamma_i v^i(\varphi) \quad (3.66)$$

Next, at $n = 1$ we find :

$$\delta^{(1)} = D_\Sigma^{(1)}h^{(0)} + D_\Sigma^{(0)}h^{(1)} \quad (3.67)$$

which, by (3.5) yields :

$$\delta^{(1)} = \int d^2x \frac{\delta S(\varphi)}{\delta \varphi^i} v^i(\varphi) = D_\Sigma^{(0)}h^{(1)} = D_\Sigma^{(0)} \int d^2x \xi^i \left(\frac{\partial v^l}{\partial \varphi^i} - F_{ik}^l(\varphi) v^k \right) \gamma_l \quad (3.68)$$

Since the left-hand side belongs to the cohomology space of $D_\Sigma^{(0)}$, while the right-hand side is a coboundary, we obtain the condition :

$$\delta^{(1)} = \int d^2x \frac{\delta S(\varphi)}{\delta \varphi^i} v^i(\varphi) \quad (3.69)$$

For $h^{(1)}$ in (3.68) we can choose the solution :

$$h^{(1)} = \int d^2x \xi^i \left(\frac{\partial v^l}{\partial \varphi^i} - F_{ik}^l(\varphi) v^k \right) \gamma_l \quad (3.70)$$

A further step at $n = 2$ gives :

$$l^{(2)} - (D_{\Sigma}^{(2)}h^{(0)} + D_{\Sigma}^{(1)}h^{(1)}) = D_{\Sigma}^{(0)}h^{(2)} \quad (3.71)$$

Let us show that the left-hand side is a $D_{\Sigma}^{(0)}$ -cocycle. Indeed :

$$D_{\Sigma}^{(0)}(l^{(2)} - D_{\Sigma}^{(2)}h^{(0)} - D_{\Sigma}^{(1)}h^{(1)}) = D_{\Sigma}^{(0)}l^{(2)} + D_{\Sigma}^{(2)}D_{\Sigma}^{(0)}h^{(0)} + D_{\Sigma}^{(1)}(D_{\Sigma}^{(1)}h^{(0)} + D_{\Sigma}^{(0)}h^{(1)}) \quad (3.72)$$

which, by (3.65) and (3.67), becomes :

$$D_{\Sigma}^{(0)}(l^{(2)} - D_{\Sigma}^{(2)}h^{(0)} - D_{\Sigma}^{(1)}h^{(1)}) = D_{\Sigma}^{(0)}l^{(2)} + D_{\Sigma}^{(1)}\delta^{(1)} \quad (3.73)$$

This vanishes because, by definition, l is a D_{Σ} -cocycle. Therefore the left-hand side of (3.71) is a $D_{\Sigma}^{(0)}$ -cocycle with filtration number 2, which ensures the possibility of writing it as $D_{\Sigma}^{(0)}h^{(2)}$.

Again, the levels $n > 2$ can be analyzed in the same way and the only condition for l to be a D_{Σ} -coboundary is (3.69), thereby concluding our proof.

4. Conclusion

We have analyzed in detail and in a regularization-independent way the ultraviolet renormalizability of the background-quantum split symmetry. In our treatment the possible renormalizations of the quantum field $\xi^i(x)$ are clearly functions of the quantum and the background fields ; indeed they are explicitly given in the form of a trivial cocycle which can always be added to the classical action. The stability analysis of the symmetry shows that the classical action is stable with respect to what we might call a choice of physical parameters, *i.e.* the (infinite) ones needed to identify the metric and the torsion potential modulo a field diffeomorphism [10]. On the other hand the split symmetry representation acquires counterterms which cannot be generated by a γ_i source redefinition ; they do not play any role in the stability analysis of the action and are dictated by the B.R.S. identity itself.

The symmetry, as expected, is anomaly free. Since we have not relied on any regularization, our treatment should apply equally well to supersymmetric models.

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